IDHAYA COLLEGE FOR WOMEN, KUMBAKONAM.

DEPARTMENT OF MATHEMATICS

CLASS	**: II M.Sc., MATHS**
SUBJECT NAME	**: ADVANCED NUMERICAL ANALYSIS**
SUBJECT CODE	**: P16MA43**
SEMESTER	**: IV**
UNIT	**: V (SINGLESTEP METHOD, RK METHOD)**
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UNIT V

1. Singlestep method:

The methods for the solution of the initial value problem.

$$u' = f(t,u),\ u(t_0)=\eta_0,\ t\in[t_0,b] \text{ -------------- } (1)$$

can be classified mainly into two types.

They are (i) singlestep methods and (ii) multistep methods.

In singlestep methods, the solution at any point is obtained using the solution at only the previous point.

Thus, a general singlestep method can be written as,

$$u_{j+1} = u_j + h\phi(t_{j+1}\ ,t_j\ ,u_{j+1}, u_j\ ,h \quad \text{ -------------- } (2)$$

where ϕ is a function of the arguments $t_j\ ,t_{j+1}\ ,u_j\ ,u_{j+1}, h$ and also depends of f .

We often write it as $\phi(t,u,h)$. This function ϕ is called the ***increment function***.

If u_{j+1} can be obtained simply by evaluating the right hand side of (2), then the method is called an ***explicit method***.

In this case, the method is of the form

$$u_{j+1} = u_j + h\phi(t_j, u_j, h) \quad \text{ ------------------------ } (3)$$

If the right hand side of (2) depends on u_{j+1} also, then it is called an ***implicit method***.

The general form in this case is as given in (2).

2. Discretization error or Local truncation error:

The true (exact) value $u(t_j)$ satisfies the equation.

$$u(t_{j+1}) = u(t_j) + h\phi(t_{j+1}\ ,t_j\ ,u(t_{j+1})\ ,u(t_j)\ ,h) + T_{j+1}$$

Where T_{j+1} is called the local truncation error or discretization error of the method. The truncation error is given by $T_{j+1} = u(t_{j+1}) - u(t_j) - h\phi(t_{j+1}\ ,t_j\ ,u(t_{j+1})\ ,u(t_j)\ ,h)$.

3. Order of a method:

The order of a method is the largest integer p for which

$$\left|\frac{1}{h}T_{j+1}\right| = O(h^p)$$

We now derive singlestep methods which have different increment functions.

4. The 2nd order implicit R.K formula:

$u_{j+1} = u_j + K_1$ where $K_1 = hf\left(t_j + \frac{1}{2}h\,, u_j + \frac{1}{2}K_1\right).$

5. Lotkin's bounds:

$$\left|\frac{\partial^{i+j} f}{\partial t^i \partial u^j}\right| < \frac{L^{i+j}}{M^{j-1}}\,,\ i,\, j = 0,1,2,......;\ \left|T_{j+1}\right| < ML^2h^3\left[4\left|\frac{1}{6} - \frac{c_2}{4}\right| + \frac{1}{3}\right].$$

1. Give the initial value problem $u' = t^2 + u^2$, $u(0) = 0$. Determine the first three non-zero terms in the Taylor series for $u(t)$ and hence obtain the value for $u(1)$.

Solution:

Given: $u' = t^2 + u^2$, where, $u(0) = 0$, $t_0 = 0$, $u_0 = 0$

$u' = t_0^2 + u_0^2 = (0)^2 + (0)^2 = 0,$

$u'' = 2t_0 + 2u_0u_0' = 2(0) + 2(0)(0) = 0,$

$u''' = 2 + 2(u_0u_0'' + u_0'u_0') = 2 + 0 = 2,$

$u^4 = 2(u_0u_0''' + u_0'u_0'' + 2u_0'u_0'') = 2(u_0u_0''' + 3u_0'u_0'') = 0,$

$u^5 = 2(u_0u_0^4 + u_0'u_0''' + 3u_0'u_0''' + 3u_0''u_0'') = 2(u_0u_0^4 + 4u_0'u_0''' + 3u_0''u_0'') = 0,$

$u^6 = 2(u_0u_0^5 + u_0'u_0^4 + 4u_0'u_0^4 + 4u_0''u_0''' + 6u_0''u_0''') = 2(u_0u_0^4 + 4u_0'u_0''' + 3u_0''u_0'') = 0,$

$u^7 = 2(u_0u_0^6 + u_0'u_0^5 + 5u_0'u_0^5 + 5u_0''u_0^4 + 10u_0''u_0^4 + 10u_0'''u_0''')$

$\quad = 2(u_0u_0^6 + 6u_0'u_0^5 + 15u_0''u_0^4 + 10u_0'''u_0''') = 2(40) = 80,$

$u^8 = 2(u_0u_0^7 + u_0'u_0^6 + 6u_0'u_0^6 + 6u_0''u_0^5 + 15u_0''u_0^5 + 15u_0'''u_0^4 + 10u_0'u_0^4)$

$\quad = 2(u_0u_0^7 + 7u_0'u_0^6 + 21u_0''u_0^5 + 35u_0'''u_0^4) = 0,$

$u^9 = 2(u_0u_0^8 + u_0'u_0^7 + 7u_0'u_0^7 + 7u_0''u_0^6 + 21u_0''u_0^6 + 21u_0'''u_0^5 + 35u_0'''u_0^5 + 35u_0^4u_0^4)$

$\quad = 2(u_0u_0^8 + 8u_0'u_0^7 + 28u_0''u_0^6 + 56u_0'''u_0^5 + 35u_0^4u_0^4) = 0,$

$u^{10} = 2(u_0u_0^9 + u_0'u_0^8 + 8u_0'u_0^8 + 8u_0''u_0^7 + 28u_0''u_0^7 + 28u_0'''u_0^6 + 56u_0'''u_0^6 + 56u_0^4u_0^5 + 70u_0^4u_0^5)$

$\quad = 2(u_0u_0^9 + 9u_0'u_0^8 + 36u_0''u_0^7 + 84u_0'''u_0^6 + 126u_0^4u_0^5) = 0,$

$u^{11} = 2(u_0u_0^{10} + u_0'u_0^9 + 9u_0'u_0^9 + 9u_0''u_0^8 + 36u_0''u_0^8 + 36u_0'''u_0^7 + 84u_0'''u_0^7 + 84u_0^4u_0^6 + 126u_0^4u_0^6 + 126u_0^5u_0^5)$

$\quad = 2(u_0u_0^{10} + 10u_0'u_0^9 + 45u_0''u_0^8 + 120u_0'''u_0^7 + 210u_0^4u_0^6 + 126u_0^5u_0^5)$

$\quad = 2(0 + 0 + 0 + 120(2)(80) + 0 + 0) = 38400.$

The Taylor series for $u(t)$ becomes,

$$u(t)=\frac{1}{3}t^3+\frac{1}{63}t^7+\frac{2}{2079}t^{11}$$

The approximate value of $u(1)$ is given by,

$$u(1)=\frac{1}{3}+\frac{1}{63}+\frac{2}{2079}=0.3502.$$

If only the first two terms are used, then the value of t is obtained from

$$\left|\frac{2}{2079}t^{11}\right|<0.5\times10^{-7}$$

Solving we get $t=0.41$.

2. Solve the initial value problem $u'=-2tu^2$, $u(0)=1$ with $h=0.2$ on the interval [0, 0.4]. Use the fourth order classical RK method.

Solution:

Given: $u'=-2tu^2$, $u(0)=1$, $t_0=0$, $u_0=1$, $h=0.2$ on the interval [0, 0.4].

The fourth order classical RK method: $u_{j+1}=u_j+\frac{1}{6}\left[K_1+2K_2+2K_3+K_4\right]$

For $j=0$**,**

$$K_1=hf(t_0,u_0)=0,$$

$$K_2=hf(t_0+\frac{h}{2},u_0+\frac{1}{2}K_1)=-0.04,$$

$$K_3=hf(t_0+\frac{h}{2},u_0+\frac{1}{2}K_2)=-0.03842,$$

$$K_4=hf(t_0+h,u_0+K_3)=-0.0739.$$

Hence $u(0.2)=0.9615$**.**

For $j=1$**,** $t_1=0.2$, $u_1=0.9615$**;**

$$K_1=hf(t_1,u_1)=-0.0739,$$

$$K_2=hf(t_1+\frac{h}{2},u_1+\frac{1}{2}K_1)=-0.1025,$$

$$K_3=hf(t_1+\frac{h}{2},u_1+\frac{1}{2}K_2)=-0.0994,$$

$$K_4=hf(t_1+h,u_1+K_3)=-0.1189.$$

Hence $u(0.4)=0.8621$.

3. Given the initial value problem $u' = -2tu^2$, $u(0) = 1$ estimate $u(0.4)$ using (i) Modified Euler-Cauchy method, and (ii) Heun method, with $h = 0.2$. Compare the results with the exact solution $u(t) = 1/(1+t^2)$.

Solution:

Given: $u' = -2tu^2$, $u(0) = 1$, $h = 0.2$

(i) **Modified Euler-Chuchy method:**

$$u_{j+1} = u_j + K_2$$

Where,

$$K_1 = hf(t_j, u_j);$$

$$K_2 = hf(t_j + \frac{h}{2}, u_j + \frac{1}{2}K_1)$$

For $j = 0, t_0 = 0, u_0 = 1$,

$$K_1 = hf(t_0, u_0) = 0$$

$$K_2 = hf(t_0 + \frac{h}{2}, u_0 + \frac{1}{2}K_1) = -0.04 \ .$$

Hence $u(0.2) = 0.96$

For $j = 1, t_1 = 0.2, u_1 = 0.96$.

$$K_1 = hf(t_1, u_1) = -0.0737 \ ,$$

$$K_2 = hf(t_1 + \frac{h}{2}, u_1 + \frac{1}{2}K_1) = -0.1023.$$

Hence $u(0.4) = 0.8577$.

(ii) **Heun method:**

$$u_{j+1} = u_j + \frac{1}{2}(K_1 + K_2)$$

Where,

$$K_1 = hf(t_j, u_j);$$

$$K_2 = hf(t_j + h, u_j + K_1).$$

For $j = 0, t_0 = 0, u_0 = 1$.

$$K_1 = hf(t_0, u_0) = 0 \ ,$$

$$K_2 = hf(t_0 + h, u_0 + K_1) = -0.4.$$

Hence $u(0.2) = 0.96$ **.**

For $j = 1, t_1 = 0.2, u_1 = 0.96$.

$$K_1 = hf(t_1, u_1) = -0.0737,$$

$$K_2 = hf(t_1 + h, u_1 + K_1) = -0.1257.$$

Hence $u(0.4) = 0.8603$ **.**

The exact solution,

$$u(t) = 1/(1+t^2),$$

$$u(0.2) = 0.9615, u(0.4) = 0.8621.$$