IDHAYA COLLEGE FOR WOMEN, KUMBAKONAM

DEPARTMENT OF MATHEMATICS

CLASS	: I M.Sc., MATHS
SUBJECT NAME	: MATHEMATICAL MODELLING
SUBJECT CODE	: P16MAE1B
SEMESTER	: II
UNIT	: V (Mathematical Modelling Through Graph)
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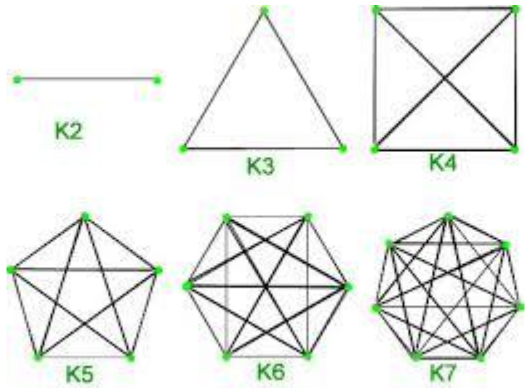
UNIT V

PART A

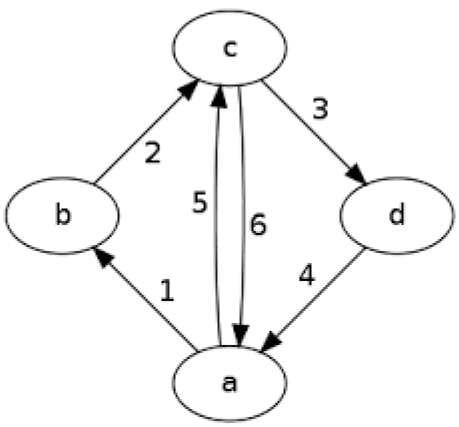
1) Define: Complete graph.

A graph is called **Complete.** if every pair of its vertices is joined by an edge.



2) Define: Digraph.

A graph is called a **Directed graph** or a **Digraph** if every edge is directed with an arrow.



3) Define: Signed graph.

A graph is called a **Signed graph** if every edge has a either a plus or minus sign associated with it.

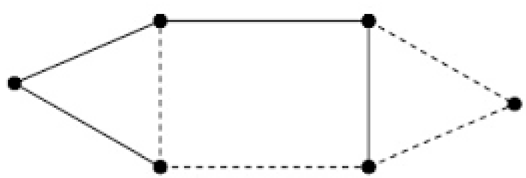
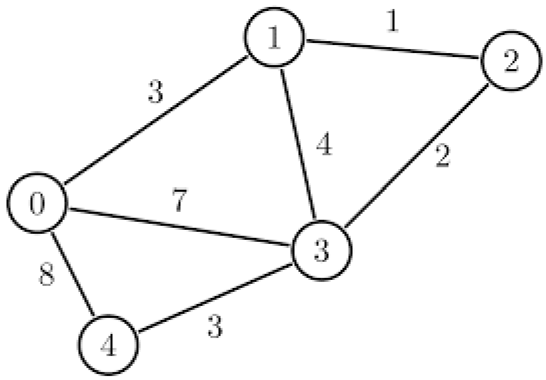


Figure 1.2: A signed graph

4) **Define: Weighted and Weighted signed Digraphs.**

A graph is called a **Weighted digraph** if every directed edge has a weight associated with it. We may also have digraphs with positive and negative numbers associated with edges. These will be called **Weighted signed Digraphs**.



5) **Define the nature of models in terms of graphs.**

We shall consider, the length of the edge joining two vertices willnot be relevant. It will not also be relevant whether the edge is straight or curved.

The relevant facts would be:

(a) Which edges are joined;
(b) Which edges are directed and in which directions;
(c) Which edges have positive or negative signs associated with them;
(d) Which edges have weights associated with them and what these weights are.

6) **Define: Food webs.**

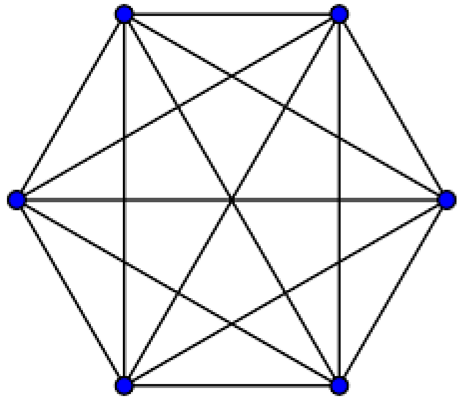
Here aSb if a eats b and we draw a directed edge from a to b. Here also ~(aSa) and aSb⇒~(bSa). However the transitive law need not hold. Thus consider the food web in figure. Here fox eats bird, bird eats grass, but fox does not eat grass.

We can however calculate measure of the status of each species in this food web by using equation h(x)=$\sum_k kn_k$.

7) **Define: Complete algebraic graph and when it is said to be balanced.**

A **Complete algebraic graph** is defined to be a complete graph such that between any two edges of it, there is a positive or negative sign. A complete algebraic graph is said to be balanced if all its triangles are balanced.

An alternative definition states that a complete algebraic graph is **balanced** if all its cycles are positive.



8) Define: Antibalance and Duobalance of a graph.

An algebraic graph is said to be **Antibalanced** if every cycle in it has an even number of positive edges. The concept can be obtained from that of a balanced graph by changing the signs of the edges.

A signed graph is said to be **Duobalanced** if it is both balanced and antibalanced.

9) Define Genetic graphs.

In a genetic graph, we draw a directed edge from A to B to indicate that B is the child of A. In general each vertex will have two incoming edges, one from the vertex representing the father and the other from the representing the mother. If the father or mother is unknown, there may be less than two incoming edges. Thus in a genetic graph, the local degree of incoming edges at each vertex must be less than or equal to two. This is a necessary condition for a directed graph to be a genetic graph, but it is not a sufficient condition.

10) Define more general weighted Digraphs.

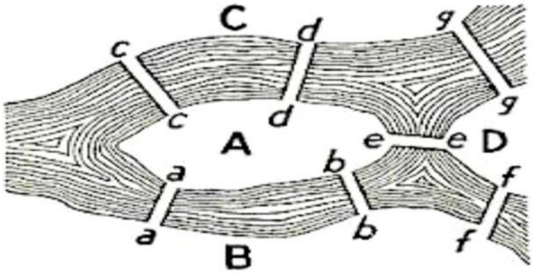
In the most general case, the weight associated with a directed edge can be positive or negative. Thus in figure means that a unit change at vertex at at time t causes changes of -2 units at vertex 2, of 2 units at vertex 4 and of 3 units at vertex at time t+1. Similarly a change of 1 unit at vertex 2 causes a change of -3 units at 3 vertex, 4 units at vertex 4 and of 2 units at vertex 5 and so on. Given the values at all vertices at time t, we can find the values at time t+1,t+2,t+3,… The process of doing this systematically is known as the **Pulse rule**.

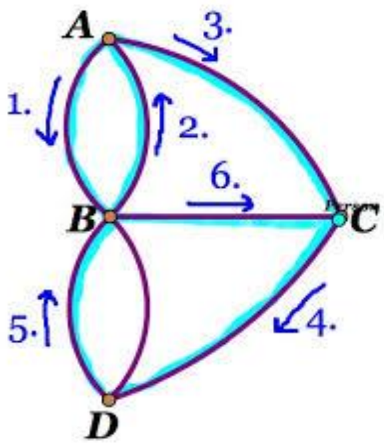
PART B

1. Write down the seven bridges problem:

There are four land masses A,B,C,D which are connected by seven bridges numbered

1 to 7 across a river. The problem is to start from any point in one of the land masses, cover each of the seven bridges once and once only and return to the starting point .





There are two ways of attacking this problem. One method is to try to solve the problem by walking over the bridges. Hundreds of people tried to do so in their evening walks and failed to find a path satisfying the conditions of the problem. A second method is to draw a scale map of the bridges on paper and try to find a path by using a pencil.

It is at this stage that concepts of mathematical modeling are useful. It is obvious that the sizes of the land masses are unimportant, the length of the bridges or whether these are straight or curved are irrelevant. What is relevant information is that A and B are connected connected two bridges 1 and 2, B and C are connected by two bridges 3 and 4, B and D are by one bridge number 5, A and D are connected by bridge number 6 and C and D are connected by bridge number 7. All these facts are represented by the graph with four vertices and seven edges in figure 7.2. if we can trace this graph in such a way that we start with any vertex and return to the same vertex and trace every edge once and once only without lifting the pencil from the paper, the problem can be solved. Again trial and error ethod cannot be satisfactorily used to show that no solution is possible.

The number of edges meeting at the vertex is called the degree of that vertex. We note that the degrees of A, B, C, D are 3,5,3,3 respectively and each of these is an odd number. If we have to start from a vertex and return to it, we need an even number of edges at that vertex. Thus it is easily seen that Konigsberg bridges problem cannot be solved.

This example also illustrates the power of mathematical modeling. We have not only disposed of the seven-bridges problem, but we have discovered a technique for solving many problems of the same type.

2. **Write a short note on structure theorem and its implications:**

Theorem: the following four conditions are equivalent:

(i) The graph is balanced i.e., every cycle in it is positive.

(ii) All closed line-sequences in the graph are positive i.e., any sequence of edges starting from a given vertex and ending on it and possibly passing through the same vertex more than once is positive.

(iii) Any two line-sequences between two vertices have the same sign.

(iv) The set of all points of the graph can be partitioned into two disjoint set such that every positive sign connects two points in the same set and every negative sign connects two points of different sets.

The last condition has an interesting interpretation with possibility of application. It states that if in a group of persons there are only two possible relationships viz., liking and disliking and if the algebraic graph representing these relationships is balanced, then the group will break up into two separate parties such that persons within a party like one another, but each persons of one party dislikes every person of the other party. If a balanced situation is regarded as stable, this theorem can be interpreted to imply that a two-party political system is stable.

3. **Write a short note on communication network**

A directed graph can be serve as a model for a communication network. Thus consider the network given in figure 7.10. if an edge is directed from *a* to *b*, it means that *a* can communicate with *b*. In the given network *e* can communicate directly with *b*, but *b* can communicate with *e* only indirectly through *c* and *d*. However every individual can communicate with every other individual.

Our problem is to determine the importance of each individual in this network. The importance can be measured by the fraction of the messages on an average that pass through him. In the absence of any other knowledge, we can assume that if an individual can send message direct to *n* individuals, he will send a message to any one of them with probability $1/n$. In the present example, the communication probability matrix is :

$$
\begin{array}{c|c}
 & \begin{matrix} a & b & c & d & e \end{matrix} \\
\begin{matrix} a \\ b \\ c \\ d \\ e \end{matrix} & \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 0 & 1/3 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}
\end{array}
$$

No individual is to send a message to himself and so all diagonal elements are zero. Since all elements of the matrix are non-negative and the sum of elements are every two is unity, the matrix is a stochastic matrix and one of its eigenvalues is unity. The corresponding normalized eigenvector is [11/45, 13,45, 3/10, 1/10, 1/15]. In the long run, these fractions of messages will pass through a, b, c, d, e respectively. Thus we can conclude that in this network, c is the most important person.

If in a network, an individual cannot communicate with every other individual either directly or indirectly, the Markov chain is not ergodic and the process of finding the importance of each individual breaks down.

4. Write a short note on general communication networks:

So for we have considered communication networks in which the weight associated with a directed edge represents the probability of communications along that edge. We can however have more general networks. E.g.,

(a) For communication of messages where the directed edge represents the channel and the weight represents the capacity of the channel say in bits per second.
(b) For communication of gas in pipelines where the weight are capacities, say in gallons per hour.
(c) Communication roads where the weights are the capacities in car per hour.

An interesting problem is to find the maximum flow rate, of whatever is being communicated, from any vertex of the communication network to any other. Useful graph-theoretic algorithms for this have been developed by Elias. Feinstein and Shannon as well as by Ford and Fulkerson.

5. Write a short note on weighted bipartitic digraphs and difference equations.

Consider the system of difference equations

$x_{t+1} = a_{11}x_t + a_{12}y_t + a_{13}z_t$

$y_{t+1} = a_{21}x_t + a_{22}y_t + a_{23}z_t$

$z_{t+1} = a_{31}x_t + a_{32}y_t + a_{33}z_t$

this can be represented by a weighted bipartitic digraph. This weights can be positive or negative.

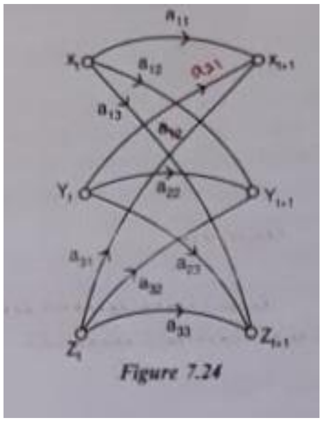


Figure 7.24

PART C

1) A) **Write a short note on Application of Directed graphs to Detection of Cliques**.

A subset of person in a sonic-psychological group will be said to forma clique if (i) every member of this subset has a symmetrical relation with every other member of this subset (ii) no other group member has a symmetric relation with all the members of the subset (iii) the subset has at least three members.

Clique can be defined as a maximal completely connected subset of the original group, containing at least three persons. This subset should not be properly contained in any larger completely connected subset.

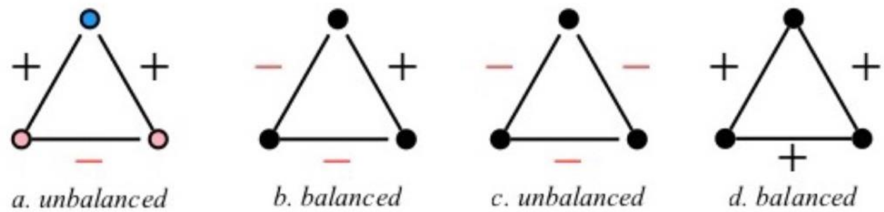
If the group consists of n persons, we can represent the group by n vertices of a graph. The structure is provided by persons knowing or being connected to other persons. If a person I knows j, we can draw a directed edge from i to j. if i knows j and j knows i, then we have a symmetrical relation between i and j.

With very small groups, we can find cliques by carefully observing the corresponding graphs. For larger groups analytical methods based on the following results are useful: (i) i is a member of a clique if the ith diagonal element of S^3 is different from zero. (ii) If there is only one clique of k members in the group, the corresponding k elements of S^3 is will be zero. (iii) If there are only two cliques with k and m members respectively and there is no element common to these cliques, then k elements of S^3 will be *(k-1)(k-2)/2*, m elements of S^3 will be *(m-1)(m-2)/2* and the rest of the elements will be zero. (iv) If there are m disjoint cliques with $k_1, k_2, \ldots, k_m$ members, then the trace of S^3 is $\frac{1}{2}\sum_{i-1}^{m} k_i(k_i-1)(k_i -$

2). (v) A member is non-cliquical if and only if the corresponding row and column of $S^2 \times S$ consists entirely of zeros.

B) **Write a short note on Balance of Signed graphs.**

A signed graph is one in which every edge has a positive or negative sign associated with it. Thus the four graphs of the figure are signed graphs. Let positive sign denote friendship and negative sign denote enemity, then in (graphi) A is a friend of both B and C and B and C are also friends. In (graph ii) A is a friend of B and A and B are both jointly enemies of C. In (graph iii) A is a friend of both B and C, but B and C are enemies. In (graph iv) A is an enemy of both B and C, but B and C are not friends.



First two graphs represents normal behaviour and are said to be balanced, while the last two graphs represent unbalanced situations since if A is a friend of both B and C and B and C are enemies, this creates a tension in the system and there is a similar tension when B and C have a common enemy A, but are not friends of each other.We define the sign of a cycle as the product of the signs of component edges.

We say that a cycle of length three or a triangle is balanced if and only if its sign is positive. A complete algebraic graph is defined to be a positive or negative sign. A complete algebraic graph is said to be balanced if all its triangles are balanced. An alternative definition states that a complete algebraic graph is balanced if all its cycles are positive.It can be shown that the two definitions are equivalent.

A graph is locally balanced at a point a if all the cycles passing through a are balanced. If a graph is locally balanced at all points of the graph, it will obviously be balanced. A graph is defined to be m-balanced if all its cycles of length m are positive. For an incomplete graph, it is preferable to define it to be balanced if all its cycles are positive. The definition in terms of triangle is not satisfactory, as there may be no triangles in the graph.

2) **Derive the Weighted digraphs and Markov chains.**

A Markovian system is characterised by a transition probability matrix. Thus if the states of a system are represented by 1,2,…,n and p_{ij} gives the probability of transition from the ith state to jth state, the system is characterised by the transition probability matrix.

$$T=\begin{matrix} p_{11} & p_{12} & ..p_{1j} & .. & p_{1n} \\ p_{21} & p_{22} & ..p_{2j} & .. & p_{2n} \\ .. & .. & .. .. & .. & .. \\ p_{i1} & p_{i2} & ..p_{ij} & .. & p_{in} \\ .. & .. & .. .. & .. & .. \\ p_{n1} & p_{n2} & ..p_{nj} & .. & p_{nn} \end{matrix}$$

Since $\sum_{i=1}^{n} p_{ij}$ represents the probability of the system going from the ith state to any other state or of remaining in the same state, this sum must be equal to unity. Thus the sum of elements of every row of a t.p.m. is unity.

Consider a set of N such Markov systems where N is large and suppose at any instant $NP_1, NP_2, ..., NP_n$ of these $(P_1+P_2+...+P_n=1)$ are in states 1,2,3,…,n respectively. After one step, let the proportions in these states be denoted by $P_1', P_2', ..., P_n'$, then

$$P_1' = P_1p_{11} + P_2p_{21} + P_3p_{31} + \cdots + P_np_{n1}$$
$$P_2' = P_2p_{12} + P_2p_{22} + P_3p_{32} + \cdots + P_np_{n2}$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$P_n' = P_1p_{1n} + P_2p_{2n} + P_3p_{3n} + \cdots + P_np_{nm}$$

Or $\quad P' = PT$

Where P and P' are row matrices representing the proportions of systems in various states before and after the step and T is the t.p.m.

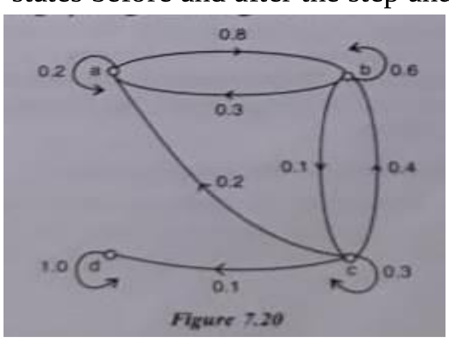
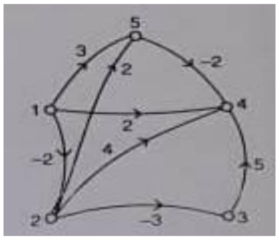


Figure 7.20

We assume that the system has been in operation for a long time and the proportions $P_1, P_2, \ldots, P_n$ have reached equilibrium values. In this case

$P=PT$ or $P(1-T) = 0$

Where I is the unit matrix. This represents a system of n equations for determining the equilibrium values of $P_1, P_2, \ldots, P_n$. If the equations are consistent, the determinant of the coefficient must vanish i.e.,$|T - I|$=0. This requires that unity must be an eigenvalue of T. However this, as we have seen already is true. This shows that an equilibrium state is always possible for a Markov chain.



A Markovian system can be represented by a weighted directed graph. Thus consider the Markovian system with the stochastic matrix

$$\begin{array}{cccc} a & b & c & d \end{array}$$
$$\begin{bmatrix} 0.2 & 0.8 & 0 & 0 \\ 0.3 & 0.6 & 0.1 & 0 \\ 0.2 & 0.4 & 0.3 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Its weighted digraph is given in figure.

In this example d is an absorbing state or a state of equilibrium. Once a system reaches the state d, it stays there forever.

It is clear from the figure, that in whichever state, the system may start, it will ultimately end in state *d*. However the number of steps that may be required to reach d depends on chance. Thus starting from c, the number of steps to reach d may be 1,2,3,4,….; starting from b the number of steps to reach d may be 2,3,4,… and starting from a, the number of steps may be 3,4,5,… In each case, we can fiund the probability that the number of steps required in n and then we can find the expected number of steps to reach it.

Thus the matrix $\begin{bmatrix} 1 & 0 \\ {}^{1}/_{3} & {}^{2}/_{3} \end{bmatrix}$

a is an absorbing state. Starting from b, we can reach a in 1,2,3,…,n steps with probabilities $({}^{1}/_{3}), ({}^{1}/_{3})({}^{2}/_{3}), ({}^{1}/_{3})({}^{2}/_{3})^{2}, \dots, ({}^{1}/_{3})({}^{2}/_{3})^{n-1}, \dots$, so that the expected number of steps is

$$\sum_{n=1}^{\infty} n\frac{1}{3}\left(\frac{2}{3}\right)^{n-1} = 3.$$

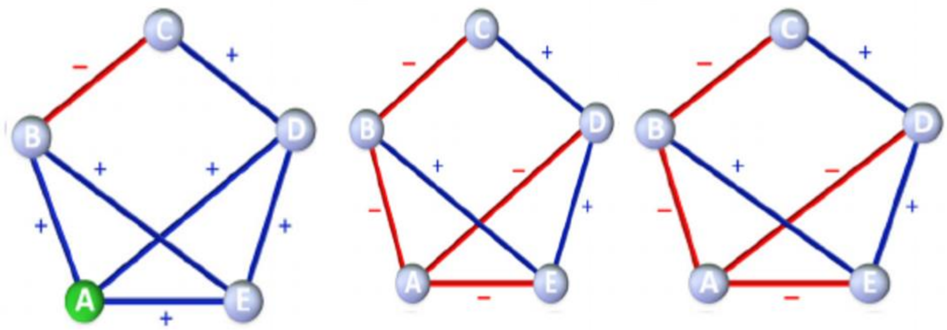
3) A) **Derive the Degree of Unbalance of a graph.**

For many purposes it is enough to know that a situation is unbalanced. We may be interested in the degree of unbalance and the possibility of a balancing process which may enable one to pass from an unbalanced to a balanced graph.

The possibility that methods of graph theory can be applied to dynamic situations also.

Cartwright and Harry define the degree of balance of a group G to be the ratio of the positive cycles of G to the total number of cycles in G. This balance index obviously lies between 0 and 1. G_1 has six negative triangles *viz (abc), (ade), (bcd), (bce), (bde), (cde)* and has four positive triangles. G_2 has four negative triangles *viz (abc0, (abd), (bce)*and *(bde)* and six positive triangles. The degree of balance of $G_1$is therefore less than the degree of balance of G_2.

However in order to get a balanced graph from G_1, we have to change the sign of only two edges *viz. bc*and *de* and similarly to make G_2balanced we have to change the signs of two edges *vizbc* and *bd*. From this point of view both G_1 and $G_2$are equally unbalanced.



Abelson and Rosenberg therefore gave an alternative definition. They defined the degree of unbalance of an algebraic graph as the number of the smallest set of edges of G whose change of sign produces a balanced graph.

The degree of an antibalanced complete algebraic graph is given by $[n(n-2)+k]/4$ where $k=1$ if n odd and $k=0$ if n is even. It has been conjectured that the degree of unbalancing of every other complete algebraic graph is less than or equal to this value.

B) Derive the Communication networks with known probabilities of communication.

In the communication graph of figure, we know that a can communicate with *b* and *c* only and in the absence of any other knowledge, we assigned equal probabilities to *a's* communicating with *b* or *c*. However we may have a priori knowledge that *a's* chances of communicating with *b* and *c* are in the ratio 3:2, then the assign probability .6 to *a's* communicating with b and .4 to *a's* communicating with c. Similarly we can associate a probability with every directed edge and we get the weighted digraph with the associated matrix

$$B = \begin{bmatrix} 0 & 0.6 & 0.4 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0.4 & 0.3 & 0 & 0.3 & 0 \\ 0 & 0 & .3 & 0 & 0.7 \\ 0 & 1.0 & 0 & 0 & 0 \end{bmatrix}$$

We denote the elements are all non-negative and the sum of the elements of every row is unity so that B is a stochastic matrix and unity is one of its eigenvalues. The eigenvector corresponding to this eigenvalues will be different from the eigenvector found already and so the relative importance of the individuals depends both on the directed edges as well as on the weights associated with the edges.

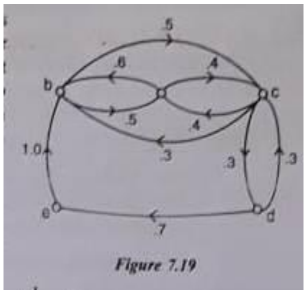


Figure 7.19