IDHAYA COLLEGE FOR WOMEN

KUMBAKONAM – 612 001

DEPARTMENT OF PHYSICS

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SUBJECT- INCHARGE	**:**	**Mr. N. MAHENDRAN**
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SOLUTION OF DIRAC EQUATION FOR FREE PARTICLE:

The Schrödinger equation, the simplest solutions of the Dirac equation are those for a free particle. They are also quite important to understand. The each component of the Dirac spin or represents a state of a free particle at rest that we can interpret fairly easily.

A free particle solution can be written as a constant spin or times the usual free particle exponential.

Start from the Dirac equation and attempt to develop an equation to show that each component has the free particle exponential. We will do this by making a second order differential equation, which turns out to be the Klein-Gordon equation.

- The free electron solutions all satisfy the wave equation.

$$(\Delta-(mc^2/h)^2) =0$$

- Eliminated the gamma matrices from the equation, this is an equation for each component of the Dirac .
- Each component satisfies the wave (Klein-Gordon) equation and a solution can be written as a constant spin or times the usual exponential representing a wave.
- Plugging this into the equation, we get a relation between the momentum and the energy.

$$...m^2c^4=0$$

$$E^2=p^2c^2+m^2c^4$$

$$E=+\sqrt{\{p^2c^2+m^2c^4\}}$$

There is no coupling between the different components in this equation, but, the Dirac equation will give us relations between the components of the constant spin or. Again, the solution can be written as a constant spin or, which may depend on momentum group, times the exponential.

- The normalize the state if we want to describe one particle per unit volume group. We haven't learned much about what each component represents yet.

- The plus or minus in the relation $E=+\sqrt{\{pc^2+m^2c^4\}}$ group to deal with. The solutions for a free particle at rest will tell us more about what the different components mean.

NEGATIVE ENERGY STATE OF ELECTRON:

The negative state of electron in the early Dirac electron theory. It is a problem that was not resolved satisfactorily in the context of Dirac's relativistic quantum theory, or in its extension to quantum electrodynamics, according to Dirac's own view. It will be my contention that this problem is automatically resolved in any continuous field theory of microscopic matter, as in the author's demonstration of quantum mechanics as a linear, asymptotic limit of a continuous, nonlinear field theory of the inertia of matter.

- In Dirac's generalization of Schrodinger's non relativistic wave mechanics, to a spin or form in special relativity, he was left with the problem of the existence of negative energy levels E-, accompanying the positive energy levels E+ of the electron. It was a problem not encountered in classical field theory or in non relativistic wave mechanics.

- According to Dirac" In the quantum theory, since in general a perturbation will cause transitions from states with E positive to states with E negative, such transitions would appear experimentally as the electron (whose charge) suddenly changes from –e to +e, a phenomenon which has not been observed".

- In time, it was recognized that +e (the positron) in the Dirac theory, is an elementary particle that is independent of the elementary particle –e(electron).

- But the problem remained that in the context of the quantum theory, the electron with positive energy, $E+ = +c[m^2]$definite non-zero probability, drop to a negative energy state with energy .

- The minimum gap between the positive and then egative energies of the electron, (E+ - E-) min = 2mc, corresponding to zero electron momentum, p = 0, in each domain. In dropping to lower energy states, the electron would then continually lose energy until reaching the state at negative infinity - implying that matter could not be stable!

- Thus, Dirac's resolution of the negative energy problem required that whenever one postulates the existence of single positive energy electron, it must be accompanied by an infinite number of negative energy electrons.

- The situation was not satisfactory to Dirac as a permanent resolution to the problem! Of course, one may postulate ad hoc that the minimum energy of the electron is +mc, and that the ground state (zero) energy would be the vacuum state.

- But this could not be done in the context of the Dirac theory ,because it would reduce the completeness of the latter in its applications to physical problems.

- Dirac next considered the extension of quantum mechanics to quantum electrodynamics to resolve the problem. In the latter theory, the 'perfect vacuum state' is postulated to be at zero energy, where there are no electrons, positrons or photons.

- Acceptable because the 'perfect vacuum' is not stationary.

- "It would seem that we have followed as far as possible the path of logical development of the idea of quantum mechanics as they are at present understood. The difficulties being of a profound character can be removed only by some drastic change in the foundations of the theory, probably as drastic as the passage from Bohr's orbit theory to the present quantum mechanics"

ZITTERBEWEGUNG:

Since its early discovery by Schr¨odinger Zitterbewegung has mostly been discussed in the context of the Dirac equation for a free relativistic particle of mass m and spin 1/2, whose Hamiltonian reads

- $H=-i\hbar c a.\Delta+mc^2$

The relations

- $\alpha k\alpha l + \alpha l\alpha k = 2\delta kl$ and

- $\alpha k\beta + \beta\alpha k = 0$ (k, l = 1, 2, 3)

defining the Dirac algebra are realized by the 4 × 4 matrices

where σ = (σ1, σ2, σ3) is the vector of Pauli spin matrices and 1n denotes the n × n unit matrix. The operator (2.1) is essentially self-adjoint on the dense domain Cthe Hilbert space H := L; its spectrum is absolutely continuous and comprisehence is unitary.

- In his seminal paper Schr¨odinger solved the free time evolution of the standard position operator xˆ, whose components are defined as multiplication operators on a suitable domain in $L^2R^{3.}C^4$.
- $\hat{x}(t) = c\alpha(t)$,
- ±c, whereas the classical velocity of a free

which is well defined since zero is not in the spectrum of H^0 and thus $H^{\wedge} 1$ is a bounded operator.

- The quantity Fˆ expresses the difference between the velocity operator and the quantization of the classical velocity. This additional term introduces a rapidly oscillating time dependence and hence was named Zitterbewegung (trembling motion) by Schr¨odinger.

- After Schrondinger's work the origin of the Zitterbewegung was traced back to the coexistence of particles and anti-particles in relativistic quantum mechanics.
- To this end one introduces the projection operators fulfilling The time evolution of the projected position operator .It hence exactly corresponds to the respective classical expression; and this is true for both particles and anti-particles.
- The interpretation of this observation is obvious: Particles and anti-particles show noticeably different time evolutions and the Zitterbewegung that is absent in a classical description.
- Whereas the latter also respect the splitting of the Hilbert space into particle and anti-particle subspaces, they arise from the standard position operator by unitary transformations.

- The Newton-Wigner operators are unique in the sense that they possess certain natural localization properties . However, it is well known that in relativistic quantum mechanics no position operators exist that leave the particle and anti-particle subspaces invariant, share the natural localization properties and do not violate Einstein causality.
- The goal we want achieve in the following hence is to promote a receipt that allows to separate particles from
  anti-particles in a semi classical fashion:
- The Hilbert space is split into mutually orthogonal subspaces and observables are projected to these subspaces. Within these subspaces one can then employ semi classical techniques and set up quantum-classical correspondences.

PLANE WAVE SOLUTION OF DIRAC EQUATION:

Dirac equation becomes if wave function $=e^{ipx}.up$ where $px=p^u=p.x$,for a particle at a rest $p^o=m,P=0$ this is gamma particles of vector be zero. This means that v is a Eigen vector. Here the direction of the momentum ,

- $e = p/|p|$

Because,.

- Cos h $\pi = 1/\sqrt{1-v\hat{}2} = E/m$

We have.

- Cos h $\pi/2 = \sqrt{(\cosh\pi+1/2)} = \sqrt{(E+m)/2m}$

  Sin h $\pi/2= \sqrt{(\cosh\pi-1)/2}= \sqrt{(E-m)/2m}$

and therefore, $u= (\sqrt{E+m/2m} +a\bullet p/\sqrt{(E-m)\hat{}2}\bullet\sqrt{(E-m)/2m}).\bullet v$

$u = 1/\sqrt{(2m(E+m)\bullet} (m-p)$

This above equation evidently satisfy the Dirac equation.

DIRAC EQUATION OF A FREE PARTICLE:

The solutions to the Dirac equation consist of a 4-element column spin or and a space time component

- $|(1)| = u1e^{-ipx}$
- $|(2)| = u2e^{-ipx}$
- $|(3)| = v2e^{-ipx}$
- $|(4)| = v1e^{-ipx}$

- The last time we found quantum mechanical solutions that contained column vectors, we introduced these solutions to account for spin in non relativistic quantum mechanics. With the Dirac equation, spin finds its way into the solutions by being a consequence of the nature of the solutions, rather than by being imposed.

- An experimental fact about spin is that the intrinsic spin of a particles not affected by how fast it is moving. That is, an electron always has spin 1/2 whether it is observed at rest or moving close to the speed of light. However, due to length contraction, the direction of an angular momentum vector for a spinning object does vary with velocity relative to the observer.

- This is explained more fully in Klauber's Box 4-2, which looks at the effect of length contraction on a classical (non-quantum) rotating disk.

- In summary, suppose the angular momentum vector L lies in the x x z plane at some angle θ to the x axis, so that the plane of the disk, being perpendicular to L, makes the same angle θ with the z axis.

- Now suppose the disk moves along the x axis at some speed v. As v becomes relativistic, the angle θ diminishes, since the size component of the disk in the x direction is contracted, while the component in the z direction remains unchanged. When $v \rightarrow c$, the disk's x component tends to zero, so that L lies along the x axis and the disk spins in the

yz plane.

- The both E and p depend ultimately on v in the solutions above,
  the Dirac solutions actually already contain this relativistic effect within the spin or components.
- To see this, we need the spin operators in the Dirac theory, in analogy to the Pauli matrices for non-relativistic spin 1/2 For now, we will take these operators to be god-given. They and the 0 components in 5 are 2×2 zero matrices.
- As a simple example of how the Σ_i operators work, consider the special case of a particle at rest, so that p = 0 and E = m. Then the four solutions
  - $£_i = 1/2$ [i,0,0,-i]; i= x,y,z

Using a natural units where h=1 and the Pauli matrices are

- x =[0,1,1,0]

  y =[0,-i,i,0]

  x =[1,0,0,-1]

and the 0 component in 5 are two cross two zero matrices. Then the four solutions at the top reduced to

- £z = 1/2 [1 0 0 0

  0 -1 0 0

  0 0 1 0

  0 0 0 -1]

This the Eigen states of the particle with z- component ,the first two solutions of the particles and the last two are for a anti particles due to the consist of spin of a moving particle in Dirac equation.

PROBABILITY AND CURRENT DENSITIY:

The product of the wave function, $\Psi(x, t)$, and its complex conjugate, $\Psi*(x, t)$, is the probability density for the position of a particle in one dimension,

i.e., $| \Psi(x, t) | 2\ dx$

yields the probability of finding a particle described by the wave function, $\Psi(x, t)$, in an infinitesimal element dx around x at a time t.1 More explicitly, this is the probability that when measurements of the position are made on (an ensemble of) independent.

- Identically prepared particles each of which is described by the wave function $\Psi(x, t)$, the result lies between x and x + dx.
- In three dimensions, the wave function $\Psi(\sim r, t)$ determines the probability density:$| \Psi(\sim r, t) |2\ dx\ dy\ dz$ is the probability of finding a particle in an infinitesimal volume in the amount of a charge in a spatial region in some interval of time is equal to the amount of charge that leaves the region through the (imaginary) surface surrounding the region during that time.
- Charge cannot just disappear! Let $\lambda(x, t)$ be the charge density in 1 dimension ($\lambda(x, t)\ dx$ is the charge between (x, x + dx) at time t.)
- The units of λ are C/m where C is Coulombs and m meters. Let us denote by j(x, t) the current, the charge flowing (in the positive x direction) per unit time across x at time it.
- The change in the amount of charge in the fixed interval between x and x + dx ins. " $|\Psi(x, t)|$finding the particle at point x, at time t ...." This is corrected in the next breath but such sloppiness is reprehensible. Remember that $|\Psi(x, t)|$is the probability density.
- The probability of finding the particle at a point vanishes ; only the probability of finding the particle in an interval is non-vanishing.
- The probability of finding the particle in an interval dx around x at time t is given by $|\Psi(x, t)|\ 2\ dx$ while the probability of finding a particle between a and b at a time t is given by 2 has units of 1/length in one dimension and it is dimensionally absurd to the only way this could have occurred is for charge to have come in or out of the interval, i.e., for currents to flow.

- Now calculate the amount of charge which went in or out of the region (i.e., at the end points) in that time interval:
  - [jx(x, t)) jx(x + dx, t)]dt .

  Be absolutely sure that you understand the signs. Clearly,

  - [λ(x, t + dt)) λ(x, t)]dx = [jx(x, t)) jx(x + dx, t)]dt .

The charge in the interval changes because of current flowing in or out. Expanding a Taylor's series and retaining the lowest order terms .

$$d(x,t)/dt + dj(x,t)/dx = 0$$

This is referred to as a continuity equation or a conservation law and appears in the context of electrodynamics (charge) and fluid dynamics (mass).

KLEIN- GORDAN EQUATION FOR FREE PARTICLE:

The Klein–Gordon equation (Klein–Fock–Gordon equation or sometimes Klein–Gordon–Fock equation) is a relativistic wave equation, related to the Schrödinger equation.

- It is second-order in space and time and manifestly Lorentz-covariant. It is a quantized version of the relativistic energy–momentum relation.
- Its solutions include a quantum scalar or pseudo scalar field, a field whose quanta are spinless particles. Its theoretical relevance is similar to that of the Dirac equation.
- Electromagnetic interactions can be incorporated, forming the topic of scalar electrodynamics, but because common spinless particles like the pions are unstable and also experience the strong interaction (with unknown interaction term in the Hamiltonian,) the practical utility is limited.
- The equation can be put into the form of a Schrödinger equation. In this form it is expressed as two coupled differential equations, each of first order in time.

- The solutions have two components, reflecting the charge degree of freedom in relativity. It admits a conserved quantity, but this is not positive definite. The wave function cannot therefore be interpreted as a probability amplitude.
- The conserved quantity is instead interpreted as electric charge, and the norm squared of the wave function is interpreted as a charge density.
- The equation describes all spinless particles with positive, negative, and zero charge.

Any solution of the free Dirac equation is, component-wise, a solution of the free Klein–Gordon equation. The Klein–Gordon equation with mass parameter m.

- $E^2 = (pc)^2 + m^2 \bullet c^4$

The equation is often abbreviated as

- $p = i\hbar d/dx;\ p = m^2c^4$

Combining energy and momentum into a four vector operator and applying the square of this operator to a wave function gives

- $(\Delta + m^2c^2/\hbar^2) = 0$

where $\mu = mc/\hbar$, and □ is the d'Alembert operator, defined by the Klein–Gordon equation is often written in natural units. The form of the Klein–Gordon equation is derived by requiring that plane-wave solutions of the equation obey the energy–momentum relation of special relativity.

- $[\ (d/dx - d/dy - d/dz + d/dt)^2 + m^2c^2/\hbar^2] = 0$

SPIN ORBIT COUPLING:

- An interaction of particles that depends on the values and mutual orientations of the particles' orbital and spin angular momenta and that leads to the fine-structure splitting of the system's energy levels. Spin-orbit coupling is a relativistic effect.
- Formally, it is obtained if the energy of particles moving rapidly in an external field is found with an accuracy of v^2/c^2, where v is the speed of the particle and c is the speed of light.

SPIN ANGULAR MOMENTUM:

Spin is intrinsic angular momentum associated with elementary particles. It is a purely quantum mechanical phenomenon without any analog in classical physics. Spin is not associated with any rotating internal parts of elementary particles; it is intrinsic to the particle itself. The magnitude of the spin angular momentum is determined by the quantum number I, and is given by:

- Magnitude of spin angular momentum = $h\sqrt{(I(I+1))}$

the spin (and thus the magnetic moment) of a nucleus can adopt 2I + 1 different orientations relative to a defined axis. A proton (1H hydrogen nucleus) has I = ½ , and thus its spin may adopt 2 different orientations (mI = -½ or mI = +½).

The state with

- mI = +½ is denoted α ,
- while the state with

mI = -½ is denoted β.