UNIT - IV

CENTRAL ORBITS

General Orbits

In this section we get the orbit of particles when their velocity components (or) their acceleration components are given. Central orbit is an orbit under a central force.

Central Force: 2m

When a particle is subject to the action of a force which is always either towards (or) away from a fixed point, the particle is said to be under the action of a central force.

The Velocity and acceleration components in the radial and transverse directions are as follows.

Velocity components : $\dot{r}, r\dot{\theta}$

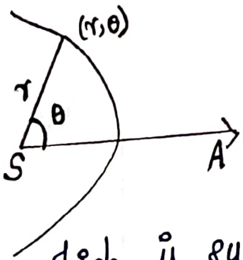
Acceleration components : $\ddot{r}-r\dot{\theta}^2, \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$

(7) 2m. **Conic :**

Now it is necessary to remember that the polar eqn. of a conic is

$$\frac{l}{r} = 1+e\cos\theta,$$

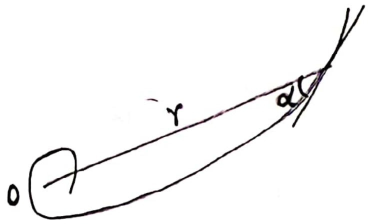
If one focus S is the pole and SA is the initial line as in the figure. l is the semi latus rectum.



(8) 2m.

Equi angular spiral :

Equiangular spiral is a curve which is such that the angle b/w the radius vector and the respective tangent is a constant, say α. Its polar eqn. is



$$r = Ae^{(\cot\alpha)\theta},$$

where A is a constant.

Eg :

(X) 5m 1. The velocities of a particle along and perpendicular to the radius vector are λr and $\mu\theta$. Find the path & s.r. the acceleration components along and $\perp^r$ to the radius vector are

$$\lambda^2 r - \frac{\mu^2\theta^2}{r} \cdot \mu\theta\left(\lambda + \frac{\mu}{r}\right).$$

Proof :

It is given that,

$$\dot{r} = \lambda r \quad (\text{or}) \quad \frac{dr}{dt} = \lambda r, \longrightarrow (1).$$

$$r\dot{\theta} = \mu\theta \quad (\text{or}) \quad r\frac{d\theta}{dt} = \mu\theta \longrightarrow (2).$$

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Dividing ① by ②, we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{\lambda r}{\mu\theta} \quad (\text{or}) \quad \frac{1}{r^2}\,dr = \frac{\lambda}{\mu}\cdot\frac{1}{\theta}.d\theta$$

On integration, we get the eqn. of the path as

$$-\frac{1}{r} = \frac{\lambda}{\mu}\log\theta + A. \quad \text{const.}$$

The acceleration along the radius vector is,

$$\ddot{r} - r\dot{\theta}^2 = \frac{d\dot{r}}{dt} - r\dot{\theta}^2$$

$$= \frac{d(\lambda r)}{dt} - r\left(\frac{\mu\theta}{r}\right)^2 \quad \text{by (1) \& (2),}$$

$$= \lambda\frac{dr}{dt} - \frac{\mu^2\theta^2}{r}$$

$$= \lambda(\lambda r) - \frac{\mu^2\theta^2}{r}$$

$$= \lambda^2 r - \frac{\mu^2\theta^2}{r}.$$

The acceleration $\perp^r$ to the radius vector is

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{r}\frac{d}{dt}\left(r^2.\frac{\mu\theta}{r}\right) \quad \text{by ②.}$$

$$= \frac{\mu}{r}\left[\dot{r}\theta + r\dot{\theta}\right]$$

$$= \frac{\mu}{r}\left[\lambda r\theta + \mu\theta\right]$$

$$= \mu\theta\left[\lambda + \frac{\mu}{r}\right].$$

2). The velocities of a particle along & $\perp^r$ to the radius vector from a fixed origin are 'a' & 'b'.

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Find the path and the accelerations along and $\perp^r$ to the radius vector.

solnr

It is given that $\dot{r} = a$, $r\dot{\theta} = b$. Dividing the first by the second,

$$\frac{\dot{r}}{r\dot{\theta}} = \frac{a}{b}. \text{ So } \frac{1}{r}\frac{dr}{d\theta} = \frac{a}{b}.$$

$$\therefore \frac{1}{r}\,dr = \frac{a}{b}\,d\theta.$$

Integration gives the eqn. of the path as

$$\log r = \frac{a}{b}\theta + \log c \quad (\text{or})$$

$$\frac{r}{c} = e^{a/b\,\theta} \quad (\text{or}) \quad r = ce^{a/b\,\theta}.$$

Which is the eqn. of an equiangular spiral.

Also $\ddot{r} = 0$ and $\dot{\theta} = b/r$.

So the acceleration components are,

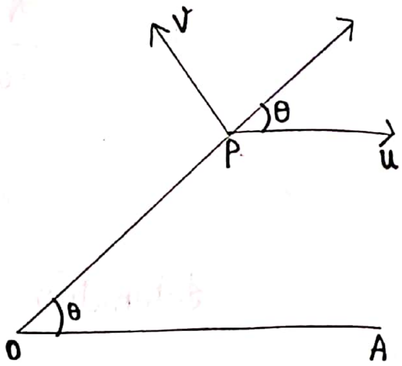
(i). $\ddot{r} - r\dot{\theta}^2 = 0 - r\dfrac{b^2}{r^2} = -\dfrac{b^2}{r}$.

(ii). $\dfrac{1}{r}\dfrac{d}{dt}(r^2\dot{\theta}) = \dfrac{1}{r}\dfrac{d}{dt}(br) = \dfrac{b\dot{r}}{r} = \dfrac{ab}{r}$.

3). Show that the path of a pty. P whose velocity is such that its components in a fixed direction and in the direction $\perp^r$ to the line joining P to a fixed point 'O' are respectively the constants u & v, is a conic with a focus at 'O' & eccentricity u/v.

<u>Proof:</u>

Let 'O' be the pole and OA, the given fixed direction, the initial line.



Then the components of the velocity 'u' in the radial and transverse directions are

$$u\cos\theta \,,\quad -u\sin\theta .$$

$$\therefore \quad \dot{r} = u\cos\theta \,,\quad r\dot{\theta} = v - u\sin\theta .$$

Dividing $\dot{r}$ by $r\dot{\theta}$, we get

$$\frac{1}{r}\frac{dr}{d\theta} = \frac{u\cos\theta}{v - u\sin\theta} .$$

Integration gives,

$$\log r = -\log(v - u\sin\theta) + \log c .$$

(i.e.,) $r = \dfrac{c}{v - u\sin\theta}$ (or) $\dfrac{c}{r} = v - u\sin\theta$.

(i.e.,) $\dfrac{c/v}{r} = 1 - \dfrac{u}{v}\sin\theta$.

So the path is a conic with its focus at 'O' and eccentricity is u/v.

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4). A pt. P describes an equiangular spiral with a constant angular velocity about the pole 'O'. S.T. its acceleration varies as OP and is in a direction making with the tangent at 'P' the same constant angle that OP makes.

Soln:

The eqn. of an equiangular spiral is $r = ae^{\theta \cot\alpha}$, where a, α are constants.

α is the angle b/w the tangent at any pt. and the radius vector to that pt.

Let the constant angular velocity be ω. Now

$$\dot{r} = ae^{\theta\cot\alpha} \cot\alpha\,\dot{\theta} = \omega\cot\alpha\, r,$$

$$\ddot{r} = \omega\cot\alpha\,\dot{r} = \omega^2\cot^2\alpha\, r.$$

Then the radial and the transverse components of acceleration are,

$$\ddot{r} - r\dot{\theta}^2 = \omega^2\cot^2\alpha\, r - r\omega^2 = \omega^2 r(\cot^2\alpha - 1)$$

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{\omega}{r}(2r\dot{r}) = \frac{\omega}{r}(2r\omega\cot\alpha\, r)$$

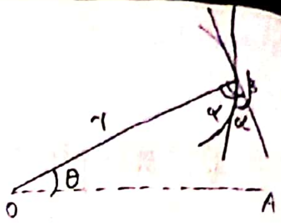
$$= 2\omega^2 r\cot\alpha.$$

$$\therefore \text{Acceleration} = \sqrt{\omega^4 r^2 \underbrace{(\cot^2\alpha - 1)^2}_{\cot^4\alpha + 1 - 2\cot^2\alpha} + 4\omega^4 r^2 \cot^2\alpha}$$

$$= \sqrt{\omega^4 r^2 (1 + \cot^2\alpha)^2}$$

$$= \omega^2 r \csc^2\alpha.$$

which is proportional to r.



If β is the angle b/w the radial direction and the direction of the acceleration, then

$$\tan\beta = \frac{\text{Transverse Component}}{\text{Radial Component}}$$

$$= \frac{2\omega^2 r \cot\alpha}{\omega^2 r(\cot^2\alpha - 1)}$$

$$= \frac{2}{\tan\alpha(\cot^2\alpha - 1)}$$

$$= \frac{2}{\cot\alpha - \tan\alpha} = \frac{2}{1/\tan\alpha - \tan\alpha}$$

$$= \frac{2\tan\alpha}{1 - \tan^2\alpha} = \tan 2\alpha.$$

So $\beta = 2\alpha$. But the angle b/w the radial direction & the tangent is α. So the angle b/w the tangent & the acceleration is α.

CENTRAL ORBIT :-

In this chapter we study the motion of a particle subject to the action of a central force.

Central Force :

When a particle is subject to the action of a force which is always either towards (or) away from a fixed pt, the particle is said to be under the action of a central force. That is, a central force is a force whose line of action always passes through a fixed point.

2m.

Centre of force :

A central force is a force whose line of action always passes through a fixed point. The fixed point is called the centre of force.

Polar Co-ordinates :

To study the central motion of a particle we use polar co-ordinates r, θ, choosing the centre of forces as the pole and a fixed line through it as the initial line.

Notation :

We shall denote the central force per unit mass by $\phi(r)\hat{r}$,

where $\hat{r}$ is the unit vector in the radial direction in the sense in which r increases and $\phi(r)$ is a function of distance r of the particle from the centre of force.

Central Orbit:

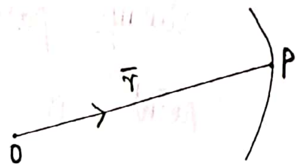
The path described by a particle under a Central force is called a Central orbit.

Book Work:

(7) 5M. To S.T. a central orbit is a plane curve.

Proof:

Let us make the following assumptions:



O : Centre of force.

P : Position of the particle at time 't'.

$\bar{r}$: $\overline{OP}$ ($r = OP$).

$\hat{r}$: Unit vector along OP.

m : Mass of the particle.

$\phi(r)\hat{r}$: Central force per unit mass.

Then the eqn. of motion of the particle is,

$$m\ddot{\bar{r}} = m\phi(r)\hat{r} \quad (\text{or}) \quad \ddot{\bar{r}} = \phi(r)\hat{r}. \longrightarrow (1)$$

Let us consider, in particular,

$$\frac{d}{dt}(\bar{r}\times\dot{\bar{r}}) = \dot{\bar{r}}\times\dot{\bar{r}} + \bar{r}\times\ddot{\bar{r}}$$

$$= \bar{0} + \bar{r}\times\phi(r)\hat{r}$$

$$= \bar{0} + \bar{0} = \bar{0}.$$

This implies that $\bar{r} \times \dot{\bar{r}}$ is a constant vector say $\bar{c}$. Then $\bar{r}$ (or) $\bar{OP}$ is always perpendicular to $\bar{c}$.

So P is always in the plane through O and $\perp^r$ to $\bar{c}$. Hence the motion of P is coplanar and the orbit is a plane curve.

V.V. (4) Differential eqn. of a central orbit :

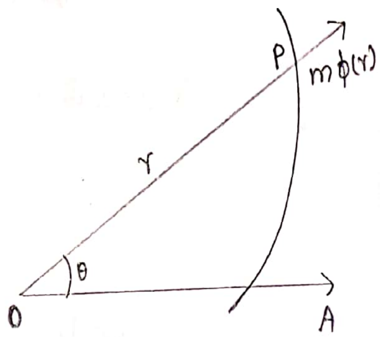
We now obtain the eqn. of a central orbit in polar co-ordinates.

Book work :

To obtain the differential eqn. of a central orbit in polar co-ordinates.

Proof :

Let us make the following assumptions :



O : Centre of force.

O : Pole.

OA : Initial line.

P(r, θ) : Position of the particle at time 't'.

$\phi(r)$: Central force per unit mass in the direction OP.

m : Mass of the particle.

The motion is a coplanar motion. We shall consider the eqns/. of motion corresponding to the radial and transverse directions.

By using the result,
We know that, in these directions, the acceleration components are

$$\ddot{r} - r\dot{\theta}^2, \quad \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}).$$

The force components in these directions are

$m\phi(r), \, 0. \rightarrow$ force.

Therefore the respective eqns/. of motions are,

$$m(\ddot{r} - r\dot{\theta}^2) = m\phi(r) \text{ (or) } \ddot{r} - r\dot{\theta}^2 = \phi(r), \longrightarrow ①.$$

$$m\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0 \text{ (or) } \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0 \rightarrow ②.$$

Of these, the second implies that $r^2\dot{\theta}$ is a constant, say h. If $1/r$ is denoted by u, then

$h = r^2\dot{\theta}$; $1/r = u$

$$\dot{\theta} = \frac{h}{r^2} = hu^2 \rightarrow ③.$$

Differentiation of $r = \frac{1}{u}$ w.r.t. "t" gives

$$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt}.$$

$$= -\frac{1}{u^2}\frac{du}{d\theta}(hu^2) = -h\frac{du}{d\theta} \text{ by (3)} \longrightarrow ④.$$

$$\therefore \frac{d^2r}{dt^2} = -h\frac{d}{dt}\left(\frac{du}{d\theta}\right)$$

$$= -h\frac{d}{d\theta}\left(\frac{du}{d\theta}\right)\frac{d\theta}{dt}$$

$$= -h\frac{d^2u}{d\theta^2}(hu^2)$$

$$= -h^2u^2\frac{d^2u}{d\theta^2} \text{ by (3).} \rightarrow (5).$$

substituting these values of $\ddot{r}$ and $\dot{\theta}$ in (1),

$$-h^2u^2\frac{d^2u}{d\theta^2} - r(hu^2)^2 = \phi(r). \rightarrow -h^2u^2\left[\frac{d^2u}{d\theta^2} + \frac{1}{u}u^2\right] = \phi(r)$$

since $r = 1/u$, this can be written as, $\rightarrow -h^2u^2\left[\frac{d^2u}{d\theta^2} + u\right] = \phi(r)$

$$\boxed{\frac{d^2u}{d\theta^2} + u = -\frac{\phi(r)}{h^2u^2}} \cdot \text{ where } h = r^2\dot{\theta}\ ;\ u = 1/r$$

This is the eqn. of the central orbit in polar co-ordinates.

Differential eqn. for an attractive central force:

(7) 10M. If the central force is an attractive one, that is, a force towards O, having a magnitude F per unit mass, then

$$\phi(r) = -F.$$

in which case the differential eqn. of the orbit becomes,

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}}$$

Apse :

If 'O' is the pole and P is a point on a curve such that OP is perpendicular to the tangent at P, then P is an apse. If P is an apse OP is a maximum (or) minimum of r. For eg., in an ellipse, if S is the pole, then the ends of the major axes are pole apses.

Maximum and minimum angular velocity :-

$\dot{\theta} = h/r^2$.

Since $r^2\dot{\theta} = h$ is a constant in a central orbit, the angular velocity of the particle about 'O' is a maximum (or) minimum according as the radius vector 'r' is a minimum (or) maximum. So, at an apse, the angular velocity of the particle is either a maximum (or) minimum. The apses are given by $du/d\theta = 0$.

2m

Force per unit mass :

The force per unit mass is

$$\phi(r)\,\hat{r} = -h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)\hat{r}. \longrightarrow (6).$$

2m. velocity v at P :- → central orbit (or) polar co-ordinates.

W.K.T. In a coplanar motion the velocity components along the radial and transverse directions are

$$\dot{r},\ r\dot{\theta}. \qquad v^2 = v_1^2 + v_2^2.$$

$$\therefore\ v^2 = (\dot{r})^2 + (r\dot{\theta})^2$$

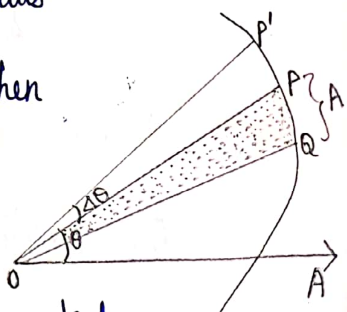
$$= h^2\left(\frac{du}{d\theta}\right)^2 + \frac{1}{u^2}(hu^2)^2 \text{ by (4), (3)} \qquad \dot{\theta} = h/r^2$$

$$= h^2\left(\frac{du}{d\theta}\right)^2 + h^2u^2 \qquad \dot{\theta} = hu^2.$$

$$= h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] \rightarrow (7).$$

(A) 2m (or) 3m. **Areal Velocity** :-

Let the initial position of the particle be Q and the position at time 't' be $P(r,\theta)$. Let the area OQP swept by the radius vector moving from OQ to OP be A. Then



$$\frac{dA}{dt}.$$

is called the areal velocity of the particle.

Let P' be the position of the particle at time $t+\Delta t$, then

$$\Delta A = \text{Area } POP' \simeq \tfrac{1}{2}\, OP \cdot OP' \cdot \sin\Delta\theta.$$

$$\simeq \tfrac{1}{2}\, r(r+\Delta r)\,\Delta\theta \quad \text{since } \Delta\theta \text{ is small}.$$

$$\therefore \quad \frac{\Delta A}{\Delta t} = \tfrac{1}{2} r(r+\Delta r)\frac{\Delta \theta}{\Delta t}.$$

So the areal velocity of P is

$$\lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{dA}{dt}$$

$$\boxed{\frac{dA}{dt} = \tfrac{1}{2} r^2 \dot{\theta}.}$$

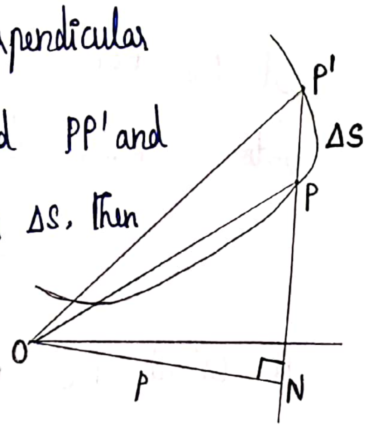
Constancy of the areal velocity in central orbit:

In a central orbit $r^2\dot{\theta}$ is a constant which we have denoted by h. So in a central orbit the areal velocity

$\tfrac{1}{2} r^2 \dot{\theta}$ is a constant $h/2$.

Alternative form for areal velocity:

If $ON(=p)$ is the perpendicular drawn from O to the chord PP' and the length of the arc PP' is Δs, then



$$\Delta A = \tfrac{1}{2} PP' \times ON$$

$$\simeq \tfrac{1}{2} \Delta s \times ON.$$

$$\therefore \quad \frac{\Delta A}{\Delta t} = \tfrac{1}{2} \frac{\Delta s}{\Delta t} \times ON.$$

$$\therefore \quad \frac{dA}{dt} = \tfrac{1}{2} \frac{ds}{dt} p = \tfrac{1}{2} vp = \tfrac{1}{2} pv,$$

where v is the velocity of the particle at P.

Remarks:

Now $\frac{1}{2} pv = \frac{h}{2}$ (or) $pv = h$. So the velocity at any pt. P is inversely proportional to the perpendicular drawn from O to the tangent at P.

Constancy of moment of momentum:

The momentum of the particle is mv which is along the tangent. Its moment about O is,

$$ON \times (mv) = m(pv) = mh.$$

Thus the moment of momentum about O, otherwise known as angular momentum about 'O', is the constant mh.

Book Work:

5M. 10M.

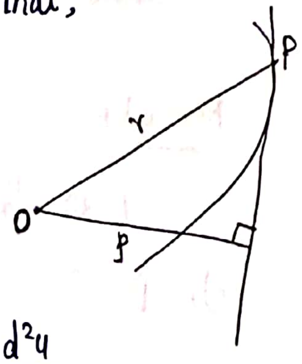
To obtain the differential eqn. of a central orbit in p-r coordinates.

Proof:

For any P on the orbit, the radius vector $r (= OP)$ and the perpendicular distance p of O from the tangent at P are the p-r co-ordinates of P. From

Areal velocity.

Differential Geometry, we know that,



$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2.$$

Differentiating this w.r.t. 'θ',

$$-\frac{2}{p^3}\frac{dp}{d\theta} = 2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}$$

$$= 2\frac{du}{d\theta}\left[u + \frac{d^2u}{d\theta^2}\right].$$

If the central force (per unit mass) in the radial direction is $\phi(r)$, then

$$\frac{d^2u}{d\theta^2} + u = -\frac{\phi(r)}{h^2u^2}.$$

$$\therefore -\frac{2}{p^3}\frac{dp}{d\theta} = 2\frac{du}{d\theta}\left[-\frac{\phi(r)}{h^2u^2}\right] \quad \text{(or)}$$

$$\frac{1}{p^3}\frac{dp}{d\theta} = \frac{\phi(r)}{h^2u^2}\cdot\frac{du}{d\theta}.$$

$$\therefore \frac{1}{p^3}\frac{dp}{dr}\cdot\frac{dr}{d\theta} = \frac{\phi(r)}{h^2u^2}\frac{d(1/r)}{d\theta} \quad \text{since } u = \frac{1}{r}.$$

$$= \frac{\phi(r)}{h^2u^2}\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)$$

$$= -\frac{\phi(r)}{h^2}\cdot\frac{dr}{d\theta}.$$

$$\boxed{\therefore \frac{h^2}{p^3}\frac{dp}{dr} = -\phi(r).}$$

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This eqn. is in terms of p and r. This is the p-r eqn. of the orbit. This is also called pedal equation of the orbit.

Eqn. for an attractive central force :-

If the central force is an attractive one of magnitude F per unit mass, then $\phi(r) = -F$ and the p-r eqn. of the orbit is,

$$\frac{h^2}{p^3} \cdot \frac{dp}{dr} = F.$$

Laws of a central force :-

When the eqn. of a central orbit is given, to obtain the force per unit mass and the speed of the particle at a distance 'r' from the centre of force, we have to calculate,

$$-h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)\hat{r}, \quad h\sqrt{\left(\frac{du}{d\theta}\right)^2 + u^2}.$$

Method to find the central orbit :-

If F is the central acceleration towards the centre of force, then to obtain the eqn. of the respective orbit, we have to solve the label

differential eqn.

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2} \longrightarrow ①.$$

Here h may be found by using $h = pv$ from the given Condition.

One method of solving the differential eqn. is by multiplying both sides of it by $2\frac{du}{d\theta}$ and then integrating w.r.t. "θ". Then we get

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = 2\int \frac{F}{h^2u^2} \frac{du}{d\theta} d\theta.$$

$$= 2\int \frac{F}{h^2u^2} du.$$

because

$$\frac{d}{d\theta}\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = 2\left(\frac{d^2u}{d\theta^2} + u\right)\frac{du}{d\theta}.$$

Thus we get a first order differential eqn.

$$\frac{du}{d\theta} = (\text{A function of } u).$$

Which can be solved as such (or) after replacing 'u' by $1/r$.

Book Work:

To find the orbit of a particle moving under an attractive central force varying inversely as the square of the distance.

For ellipse, parabola, hyperbola. conditions. in (circ)

Proof:

Let the pole be the centre of force. Let the force per unit mass be $\frac{\mu}{r^2}$ ($F \propto 1/r^2$). Then the differential eqn. of the orbit is,

$$\frac{d^2u}{d\theta^2} + u = \frac{1}{h^2u^2} \frac{\mu}{r^2} \quad (or)$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \rightarrow ①.$$

Using the operator $\frac{d}{d\theta} = D$, we have

$$(D^2+1)u = \frac{\mu}{h^2}.$$

$m^2 = -1$
$m = \pm i$ = $m = \pm i$.

C.f = $A \cos \beta x + B \sin \beta x$.
$= C_1 \cos\theta + C_2 \sin\theta$.
$= A \cos B \cos\theta - A \sin B \sin\theta$.

$$C.F. = A\cos(\theta+B); \quad P.I = \frac{1}{D^2+1}\left(\frac{\mu}{h^2}\right) = \frac{\mu}{h^2}.$$

So the general soln. is

$$u = A\cos(\theta+B) + \frac{\mu}{h^2} \rightarrow ②.$$

$\div$ by h^2/μ,

$$\frac{du}{d\theta} = -A\sin(\theta+B)$$

(i.e.,) $$\frac{h^2u}{\mu} = 1 + \frac{h^2A}{\mu}\cos(\theta+B)$$

(i.e.,) $$\frac{(h^2/\mu)}{r} = 1 + \frac{h^2A}{\mu}\cos(\theta+B).$$

It represents a conic whose semi latus rectum and eccentricity are

$$l = \frac{h^2}{\mu}, \quad e = \frac{h^2 A}{\mu} \rightarrow (3).$$

Nature of the Orbit:

Multiplying both sides of (1) by $2\frac{du}{d\theta}$ and integrating w.r.t. "θ", we get

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = 2\int \frac{\mu}{h^2}\, du \quad \text{(or)}$$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = 2\frac{\mu u}{h^2} + c.$$

$$\therefore \quad h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = 2\mu u + c. \rightarrow (4).$$

In this LHS is the square of the velocity.

Therefore,

$$v^2 = 2\mu u + c \rightarrow (5).$$

substituting in (4) the value of u obtained from (2)

$$h^2\left[A^2\sin^2(\theta+B) + \frac{\mu^2}{h^4} + 2\cdot\frac{\mu}{h^2}\cdot A\cos(\theta+B) + A^2\cos^2(\theta+B)\right] = 2\mu u + c.$$

$$\text{(or)} \quad h^2\left[A^2 + \frac{\mu^2}{h^4} + \frac{2\mu A}{h^2}\cos(\theta+B)\right] = 2\mu u + c.$$

(or). $h^2A^2 + \frac{\mu^2}{h^2} + 2\mu\left[u - \frac{\mu}{h^2}\right] = 2\mu u + c$ by ②.

$h^2A^2 + \mu^2/h^2 + \cancel{2\mu u} - 2\mu^2/h^2 = \cancel{2\mu u} + c.$

(or). $h^2A^2 - \frac{\mu^2}{h^2} = c$ (or) $h^2A^2 = \frac{\mu^2}{h^2} + c.$

$\div \mu^2,$

$\therefore \frac{h^4A^2}{\mu^2} = 1 + \frac{ch^2}{\mu^2}$ (or) $e^2 = 1 + \frac{ch^2}{\mu^2}$ by ③.

$e = \frac{h^2A}{\mu}.$

Thus the orbit is an ellipse (or) parabola (or) hyperbola according as

Conditions (2m).

	Ellipse	Parabola	Hyperbola
	$e<1$	$e=1$	$e>1$
	$c<0$	$c=0$	$c>0$
(or) from (5)	$v^2-2\mu u<0$	$v^2-2\mu u=0$	$v^2-2\mu u>0.$
(or)	$v^2-\frac{2\mu}{r}<0$	$v^2-\frac{2\mu}{r}=0$	$v^2-\frac{2\mu}{r}>0$
(or)	$v^2<\frac{2\mu}{r}$	$v^2=\frac{2\mu}{r}$	$v^2>\frac{2\mu}{r}$
(or)	$v<\sqrt{\frac{2\mu}{r}}$	$v=\sqrt{\frac{2\mu}{r}}$	$v>\sqrt{\frac{2\mu}{r}}.$

Therefore the nature of the orbit depends on the velocity with which the particle is projected from the point whose distance from the centre of force is "r".

(17) 2m.

Critical Velocity :

The quantity $\sqrt{\frac{2\mu}{r}}$ is called the critical velocity at the distance 'r'. So the nature of the orbit depends on the critical velocity.

The critical velocity can be seen to be the velocity that would be acquired by a particle, in the same force field, in reaching the pt. in the question, starting from rest at the pt. at infinity. The eqn. for the motion from ∞ towards the pole is,

$$\ddot{r} = -\frac{\mu}{r^2}.$$

Multiplying both sides by $2\dot{r}$ and then integrating w.r.t. "t", we get

$$2\int \dot{r}\,\ddot{r}\,dt = -2\mu \int \frac{1}{r^2}\,dr \quad (\text{or})$$

$$\dot{r}^2 = 2\mu\frac{1}{r} + c.$$

$$\therefore \quad \left[\dot{r}^2\right]_0^v = 2\mu\left[\frac{1}{r}\right]_\infty^r \quad (\text{or})$$

$$v^2 = \frac{2\mu}{r}.$$

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Alternative Method :

The orbit and its nature can also be obtained from the differential eqn.

$$\frac{h^2}{p^3}\frac{dp}{dr} = f.$$

which is in p-r Co-ordinates. Now it is

$$\frac{h^2}{p^3}\frac{dp}{dr} = \frac{\mu}{r^2} \quad (\text{or}) \quad h^2\int\frac{dp}{p^3} = \mu\int\frac{dr}{r^2}.$$

$$\frac{h^2}{-2p^2} = \frac{\mu}{-r} + A \quad (\text{or}) \quad \frac{h^2}{p^2} = \frac{2\mu}{r} + B \rightarrow (6).$$

But w.k.t. the p-r eqns. of an ellipse, a parabola, the nearer branch of a hyperbola are

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1, \quad p^2 = ar, \quad \frac{b^2}{p^2} = \frac{2a}{r} + 1,$$

So the orbit is an ellipse (or) a parabola (or) a hyperbola according as

$$B<0, \quad B=0, \quad B>0. \rightarrow (7).$$

Also w.k.t. $pv = h$. Hence (6) becomes,

$$v^2 = \frac{2\mu}{r} + B \quad (\text{or}) \quad B = v^2 - \frac{2\mu}{r}.$$

Thus the Conditions (7) become that the orbit is an ellipse, a parabola, a hyperbola if $v < \sqrt{2\mu/r}$, $v = \sqrt{\frac{2\mu}{r}}$, $v > \sqrt{2\mu/r}$.

Book Work:

To find the orbit of a particle moving under an attractive force varying as the distance.

Proof:

Let the P.V. of the particle of mass m, at time t, be $\bar{r}$.

If $n^2 r$ is the force per unit mass, then the eqn. of motion is

$\ddot{m\bar{r}} = -mn^2 r \hat{r}$ (or) $\ddot{\bar{r}} = -n^2 \bar{r}$.

If $\bar{r} = x\bar{i} + y\bar{j}$, then we get

$\ddot{x} = -n^2 x$, $\ddot{y} = -n^2 y$.

The general solutions of these differential eqns. are

$x = A\cos nt + B\sin nt$,

$y = C\cos nt + D\sin nt$,

where the constants A, B, C, D depend upon the initial conditions. Solving these two eqns. for $\cos nt$, $\sin nt$

$$\frac{\cos nt}{\begin{vmatrix} B & -x \\ D & -y \end{vmatrix}} = \frac{-\sin nt}{\begin{vmatrix} A & x \\ C & y \end{vmatrix}} = \frac{1}{\begin{vmatrix} A & B \\ C & D \end{vmatrix}}$$

$\therefore$ $\cos nt = \dfrac{-By + Dx}{AD - BC}$, $\sin nt = \dfrac{-Ay + Cx}{AD - BC}$.

squaring and adding, we get the eqn. of the path as

$$(Cx - Ay)^2 + (Dx - By)^2 = (AD - BC)^2.$$

which being a second degree eqn. represents a conic. Further it satisfies "$h^2 - ab < 0$". Hence it represents an ellipse.

Since $(-x, -y)$ satisfies the equation, the ellipse is symmetrical about the origin.

So the centre of force is at the centre of the ellipse.

Eg:

1). S.T. the force towards the pole under which a particle describes the curve $r^n = a^n \cos n\theta$ varies inversely as the $(2n+3)^{th}$ power of the distance of the particle from the pole.

V.V.(I) 5M. 10M

Proof:

Expressing the eqn. in terms of u,

$$\frac{1}{u^n} = a^n \cos n\theta.$$

Taking logarithm and differentiating w.r.t. "θ";

$$\log\left(\frac{1}{u^n}\right) = \log(a^n \cos n\theta).$$

$\log(1/u^n)$ $\log 1 - \log u^n = \log a^n + \log \cos n\theta$.

$-n \log u$ $\quad -\frac{1}{u^n}(nu^{n-1})\frac{du}{d\theta} = 0 + \frac{1}{\cos n\theta}(-\sin n\theta)n$

$-n \, ^1/_u \, ^{du}/_{d\theta}$.

$$\frac{n}{u}\frac{du}{d\theta} = n\tan n\theta.$$

$$\frac{du}{d\theta} = u\tan n\theta.$$

$$\frac{d^2u}{d\theta^2} = u\sec^2 n\theta\,(n) + \tan n\theta \frac{du}{d\theta}$$

$$= nu\sec^2 n\theta + \tan n\theta\,(u\tan n\theta)$$

$$= u(\tan^2 n\theta + n\sec^2 n\theta).$$

$$h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = h^2u^3(\tan^2 n\theta + n\sec^2 n\theta + 1)$$

$(\because 1 + \tan^2 n\theta = \sec^2 n\theta)$

$$= h^2u^3(n+1)\sec^2 n\theta$$

$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}$

$$= h^2 \cdot \frac{1}{r^3}(n+1)\frac{a^{2n}}{r^{2n}}.$$

$r^n = a^n \cos n\theta$

$\cos n\theta = \frac{r^n}{a^n}$

$h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = F$.

$\sec n\theta = \frac{a^n}{r^n}$.

$$= \frac{h^2(n+1)a^{2n}}{r^{2n+3}}$$

$F = h^2(n+1)a^{2n}\frac{1}{r^{2n+3}}$ //.

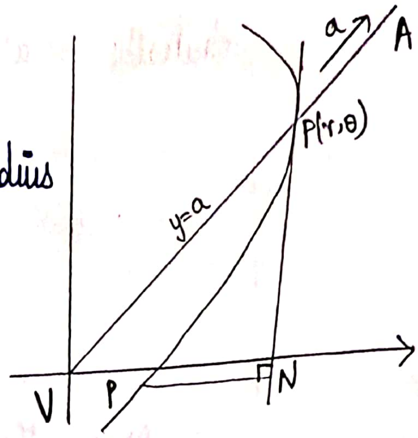
$\therefore F \propto \frac{1}{r^{2n+3}}$.

This is the central attractive force which varies inversely as r^{2n+3}.

2). A particle moves with a central acceleration μr^{-7} and starts from an apse at a distance 'a' with a velocity equal to the velocity which would be acquired by the particle travelling from rest at infinity to the apse. S.T. the eqn. of its orbit is $r^2 = a^2\cos 2\theta$.

Proof :

For the motion along the radius vector, from ∞ to any point $P(r,\theta)$ on the orbit, we have



$$\ddot{r} = -\frac{\mu}{r^7} \rightarrow (1).$$

Let v be the velocity attained by the particle, travelling from rest at ∞ to P. Then multiplying both sides of (1) by $2\dot{r}$ and integrating w.r.t "t",

$$2\dot{r}\ddot{r} = \frac{-\mu}{r^7} 2\dot{r}.$$

$$\int d(\dot{r})^2 = -2\mu \int \frac{1}{r^7} \frac{dr}{dt}\, dt.$$

$$\left[\dot{r}^2\right]_0^v = -\int_\infty^r \frac{\mu}{r^7}\left(2\frac{dr}{dt}\right)dt$$

$$= -2\mu \int_\infty^r \frac{1}{r^7}\, dr.$$

(i.e.,) $$v^2 = \frac{\mu}{3r^6}.$$

So the square of velocity v^2 of projection at the apse $r=a$ is

$$v^2 = \frac{\mu}{3a^6}.$$

Next we shall find the value of the constant h^2.

Initially at the apse $r = p = a$ and, from $h = pv$,

$$h^2 = p^2v^2 = a^2\left(\frac{\mu}{3a^6}\right)$$

$$= \frac{\mu}{3a^4} \rightarrow ②.$$

Now the differential eqn. of the orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2u^2r^7}.$$

Eliminating h^2 by (2) & simplifying,

$$\frac{d^2u}{d\theta^2} + u = 3a^4u^5.$$

$$\frac{d^2u}{d\theta^2} + u = \frac{\cancel{\mu}\, 3a^4}{\cancel{\mu}\, u^2 r^7}. \quad \left(\because r = \frac{1}{u}\right)$$

$$= 3a^4u^7.u^{-2} = 3a^4u^5.$$

X^{ly} both sides by $2\frac{du}{d\theta}$ & integrating w.r.t. "θ", we get

$$\int\left(2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}\right)d\theta + \int\left(2u\frac{du}{d\theta}\right)d\theta =$$

$$\int\left(3a^4u^5\, 2\frac{du}{d\theta}\right)d\theta.$$

$$\Rightarrow \int d\left(\frac{du}{d\theta}\right)^2 + \int d(u^2) = 6a^4\int u^5\,du.$$

$$\left(\frac{du}{d\theta}\right)^2 + u^2 = \cancel{6}a^4\frac{u^6}{\cancel{6}} + c$$

(or) $$\left(\frac{du}{d\theta}\right)^2 + u^2 = a^4u^6 + c \Rightarrow \frac{1}{\cancel{a^2}} = \cancel{a^4}\frac{1}{\cancel{a^6}} + c.$$

$$1 + c = 1$$

$$\boxed{c = 0}$$

But, initially $\frac{du}{d\theta} = 0,\ u = \frac{1}{a}$.

$\therefore$ $c = 0$ and $$\left(\frac{du}{d\theta}\right)^2 = a^4u^6 - u^2$$

since $u = \frac{1}{r}$, $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$. Therefore

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{a^4}{r^6} - \frac{1}{r^2} \quad \text{(or)}$$

$$\frac{dr}{d\theta} = \frac{\sqrt{a^4 - r^4}}{r}$$

$$\therefore \quad \frac{r}{\sqrt{a^4 - r^4}}\, dr = d\theta .$$

setting $r^2 = y$ and integrating, $\Rightarrow 2r\,dr = dy$.

$$\frac{1}{2}\frac{1}{\sqrt{a^4 - y^2}}\, dy = d\theta, \quad \sin^{-1}\frac{y}{a^2} = 2\theta + c.$$

$$\therefore \quad \sin^{-1}\frac{r^2}{a^2} = 2\theta + c.$$

If the initial line is chosen through the apse so that the apse is $(a, 0)$, Then, when $r = a$, $\theta = 0$.

This gives $c = \pi/2$.

$$\therefore \quad \frac{r^2}{a^2} = \sin\left(2\theta + \pi/2\right) \quad \text{(or)} \quad r^2 = a^2\cos 2\theta .$$

CONIC AS A CENTRAL ORBIT:

V.V. (7.)

Book work :

When a central orbit is a conic with the centre of the force at one focus, to find the law of force and the speed of the particle.

Force.

Choosing the focus S as the pole we get the polar eqn. of the conic as

$$\frac{l}{r} = 1 + e\cos\theta \quad (or) \quad u = \frac{1}{l} + \frac{e}{l}\cos\theta,$$

where l is the semilatus rectum. Therefore,

$$\frac{du}{d\theta} = -\frac{e}{l}\sin\theta, \quad \frac{d^2u}{d\theta^2} = -\frac{e}{l}\cos\theta.$$

$$\therefore h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right) = h^2u^2\left[-\frac{e}{l}\cos\theta + \frac{1}{l} + \frac{e}{l}\cos\theta\right]$$

$$= h^2u^2\frac{1}{l} = \frac{h^2}{l}\cdot\frac{1}{r^2}.$$

Thus the force per unit mass in the radial direction but towards the pole is

$$\frac{h^2}{l}\cdot\frac{1}{r^2} \rightarrow (1). \qquad F \propto 1/r^2 \rightarrow \text{Inverse square law.}$$

Thus, the force per unit mass in the radial direction but towards the pole is

$\frac{h^2}{}$

which is inversely proportional to the square of the distance from the pole. It is an attractive central force.

Inverse square law :

In the above book work the force (1) is,

$$\frac{h^2}{l} \cdot \frac{1}{r^2},$$

which is inversely proportional to the square of the distance. This rule of force is called inverse square law.

Velocity :

The square of the speed of the particle at a distance 'r' is,

$$v^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right].$$

$$= h^2\left[\left(-\frac{e}{l}\sin\theta\right)^2 + \left(\frac{1}{l} + \frac{e}{l}\cos\theta\right)^2\right]$$

$$= \frac{h^2}{l^2}\left[e^2\sin^2\theta + 1 + 2e\cos\theta + e^2\cos^2\theta\right)$$

$$= \frac{h^2}{l^2}\left(e^2+1+2e\cos\theta\right) = \frac{h^2}{l^2}\left[e^2+1+2\left[\frac{l}{r}-1\right)\right]$$

$$= \frac{h^2}{l^2}\left(e^2+1+2\frac{l}{r}-2\right)$$

$$= \frac{h^2}{l^2}\left(e^2-1+\frac{2l}{r}\right)$$

$$= \frac{h^2}{l}\left[\frac{e^2-1}{l} + \frac{2}{r}\right]$$

$$= \mu\left[\frac{2}{r} + \frac{e^2-1}{l}\right].$$

where $\mu = \frac{h^2}{l}$.

Parabola:

When the path is a parabola, $e=1$ and therefore

$$v^2 = \mu \frac{2}{r}.$$

Ellipse:

When the path is an ellipse,

$$v^2 = \mu\left[\frac{2}{r} + \frac{e^2-1}{l}\right]$$

$$= \mu\left[\frac{2}{r} + \frac{e^2-1}{b^2/a}\right] \to l.$$

$$= \mu\left[\frac{2}{r} + \frac{a(e^2-1)}{b^2}\right]$$

$$= \mu\left[\frac{2}{r} + \frac{a^2(e^2-1)}{ab^2}\right]$$

$$= \mu\left[\frac{2}{r} - \frac{b^2}{ab^2}\right]$$ since $b^2 = a^2(1-e^2)$

$b^2 = (a^2(e^2-1)) \rightarrow$ hyperbola.

$$v^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right].$$

Maximum & minimum velocities:

If A, A' are the ends of the major axis such that A is closer to S (the pole) than A', r is a minimum at A and maximum at A'. But the velocity is a maximum when "r" is a minimum. Thus the velocity is a maximum at A and a minimum at "A'".

Periodic time :

When the orbit is an ellipse the periodic time of the particle is the total area divided by the constant areal velocity. So it is

$$\frac{\pi ab}{h/2} = \frac{\pi ab}{\frac{1}{2}\sqrt{\mu l}}$$

$$= \frac{2\pi ab\sqrt{a}}{\sqrt{\mu}\, b} \qquad l = b^2/a.$$

$$= \frac{2\pi}{\sqrt{\mu}} a^{3/2}.$$

Hyperbola :

When the path is the branch of the hyperbola nearer to the centre of force,

$$b^2 = a^2(e^2-1) \quad (\text{or}) \quad e^2 - 1 = \frac{b^2}{a^2}.$$

and consequently

$$v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right).$$

Remark :

It is important to note that when the path is the branch of the hyperbola not nearer to the centre of force, its eqn. is

$$\frac{l}{r} = -1 + e\cos\theta$$

In this case the force per unit mass is

$$\frac{h^2}{l} \cdot \frac{1}{r^2}\hat{r}.$$

It is a repulsive force as we have in the case of like charges.

V.V (I) SB (or) 2m.

KEPLER'S LAWS OF PLANETARY MOTION :-

Before the development of mathematical theory of planetary motion by Newton, with the assumption that any two particles attract each other with a force $\gamma m_1 m_2 / r^2$, where γ is a universal constant, m_1 & m_2 are the masses of the particles and r is the distance b/w them, Kepler propounded the following three laws on the basis of astronomical observations:-

K.1. :- The planets describe ellipses about the sun as focus.

K.2. :- The radius vector drawn from the sun to a planet sweeps out equal areas in equal times.

K.3. :- The squares of the periodic times of the planets are proportional to the cubes of the semi major axis of their respective orbits.

It is obvious that K.2. implies that for each planet the areal velocity $\frac{1}{2}r^2\dot{\theta}$ is a constant and the transverse component of the force acting on the planet is zero.

So the force acting on the planet is along the radius vector and thus it is a central force, the sun being the centre of force.

K.1. implies that the force of attraction on the planet is inversely proportional to the square of its distance from the sun.

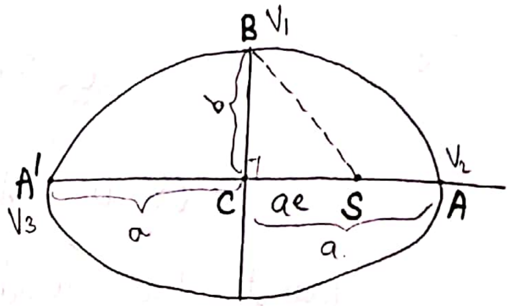
So Newton's work is the assumption of inverse square law establishing the truth of Kepler's statement mathematically.

Eg :-

(ⓘ 5 B). A particle describes an elliptic orbit under a central force towards one focus S. If v_1 is the speed at the end B of the minor axis and v_2, v_3 the speeds at the ends A, A' of the major axis, S.T. $V_1^2 = V_2 V_3$.

Proof:

In the present case the velocity formula is



$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right).$$

Now the distances r of the points A, A', B from S are

$$SA = CA - CS = a - ae = a(1-e).$$

$$SA' = SC + CA' = ae + a = a(1+e).$$

$$SB = \sqrt{(ae)^2 + b^2} = a. \quad \to \text{Pythagoras Thm.}$$

$$= \sqrt{a^2e^2 + a^2(1-e^2)}$$

$$= \sqrt{a^2\cancel{e^2} + a^2 - a^2\cancel{e^2}}$$

$$= a.$$

$$\therefore \quad V_1^2 = \mu\left(\frac{2}{a} - \frac{1}{a}\right) = \frac{\mu}{a},$$

$$V_2^2 = \mu\left\{\frac{2}{a(1-e)} - \frac{1}{a}\right\}$$

$$= \frac{\mu}{a}\,\frac{2-1+e}{1-e} = \frac{\mu}{a}\,\frac{1+e}{1-e}.$$

$$V_3^2 = \mu\left\{\frac{2}{a(1+e)} - \frac{1}{a}\right\}$$

$$= \frac{\mu}{a}\,\frac{2-1-e}{1+e} = \frac{\mu}{a}\,\frac{1-e}{1+e}.$$

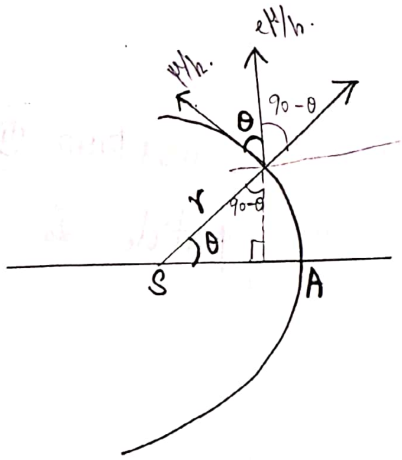
$$V_2^2 V_3^2 = \frac{\mu^2}{a^2} = V_1^4 \text{ (or) } V_2 V_3 = V_1^2.$$

V.V.I.Q. 10m.

Q. S.T. the velocity of a particle moving in an ellipse about the centre of force at a focus is compounded of two constant velocities, namely,

(i). $\frac{\mu}{h}$ perpendicular to the radius vector.

(ii). $\frac{e\mu}{h}$ perpendicular to the major axis.



Proof :- (iii).

If the eqn. of the ellipse is

$$\frac{l}{r} = 1+e\cos\theta , \rightarrow ①$$

Then we know that

$$\mu = \frac{h^2}{l} \text{ (or) } l = \frac{h^2}{\mu} \rightarrow ②.$$

Also we have,

$$r^2\dot{\theta} = h \rightarrow ③.$$

The velocity components in the radial & transverse directions are

$$\dot{r}, r\dot{\theta}.$$

Differentiating the eqn. ① w.r.t. "t",

$$-\frac{l}{r^2}\dot{r} = -e\sin\theta\,\dot{\theta} . \dot{r} = r^2 e\sin\theta\,\dot{\theta}/l$$
$$= \frac{e\sin\theta\,(h)}{l}.$$

By ③,

$$\therefore \quad \dot{r} = \frac{e\sin\theta}{l}(r^2\dot{\theta}) = \frac{eh}{l}\sin\theta \text{ by } ③,$$

$$= \frac{e\mu}{h} \sin\theta \text{ by } ② \longrightarrow ④.$$

$$r\dot{\theta} = \frac{r^2\dot{\theta}}{r} = \frac{h}{r} \text{ by } ③.$$

$$= h\,\frac{1+e\cos\theta}{l} = \frac{\mu}{h}(1+e\cos\theta) \text{ by } ②,$$

$$= \frac{\mu}{h} + \frac{e\mu}{h}\cos\theta. \longrightarrow ⑤.$$

From ④ & ⑤, we can say that the velocity of the particle is composed of three velocities, namely,

$\frac{e\mu}{h}\sin\theta$ in the radial direction, $\rightarrow$ ⑥.

$\frac{e\mu}{h}\cos\theta$ in the transverse direction, $\rightarrow$ ⑦.

$\frac{\mu}{h}$ in the transverse direction $\rightarrow$ ⑧.

From the figure it is clear that the radial & transverse directions are inclined to the perpendicular to the major axis at an angle θ and $90°-\theta$. So the resultant of ⑥ & ⑦ is the velocity

$\frac{e\mu}{h}$ perpendicular to the major axis. $\rightarrow$ ⑨.

Now ⑧ & ⑨ constitute the required result.

Hence. $\frac{e\mu}{h}$ $\perp^r$ to major axis & μ/h $\perp^r$ to radius vector.