28.11.2019

UNIT-III

LAPLACE TRANSFORMS.

1] $L(k) = \frac{k}{s}$; if $s > 0$.

2] $L(e^{at}) = \frac{1}{s-a}$

3] $L(e^{-at}) = \frac{1}{s+a}$

4] $L(\sin at) = \frac{a}{s^2+a^2}$

5] $L(\cos at) = \frac{s}{s^2+a^2}$

6] $L(t^n) = \frac{n!}{s^{n+1}}$

7] $L(\sinh at) = \frac{a}{s^2-a^2}$

8] $L(\cosh at) = \frac{s}{s^2-a^2}$

Laplace transform (definition):-

Let $f(t)$ be a ~~continuous~~ function of small t. defined for $t \geq 0$: multiply $f(t)$ by e^{-st}, thus getting $e^{-st} f(t)$.

$$F(s) = \int_0^{\infty} e^{-st} \cdot f(t)\, dt.$$

Definition:-

If a function $f(t)$ define for all positive values of the variables and of $\int_0^{\infty} e^{-st} f(t)\, dt$ exists and it is equal to $F(s)$, then $F(s)$ is called Laplace transform of $f(t)$ and it is denoted by the Symbol $L\{f(t)\}$. Hence.

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt = F(s)$$

The operator L that transform $f(t)$. $F(s)$ is is called the Laplace transform operator.

Note:-

$$\underset{s \to 0}{Lt} F(s) = 0$$

Piecewise continuity:-

A function $f(t)$ is said to be piecewise continuous in a $[a,b]$, if it is defined on that interval and is such that the interval can be broken up into a finite number of such interval in each of which $f(t)$ is continuous. $f(t)$ can have only ordinary finite discontinuity in the interval.

Sufficient condition for the existence of the laplace transform:-

i) $f(t)$ is continuous (or) piecewise continuous in the $[a,b]$ where $a>0$.

ii) It is of exponential order.

iii) $t^n f(t)$ is bounded near, $t=0$ for some number $n>1$.

Scanned with CamScanner

Results :-

i) $L\{c f(t)\} = c L\{f(t)\}$ where c is a constant.

Proof:-

we have

$$L\{c f(t)\} = \int_0^\infty e^{-st} c f(t)\, dt.$$

$$= c \int_0^\infty e^{-st} f(t)\, dt.$$

where c is constant

$L\{c f(t)\} = c L\{f(t)\}$ [$\because$ By the definition]

ii) $L\{e^{at}\} = \frac{1}{s-a}$ provided $s-a > 0$.

Proof :-

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

$$L\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\, dt.$$

$$= \int_0^\infty e^{-t(s-a)}\, dt.$$

$$= \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^\infty$$

Scanned with CamScanner

$$= \frac{e^{-\infty}}{-(s-a)} - \frac{e^{0}}{-(s-a)}$$

$$= 0 + \frac{1}{s-a}\left[\because \begin{matrix} e^{0} = 1 \\ e^{-\infty} = 0 \end{matrix}\right]$$

$$L\{e^{at}\} = \frac{1}{s-a} > 0.$$

iii) $L\{e^{-at}\} = \frac{1}{s+a}$ provided $s+a > 0$.

Proof:-

$$L\{f(t)\} = \int_0^{\infty} e^{-st} f(t)\, dt.$$

$$L\{e^{-at}\} = \int_0^{\infty} e^{-st} e^{-at}\, dt.$$

$$= \int_0^{\infty} e^{-(s+a)t}\, dt.$$

$$= \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_0^{\infty}$$

$$= \frac{e^{-\infty}}{-(s+a)} - \frac{e^{0}}{-(s+a)}$$

$$= \frac{1}{s+a}\left[\because \begin{matrix} e^{0} = 1 \\ e^{-\infty} = 0 \end{matrix}\right]$$

$$L\{e^{-at}\} = \frac{1}{s+a}, \quad s+a > 0.$$

iv) $L(1) = \frac{1}{s}$, if $s > 0$.

Proof:-

w.k.t

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

$$L(1) = \int_0^\infty e^{-st} (1)\, dt.$$

$$= \left[\frac{e^{-st}}{-s}\right]_0^\infty$$

$$= \left[\frac{e^{-\infty}}{-s} + \frac{e^{0}}{s}\right]$$

$$= 0 + 1/s = 1/s$$

v) $L(t) = \frac{1}{s^2}$, if $s > 0$.

Proof:-

w.k.t

$$L(t) = \int_0^\infty e^{-st}. t\, dt \qquad \int u\,dv = uv - \int v\,du.$$

$$u = t \qquad dv = e^{-st} dt$$

$$du = dt \qquad v = \frac{e^{-st}}{-s}$$

$$L\{t\} = \left[t \cdot \frac{e^{-st}}{-s}\right]_0^{\infty} + \int_0^{\infty} \frac{e^{-st}}{s}\, dt$$

$$= (0) + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_0^{\infty}$$

$$= \frac{1}{s}\left\{\frac{e^{-\infty}}{-s} + \frac{e^{0}}{s}\right\}$$

$$= \frac{1}{s}\left[\frac{1}{s}\right]$$

$$L\{t\} = \frac{1}{s^2}$$

Laplace transform of sinhat

$$L\{\sinh at\} = \int_0^{\infty} e^{-st} \sinh at\, dt.$$

$$= \int_0^{\infty} e^{-st} \cdot \frac{e^{at} - e^{-at}}{2}\, dt.$$

$$= \frac{1}{2} \cdot \int_0^{\infty} e^{-(s-a)t} dt - \frac{1}{2}\int_0^{\infty} e^{-(s+a)t} dt.$$

$$\frac{1}{2(s-a)} - \frac{1}{2(s+a)} \text{ if } s > |a|$$

$$= \frac{s+a-s+a}{2(s^2-a^2)} = \frac{a}{s^2-a^2}$$

Laplace transform of coshat.

$$L\{\cosh at\} = \int_0^{\infty} e^{-st} \cosh at\, dt.$$

$$= \int_0^{\infty} e^{-st} \cdot \left(\frac{e^{at} + e^{-at}}{2}\right) dt.$$

$$= \frac{1}{2}\int_0^\infty e^{-(s-a)t}\,dt + \frac{1}{2}\int_0^\infty e^{-(s+a)t}\,dt.$$

$$= \frac{1}{2}\cdot\frac{1}{s-a} + \frac{1}{2(s+a)}$$

$$= \frac{s}{s^2-a^2} \text{ if } s > |a|.$$

I. Linearity property:-

If $f_1(t)$ and $f_2(t)$ are two functions of t defined for positive values of t. if c is a constant then,

$$L\{f_1(t)+f_2(t)\} = L\{f_1(t)\} + L\{f_2(t)\}$$

and $L\{cf(t)\} = cL\{f(t)\}$

Proof:-

$L\{f_1(t)+f_2(t)\}$

$$= \int_0^\infty e^{-st}\,[f_1(t)+f_2(t)]\,dt.$$

$$= \int_0^\infty e^{-st} f_1(t)\,dt + \int_0^\infty e^{-st} f_2(t)\,dt.$$

$$= L\{f_1(t)\} + L\{f_2(t)\}$$

Scanned with CamScanner

$$L\{cf_1(t)\} = \int_0^\infty e^{-st} c f_1(t)\, dt.$$

$$= c\int_0^\infty e^{-st} f_1(t)\, dt.$$

$$= cL\{f_1(t)\}$$

ii) Shifting property :-

If $L\{f(t)\} = \bar{f}(s)$ then

$$L\{e^{-at} f(t)\} = \bar{f}(s+a)$$

Proof :-

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \bar{f}(s)$$

$$L\{e^{-at} f(t)\} = \int_0^\infty e^{-st}\, e^{-at} f(t)\, dt.$$

$$= \int_0^\infty e^{-(s+a)t} f(t)\, dt.$$

$$= \bar{f}(s+a)$$

Note :- $L\{e^{at} f(t)\} = \bar{f}(s-a)$

iii) Change of scale property:-

$$\text{If } L\{f(t)\} = \bar{f}(s) \text{ then}$$

$$L\{f(at)\} = \frac{1}{a}\bar{f}\left[\frac{s}{a}\right]$$

Proof:-

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt.$$

$$L\{f(at)\} = \int_0^\infty e^{-st} f(at)\, dt.$$

Put $at = t'$

$a\,dt = dt'$

(or) $dt = \frac{1}{a} dt'$

$\therefore L\{f(at)\}$

$$= \int_0^\infty e^{-st} f(at)\, dt.$$

$$= \int_0^\infty e^{-\frac{s}{a}t'} f(t') \frac{1}{a} dt'$$

$$= \frac{1}{a}\int_0^\infty e^{-s/a\, t'} f(t')\, dt'$$

$$= \frac{1}{a}\int_0^\infty e^{-\frac{s}{a}t} f(t)\, dt.$$

$$= \frac{1}{a}\bar{f}\left(\frac{s}{a}\right)$$

Ex:1

Find the Laplace transform:

i) t^3 ii) e^{-4t} iii) $\sin 4t$ iv) $\cos t/2$.

Soln:-

i) $L\{t^3\} = \frac{3!}{s^4} = \frac{6}{s^4}$

ii) $L\{e^{-4t}\} = \frac{1}{s+4}$

iii) $\sin 4t = \frac{4}{s^2+16}$

iv) $L\{\cos t/2\} = \frac{s}{s^2+\frac{1}{4}}$

$= \frac{4s}{4s^2+1}$

Ex:2

i) $\sin^2 3t$

$= \frac{1-\cos 6t}{2}$

$L\{\sin^2 3t\} = L\{\frac{1}{2} - \frac{1}{2}\cos 6t\}$

Scanned with CamScanner

$= \frac{1}{2}L\{1\} - \frac{1}{2}L\{\cos 6t\}$

$= \frac{1}{2s} - \frac{s}{2(s^2+36)} = \frac{1}{2}\left[\frac{s^2+36-s^2}{s(s^2+36)}\right]$

ii) $\cos^3 t$

$= \frac{1}{2}\left(\frac{36}{s(s^2+36)}\right) = \frac{18}{s(s^2+36)}$

$\cos 3t = 4\cos^3 t - 3\cos t$

$4\cos^3 t = \frac{\cos 3t + 3\cos t}{4}$

$L\{\cos^3 t\} = \frac{1}{4}L\{\cos 3t\} + \frac{3}{4}L\{\cos t\}$

$= \frac{1}{4}\frac{s}{s^2+9} + \frac{3s}{4(s^2+1)}$

$= \frac{s[s^2+1] + 3[s^2+9]}{4(s^2+9)(s^2+1)}$

$= \frac{s[4s^2+28]}{4(s^2+1)(s^2+9)}$

$= \frac{s(s^2+7)}{(s^2+1)(s^2+9)}$

iii) $\sin^3 2t$.

$\sin 3A = 3\sin A - 4\sin^3 A$

$4\sin^3 A = 3\sin A - \sin 3A$

$\sin^3 A = \frac{3\sin A - \sin 3A}{4}$

$\sin^3 2t = \frac{3\sin 2t - \sin 6t}{4}$

$\therefore L\{\sin^3 2t\} = \frac{3}{4}L\{\sin 2t\} - \frac{1}{4}L\{\sin 6t\}$

$= \frac{3}{4}\frac{2}{s^2+4} - \frac{1}{4}\cdot\frac{6}{s^2+36}.$

$$= \frac{6[s^2+36-(s^2+4)]}{4(s^2+4)(s^2+36)}$$

$$= \frac{\frac{8}{4}}{(s^2+4)(s^2+36)}$$

iv) $\cos^4 t$

$$\cos^4 t = (\cos^2 t)^2$$

$$= \left(\frac{1+\cos 2t}{2}\right)^2$$

$$= \frac{1}{4}\left[1+2\cos 2t+\cos^2 2t\right]$$

$$= \frac{1}{4}\left[1+2\cos 2t+\frac{1+\cos 4t}{2}\right]$$

$$= \frac{1}{8}\left[2+4\cos 2t+1+\cos 4t\right]$$

$$= \frac{1}{8}\left[3+4\cos 2t+\cos 4t\right]$$

$$L\{\cos^4 t\} = \frac{1}{8}\left[[3L(1)]+4L\{\cos 2t\}+L\{\cos 4t\}\right]$$

$$= \frac{1}{8}\left[\frac{3}{s}+4\cdot\frac{s}{s^2+4}+\frac{s}{s^2+16}\right]$$

Ex: 3

i) $\cos 4t \sin 3t$

$$= \frac{1}{2}\left[\sin 7t - \sin t\right]$$

$= \frac{1}{2} L\{\sin 7t - \sin t\}$

$= \frac{1}{2} L\{\sin 7t\} - \frac{1}{2} L\{\sin t\}$

$= \frac{1}{2} \frac{7}{s^2+49} - \frac{1}{2} \cdot \frac{1}{s^2+1}$ //

$L(\sin at) = \frac{a}{s^2+a^2}$, if $s>0$.

$\longrightarrow$

$L\{\sin at\} = \int_0^\infty e^{-st} \sin at \, dt.$

w.r.t

$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx) + k.$

$a = -s$
$b = a$
$x = t.$

$L\{\sin at\} = \left[\frac{e^{-st}}{s^2+a^2}(-s \sin at - a \cos at)\right]_0^\infty$

$= \left\{0 - \left[\frac{1}{s^2+a^2}(-a)\right]\right\}$

$L\{\sin at\} = \frac{a}{s^2+a^2}$

$L(\cos at) = \frac{s}{s^2+a^2}$, if $s > 0$.

$\rightarrow$

$L(\cos at) = \int_0^\infty e^{-st} \cos at \, dt.$

w.r.t.

$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) + k.$

$$L\{\cos at\} = \left[\frac{e^{-st}}{s^2+a^2}(-s\cos at + a\sin at)\right]_0^\infty$$

$$= \left\{0 - \frac{1}{(s^2+a^2)}(-s)\right\}$$

$$L\{\cos at\} = \frac{s}{s^2+a^2}$$

Ex: 3

2] $\sin^2 t \cos^3 t$.

→

$$\sin^2 t \cos^3 t = \left(\frac{1-\cos 2t}{2}\right)\left(\frac{\cos 3t + 3\cos t}{4}\right)$$

$$= \frac{1}{8}(\cos 3t + 3\cos t - \cos 2t\cos 3t - 3\cos 2t\cos t)$$

$$= \frac{1}{8}\left\{\cos 3t + 3\cos t - \frac{1}{2}(\cos 5t + \cos t) - 3\left(\frac{1}{2}(\cos 3t + \cos t)\right)\right\}$$

$$= \frac{1}{8}\left\{\cos 3t + 3\cos t - \frac{1}{2}\cos 5t - \frac{1}{2}\cos t - \frac{3}{2}\cos 3t - \frac{3}{2}\cos t\right.$$

$$= \frac{1}{8}\left(-\frac{1}{2}\cos 3t + \cos t - \frac{1}{2}\cos 5t\right)$$

$$L(\sin^2 t\cos^3 t) = \frac{1}{8}\left\{-\frac{1}{2}L(\cos 3t) + L(\cos t) - \frac{1}{2}L(\cos 5t)\right\}$$

$$= \frac{1}{8}\left\{-\frac{1}{2}\left(\frac{s}{s^2+9}\right) + \frac{s}{s^2+1} - \frac{1}{2}\left(\frac{s}{s^2+25}\right)\right.$$

$$= \frac{1}{16}\left\{\frac{2s}{s^2+1} - \frac{s}{s^2+9} - \frac{s}{s^2+25}\right\}$$

3] $t^4 - t^2 - t + \sin\sqrt{2}t$.

$\rightarrow$

$$L(t^4) = \frac{4!}{s^5} = \frac{4\times3\times2\times1}{s^5} = \frac{24}{s^5}$$

$$L(t^2) = \frac{2!}{s^3} = \frac{2\times1}{s^3} = \frac{2}{s^3}$$

$$L(t) = \frac{1}{s^2}$$

$$L(\sin\sqrt{2}t) = \frac{\sqrt{2}}{s^2+2}.$$

$$L\{t^4 - t^2 - t + \sin\sqrt{2}t\} = \frac{24}{s^5} - \frac{2}{s^3} - \frac{1}{s^2} + \frac{\sqrt{2}}{s^2+2}.$$

4] find the laplace transform of the following function

$$f(t) = \begin{cases} \sin t & 0 < t < \pi \\ 0 & t > \pi \end{cases}$$

$\rightarrow$

$$L(f(t)) = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\pi e^{-st} \sin t\, dt + \int_\pi^\infty e^{-st} (0)\, dt$$

$$\int e^{ax} \sin bx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$= \left[\frac{e^{-st}}{s^2+1}(-s\sin t - \cos t)\right]_0^\pi$$

$$= \frac{e^{-s\pi}}{s^2+1}(1) - \frac{1}{s^2+1}(-1)$$

$$= \frac{(e^{-s\pi}+1)}{s^2+1}$$

2) $f(t) = \begin{cases} e^{2t} & 0 < t < 3 \\ 1 & t > 3. \end{cases}$

$\rightarrow$

$$L(f(t)) = \int_0^\infty e^{-st} f(t)\,dt.$$

$$= \int_0^3 e^{-st} \cdot e^{2t}\,dt + \int_3^\infty e^{-st}(1)\,dt.$$

$$= \int_0^3 e^{-(s-2)t}\,dt + \int_3^\infty e^{-st}\,dt.$$

$$= \left[\frac{e^{-(s-2)t}}{-(s-2)}\right]_0^3 + \left[\frac{e^{-st}}{-s}\right]_3^\infty$$

$$= \left[\frac{e^{-3(s-2)}}{-(s-2)}\right] - \left[\frac{1}{-(s-2)}\right] + \left[0 + \frac{e^{-3s}}{s}\right]$$

$$= \frac{1-e^{-3(s-2)}}{s-2} + \frac{e^{-3s}}{s}$$

1] Multiplication of t.

$$L\{t f(t)\} = -\frac{d}{ds}\bar{f}(s)$$

$$L\{t^2 f(t)\} = (-1)^2 \frac{d^2}{ds^2}\bar{f}(s)$$

$\vdots \qquad \vdots$

$$L\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\bar{f}(s)$$

2] Division of "t"

$$L\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s)\,ds$$

1] Find the Laplace transform of the following function.

i) $e^{7t}\sin^2 t$

$\rightarrow$

$$L(e^{7t}\sin^2 t) = L\left(e^{7t}\left(\frac{1-\cos 2t}{2}\right)\right)$$

$$= \frac{1}{2}L(e^{7t}) - \frac{1}{2}L(e^{7t}\cos 2t)$$

$$= \frac{1}{2}\left(\frac{1}{s-7}\right) - \frac{1}{2}\left\{e^{7t}\cdot\frac{s}{s^2+4}\right\}$$

$$= \frac{1}{2}\left(\frac{1}{s-7}\right) - \frac{1}{2}\left\{\frac{s-7}{(s-7)^2+4}\right\}$$

2) $e^{-3t}\cos 2t$.

$$L(e^{-3t}\cos 2t) = L(e^{-3t}\cos 2t)$$

$$= L(e^{-3t}).L(\cos 2t)$$

$$= \frac{1}{s+3}\left(\frac{s}{s^2+4}\right)$$

3) $e^{-2t}\sin 2t + e^{3t}$.

$$L(e^{-2t}\sin 2t + e^{3t}) = L(e^{-2t})L(\sin 2t) + L(e^{3t})$$

$$= L(e^{-2t})L\left(\frac{a}{s^2+a^2}\right) + \left(\frac{1}{s-3}\right)$$

4] $2t^2 e^{-t} - t + \cos 4t.$

$\rightarrow$

$$L[f(t)] = 2L(t^2 e^{-t}) - L(t) + L(\cos 4t)$$

$$= 2L(e^{-t})\left(\frac{2}{s^3}\right) - \frac{1}{s^2} + \frac{1}{s^2+16}$$

$$= 2\left[\frac{2}{(s+1)^3}\right] - \frac{1}{s^2} + \frac{s}{s^2+16}$$

$$= \frac{4}{(s+1)^3} - \frac{1}{s^2} + \frac{s}{s^2+16}$$

5] $t e^{2t} \cos 5t$

$\rightarrow$

$$L(\cos 5t) = \frac{s}{s^2+25}$$

$$L(t\cos 5t) = -\frac{d}{ds}\left(\frac{s}{s^2+25}\right)$$

$$= -\left\{\frac{(s^2+25)(1) - s(2s)}{(s^2+25)^2}\right\} \quad \frac{u}{v}$$

$$= -\left\{\frac{s^2+25-2s^2}{(s^2+25)^2}\right\}$$

$$L(t\cos 5t) = \frac{s^2-25}{(s^2+25)^2}$$

$$L(t e^{2t} \cos 5t) = \frac{(s-2)^2 - 25}{((s-2)^2+25)^2}$$

$$= \frac{s^2-4s+4-25}{(s^2-4s+4+25)^2}$$

$$= \frac{s^2-4s-21}{(s^2-4s+29)^2}$$

6] $(1+e^{-2t})^2$

$\rightarrow$

$L(1+e^{-2t})^2 = L(1+2e^{-2t}+e^{-4t})$

$= L(1) + 2L(e^{-2t}) + L(e^{-4t})$

$= \frac{1}{s} + 2\frac{1}{s+2} + \frac{1}{s+4}$

$= \frac{1}{s} + \frac{2}{s+2} + \frac{1}{s+4}$

7] $t\sin^2 t$.

$\rightarrow$

$L(t\sin^2 t) = L\left[t\left(\frac{1-\cos 2t}{2}\right)\right]$

$= \frac{1}{2}\left\{L(t - t\cos 2t)\right\}$

$= \frac{1}{2}L(t) - \frac{1}{2}L(t\cos 2t)$

$= \frac{1}{2}\cdot\frac{1}{s^2} - \frac{1}{2}\left(\frac{-d}{ds}\left(\frac{s}{s^2+4}\right)\right) \frac{u}{v}$

$= \frac{1}{2s^2} + \frac{1}{2}\left[\frac{(s^2+4)(1) - s(2s)}{(s^2+4)^2}\right]$

$= \frac{1}{2s^2} + \frac{1}{2}\left[\frac{4-s^2}{(s^2+4)^2}\right]$

8] $t^2\cos 4t$.

$\rightarrow$

$L(t^2\cos 4t) = \frac{d^2}{ds^2} L(\cos 4t)$

$= \frac{d^2}{ds^2}\left(\frac{s}{s^2+16}\right)$

Scanned with CamScanner

$$= \frac{d}{ds}\left[\frac{d}{ds}\left(\frac{s}{s^2+16}\right)\right]$$

$$= \frac{d}{ds}\left(\frac{(s^2+16)(1) - (s)(2s)}{(s^2+16)^2}\right)$$

$$= \frac{d}{ds}\left(\frac{16-s^2}{(s^2+16)^2}\right)$$

$$= \frac{(s^2+16)^2(-2s) - (16-s^2)(2(s^2+16)(2s))}{(s^2+16)^4}$$

$$= \frac{(2s)\cancel{(s^2+16)}\{-s^2-16-32+2s^2\}}{(s^2+\cancel{16})^{\cancel{4}}}$$

$$= \frac{(2s)(s^2-48)}{(s^2+16)^3}$$

$$= \frac{2s^3-96s}{(s^2+16)^3} \,//$$

9] $t^2 \cosh at$.

$\rightarrow$

$$L(t^2 \cosh at) = L\left[t^2\left(\frac{e^{at}+e^{-at}}{2}\right)\right]$$

$$= \frac{1}{2}\left\{L(t^2 e^{at}) + L(t^2 e^{-at})\right\}$$

$$= \frac{1}{2}\left\{L(e^{at})\left(\frac{2}{s^3}\right) + L\left(e^{-at}\right)\left(\frac{2}{s^3}\right)\right\}$$

$$= \frac{1}{2}\left\{\frac{2}{(s-a)^3} + \frac{2}{(s+a)^3}\right\}$$

$$= \frac{1}{(s-a)^3} + \frac{1}{(s+a)^3} \,//$$

Find the Laplace transform of the following function:

1] $\dfrac{\sin^2 t}{t}$

$\rightarrow$

$L(\sin^2 t) = L\left(\dfrac{1-\cos 2t}{2}\right)$

$= \dfrac{1}{2}\left[L(1) - L(\cos 2t)\right]$

$L(\sin^2 t) = \dfrac{1}{2}\left(\dfrac{1}{s} - \dfrac{s}{s^2+4}\right)$

$L\left(\dfrac{\sin^2 t}{t}\right) = \dfrac{1}{2}\displaystyle\int_s^\infty \left(\dfrac{1}{s} - \dfrac{s}{s^2+4}\right) ds$

$= \dfrac{1}{2}\left[\log s - \dfrac{1}{2}\log(s^2+4)\right]_s^\infty$

$= \dfrac{1}{2}\left\{0 - \left[\log s - \log(s^2+4)^{1/2}\right]\right\}$

$= \dfrac{1}{2}\log\left[\dfrac{\sqrt{s^2+4}}{s}\right]$ //

2] $\dfrac{\cos 3t - \cos 2t}{t}$

$\rightarrow$

$L(\cos 3t - \cos 2t) = \dfrac{s}{s^2+9} - \dfrac{s}{s^2+4}$

$L\left[\dfrac{\cos 3t - \cos 2t}{t}\right] = \displaystyle\int_s^\infty \left(\dfrac{s}{s^2+9} - \dfrac{s}{s^2+4}\right) ds$

$= \left[\dfrac{1}{2}\log(s^2+9) - \dfrac{1}{2}\log(s^2+4)\right]_s^\infty$

$$= \frac{1}{2}\left\{\log\frac{s^2+a}{s^2+4}\right\}_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2(1+\frac{a}{s^2})}{s^2(1+\frac{4}{s^2})}\right]_s^\infty$$

$$= \frac{1}{2}\left[0 - \log\frac{(1+\frac{a}{s^2})}{(1+\frac{4}{s^2})}\right]$$

$$= \frac{1}{2}\left[\log\frac{(1+\frac{4}{s^2})}{(1+\frac{a}{s^2})}\right]$$

$$= \frac{1}{2}\log\left[\frac{s^2+4}{s^2+a}\right]$$

3] $\frac{e^{3t}-e^{-2t}}{t}$

→

$$L(e^{3t}-e^{-2t}) = \frac{1}{s-3} - \frac{1}{s+2}.$$

$$L\left(\frac{e^{3t}-e^{-2t}}{t}\right) = \int_s^\infty\left[\frac{1}{s-3} - \frac{1}{s+2}\right]ds.$$

$$= \left[\log(s-3) - \log(s+2)\right]_s^\infty$$

$$= \left[\log\frac{(s-3)}{(s+2)}\right]_s^\infty$$

$$= \left[\log\frac{\cancel{s}(1-\frac{3}{s})}{\cancel{s}(1+\frac{2}{s})}\right]_s^\infty$$

~~log~~

$$= \left[0 - \log\left(\frac{\left(1 - \frac{3}{s}\right)}{\left(1 + \frac{3}{s}\right)} \right) \right]$$

$$= \log\left[\frac{(s+2)}{(s+3)} \right],$$