UNIT-4

INVERSE LAPLACE TRANSFORMS.

1] $L^{-1}\left(\frac{1}{s}\right) = 1$

2] $L^{-1}\left(\frac{1}{s^{n+1}}\right) = \frac{t^n}{n!}$

3] $L^{-1}\left(\frac{1}{s+a}\right) = e^{-at}$

4] $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

5] $L^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{\sin at}{a}$

6] $L^{-1}\left(\frac{s}{s^2+a^2}\right) = \cos at$

7] $L^{-1}\left(\frac{1}{s^2-a^2}\right) = \frac{\sinh at}{a}$

8] $L^{-1}\left(\frac{s}{s^2-a^2}\right) = \cosh at.$

1] Find $L^{-1}\left[\frac{7s-1}{(s+1)(s+2)(s+3)}\right]$

→

Let $\frac{7s-1}{(s+1)(s+2)(s+3)} = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3} \rightarrow ①$

$7s-1 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2)$

Put $s=-1 \Rightarrow -8 = A(1)(2) \Rightarrow 2A = -8 \Rightarrow \boxed{A=-4}$

Put $s=-2 \Rightarrow -15 = B(-1)(1) \Rightarrow \boxed{B=15}$

Put $s=-3 \Rightarrow -22 = C(-2)(-1) \Rightarrow 2C = -22 \Rightarrow \boxed{C=-11}$

① $\Rightarrow \frac{7s-1}{(s+1)(s+2)(s+3)} = \frac{-4}{s+1} + \frac{15}{s+2} - \frac{11}{s+3}$

$$L^{-1}\left[\frac{-4}{s+1}+\frac{15}{s+2}-\frac{11}{s+3}\right] = -4L^{-1}\left[\frac{1}{s+1}\right]+15L\left[\frac{1}{s+2}\right]-11L^{-1}\left[\frac{1}{s+3}\right]$$

$$= -4e^{-t}+15e^{-2t}-11e^{-3t}$$

2) $L^{-1}\left[\frac{s^2+9s+2}{(s-1)^2(s+2)}\right]$

→

Consider $\frac{s^2+9s+2}{(s-1)^2(s+2)} = \frac{A}{(s-1)}+\frac{B}{(s-1)^2}+\frac{C}{(s+2)} \rightarrow ①$

$$s^2+9s+2 = A(s-1)(s+2)+B(s+2)+C(s-1)^2$$

Put $s=1 \Rightarrow 12 = B(3) \Rightarrow \boxed{B=4}$

Put $s=-2 \Rightarrow -12 = C(9) \Rightarrow \boxed{C=-\frac{4}{3}}$

Put $s=0 \Rightarrow 2 = -2A+2B+C$

$2 = -2A+8-\frac{4}{3}$

$2A = 6-\frac{4}{3} = \frac{14}{3}$

$\boxed{A=\frac{7}{3}}$

$① \Rightarrow \frac{s^2+9s+2}{(s-1)^2(s+2)} = \frac{7}{3(s-1)}+\frac{4}{(s-1)^2}-\frac{4}{3(s+2)}$

$$L^{-1}\left[\frac{7}{3(s-1)}+\frac{4}{(s-1)^2}-\frac{4}{3(s+2)}\right] = \frac{7}{3}L^{-1}\left[\frac{1}{(s-1)}\right]+4L^{-1}\left[\frac{1}{(s-1)^2}\right]-\frac{4}{3}L^{-1}\left[\frac{1}{(s+2)}\right]$$

$$= \frac{7}{3}e^{t}+4e^{t}t-\frac{4}{3}e^{-2t}.$$

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3) $L^{-1}\left(\frac{s^2}{(s^2+4)(s^2+9)}\right)$

$$\frac{s^2}{(s^2+4)(s^2+9)} = \frac{A}{s^2+4} + \frac{B}{s^2+9}$$

$$s^2 = A(s^2+9) + B(s^2+4)$$

Put $s^2 = -4$

$-4 = 5A$

$A = -\frac{4}{5}$ Put $s^2 = -9$

$-9 = -5B$

$\boxed{B = \frac{9}{5}}$

$$\frac{s^2}{(s^2+4)(s^2+9)} = \frac{-4}{5(s^2+4)} + \frac{9}{5(s^2+9)}$$

$$L^{-1}\left(\frac{s^2}{(s^2+4)(s^2+9)}\right) = -\frac{4}{5} L^{-1}\left(\frac{1}{s^2+4}\right) + \frac{9}{5} L^{-1}\left(\frac{1}{s^2+9}\right)$$

$$= -\frac{4}{5}\,\frac{\sin 2t}{2} + \frac{9}{5}\,\frac{\sin 3t}{3}$$

$$= -\frac{2}{5}\sin 2t + \frac{9}{15}\sin 3t.$$

4] Find $L^{-1}\left(\frac{s+2}{(s-4)(s^2+1)}\right)$

Soln:-

$$\frac{s+2}{(s-4)(s^2+1)} = \frac{A}{(s-4)} + \frac{Bs+C}{s^2+1}$$

$$s+2 = A(s^2+1) + (Bs+C)(s-4)$$

$$s+2 = As^2 + A + Bs^2 - 4B + Cs - 4C$$

Put $s=4$

$$6 = 17A$$

$$\boxed{A = 6/17}$$

Equating the coefficient of s^2

$$A + B = 0$$

$$\boxed{B = -\frac{6}{17}}$$

Equate the constant term

$$A - 4C = 2$$

$$4C = A - 2$$

$$= \frac{6}{17} - 2$$

$$4C = -\frac{28}{17} \quad C = -\frac{7}{17}$$

$$\frac{s+1}{(s-4)(s^2+1)} = \frac{6}{17} L^{-1}\left(\frac{1}{s-4}\right) - \frac{6}{7} L^{-1}\left(\frac{s}{s^2+1}\right) - \frac{7}{17} L^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= \frac{6}{17} e^{4t} - \frac{6}{17} \cos t - \frac{7}{17} \sin t$$

5) $L^{-1}\left[\frac{1}{s(s^2+a^2)}\right]$

Let

$$\frac{1}{s(s^2+a^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+a^2} \rightarrow ①$$

$$1 = A(s^2+a^2) + (Bs+C)s$$

Put $s=0$

$$1 = Aa^2 \Rightarrow \boxed{A = 1/a^2}$$

Coeff. of. s^2,

$$0 = A + B$$

$$0 = 1/a^2 + B$$

$$\boxed{B = -1/a^2}$$

Coeff. of. s

$$\boxed{C = 0}$$

$$① : \frac{1}{s(s^2+a^2)} = \frac{1/a^2}{s} + \frac{-1/a^2 s}{s^2+a^2}$$

$$L^{-1}\left[\frac{1}{s(s^2+a^2)}\right] = \frac{1}{a^2}L^{-1}\left[\frac{1}{s}\right] - \frac{1}{a^2}L^{-1}\left[\frac{s}{s^2+a^2}\right]$$

$$= \frac{1}{a^2} - \frac{1}{a^2}(\cos at)$$

6] $L^{-1}\left[\log\left(\frac{1+s}{s}\right)\right]$

→

$$\log\left(\frac{1+s}{s}\right) = \log(1+s) - \log(s)$$

$$\frac{-d}{ds}\left[f(s)\right] = -\left[\frac{1}{1+s} - \frac{1}{s}\right]$$

$$L^{-1}\left[\frac{-d}{ds} f(s)\right] = L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{1+s}\right]$$

$$tf(t) = L^{-1}\left(\frac{1}{s}\right) - L^{-1}\left(\frac{1}{s+1}\right)$$

$$tf(t) = 1 - e^{-t}$$

$$f(t) = \frac{1-e^{-t}}{t}$$

7] Find the inverse laplace transform of

$$\log\left(\frac{s^2+9}{s^2+1}\right)$$

→

$$\log\left(\frac{s^2+9}{s^2+1}\right) = \log(s^2+9) - \log(s^2+1)$$

$$-\frac{d}{ds} f(s) = -\left[\frac{2s}{s^2+9} - \frac{2s}{s^2+1}\right]$$

$$L^{-1}\left[\frac{-d}{ds} f(s)\right] = 2L^{-1}\left[\frac{s}{s^2+1}\right] - 2L^{-1}\left[\frac{s}{s^2+9}\right]$$

$$tf(t) = 2\cos t - 2\cos 3t.$$

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$$f(t) = \frac{2(\cos t - \cos 3t)}{t}.$$

8) $\frac{6}{(s+2)^4}$

→

$$L^{-1}\left[\frac{6}{(s+2)^4}\right] = 6e^{-2t} L^{-1}\left[\frac{1}{s^4}\right]$$

$$= 6e^{-2t}\left[\frac{t^3}{3!}\right]$$

$$= t^3 e^{-2t}.$$

9) $\frac{2s^2+10s}{(s^2-2s+5)(s+1)}$

→

$$= L^{-1}\left[\frac{2s^2+10s}{[(s-1)^2+4][(s-1)+2]}\right]$$

$$= e^t L^{-1}\left[\frac{2s^2+10s}{(s^2+4)(s+2)}\right]$$

Consider,

$$\left[\frac{2s^2+10s}{(s^2+4)(s+2)}\right]$$

$$\frac{2s^2+10s}{(s+2)(s^2+4)} = \frac{A}{(s+2)} + \frac{Bs+C}{(s^2+4)}$$

$$\frac{2s^2+10s}{(s+2)\cancel{(s^2+4)}} = \frac{A(s^2+4) + Bs+C(s+2)}{(s+2)\cancel{(s^2+4)}}$$

Put $s = -2$

$8 - 20 = 8A = -\frac{3}{2} = A.$ $\boxed{A = -3/2}$

$2 = A+B.$

Equate coefficient of s^2

$$A+B = 2$$

$$-\frac{3}{2} + B = 2$$

$$B = 2 + \frac{3}{2} = \frac{7}{2}$$

Equate coefficient of constant

$$4A + 2C = 0$$

$$4\left(-\frac{3}{2}\right) + 2C = 0$$

$$-\frac{12}{2} + 2C = 0$$

$$-6 + 2C = 0$$

$$2C = 6$$

$$\boxed{C = 3}$$

$$\left[\frac{2s^2+10s}{(s^2+2)(s+2)}\right] = \frac{-3}{2(s+2)} + \frac{7(s+3)}{2(s^2+4)}$$

$$e^t L^{-1}\left[\frac{2s^2+10s}{(s^2+2)(s+2)}\right] = e^t L^{-1}\left(\frac{-3}{2(s+2)}\right) + e^t L^{-1} \frac{7(s+3)}{2(s^2+4)}$$

Solving differential equations using Laplace:-

$L(y'') = L\left\{\frac{d^2y}{dt^2}\right\} = s^2 L(y) - sy(0) - y'(0)$

$L(y') = L\left\{\frac{dy}{dt}\right\} = sL(y) - y(0)$

1) Solve $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$ given $y(0) = -2$, $y'(0) = 5$

Solution:-

Given $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 2y = 0$

Taking Laplace on both sides.

$L\left\{\frac{d^2y}{dt^2}\right\} - L\left\{\frac{dy}{dt}\right\} - 2L\{y\} = 0$

$s^2L\{y\} - sy(0) - y'(0) - [sL\{y\} - y(0)] - 2L\{y\} = 0$

$(s^2 - s - 2)L\{y\} - sy(0) - y'(0) + y(0) = 0$

$(s^2 - s - 2)L\{y\} - s(-2) - 5 - 2 = 0$

$(s^2 - s - 2)L\{y\} + 2s - 7 = 0$

$(s^2 - s - 2)L\{y\} = 7 - 2s$

$L\{y\} = \frac{7-2s}{s^2 - s - 2}$

$$y = L^{-1}\left[\frac{7s - 2s}{s^2 - s - 2}\right]$$

Consider,

$$\frac{7-2s}{(s^2-s-2)} = \frac{7-2s}{(s+1)(s-2)}$$

$$\frac{7-2s}{(s^2-s-2)} = \frac{A}{(s+1)} + \frac{B}{s-2}$$

$$7-2s = A(s-2) + B(s+1)$$

Put $s = -1$

$$7+2 = -3A$$

$$9 = -3A$$

$$\boxed{A = -3}$$

Put $s = 2$

$$7-4 = 3B$$

$$\boxed{B = 1}$$

$\cancel{7-2s}$

$$y = L^{-1}\left[\frac{-3}{(s+1)} + \frac{1}{(s-2)}\right]$$

$$= -3L^{-1}\left[\frac{1}{(s+1)}\right] + L^{-1}\left[\frac{1}{(s-2)}\right]$$

$$= -3e^{-t} L^{-1}[1/s] + e^{2t} L^{-1}[1/s]$$

$$y = -3e^{-t} + e^{2t}$$

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2] Solve:-

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} + 5y = 4e^{3t} \text{ given } y(0) = 2 ; \ y'(0) = 7.$$

$$L\left\{\frac{d^2y}{dt^2}\right\} - 4L\left\{\frac{dy}{dt}\right\} + 5L\{y\} = 4L\{e^{3t}\}$$

$$s^2L\{y\} - s(y)(0) - y'(0) - 4\left[sL\{y\} - y(0)\right] + 5L\{y\} = 4\left(\frac{1}{s-3}\right)$$

$$(s^2 - 4s + 5)L\{y\} - s(2) - 7 + 4(2) = \frac{4}{s-3}$$

$$(s^2 - 4s + 5)L\{y\} - 2s + 1 = \frac{4}{s-3}$$

$$(s^2 - 4s + 5)L\{y\} = \frac{4}{s-3} + (2s-1)$$

$$L\{y\} = \frac{4 + (2s-1)(s-3)}{(s^2 - 4s + 5)(s-3)}$$

$$y = L^{-1}\left[\frac{4 + (2s-1)(s-3)}{(s^2 - 4s + 5)(s-3)}\right]$$

$$\frac{4 + (2s-1)(s-3)}{(s^2 - 4s + 5)(s-3)} = \frac{A}{(s-3)} + \frac{Bs + C}{s^2 - 4s + 5}$$

$$4 + (2s-1)(s-3) = A(s^2 - 4s + 5) + (Bs + C)(s-3)$$

Put $s = 3$

$$4 = A(9 - 12 + 5)$$

$$4 = A(14 - 12)$$

$$\boxed{A = 2}$$

Puts: coeff. of s^2

$\cancel{0 = A + B}$ $\quad 2 = 2 + B$

$\cancel{0 = 2 + B}$ $\quad \boxed{B = 0}$

$\cancel{B = 3}$

coeff. of s

$-7 = -4 + C - 3B$

$C = \cdot$

$$y = 2L^{-1}\left(\frac{1}{s-3}\right) + L^{-1}\left(\frac{1}{s^2-4s+5}\right)$$

$$y = 2e^{3t} + L^{-1}\frac{1}{(s^2-4s+4-4+5)}$$

$$= 2e^{3t} + L^{-1}\left[\frac{1}{(s-2)^2+1}\right]$$

$$= 2e^{3t} + e^{2t}L^{-1}\left[\frac{1}{s^2+1}\right]$$

$$= 2e^{3t} + e^{2t}\sin t.$$

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8] Solve the differential Equation $\frac{d^2y}{dt^2} - 4\frac{dy}{dt} - 5y = te^t$, $y(0) = 0$, $y'(0)$

Given :-

$$\frac{d^2y}{dt^2} - 4\frac{dy}{dt} - 5y = te^t$$

Taking Laplace on both side:-

$$L\left\{\frac{d^2y}{dt^2}\right\} - 4L\left\{\frac{dy}{dt}\right\} - 5L\{y\} = L[te^t]$$

$$s^2L\{y\} - sy(0) - y'(0) - 4[sL\{y\} - y(0)] - 5L\{y\} = \frac{1}{(s-1)^2}$$

$$(s^2 - 4s - 5)L\{y\} = \frac{1}{(s-1)^2}$$

$$L\{y\} = \frac{1}{(s^2-4s-5)(s-1)^2}$$

$$L\{y\} = \frac{1}{(s+1)(s-5)(s-1)^2}$$

$$y = L^{-1}\left[\frac{1}{(s+1)(s-5)(s-1)^2}\right] \rightarrow A$$

Consider,

$$\frac{1}{(s+1)(s-5)(s-1)^2} = \frac{A}{s+1} + \frac{B}{s-5} + \frac{C}{s-1} + \frac{D}{(s-1)^2} \rightarrow 1$$

$$1 = A(s-5)(s-1)^2 + B(s+1)(s-1)^2 + C(s+1)(s-5)(s-1) + D(s+1)(s-5)$$

Put $s = 5$

$$1 = 0 + B(6)(16)$$

$$1 = 96B$$

$$\boxed{B = \frac{1}{96}}$$

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Puts = 1

$1 = D(2)(-4)$

$1 = -8D$

$$\boxed{D = \frac{-1}{8}}$$

Puts = -1

$1 = A(4)(-6)$

$1 = -24A$

$$\boxed{A = \frac{1}{-24}}$$

Puts = 0

$1 = A(-5)(1) + B($

$$\boxed{C = \frac{1}{32}}$$

$$① \Rightarrow \frac{1}{(s+1)(s-5)(s-1)^2} = \frac{-1/24}{(s+1)} + \frac{1/96}{(s-5)} + \frac{1/32}{(s-1)} - \frac{\frac{1}{8}}{(s-1)^2}$$

Substitute in ④, we get.

$$y = -\frac{1}{24} L^{-1}\left[\frac{1}{s+1}\right] + \frac{1}{96} L^{-1}\left[\frac{1}{s-5}\right] + \frac{1}{32} L^{-1}\left[\frac{1}{s-1}\right] - \frac{1}{8} L^{-1}\left[\frac{1}{(s-1)^2}\right]$$

$$= -\frac{1}{24} e^{-t} + \frac{1}{96} e^{5t} + \frac{1}{32} e^{t} - \frac{1}{8} t e^{t}$$

4) Using the method of Laplace transform $\frac{dx}{dt} + 2x - 3y = 2t$

$\frac{dy}{dt} - 3x + 2y = e^{2t}$ given $x(0) = 0$, $y(0) = 0$.

Given :-

$$\frac{dx}{dt} + 2x - 3y = 2t$$

Taking Laplace on both side.

$$L\left\{\frac{dx}{dt}\right\} + 2L\{x\} - 3L\{y\} = 2L\{t\}$$

$$sL\{x\} - x(0) + 2L\{x\} - 3L\{y\} = 2\left(\frac{1}{s^2}\right)$$

$$(s+2)L\{x\} - 3L\{y\} = \frac{2}{s^2} \rightarrow (1)$$

Given :- $\frac{dy}{dt} - 3x + 2y = e^{2t}$ Taking Laplace on both side.

$$L\left\{\frac{dy}{dt}\right\} - 3L\{x\} + 2L\{y\} = L\{e^{2t}\}$$

$$sL\{y\} - y(0) - 3L\{x\} + 2L\{y\} = \frac{1}{s-2}$$

$$(s+2)L\{y\} - 3L\{x\} = \frac{1}{s-2} \rightarrow (2)$$

~~Using Cramer's rule~~

(1) $\Rightarrow (s+2)L\{x\} - 3L\{y\} = \frac{2}{s^2}$

(2) $\Rightarrow (s+2)L\{y\} - 3L\{x\} = \frac{1}{s-2}$

Using Cramer's rule

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad \Delta x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

$$\Delta y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$L(x) = \bar{x} = \frac{\Delta x}{\Delta} ; L(y) = \bar{y} = \frac{\Delta y}{\Delta}$

$$(s+2)L\{x\} - 3L\{y\} = \frac{2}{s^2} \rightarrow ①$$

$$-3L\{x\} + (s+2)L\{y\} = \frac{1}{s-2} \rightarrow ②$$

$$\Delta = \begin{vmatrix} s+2 & -3 \\ -3 & s+2 \end{vmatrix} = (s+2)^2 - 9 = s^2+4s+4-9 = s^2+4s-5.$$

$$\Delta x = \begin{vmatrix} 2/s^2 & -3 \\ 1/s^2 & s+2 \end{vmatrix} = \frac{2}{s^2}(s+2) + \frac{3}{s-2} = \frac{(2s+4)(s-2)+3s^2}{s^2(s-2)}$$

$$= \frac{2s^2-4s+4s-8+3s^2}{s^2(s-2)}$$

$$= \frac{5s^2-8}{s^2(s-2)}$$

$$\Delta y = \begin{vmatrix} s+2 & 2/s^2 \\ -3 & \frac{1}{s-2} \end{vmatrix} = \frac{s+2}{s-2} + \frac{6}{s^2} = \frac{s^3+2s^2+6s-12}{s^2(s-2)}$$

$$L(x) = \bar{x} = \frac{\Delta x}{\Delta}$$

$$= \frac{\frac{5s^2-8}{s^2(s-2)}}{s^2+4s-5}$$

$$L(x) = \frac{5s^2-8}{s^2(s-2)(s-1)(s+5)}$$

$$x = L^{-1}\left[\frac{5s^2-8}{s^2(s-1)(s-2)(s+5)}\right]$$

Consider,

$$\frac{5s^2-8}{s^2(s-1)(s-2)(s+5)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{(s-1)} + \frac{D}{(s-2)} + \frac{E}{(s+5)}$$

$$5s^2-8 = A(s)(s-1)(s-2)(s+5) + B(s-1)(s-2)(s+5) + C(s^2)(s-2)(s+5)$$
$$+ D(s^2)(s-1)(s+5) + E(s^2)(s-1)(s-2)$$

Put s=0

$$-8 = B(-1)(-2)(5)$$

$$-8 = 10B$$

$$B = -\frac{8}{10} \qquad \boxed{B = -\frac{4}{5}}$$

Put s=1

$$5-8 = C(1)(-1)(6)$$

$$-3 = C(-\overset{6}{\cancel{12}})$$

$$\boxed{C = 1/2}$$

Put s=2

$$20-8 = D(4)(1)(7)$$

$$12 = 28D$$

$$D = \frac{12}{28}$$

$$\boxed{D = 4/7}$$

Put $s = -5$

$$125 - 8 = E(5)^2(-6)(-7)$$

$$117 = 1050E$$

$$\boxed{E = \frac{39}{350}}$$

Equating the co-efficient of s^4

$$A + C + D + E = 0$$

$$A = -\frac{8}{7} - \frac{39}{350} - \frac{1}{2}$$

$$= \frac{-150 - 39 - 175}{350}$$

$$= -\frac{364}{350}$$

$$= -\frac{182}{175}$$

From I,

$$x = \frac{-182}{175} - \frac{4}{5}t + \frac{3}{7}e^{2t} + \frac{39}{350}e^{-5t} + \frac{1}{2}e^{t}$$

$$8y = \frac{dx}{dt} + 2x - 2t$$

$$= -\frac{4}{5} + \frac{6}{7}e^{2t} - \frac{195}{350}e^{-5t} + \frac{1}{2}e^{t} - \frac{364}{175} - \frac{8}{5}t + \frac{6}{7}e^{2t} + \frac{78}{350}e^{-5t} + e^{t} - 2t.$$

$$= -\frac{504}{175} - \frac{18}{5}t + \frac{12}{7}e^{2t} - \frac{117}{350}e^{-5t} + \frac{3}{2}e^{t}$$

$$\therefore y = \frac{-168}{175} - \frac{18}{5}t + \frac{12}{7}e^{2t} - \frac{391}{350}e^{-5t} + \frac{1}{2}e^{t}.$$

5] Solve the simultaneous Equation

$$3\frac{dx}{dt} + \frac{dy}{dt} + 2x = 1 \rightarrow ①$$

$$\frac{dx}{dt} + 4\frac{dy}{dt} + 3y = 0 \rightarrow ②$$

Given $x(0) = Y$ at $t = 0$.

$$3\frac{dx}{dt} + \frac{dy}{dt} + 2x = 1$$

$$\frac{dx}{dt} + 4\frac{dy}{dt} + 3y = 0$$

Taking Laplace, we get.

$$3L\left\{\frac{dx}{dt}\right\} + L\left\{\frac{dy}{dt}\right\} + 2L(x) = L(1)$$

$$L\left\{\frac{dx}{dt}\right\} + 4L\left\{\frac{dy}{dt}\right\} + 3L(y) = 0$$

(ie) $3sL(x) + sL(y) + 2L(x) = 1/s$

$sL(x) + 4sL(y) + 3L(y) = 0$

$s(3s+2)L(x) + s^2L(y) = 1 \rightarrow ①$

$sL(x) + (4s+3)L(y) = 0 \rightarrow ②$

From ① and ②

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$$s(3s+2)L(x) - \frac{s^3}{4s+3}L(x) = 1$$

$$\frac{s[(3s+2)(4s+3) - s^2]}{4s+3} L(x) = 1$$

$$\frac{s(11s^2+17s+6)}{4s+3} L(x) = 1$$

$$L(x) = \frac{4s+3}{s(11s^2+17s+6)}$$

$$= \frac{4s+3}{s(11s+6)(s+1)}$$

Let, $\frac{4s+3}{s(11s+6)(s+1)} = \frac{A}{s} + \frac{B}{11s+6} + \frac{C}{s+1}$

$$4s+3 = A(11s+6)(s+1) + Bs(s+1) + Cs(11s+6)$$

Put $s=0$,

$$3 = 6A \quad \therefore A = 1/2$$

Put $s=-1$,

$$-1 = C(-1)(-5)$$

$$C = -1/5$$

Put $s = -6/11$

$$-\frac{24}{11} + 3 = B\left(\frac{-6}{11}\right)\left(\frac{5}{11}\right)$$

$$\frac{9}{11} = \frac{-30B}{121}$$

$$B = \frac{-33}{10}$$

$$x = \frac{1}{2}L^{-1}\left(\frac{1}{s}\right) - \frac{33}{10}L^{-1}\left(\frac{1}{11s+6}\right) - \frac{1}{5}L^{-1}\left(\frac{1}{s+1}\right)$$

$$= \frac{1}{2} - \frac{33}{10}e^{-\frac{6}{11}t} - \frac{1}{5}e^{-t}$$

$$L(y) = -\frac{s}{4s+3}L(x)$$

$$= \frac{-s}{4s+3}\;\frac{4s+3}{s(11s+6)(s+1)}$$

$$= -\frac{1}{(11s+6)(s+1)}$$

$$= \frac{D}{11s+6} + \frac{E}{s+1}$$

$$-1 = D(s+1) + E(11s+6)$$

Put $s=-1$,

$-1 = -5E$

$\boxed{E = 1/5}$

Put $s = -\frac{6}{11}$

$-1 = D\left(\frac{5}{11}\right)$

$\boxed{D = -11/5}$

$$\therefore y = -\frac{11}{5}L^{-1}\left(\frac{1}{11s+6}\right) + \frac{1}{5}L^{-1}\left(\frac{1}{s+1}\right)$$

$$= -\frac{11}{5}e^{-\frac{6}{11}t} + \frac{1}{5}e^{-t}.$$

Aliter:-

Given:-

$$\frac{3dx}{dt} + \frac{dy}{dt} + 2x = 1 \rightarrow ①$$

$$\frac{dx}{dt} + 4\frac{dy}{dt} + 3y = 0 \rightarrow (2)$$

Taking Laplace on both side

(1) $\Rightarrow$ $3L\left\{\frac{dx}{dt}\right\} + L\left\{\frac{dy}{dt}\right\} + 2L\{x\} = L\{1\}$

$$3[sL\{x\} - \cancel{x(0)}^{0}] + sL\{y\} - \cancel{y(0)}^{0} + 2L\{x\} = \frac{1}{s}$$

$$(3s+2)L\{x\} + sL\{y\} = \frac{1}{s} \rightarrow (3)$$

(2) $\Rightarrow$ $L\left\{\frac{dx}{dt}\right\} + 4L\left\{\frac{dy}{dt}\right\} + 3L\{y\} = 0$

$$sL\{x\} - \cancel{x(0)}^{0} + 4[sL\{y\} - \cancel{y(0)}^{0}] + 3L\{y\} = 0$$

$$sL\{x\} + (4s+3)L\{y\} = 0 \rightarrow (4)$$

(3) $\Rightarrow$ $(3s+2)L\{x\} + sL\{y\} = \frac{1}{s}$

(4) $\Rightarrow$ $sL\{x\} + (4s+3)L\{y\} = 0$

Using Cramer's rule

$$\Delta = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \qquad \Delta x = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$$

$$\Delta y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

$$L(x) = \bar{x} = \frac{\Delta x}{\Delta}; \quad L(y) = \frac{\Delta y}{\Delta}$$

$$(3s+2)L(x) + sL(y) = \frac{1}{s} \rightarrow (3)$$

$$sL(x) + (4s+3)L(y) = 0 \rightarrow (4)$$

$$\Delta = \begin{vmatrix} 3s+2 & s \\ s & 4s+3 \end{vmatrix} = (3s+2)(4s+3) - s^2$$

$$= 12s^2 + 9s + 8s + 6 - s^2$$

$$= 11s^2 + 17s + 6.$$

$$= (s+1)(11s+6)$$

$$\Delta x = \begin{vmatrix} \frac{1}{s} & s \\ 0 & 4s+3 \end{vmatrix} = \left(\frac{1}{s}\right)(4s+3) = \frac{4s+3}{s}$$

$$\Delta y = \begin{vmatrix} 3s+2 & 1/s \\ s & 0 \end{vmatrix} = -1$$

$$L(x) = \frac{\Delta x}{\Delta}$$

$$L(x) = \frac{4s+3}{s(s+1)(11s+6)} \rightarrow (A)$$

Consider

$$\frac{4s+3}{s(s+1)(11s+6)} = \frac{A}{s} + \frac{B}{(s+1)} + \frac{C}{(11s+6)}$$

$$4s+3 = A(s+1)(11s+6) + B(s)(11s+6) + C(s+1)(s)$$

Put $s = -1$

$$-1 = B(-1)(-5)$$

$$\boxed{B = -\frac{1}{5}}$$

Put s = 0

$$3 = A(1)(6)$$

$$\boxed{A = \frac{1}{2}}$$

Put s = 1

$$7 = A(2)(17) + B(17) + C(2)$$

$$7 = \cancel{1/2}(\cancel{2})(17) + \frac{1}{5}(17) + C(2)$$

$$7 - 17 = -\frac{1}{5}(17) + C(2)$$

$$-10 = -\frac{17}{5} + 2C$$

$$-10 + \frac{17}{5} = 2C$$

$$\frac{-50+17}{5} = 2C$$

$$\frac{-67}{10} = C$$

$$L(x) = \frac{\frac{1}{2}}{s} - \frac{\frac{1}{5}}{s+1} - \frac{33/10}{11s+6}$$

$$x = \frac{1}{2}L^{-1}\left[\frac{1}{s}\right] - \frac{1}{5}L^{-1}\left[\frac{1}{s+1}\right] - \frac{\overset{3}{\cancel{33}}}{10} \times \frac{1}{\cancel{11}} L^{-1}\left[\frac{1}{(s+\frac{6}{11})}\right]$$

$$x = \frac{1}{2}\cancel{s} - \frac{1}{5}e^{-t} - \frac{3}{10}e^{-6/11 t}$$

$$L(y) = \frac{\Delta y}{\Delta}$$

$$L(y) = \frac{-1}{(s+1)(11s+6)} + (8)$$

Consider

$$\frac{-1}{(s+1)(11s+6)} = \frac{A}{s+1} + \frac{B}{11s+6}$$

$$-1 = A(11s+6) + B(s+1)$$

Put $s=-1$

$$-1 = A(-5)$$

$$\boxed{A = \frac{1}{5}}$$

Put $s=0$

$$-1 = A(6) + B$$

$$-1 = \frac{6}{5} + B$$

$$\boxed{-\frac{11}{5} = B}$$

$$\cancel{B} \ L(y) = \frac{\frac{1}{5}}{s+1} - \frac{11/5}{(11s+6)}$$

$$y = \frac{1}{5}L^{-1}\left[\frac{1}{s+1}\right] - \frac{\cancel{11}}{5 \times \cancel{11}} L^{-1}\left[\frac{1}{s+\frac{6}{11}}\right]$$

$$y = \frac{1}{5}e^{-t} - \frac{1}{5}e^{-\frac{6}{11}t}$$