Chapter 15

Chi – Square Distribution

15.1 * z is called standard normal variate.

Define χ^2 distribution 2M

The square of a standard normal variate z is called Chi – Square Distribution with degree of freedom 1.

If x is normally distributed over mean μ & variance σ^2, then $x \sim N(\mu, \sigma^2)$

$$z = \frac{x-\mu}{\sigma}$$

Squaring on both sides, we get

$$\chi^2 = \left(\frac{x-\mu}{\sigma}\right)^2 \Rightarrow \text{chi-square variate with 1 d.f.}$$

If x can be generalised to n independent normal variates with mean μ_i & variance σ_i^2

(ie) x_i, $i = 1, 2, \ldots, n$, then

$$\chi^2 = \sum_{i=1}^{n} \left(\frac{x_i - \mu_i}{\sigma_i}\right)^2$$ is a chi-square variate of n d.f.

15.2 Derivation of p.d.f. of χ^2 – distribution:
(First method only enough)

* Deriving pdf of χ^2 – distribution using Moment Generating Function

If X_i, $(i = 1, 2, \ldots, n)$ are independent $N(\mu, \sigma^2)$, then we have to derive the distribution of

$$\chi^2 = \sum_{i=1}^{n}\left(\frac{x_i - \mu_i}{\sigma_i}\right)^2 \quad (i = 1, 2, \ldots, n)$$

Let $u_i = \frac{x_i - \mu_i}{\sigma_i} \sim N(0,1)$. Then

$$\chi^2 = \sum_{i=1}^{n} u_i^2$$

Since x_i are independent, then u_i is also independent.

u_i's are independent & identically distributed random variable over $(0,1)$.

$\therefore$ Moment Generating Function is given by

$$M_{\chi^2}(t) = M_{\sum u_i^2}(t) \longrightarrow \text{(summation of } n \text{ terms)}$$

(Multiplying)

$$= \prod_{i=1}^{n} M_{u_i^2}(t) = \left[M_{u_i^2}(t)\right]^n \longrightarrow ①$$

[Since $\prod$ is called capital 'pi'. It denotes product operator.]

We know that

$$M_x(t) = E(e^{xt}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

[Refer pg: 9.8 Normal Distribution]

$$M_{u_i^2}(t) = E\left[\exp(u_i^2 t)\right]$$

$$= \int_{-\infty}^{\infty} \exp(u_i^2 t)\, f(x_i)\, dx_i$$

$$\left[\because f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}\right]$$ (Refer page: 9.5)

$$M_{u_i^2}(t) = \int_{-\infty}^{\infty} \exp(t u_i^2) \frac{1}{\sigma\sqrt{2\pi}} \exp\left[\frac{-(x_i-\mu)^2}{2\sigma^2}\right] dx_i$$

substitute $u_i = \frac{x_i - \mu}{\sigma}$ in the above equation.

Differentiating it, we get

$$du_i = \frac{1}{\sigma} dx_i \Rightarrow \boxed{dx_i = \sigma\, du_i}$$

$$M_{u_i^2}(t) = \int_{-\infty}^{\infty} \exp(t u_i^2) \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{\frac{-(x_i-\mu)^2}{2\sigma^2}\right\} dx_i$$

$$= \int_{-\infty}^{\infty} \exp(t u_i^2) \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2\right] dx_i$$

$$= \int_{-\infty}^{\infty} \exp(t u_i^2) \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \exp\left(\frac{-u_i^2}{2}\right) \cancel{\sigma}\, du_i$$

$$M_{u_i^2}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(t u_i^2) \exp\left(\frac{-u_i^2}{2}\right) du_i$$

(Adding the powers) & taking LCM

$$\left[e^{t u_i^2} \times e^{\frac{-u_i^2}{2}} = e^{t u_i^2 + \left(\frac{-u_i^2}{2}\right)} = e^{-u_i^2\left(\frac{1-2t}{2}\right)} \right]$$

$$M_{u_i^2}(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\left\{-\left(\frac{1-2t}{2}\right) u_i^2\right\} du_i \quad -(i)$$

We know that by gamma function,

$$\int_{-\infty}^{\infty} e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{a}$$, where a^2 is coefficient of x^2 in the exponential term

Here in equation(i) $a = \left(\frac{1-2t}{2}\right)^{1/2}$; $x^2 = u_i^2$.

$$\therefore M_{u_i^2}(t) = \frac{1}{\sqrt{2\pi}}\left[\frac{\sqrt{\pi}}{\left(\frac{1-2t}{2}\right)^{1/2}}\right]$$

$$= \frac{1}{\sqrt{2}\,\cancel{\sqrt{\pi}}} \times \frac{\cancel{\sqrt{\pi}}}{\left(\frac{1-2t}{2}\right)^{1/2}}$$

$$= \frac{1}{\cancel{\sqrt{2}}} \times \frac{\cancel{\sqrt{2}}}{\sqrt{1-2t}}$$ (Taking reciprocal)

$$M_{u_i^2}(t) = \frac{1}{\sqrt{1-2t}} = (1-2t)^{-1/2} \rightarrow (2)$$

Sub (2) in (1)

$$M_{\chi^2}(t) = \left[M_{u_i^2}(t)\right]^n = \left[(1-2t)^{-1/2}\right]^n = (1-2t)^{-n/2}$$

$$\therefore \boxed{M_{\chi^2}(t) = (1-2t)^{-n/2}}$$

This is the M.G.F. of a gamma variate with parameters $\frac{1}{2}$ & $\frac{n}{2}$

By uniqueness theorem of M.G.F's

$$\chi^2 = \sum_{i=1}^{n} \left(\frac{x_i - \mu_i}{\sigma_i} \right)^2$$ is a gamma variate

Differentiating,

$$\therefore dP(\chi^2) = \frac{\left(\frac{1}{2}\right)^{n/2}}{\Gamma(n/2)} \left[\exp\left(-\frac{1}{2}\chi^2\right)\right] (\chi^2)^{\left(\frac{n}{2}-1\right)} d\chi^2$$

$$dP(\chi^2) = \frac{(1)^{n/2}}{(2^{n/2})\,\Gamma(n/2)} \exp\left(-\frac{\chi^2}{2}\right) (\chi^2)^{\left(\frac{n}{2}-1\right)} d\chi^2$$

This is the required pdf of chi-square distribution with n degrees of freedom.