By uniqueness theorem of M.G.F's

$$\chi^2 = \sum_{i=1}^{n} \left(\frac{x_i - \mu_i}{\sigma_i} \right)^2 \text{ is a gamma variate}$$

Differentiating,

$$\therefore dP(\chi^2) = \frac{\left(\frac{1}{2}\right)^{n/2}}{\Gamma(n/2)} \left[\exp\left(-\frac{1}{2}\chi^2\right)\right] (\chi^2)^{\left(\frac{n}{2}-1\right)} d\chi^2$$

$$dP(\chi^2) = \frac{(1)^{n/2}}{\left(2^{n/2}\right)\Gamma(n/2)} \exp\left(\frac{-\chi^2}{2}\right) (\chi^2)^{\left(\frac{n}{2}-1\right)} d\chi^2 \rightarrow (A)$$

This is the required pdf of chi-square distribution with n degrees of freedom.

15.3 Derive M.G.F. of chi-square distribution

Let X is normally distributed over χ^2-distribution of n independent variate.
(ie) $X \sim \chi^2_{(n)}$.

Then

$$M_x(t) = E(e^{tx}) = \int_0^{\infty} e^{tx} f(x)\, dx$$

$$M_x(t) = \int_0^{\infty} e^{tx} \left(\frac{1}{2^{n/2}\Gamma_{n/2}} e^{-x^2/2} x^{\frac{n}{2}-1} dx \right).$$

Sub (or) replace x^2 as x in pdf of χ^2-dist
(ie) equn (A)
Here variable is 'x'.

$$M_x(t) = \frac{1}{2^{n/2}\,\Gamma\frac{n}{2}} \int_0^\infty e^{tx}\, e^{-x/2}\, x^{\frac{n}{2}-1}\, dx$$

[Take constant value out of the integral]

$$e^{tx} \cdot e^{-\frac{x}{2}} = e^{tx - \frac{x}{2}}$$

$$= e^{x\left[t - \frac{1}{2}\right]}$$

$$e^{tx} \cdot e^{-x/2} = e^{x\left(\frac{2t-1}{2}\right)}$$

[Multiply the powers & taking LCM]

$$\therefore\ M_x(t) = \frac{1}{2^{n/2}\,\Gamma\frac{n}{2}} \int_0^\infty e^{x\left(\frac{2t-1}{2}\right)}\, x^{\frac{n}{2}-1}\, dx$$

By using Gamma integral,

$$\int_0^\infty e^{-x\left(\frac{1-2t}{2}\right)}\, x^{\frac{n}{2}-1}\, dx = \frac{\Gamma\frac{n}{2}}{\left[\frac{1-2t}{2}\right]^{n/2}}$$

$$= \frac{\Gamma\frac{n}{2}}{(1-2t)^{n/2}} \times 2^{n/2}$$

$$M_x(t) = \frac{1}{\cancel{2^{n/2}}\,\cancel{\Gamma\frac{n}{2}}} \times \frac{\cancel{\Gamma\frac{n}{2}}\ \cancel{2^{n/2}}}{(1-2t)^{n/2}} = \frac{1}{(1-2t)^{n/2}}$$

$$\therefore\ \boxed{M_x(t) = (1-2t)^{-n/2}}$$

This is the required mgf of χ^2- distribution

15.3.1 Derive (or) obtain Cumulant Generating Function of χ^2- distribution (CGF)

(First we have to know the formula for CGF)

This part u should know the formula. Need not write in exam

Cumulants generating function $K(t)$ is defined as

$$K_x(t) = \log_e M_x(t)$$

where the RHS can be expressed in convergent power series of t.

Thus,

symbol called 'kappa'. U write as small k

$$K_x(t) = k_1 t + k_2 \frac{t^2}{2!} + \dots + k_r \frac{t^r}{r!}$$

$$K_x(t) = \log M_x(t) \rightarrow (1)$$

(Logarithmic Series)

$M_x(t)$, Moment Generating Function formula is

$$\boxed{\begin{aligned} M_x(t) &= 1 + t\mu_r' + \frac{t^2}{2!}\mu_2' + \dots + \frac{t^r}{r!}\mu_r' + \dots \\ M_x(t) &= \sum_{r=0}^{\infty} \frac{t^r}{r!}\mu_r' \end{aligned}}$$

$\therefore$ Equation (1) becomes

$$K_x(t) = \log\left(1 + \mu_1' t + \mu_2' \frac{t^2}{2!} + \dots + \mu_r' \frac{t^r}{r!} + \dots\right)$$

where K_r = coefficient of $\frac{t^r}{r!}$ in $K_x(t)$

is called the rth cumulant

$8t^3 / 3 \times 2$ $\quad \frac{4t^2}{4}$

$\therefore$ 'Mean' can be given by K_1

$\mu_2 = K_2 = \text{variance}$

$\mu_3 = K_3$

$\mu_4 = K_4 + 3K_2^2$

Derive CGF of χ^2 - distribution

CGF can be given by

$$K_x(t) = \log M_x(t) = \log\left[(1-2t)^{-n/2}\right]$$

(From previous section)

$$K_x(t) = \log\left[(1-2t)^{-n/2}\right] = \frac{-n}{2}\log(1-2t)$$

$$= \frac{-n}{2}\left[2t + \frac{(2t)^2}{2} + \frac{(2t)^3}{3} + \frac{(2t)^4}{4} + \dots\right]$$

$$= -nt - nt^2 - \frac{4t^3}{3} - \dots$$

K_1 = coefficient of t in $K_x(t)$

$K_1 = n$

K_2 = coefficient of $\frac{t^2}{2!}$ in $K(t) = 2n$

K_3 = coefficient of $\frac{t^3}{3!}$ in $K(t) = 8n$

K_4 = coefficient of $\frac{t^4}{4!}$ in $K(t) = 48n$

In general, K_r = coefficient of $\frac{t^r}{r!}$ in $K(t)$

$$K_r = n\,2^{r-1}(r-1)!$$

Hence, Mean $= k_1 = n$

variance $= \mu_2 = k_2 = 2n$

$\mu_3 = k_3 = 8n$

$\mu_4 = k_4 + 3k_2^2 = 48n + 12n^2$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\cancel{64}^{8} \cancel{n^2}}{\cancel{8} n^{\cancel{3}}} = \frac{8}{n}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{48n + 12n^2}{4n^2} = \frac{\cancel{48}^{12} \cancel{n}}{\cancel{4} n^{\cancel{2}}} + \frac{\cancel{12}^{3} \cancel{n^2}}{\cancel{4} \cancel{n^2}}$$

$$\therefore \boxed{\beta_2 = \frac{12}{n} + 3}$$