Hence, Mean $= k_1 = n$

variance $= \mu_2 = k_2 = 2n$

$$\mu_3 = k_3 = 8n$$

$$\mu_4 = k_4 + 3k_2^2 = 48n + 12n^2$$

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{\overset{8}{\cancel{64}}\cancel{n^2}}{\cancel{8}n^{\cancel{3}}} = \frac{8}{n}$$

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{48n + 12n^2}{4n^2} = \frac{\overset{12}{\cancel{48}}\cancel{n}}{\cancel{4}n^{\cancel{2}}} + \frac{\overset{3}{\cancel{12}}\cancel{n^2}}{\cancel{4}\cancel{n^2}}$$

$$\therefore \boxed{\beta_2 = \frac{12}{n} + 3}$$

15.3.2 Limiting Form of χ^2 - Distribution for Large Degrees of Freedom

We know that, if $X \sim \chi^2_{(n)}$, then

$$M_X(t) = (1-2t)^{-n/2}, \quad |t| < \frac{1}{2}$$

$\therefore$ The mgf of standard χ^2-variate Z is

$$M_Z(t) = M_{\frac{X-\mu}{\sigma}}(t) = e^{-\mu t/\sigma} M_X\left(\frac{t}{\sigma}\right)$$

$$\left[\text{since here } \mu = n, \ \sigma^2 = 2n\right]$$

$$M_Z(t) = e^{-\mu t/\sigma}\left(1 - \frac{2t}{\sigma}\right)^{-n/2}$$

$$\left[\sigma^2 = 2n \Rightarrow \sigma = \sqrt{2n}\right]$$

MGF $\Rightarrow$

$$M_Z(t) = e^{-\mu t/\sqrt{2n}}\left(1 - \frac{2t}{\sqrt{2n}}\right)^{\frac{-n}{2}} = e^{-\cancel{n}t/\sqrt{2}\sqrt{\cancel{n}}}\left(1 - \frac{2t}{\sqrt{2}\sqrt{n}}\right)^{-n/2}$$

$$M_Z(t) = e^{-\sqrt{\frac{n}{2}}t}\left(1 - \sqrt{\frac{2}{n}}\,t\right)^{-n/2}$$

CGF =)

$$K_z(t) = \log M_z(t) =$$

$$= \log \left[e^{-\sqrt{\frac{n}{2}}\, t} \left(1 - \sqrt{2/n}\, t\right)^{-n/2} \right]$$

$$= \log \cancel{e}^{-\sqrt{\frac{n}{2}}\, t} + \log \left(1 - \sqrt{\frac{2}{n}}\, t\right)^{-n/2}$$

$$= -\sqrt{\frac{n}{2}}\, t - \frac{n}{2} \log \left(1 - \sqrt{\frac{2}{n}}\, t\right)$$

$$= -t\sqrt{\frac{n}{2}} + \frac{n}{2}\left[t \cdot \sqrt{\frac{2}{n}} + \frac{t^2}{2} \cdot \frac{2}{n} + \frac{t^3}{3}\left(\frac{2}{n}\right)^{3/2} + \dots \right]$$

$$\left[\log(1-x) = -x - \frac{x^2}{2!} - \frac{2x^3}{3!} - \dots \right.$$
$$\left. = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} \dots \right]$$

$$K_z(t) = -t\sqrt{\frac{n}{2}} + t\left(\frac{\cancel{n}}{\cancel{2}}\right)\left(\frac{\sqrt{2}}{\sqrt{n}}\right) + t^2\left(\frac{\cancel{n}}{2}\right)\left(\frac{1}{\cancel{n}}\right) + O(n^{-1/2})$$

Higher orders of n

$$= \cancel{-t\sqrt{\frac{n}{2}}} + \cancel{t \cdot \sqrt{\frac{n}{2}}} + \frac{t^2}{2} + O(n^{-1/2})$$

(Same terms with different signs get cancel)

$$K_z(t) = \frac{t^2}{2} + O(n^{-1/2})$$

Taking limit $n \to \infty$ on both sides

$$\lim_{n\to\infty} K_z(t) = \lim_{n\to\infty} \left(\frac{t^2}{2} + O(n^{-1/2})\right)$$

$$\lim_{n\to\infty} K_z(t) = \frac{t^2}{2} + 0 = \frac{t^2}{2}$$

(other elements become zero)

$$\boxed{\lim_{n\to\infty} K_z(t) = \frac{t^2}{2}}$$

$\therefore$ MGF becomes

$$\boxed{\lim_{n\to\infty} M_z(t) = e^{t^2/2}}$$

This is the required χ^2-distribution.

15.3.3 characteristic Function of χ^2-distribution

If $X \sim \chi^2_{(n)}$, then

$$\phi_X(t) = E\left[\exp(itX)\right]$$

$$= \int_0^\infty \exp(itx) f(x)\, dx$$

similar to MGF derivations Sec: 15.2

$$= \frac{1}{2^{n/2}\Gamma\frac{n}{2}} \int_0^\infty \exp\left[-\left(\frac{1-2it}{2}\right)x\right] x^{\frac{n}{2}-1}\, dx$$

$$\boxed{\phi_X(t) = (1-2it)^{-n/2}}$$

15.3.4 Mode & Skewness of χ^2- distribution:

Let $X \sim \chi^2_{(n)}$, so that

$$f(x) = \frac{1}{2^{n/2}\Gamma\frac{n}{2}} e^{-x/2} x^{\frac{n}{2}-1}, \quad 0 \le x < \infty \rightarrow (1)$$

Mode:- Mode is the value of x for which $f(x)$ is maximum, (ie) mode is the solution of

Refer page: 9.

$$f'(x) = 0 \ \& \ f''(x) < 0$$

Taking differentiation & log on both sides of equn (1)

$$\frac{f'(x)}{f(x)} = 0 - \frac{1}{2} + \left(\frac{n}{2} - 1\right) \cdot \frac{1}{x}$$

$$= -\frac{1}{2} + \frac{n-2}{2x}$$

$$\frac{f'(x)}{f(x)} = \frac{n-2-x}{2x} \rightarrow (2)$$

Since $f(x) \neq 0$, $f'(x) = 0$.

$n-2-x = 0 \Rightarrow x = n-2$

$\therefore f''(x) < 0$.

$\therefore$ Mode of the χ^2- distribution with n degree of freedom is $(n-2)$

$\therefore$ Karl Pearson's coefficient of skewness is

$$\text{Skewness} = \frac{\text{Mean} - \text{Mode}}{\text{S.D.}} = \frac{n - \cancel{n} + 2}{\sqrt{2n}} = \frac{\cancel{2}}{\sqrt{\cancel{2}}\sqrt{n}}$$

$$= \frac{\sqrt{2}}{\sqrt{n}} = \sqrt{\frac{2}{n}}$$

15.3.5 Additive property of χ^2 - variates

Statement :

The sum of independent chi-square variates is also a χ^2 - variates. More precisely, if X_i, $i=1,2,\ldots,k$, are independent χ^2 - variate with n_i d.f. respectively, then the sum $\sum_{i=1}^{k} X_i$ is also a chi-square variate with $\sum_{i=1}^{k} n_i$ d.f.

Proof :

We know that

$$M_{X_i}(t) = (1-2t)^{-n_i/2}, \quad i=1,2,\ldots,k$$

The mgf of the sum $\sum_{i=1}^{k} X_i$ is given by :

$$M_{\Sigma X_i}(t) = \cancel{M_{\Sigma X}(t)\, M} \prod_{i=1}^{k} M_{X_i}(t)$$

[similar to χ^2-dist. pdf]

$$= M_{X_1}(t) \cdot M_{X_2}(t) \cdots M_{X_k}(t)$$

$$= \left[(1-2t)^{-n_1/2}\right]\left[(1-2t)^{-n_2/2}\right]\cdots\left[(1-2t)^{-n_k/2}\right]$$

(Adding the powers of all the elements)

$$\boxed{M_{\Sigma X_i}(t) = (1-2t)^{-(n_1+n_2+\cdots+n_k)/2}}$$

This is the m.g.f. of χ^2 - variate with $(n_1+n_2+\ldots+n_k)$ d.f.

Hence by uniqueness theorem of mgf $\sum_{i=1}^{k} X_i$ is a χ^2 - variate with $\sum_{n=1}^{k} n_i$ d.f.

x ———— x