PART- C

State and Prove Cayley's theorem.

Any finite group is isomorphic to a group of permuta

Proof: T.P.T: G' is a group of permutations.

Step1: Let G be a finite group of order n.

Let $a \in G$, $f_a : G \to G$ by $f_a(x) = ax$

$f_a(y) = ay$

$$\therefore f_a(x) = f_a(y)$$

$$ax = ay$$

$$x = y \qquad (\because L.C.L)$$

$$\therefore f_a \text{ is } 1-1$$

$$f_a(a^{-1}y) = a(a^{-1}y)$$

$$= (aa^{-1})y$$

$$= ey$$

$$= y$$

$$\therefore f_a \text{ is onto}$$

$$\therefore f_a \text{ is bijection.}$$

Since G has n elements, f_a is just a permutation on n symbols. Let $G' = \{f_a / a \in G\}$

Step2: T.P.T: G' is a group

closure: Let $f_a, f_b \in G'$

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iii) Inverse: let $f_a, f_a^{-1} \in G'$

$(f_a \circ f_a^{-1})(x) = f_a(f_a^{-1}(x)) = f_a(a^{-1}x) = aa^{-1}x = ex = f_e(x) \rightarrow (1)$

$(f_a^{-1} \circ f_a)(x) = f_a^{-1}(f_a(x)) = f_{a^{-1}}(ax) = a^{-1}ax = ex = f_e(x) \rightarrow (2)$

$(1) \& (2) \Rightarrow f_a \circ f_{a^{-1}} = f_{a^{-1}} \circ f_a = f_e$

the inverse of f_a in G' is $f_{a^{-1}}$

G' is a group

Step 5: T.P.T : $G \cong G'$

$\phi : G \rightarrow G'$ by $\phi(a) = f_a$, $\phi(b) = f_b$

$\phi(a) = \phi(b)$

$f_a = f_b$

$f_a(x) = f_b(x)$

$ax = bx$ (∵ R C L)

$a = b$

∴ ϕ is 1-1

$f(a) = b$

$a = b$

∴ ϕ is onto

ϕ is bijection.

Also, $\phi(ab) = f_{ab} = f_a \circ f_b = \phi(a) \circ \phi(b)$

Hence ϕ is an isomorphism.

State and prove Fundamental theorem of homomorphism.

Let $f : G \rightarrow G'$ be an epimorphism. Let K be the kernel of f. Then $G/K \cong G'$.

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$(f_a \circ f_b)(x) = f_a(f_b(x))$

$= f_a(bx)$

$= a(bx)$

$= (ab)x$

$= f_{ab}(x)$

$\therefore f_a \circ f_b = f_{ab} \in G'$

$\therefore$ G' is closure

ii) Associative: let $f_a, f_b, f_c \in G'$

$((f_a \circ f_b) \circ f_c)(x) = (f_a \circ f_b(x)) \circ f_c = f_a(bx) \circ f_c$

$= f_c(f_a(bx)) = f_c(abx) = (abc)(x)$

$= f_{abc}(x) \longrightarrow ①$

$(f_a \circ (f_b \circ f_c))(x) = f_a \circ f_b(f_c(x)) = f_a \circ f_b(cx)$

$= f_a(f_b(cx)) = f_a(bcx)$

$= (abc)(x) = f_{abc}(x) \longrightarrow ②$

$\therefore$ ① & ② $\Rightarrow ((f_a \circ f_b) \circ f_c)(x) = (f_a \circ (f_b \circ f_c))(x)$

$\therefore$ G' is associative

iii) Identity: let $f_a, f_e \in G'$

$(f_a \circ f_e)(x) = f_a(f_e(x)) = f_a(ex) = aex = ax = f_a(x) \rightarrow ①$

$(f_e \circ f_a)(x) = f_e(f_a(x)) = f_e(ax) = eax = ax = f_a(x) \rightarrow ②$

① & ② $\Rightarrow$ $f_a \circ f_e = f_e \circ f_a = f_a$

$\therefore f_e$ is the identity element.

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Proof: Define $\phi: G/K \to G'$ by $\phi(Ka) = f(a)$

Step(i) T.P.T: ϕ is well defined

let $Kb = Ka$ $\forall b \in Ka$

$\therefore b = ka$ where $k \in K$.

$$f(b) = f(ka) = f(k) f(a) = e' f(a) = f(a)$$

$$\therefore \phi(Kb) = f(b) = f(a) = \phi(Ka)$$

$$\therefore \phi(Kb) = \phi(Ka)$$

$\therefore$ ϕ is well defined

Step ii: T.P.T: ϕ is 1-1

$$\therefore \phi(Ka) = \phi(Kb)$$

$$f(a) = f(b)$$

$$\Rightarrow f(a)[f(b)]^{-1} = f(ab^{-1}) = e'$$

$$\therefore ab^{-1} \in K$$

$$a \in Kb$$

$$\therefore Kb = Ka$$

$\therefore$ ϕ is 1-1

Step iii: T.P.T: ϕ is onto

let $a' \in G'$

Since f is onto, there exists $a \in G$ Such that $f(a) = a'$

$$\therefore \phi(Ka) = f(a) = a'$$

$\therefore$ ϕ is onto

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Step (iv): T.P.T: ϕ is homomorphism.

$$\phi(KaKb) = \phi(Kab) = f(ab) = f(a)f(b) = \phi(Ka)\phi(Kb)$$

$\therefore$ ϕ is homomorphism.

$\therefore$ ϕ is an isomorphism from G/K onto G'

$$\therefore G/K \cong G'$$

PART-B

6. Any infinite cyclic group G is isomorphic to $(Z,+)$

Proof: Let G be a infinite cyclic group with generator a then

$$G = \{a^n / n \in Z\}.$$

$\therefore f: Z \to G$ by $f(n) = a^n$

$\therefore$ G is infinite, $f(m) = a^m$

$$\therefore f(n) \neq f(m)$$

$$a^n \neq a^m$$

$$n \neq m$$

$\therefore f$ is 1-1

$$f(n) = m$$

$$a^n = m$$

Now, f is onto

$$f(n+m) = a^{n+m} = a^n \cdot a^m = f(n) \cdot f(m)$$

$\therefore f$ is an isomorphism.

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7. For any group G, (i) $Aut\, G$ is a group under Composition of functions (ii) $I(G)$ is a normal Subgroup of G.

Proof:

Step 1: T.P.T: $Aut\, G$ is a group.

Let $f, g \in Aut\, G$

$\therefore$ f and g are isomorphism of G to itself

$\therefore$ $f \circ g$ is an isomorphism of G to itself

$\therefore$ $f \circ g \in Aut\, G$.

$f \in Aut\, G \Rightarrow f^{-1} \in Aut\, G$.

$\therefore$ Composition of function is associative

Hence $Aut\, G$ is a group.

Step 2: T.P.T: $I(G)$ is normal Subgroup of $Aut(G)$

Let $\phi_a, \phi_b \in I(G)$

Then, $(\phi_a \phi_b)(x) = \phi_a(bxb^{-1}) = a(bxb^{-1})a^{-1}$

$= (ab)x(ab)^{-1}$

$= \phi_{ab}(x)$

$\therefore$ $\phi_a \phi_b = \phi_{ab} \in I(G)$

$\therefore$ ϕ_e is an identity element of $I(G)$

$\therefore$ $\phi_{a^{-1}}$ is an inverse of $I(G)$

$\therefore$ $I(G)$ is a Subgroup of $Aut(G)$

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I(G)

let $\alpha \in Aut(G)$ and $\phi_a \in I(G)$ Then

$$(\alpha\phi_a\alpha^{-1})(x) = \alpha\phi_a(\alpha^{-1}(x))$$
$$= \alpha(a\alpha^{-1}(x)a^{-1})$$
$$= \alpha(a)\alpha\alpha^{-1}(x)\alpha(a^{-1})$$
$$= \alpha(a)x[\alpha(a)]^{-1}$$
$$= \phi_{\alpha(a)}(x)$$

$$\alpha\phi_a\alpha^{-1} = \phi_{\alpha(a)} \in I(G)$$

$I(G)$ is normal Subgroup of Aut G.

8. Let H be a Subgroup of G. let $a \in G$ Then aHa^{-1} is a Subgroup of G.

Soln: T.P.T : aHa^{-1} is Subgroup of G

$$\therefore e = aea^{-1} \in aHa^{-1}$$
$$\therefore aHa^{-1} \neq \phi$$

let $x, y \in aHa^{-1}$

$x = ah_1a^{-1}$, $y = ah_2a^{-1}$ where $h_1, h_2 \in H$

$\therefore$ T.P.T : $xy^{-1} \in aHa^{-1}$

Now, $xy^{-1} = (ah_1a^{-1})(ah_2a^{-1})^{-1}$

$= (ah_1a^{-1})(ah_2^{-1}a^{-1})$ $\quad (\because (ab)^{-1} = (b^{-1})a^{-1})$

$= ah_1(a^{-1}a)h_2^{-1}a^{-1}$

$= ah_1h_2^{-1}a^{-1}$

$= a(h_1h_2^{-1})a^{-1} \in aHa^{-1}$

$\therefore aHa^{-1}$ is a Subgroup of G.

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10) Isomorphism is an equivalent relation among groups.

Proof: Step 1: For any group G,

$i_G : G \to G$ is an isomorphism.

$G \cong G$

$\therefore$ The relation is reflexive

Step 2: let $G \cong G'$ and let $f : G \to G'$ be an isomorphism. Then f is bijection.

$\therefore f^{-1} : G' \to G$ is also a bijection

let $x', y' \in G'$

let $f^{-1}(x') = x$ and $f^{-1}(y') = y$

Then $f(x) = x'$ and $f(y) = y'$

$\therefore f(xy) = f(x) f(y) = x'y'$

$\therefore f^{-1}(x'y') = xy = f^{-1}(x') f^{-1}(y')$

Hence f^{-1} is an isomorphism $G' \cong G$

$\therefore$ The relation is Symmetric

Step 3: let $G \cong G'$ and $G' \cong G''$

$\therefore f : G \to G'$ and $g : G' \to G''$

Since, f, g are bijection.

$g \circ f : G \to G''$ is also bijection.

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Let $x, y \in G$

Then $(gof)(xy) = g(f(xy))$

$= g(f(x) f(y))$ (f is isomorphism)

$= g[f(x)] g[f(y)]$ ($\because g$ is isomorphism)

$= gof(x) \, gof(y)$

Hence gof is isomorphism

$G \cong G''$

The relation is transitive

$\therefore$ Isomorphism is an equivalent relation among groups

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PART - A

1. Define Normal Subgroup with example.

A Subgroup H of G is called a normal Subgroup of G if $aH = Ha$ $\forall a \in G$.

Eg: For any group G, $\{e\}$ and G is a normal Subgroup.

$\because$ G is a group

$\{e\}$ is a Subgroup of G.

$a\{e\} = \{e\}a$

$\therefore$ $\{e\}$ is a normal Subgroup of G.

$\therefore$ G is also normal Subgroup.

2. Isomorphic Image of an abelian group is a abelian group.

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Also $\phi_a(a^{-1}xa) = a(a^{-1}xa)a^{-1}$

$= (aa^{-1})x(aa^{-1})$

$= exe$

$= x$

$\therefore a^{-1}xa$ is the preimage of x under ϕ_a

$\phi_a(xy) = axya^{-1}$

$= (axa^{-1})(aya^{-1})$

$= \phi_a(x)\phi_a(y)$

Thus ϕ_a is an automorphism of G.

4. Every Subgroup of an abelian group is a normal Subgroup.

Proof: let G be an abelian group and H be a Subgroup of G. let $a \in G$.

T.P.T: A is normal Subgroup

$\therefore aH = Ha$.

let $x \in aH$ then

$x = ah$ for some $h \in H$

$= ha$ (G is abelian by Commutative property)

$x \in Ha$

Hence $aH \subseteq Ha \rightarrow (1)$

let $x \in \cancel{H} aH$

$x = ha$

$= ah \in aH$

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Proof: Let $a', b' \in G'$ Then their exist $a, b \in G$

$\therefore f(a) = a'$ & $f(b) = b'$

$a'b' = f(a)f(b) = f(ab) = f(ba) = f(b)f(a)$

$= b'a'$

$\therefore$ Hence G' is abelian.

3. Define inner automorphism with example.

Let G be a group. The set of all automorphisms of G is denoted by Aut G. The set of all inner automorphism of G is denoted by $I(G)$. The automorphism $\phi_a : G \to G'$ is called inner automorphism of the group G.

Eg: Let G be any group. Let $a \in G$ Then $\phi_a : G \to G$ defined by $\phi_a(x) = axa^{-1}$ is an automorphism of $G \Rightarrow$

Let $x, y \in G$

$\phi_a(x) = \phi_a(y) \Rightarrow axa^{-1} = aya^{-1}$

$x = y$ (by R.C.L & L.C.L)

$\therefore \phi_a$ is 1-1

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Hence $Ha \subseteq aH \rightarrow (2)$

From (1) & (2) $\Rightarrow$ $\therefore aH = Ha$

$\therefore$ H is a normal Subgroup of G

5. Isomorphism with example:

Let G and G' be two groups. A map $f: G \to G'$ is called isomorphism. If

i) f is a bijection

ii) $f(xy) = f(x)f(y) \ \forall x, y \in G$

If G and G'' be isomorphic, $f: G \to G'$ it denoted by $G \cong G'$

Eg: $(z, +) \cong (2z, +)$

Consider $f: z \to 2z$, $f(x) = 2x$.

$\therefore$ f is bijection.

$\therefore f(x+y) = 2(x+y) = 2x+2y = f(x)+f(y)$

$\therefore$ f is an isomorphism

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