# Abstract Algebra

## Unit III

Normal subgroups and Quotient Groups - Isomorphism - Homomorphism.

### 3.9 Normal Subgroups and Quotient Groups

**Definition**

A subgroup $H$ of $G$ is called a normal subgroup of $G$ if $aH = Ha$ all $a \in G$.

**Example**

For any Group $G$, $\{e\}$ and $G$ are normal subgroup.

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**Theorem 3.39**

Every subgroup of an abelian group is a normal subgroup.

**proof**

Let $G$ be an abelian group and let $H$ be a subgroup of $G$. Let $a \in G$

we claim that $aH = Ha$.

Let $x \in aH$. Then $x = ah$ for some $h \in H$.

$= ha$ ($\because G$ is abelian)

$\therefore x \in Ha$. Hence $aH \subseteq Ha$. Similarly $Ha \subseteq aH$.

$\therefore aH = Ha$ and hence $H$ is a normal subgroup of $G$.

**Example**

(i) $nZ$, is a normal subgroup of $(Z,+)$

(ii) Every subgroup of $(Z_n, \oplus)$ is normal

(iii) Since any cyclic group is abelian any subgroup of a cyclic group is normal.

## Theorem 3.40

Let $H$ be a subgroup of index 2 in a group $G$. Then $H$ is a normal subgroup of $G$.

**Proof**

If $a \in H$ then $H = aH = Ha$.

If $a \notin H$, then $aH$ is a left coset different from $H$. Hence $H \cap aH = \phi$. Further, since index of $H$ in $G$ is 2, $H \cup aH = G$. Hence $aH = G - H$. Similary $Ha = G - H$.

$\therefore aH = Ha$.

Hence $H$ is a normal subgroup of $G$.

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## Theorem 3.41

Let $N$ be a subgroup of $G$. Then the following are equivalent

(i) $N$ is a normal subgroup of $G$.

(ii) $aNa^{-1} = N \ \forall a \in G$.

(iii) $aNa^{-1} \subseteq N \ \forall a \in G$.

(iv) $ana^{-1} \in N \ \forall n \in N$ and $a \in G$.

**Proof :**

(i) $\Rightarrow$ (ii)

Suppose $N$ is a normal subgroup of $G$.

$\therefore aN = Na \ \forall a \in G$.

$\therefore aNa^{-1} = Naa^{-1} = Ne = N$.

(ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (iv) are obvious.

(iv) $\Rightarrow$ (i)

Suppose that $ana^{-1} \in N \ \forall n \in N$ and $a \in G$.

We claim that $aN = Na$

Let $x \in aN$

$\therefore x = an$ for some $n \in N$

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$x = (ana^{-1})a \in Na \; (\because ana^{-1} \in N)$

$aN \subseteq Na \rightarrow (1)$

Now, let $x \in Na$

$\therefore x = na$ for some $n \in N$

$\therefore x = a(a^{-1}na) = a(a^{-1}n(a^{-1})^{-1}) \in aN$

$Na \subseteq aN \rightarrow (2)$

From (1) & (2) we get $Na = aN$.

Hence $N$ is a normal subgroup of $G$.

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## Solved problems

### Problem 1

Prove that the intersection of two normal subgroups of a group $G$ is a normal subgroup of $G$.

**Solution**

Let $H$ and $K$ be two normal subgroups of $G$.

Then $H \cap K$ is a subgroup of $G$.

Now, let $a \in G$ and $x \in H \cap K$. Then $x \in H$ and $x \in K$. $\because$ $H$ and $K$ are normal $axa^{-1} \in H$ and $axa^{-1} \in K$.

Hence $axa^{-1} \in H \cap K$. Thus $H \cap K$ is a normal subgroup of $G$.

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### Problem 2

The centre $H$ of a group $G$ is a normal subgroup of $G$.

**Solution.**

The centre $H$ of $G$ is given by

$H = \{a \,/\, a \in G, ax = xa \;\forall x \in G\}$

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Now let $x \in H$ and $a \in G$. Hence $ax = xa$.

$\therefore x = axa^{-1} \in H$.

Hence $H$ is a normal subgroup of $G$.

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Problem 3

Let $H$ be a subgroup of $G$. Let $a \in G$. Then $aHa^{-1}$ is a subgroup of $G$.

Solution.

$e = aea^{-1} \in aHa^{-1}$ and hence $aHa^{-1} \neq \phi$

Now, let $x, y \in aHa^{-1}$

Then $x = ah_1a^{-1}$ and $y = ah_2a^{-1}$

where $h_1, h_2 \in H$.

Now,

$$xy^{-1} = (ah_1a^{-1})(ah_2a^{-1})^{-1}$$

$$= (ah_1a^{-1})(ah_2^{-1}a^{-1})$$

$$= a(h_1h_2^{-1})a^{-1} \in aHa^{-1}$$

$\therefore aHa^{-1}$ is a subgroup of $G$.

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Problem 4

Show that if a group $G$ has exactly one subgroup $H$ of given order, then $H$ is a normal subgroup of $G$.

Solution

Let the order of $H$ be $m$.

Let $a \in G$. Then by solved problem 3, $aHa^{-1}$ is also a subgroup of $G$.

We claim that $|H| = |aHa^{-1}| = m$.

Now, consider $f : H \to aHa^{-1}$ defined by $f(h) = aha^{-1}$

$f$ is 1-1, for $f(h_1) = f(h_2)$

$\Rightarrow ah_1a^{-1} = ah_2a^{-1}$

$\Rightarrow h_1 = h_2$

$f$ is onto, for, let $x = aha^{-1} \in aHa^{-1}$

Then $f(h) = x$, Thus $f$ is a bijection.

$\therefore |H| = |aHa^{-1}| = m$.

But $H$ is the only subgroup of $G$ of order $m$.

$\therefore aHa^{-1} = H$. Hence $aH = Ha$.

$\therefore H$ is a normal subgroup of $G$.

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## Problem 5

If $H$ is a subgroup of $G$ and $N$ is a normal subgroup of $G$ then $HN$ is a subgroup of $G$.

### Solution

To prove that $HN$ is a subgroup of $G$, it is enough if we prove that $HN = NH$.

Let $x \in HN$. Then $x = hn$ where $h \in H$ & $n \in N$.

$\therefore x \in hN$.

But $hN = Nh$ ($\because N$ is normal)

$\therefore x \in Nh$. Hence $x = n_1h$ where $n_1 \in N$.

$\therefore x \in NH$. Hence $HN \subseteq NH$.

Similarly $NH \subseteq HN$

$\therefore HN = NH$.

Hence $HN$ is a subgroup of $G$.

Problem 6

M and N are normal subgroups of a group G such that $M \cup N = \{e\}$. Show that every element of M commutes with every element of N.

Solution

Let $a \in M$ and $b \in N$.

We claim that $ab = ba$.

Consider the element $aba^{-1}b^{-1}$.

$\because a^{-1} \in M$ and $M$ is normal, $ba^{-1}b^{-1} \in M$.

Also $a \in M$, so that $aba^{-1}b^{-1} \in M$.

Again, $\because b \in N$ and $N$ is normal, $aba^{-1} \in N$.

Also $b^{-1} \in N$, so that $aba^{-1}b^{-1} \in N$.

Thus $aba^{-1}b^{-1} \in M \cap N = \{e\}$.

$\therefore aba^{-1}b^{-1} = e$, so that $ab = ba$.

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Problem 7

Show that if H and N are subgroups of a group G and N is normal in G, then $H \cap N$ is normal in H. Show by an example that $H \cap N$ need not be normal in G.

Solution

Let $x \in H \cap N$. and $a \in H$

We claim that $axa^{-1} \in H \cap N$

Now, $x \in N$ and $a \in H \Rightarrow axa^{-1} \in N$ ($\because N$ is a normal subgroup)

Also $x \in H$ and $a \in H \Rightarrow axa^{-1} \in H$ ($\because H$ is a group)

Hence $axa^{-1} \in H \cap N$.

$\therefore H \cap N$ is a normal subgroup of H.

The following example shows that $H \cap N$ need not be normal in $G$.

Let $G = S_3$. Take $N = G$ and $H = \{e, p_3\}$.

Now $H \cap N = H$ which is not normal in $G$.

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**Theorem 3.42**

A subgroup $N$ of $G$ is normal iff the product of two cosets of $N$ is again a right coset of $N$.

**Proof**

Suppose $N$ is a normal subgroup of $G$.

Then

$$NaNb = N(aN)b$$
$$= N(Nab) \quad (\because aN = Na)$$
$$= NNab$$
$$= Nab \quad (\because NN = N)$$

Conversely suppose that the product of any two right cosets of $N$ is again a right coset of $N$.

Then $NaNb$ is a right coset of $N$.

Further $ab = (ea)(eb) \in NaNb$

Hence $NaNb$ is the right coset containing $ab$

$\therefore NaNb = Nab$

Now, we prove that $N$ is a normal subgroup of $G$.

Let $a \in G$ and $n \in N$. Then

$ana^{-1} = eana^{-1} \in NaNa^{-1} = Naa^{-1} = N$

$\therefore ana^{-1} \in N$

Hence $N$ is a normal subgroup of $G$.

## Theorem 3.43

Let $N$ be a normal subgroup of a group $G$. Then $G/N$ is a group under the operations defined by $NaNb = Nab$.

**Proof**

By theorem 3.42 the operation given by $NaNb = Nab$ is a well defined binary operation in $G/N$.

Now, let $Na, Nb, Nc \in G/N$

Then
$$Na(NbNc) = Na(Nbc)$$
$$= Na(bc)$$
$$= N(ab)c$$
$$= (NaNb)Nc$$

$\therefore$ The binary operation is associative.

$Ne = N \in G/N$ and $NaNe = Nae = Na = NeNa$

$\therefore Ne$ is the identity element.

Also $NaNa^{-1} = Naa^{-1} = Ne = Na^{-1}Na$

$\therefore Na^{-1}$ is the inverse of $Na$.

$\therefore G/N$ is a group.

**Definition**

Let $N$ be a normal subgroup of $G$. Then the group $G/N$ is called the quotient group (factor group) of $G$ modulo $N$.

**Example**

$3Z$ is a normal subgroup of $(Z, +)$.

Then the quotient group $Z/3Z = \{3Z+0, 3Z+1, 3Z+2\}$

Hence $Z/3Z$ is a group of order 3.

Exercises

problem.

prove that if $G$ is abelian and $H$ is a subgroup of $G$, then $G/H$ is abelian.

solution:

Let $G$ be abelian and

let $G/H = \{gH : g \in G\}$.

Let $g_1H, g_2H \in G/H$.

Then $(g_1H)(g_2H) = g_1g_2H = g_2g_1H$ ($G$ is abelian)

$= (g_2H)(g_1H)$

$\therefore G/H$ is abelian.

3.10 Isomorphism.

Definition

Let $G$ and $G'$ be two groups. A map $f: G \to G'$ is called an isomorphism if

(i) $f$ is a bijection

(ii) $f(xy) = f(x)f(y)\ \forall\ x, y \in G$.

Theorem 3.44

Isomorphism is an equivalence relation among groups.

proof For any group $G$, $i_G : G \to G$ is clearly an isomorphism.

Hence $G \cong G$. $\therefore$ the relation is reflexive.

Now, let $G \cong G'$ and let $f: G \to G'$ be an isomorphism.

Then $f$ is a bijection. $\therefore f^{-1}: G' \to G$ is also a bijection.

Now, let $x', y' \in G'$.

Let $f^{-1}(x') = x$ & $f^{-1}(y') = y$

Then $f(x) = x'$ and $f(y) = y'$

$\therefore f(xy) = f(x)f(y) = x'y'$

$\therefore f^{-1}(x'y') = xy = f^{-1}(x')f^{-1}(y')$

Hence $f^{-1}$ is an isomorphism.

Thus $G' \cong G$ and hence the relation is symmetric

Now, let $G \cong G'$ and $G' \cong G''$

Then $\exists$ isomorphism $f: G \to G'$ and $g: G' \to G''$

$\because f$ and $g$ are bijections. $g \circ f : G \to G''$ is also a bijection.

Now, let $x, y \in G$. Then

$$(g \circ f)(xy) = g[f(xy)]$$
$$= g[f(x)f(y)] \quad [\because f \text{ is an isomorphism}]$$
$$= g[f(x)]\, g[f(y)] \quad [\because g \text{ is an isomorphism}]$$
$$= (g \circ f)(x)\,(g \circ f)(y)$$

Hence $g \circ f$ is an isomorphism.

Thus $G \cong G''$ and hence the relation is transitive

$\therefore$ Isomorphism is an equivalence relation among groups.

**Example**

$(\mathbb{Z}, +) \cong (2\mathbb{Z}, +)$

Consider $f: \mathbb{Z} \to 2\mathbb{Z}$ given by $f(x) = 2x$

Clearly $f$ is a bijection.

Also $f(x+y) = 2(x+y) = 2x + 2y = f(x) + f(y)$

Hence $f$ is an isomorphism.

## Theorem 3.45

Let $f: G \to G'$ be an isomorphism. Then

(i) $f(e) = e'$ where $e$ and $e'$ are the identity elements of $G$ and $G'$ respectively.

(ii) $f(a^{-1}) = [f(a)]^{-1}$

**proof**

(i) To prove that $f(e) = e'$ it is enough if we prove that $a' f(e) = f(e) a' = a' \ \forall a' \in G'$

Let $a' \in G'$. since $f: G \to G'$ is a bijection, $\exists$ such that $a \in G$ such that $f(a) = a'$.

$\therefore a' f(e) = f(a) f(e) = f(ae) = f(a) = a'$

Similarly $f(e) a' = a'$.

$\therefore f(e) = e'$.

(ii) It is enough to prove that

$f(a) f(a^{-1}) = f(a^{-1}) f(a) = e'$

Now, $f(a) f(a^{-1}) = f(a a^{-1}) = f(e) = e'$

Also, $f(a^{-1}) f(a) = f(a^{-1} a) = f(e) = e'$

$\therefore f(a) f(a^{-1}) = f(a^{-1}) f(a) = e'$

$\therefore [f(a)]^{-1} = f(a^{-1})$

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## Theorem 3.46

Let $f: G \to G'$ be an isomorphism. If $G$ is abelian, then $G'$ is also abelian.

**proof**

Let $a', b' \in G'$, $\exists \ a, b \in G$ such that

$f(a) = a'$ & $f(b) = b'$

Now, $a'b' = f(a) f(b) = f(ab) = f(ba)$

$= f(b) f(a) = b'a'$

Hence $G'$ is abelian.

## Theorem 3.47

Let $f: G \to G'$ be an isomorphism. Let $a \in G$. Then the order of $a$ is equal to the order of $f(a)$.

**Proof**

Suppose the order of $a$ is $n$. Then $n$ is the least positive integer such that $a^n = e$.

Now,

$[f(a)]^n = f(a) \cdots f(a)$ ( $f(a)$ written $n$ times)

$= f(a^n)$

$= f(e) = e'$

Now, if possible let $m$ be a positive integer such that $0 < m < n$ and $[f(a)]^m = e'$

Then $f(a^m) = [f(a)]^m = e'$

But $f(e) = e'$. Since $f$ is 1-1 we've $a^m = e$ which contradicts the definition of the order of $a$.

$\therefore$ $n$ is the least positive integer such that $[f(a)]^n = e'$.

$\therefore$ The order of $f(a)$ is $n$.

## Theorem 3.48

Let $f: G \to G'$ be an isomorphism. If $G$ is cyclic then $G'$ is also cyclic.

**Proof**

Let $a$ be a generator of the group $G$. We shall prove that $f(a)$ is a generator of the group $G'$.

Let $x' \in G'$. Since $f$ is a bijection, $\exists\, x \in G$ such that $f(x) = x'$.

Now, since $G = \langle a \rangle$, $x = a^n$ for some integer $n$.

Hence $x' = f(x) = f(a^n) = [f(a)]^n$.

$\because x' \in G'$ is arbitrary every element of $G'$ is of the form $[f(a)]^n$ so that $G' = \langle f(a) \rangle$.

Hence $G'$ is cyclic.

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**Theorem 3.49**

Any infinite cyclic group $G$ is isomorphic to $(\mathbb{Z}, +)$

**proof**

Let $G$ be an infinite cyclic group with generator $a$. Then $G = \{a^n \mid n \in \mathbb{Z}\}$.

Define $f : \mathbb{Z} \to G$ by $f(n) = a^n$.

$\because G$ is infinite, $n \neq m \Rightarrow a^n \neq a^m$.

Hence $f$ is 1-1. obviously $f$ is onto.

Now $f(n+m) = a^{n+m} = a^n \cdot a^m = f(n)\, f(m)$

Hence $f$ is an isomorphism.

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**Theorem 3.50**

Any finite cyclic group of order $n$ is isomorphic to $(\mathbb{Z}_n, \oplus)$

**proof**

Let $G$ be a cyclic group of order $n$ with generator $a$. Then $G = \{e, a, a^2 \ldots a^{n-1}\}$

Define $f : \mathbb{Z}_n \to G$ by $f(r) = a^r$

Clearly $f$ is a bijection.

Now, let $r, s \in \mathbb{Z}_n$. let $r \oplus s = t$

Then $r + s = qn + t$, where $0 \leq t < n$

$\therefore f(r \oplus s) = a^{r \oplus s} = a^t$ — (1)

Also, $f(r) f(s) = a^r a^s = a^{r+s} = a^{qn+t} = a^{qn} \cdot a^t = (a^n)^q a^t = e a^t = a^t$
$\to (2)$

From (1) & (2), we get $f(r \oplus s) = f(r) f(s)$.

Hence $f$ is an isomorphism.

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Theorem: 3.51 ~~Cayl~~

Cayley's Theorem:

Any finite group is isomorphic to a group of permutations.

Proof

We shall prove this theorem in 3 steps.

We shall first find a set $G'$ of permutations. Then we prove that $G'$ is a group of permutations and finally we exhibit an isomorphism $\phi : G \to G'$

Step 1

Let $G$ be a finite group of order $n$.

Let $a \in G$. Define $f_a : G \to G$ by $f_a(x) = ax$.

Now, $f_a$ is 1-1. $\because f_a(x) = f_a(y) \Rightarrow ax = ay \Rightarrow x = y$.

$f_a$ is onto. Thus $f_a$ is a bijection.

$\because G$ has $n$ elements, $f_a$ is just a permutation on $n$ symbols.

Let $G' = \{ f_a \mid a \in G \}$.

Step 2

We prove $G'$ is a group. Let $f_a, f_b \in G'$

$(f_a \circ f_b)(x) = f_a(f_b(x)) = f_a(bx) = a(bx) = (ab)x$
$= f_{ab}(x)$

Hence $f_a \circ f_b = f_{ab}$. Hence $G'$ is closed under composition of mappings. $f_e \in G'$ is the identity element. The inverse of $f_a$ in $G'$ is $f_{a^{-1}}$

Step 3

we prove $G \cong G'$

Define $\phi : G \to G'$ by $\phi(a) = f_a$

$\phi(a) = \phi(b) \Rightarrow f_a = f_b \Rightarrow f_a(x) = f_b(x) \Rightarrow ax = bx \Rightarrow a = b$

Hence $\phi$ is 1-1. obviously $\phi$ is onto.

Also $\phi(ab) = f_{ab} = f_a \circ f_b = \phi(a) \circ \phi(b)$

Hence $\phi$ is an isomorphism.

**Definition**

An isomorphism of a group $G$ to itself is called an automorphism of $G$.

The set of all automorphism of $G$ is denoted by Aut $G$.

**Example**

Any group $G$ has atleast one automorphism namely $i_G$.

**Definition**

The automorphism $\phi_a : G \to G$ defined by $\phi_a(xy) = \phi_a(x)\phi_a(y)$ is called an inner automorphism of the group $G$. The set of all inner automorphism of $G$ is denoted by $I(G)$.

**Theorem 3.52**

For any group $G$,

(i) Aut $G$ is a group under composition of functions

(ii) $I(G)$ is a normal subgroup of Aut $G$.

**Proof**

(i) Let $f, g \in$ Aut $G$

$\therefore$ $f$ and $g$ are isomorphisms of $G$ to itself.

$\therefore$ $f \circ g$ is an isomorphism of $G$ to itself.

$\therefore$ $f \circ g \in$ Aut $G$.

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$f \in Aut\, G \Rightarrow f^{-1} \in Aut\, G$

Clearly composition of functions is associative.

Hence $Aut\, G$ is a group.

(ii) Let $\phi_a, \phi_b \in I(G)$. Then

$$(\phi_a \phi_b)(x) = \phi_a(bxb^{-1})$$
$$= a(bxb^{-1})a^{-1}$$
$$= (ab)x(ab)^{-1}$$
$$= \phi_{ab}(x)$$

Hence $\phi_a \phi_b = \phi_{ab} \in I(G)$

$\phi_e$ is the identity element of $I(G)$ and the inverse of $\phi_a$ is $\phi_{a^{-1}}$

$\therefore I(G)$ is a subgroup of $Aut\, G$.

We now prove that $I(G)$ is a normal subgroup of $Aut(G)$.

Let $\alpha \in Aut(G)$, $\phi_a \in I(G)$. Then

$$(\alpha \phi_a \alpha^{-1})(x) = \alpha \phi_a(\alpha^{-1}(x))$$
$$= \alpha(a\alpha^{-1}(x)a^{-1})$$
$$= \alpha(a)\alpha\alpha^{-1}(x)\alpha(a^{-1})$$
$$= \alpha(a)\, x\, [\alpha(a)]^{-1}$$
$$= \phi_{\alpha(a)}(x)$$

$\therefore \alpha \phi_a \alpha^{-1} = \phi_{\alpha(a)} \in I(G)$.

Hence $I(G)$ is a normal subgroup of $Aut\, G$.

## 3.11. Homomorphisms

**Definition.**

A map $f$ from a group $G$ into a group $G'$ is called a homomorphism if $f(ab)=f(a)f(b)$ $\forall\ a,b \in G$.

**Example**

$f : (R^*, \cdot) \to (R^+, \cdot)$ defined by $f(x) = |x|$ is a homomorphism.

For, $f(xy) = |xy| = |x||y| = f(x)f(y)$.

This homomorphism is onto.

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**Definition.**

Let $f : G \to G'$ be a homomorphism.

(i) If $f$ is onto, then it is called an epimorphism.

(ii) If $f$ is 1-1, then it is called a monomorphism.

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**Definition**

Let $f : G \to G'$ be a homomorphism.

Let $K = \{x \mid x \in G, f(x) = e'\}$. Then $K$ is called the kernel of $f$ and is denoted by $\ker f$.

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**Theorem**

Let $f : G \to G'$ be a homomorphism. Then the kernal $K$ of $f$ is a normal subgroup of $G$.

**proof**

$\{e'\}$ is a normal subgroup of $f(G)$.

Hence $\ker f = f^{-1}(\{e'\})$ is a normal subgroup of $G$.

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## Theorem

### Fundamental Theorem of Homomorphism

Let $f : G \to G'$ be an epimorphism. Let $K$ be the kernel of $f$. Then $G/K \cong G'$.

**Proof**

Define $\phi : G/K \to G'$ by $\phi(Ka) = f(a)$

**Step(i)**

$\phi$ is well defined.

Let $Kb = Ka$. Then $b \in Ka$

Hence $b = ka$ where $k \in K$

Now, $f(b) = f(ka) = f(k)f(a) = e' f(a) = f(a)$.

$\therefore \phi(Kb) = f(b) = f(a) = \phi(Ka)$

Hence $\phi(Ka) = \phi(Kb)$

**Step(ii)**

$\phi$ is 1-1

For: $\phi(Ka) = \phi(Kb) \Rightarrow f(a) = f(b)$

$\Rightarrow f(a)[f(b)]^{-1} = e'$

$\Rightarrow f(ab^{-1}) = e'$

$\Rightarrow ab^{-1} \in K$

$\Rightarrow a \in Kb$

$\Rightarrow Ka = Kb$

**Step(iii)**

$\phi$ is onto

let $a' \in G'$. $\because f$ is onto, $\exists$ $a \in G$ such that $f(a) = a'$.

Hence $\phi(Ka) = f(a) = a'$

**Step (iv)**

$\phi$ is a homomorphism.

$\phi(Ka\,Kb) = \phi(Kab) = f(ab) = f(a)f(b) = \phi(Ka)\,\phi(Kb)$

Thus $\phi$ is an isomorphism from $G/K$ onto $G'$

$\therefore G/K \cong G'$

Problem

Show that the map $f : (\mathbb{C}, +) \to (\mathbb{R}, +)$. defined by $f(x+iy) = y$ is an epimorphism and $\ker f = \mathbb{R}$. Deduce that $\mathbb{C}/\mathbb{R} \cong \mathbb{R}$

Solution

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$

Then $z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

$\therefore f(z_1 + z_2) = y_1 + y_2 = f(z_1) + f(z_2)$

Hence $f$ is a homomorphism.

Clearly $f$ is onto.

Now,

$\ker f = f\{x+iy \mid f(x+iy) = 0\}$

$= \{x+iy \mid y = 0\}$

$= \mathbb{R}$

$\therefore$ By the fundamental theorem of Homomorphism. $\mathbb{C}/\mathbb{R} \cong \mathbb{R}$.