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Complex Analysis
Unit 5

Evaluation of Definite Integral

Type 1:

1) **Evaluate :-** $\displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$

Solution:-

Let $I = \displaystyle\int_0^{2\pi} \frac{d\theta}{5+4\sin\theta} \rightarrow (1)$

Put $z = e^{i\theta}$.

Differentiating it we get

$dz = i\underline{\underline{e^{i\theta}}}\, d\theta$

$dz = iz\,d\theta \Rightarrow d\theta = \dfrac{dz}{iz}$

Also, we know that

$\sin\theta = \dfrac{z-z^{-1}}{2i}$

$\therefore$ Equation (1) becomes

$$I = \int_C \frac{dz}{\left[5+\cancel{4}^{2}\left(\frac{z-z^{-1}}{\cancel{2}i}\right)\right] iz}$$

$$= \int_C \frac{dz}{\left[5-2i(z-z^{-1})\right] iz} = \int_C \frac{dz}{5iz+2z^2-2}$$

Let us consider $f(z) = \frac{1}{2z^2+5iz-2}$

$$f(z) = \frac{1}{2(z+2i)(z+\frac{i}{2})}$$

$\therefore -2i$ and $-\frac{i}{2}$ are the simple poles of $f(z)$

$$\therefore \text{Res}\left\{f(z); -\frac{i}{2}\right\} = \lim_{z \to -\frac{i}{2}} \frac{1}{2(z+2i)}$$

$$= \frac{1}{2\left(-\frac{i}{2}+2i\right)} = \frac{1}{-i+4i} = \frac{1}{3i}$$

$\left[\because \text{since} \quad \text{Res}\{f(z); a\} = \lim_{z \to a} (z-a) f(z)\right]$

By Cauchy's Residue Theorem,

$$I = 2\pi i \sum_{j=1}^{n} \text{Res}\{f(z); z_i\}$$

$$= 2\pi \cancel{i} \times \frac{1}{3\cancel{i}} = \frac{2\pi}{3}$$

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2) Using Contour integration, evaluate $\int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$

Let $I = \int_0^{2\pi} \frac{d\theta}{13+5\sin\theta}$

Put $z = e^{i\theta}$

$dz = ie^{i\theta} d\theta$

$dz = iz\,d\theta$

$\therefore d\theta = \frac{dz}{iz}$; Also $\sin\theta = \frac{z-z^{-1}}{2i}$

$\therefore$ The given integral can be written as

$$I = \int_C \frac{dz}{iz\left[13 + 5\left(\frac{z - z^{-1}}{2i}\right)\right]} \quad \text{(where C is the circle } |z| = 1\text{)}$$

$$I = \int_C \frac{dz}{iz\left[13+5\left(\frac{z - \frac{1}{z}}{2i}\right)\right]} = \int_C \frac{dz}{iz\left[13 + 5\left(\frac{z^2 - 1}{2iz}\right)\right]}$$

Simplifying we get

$$I = \int_C \frac{2\,dz}{5z^2 + 26iz - 5}$$

Let us consider

$$f(z) = \frac{2}{5z^2 + 26iz - 5}$$

By factorizing the denominator we get

$$f(z) = \frac{2}{(z+5i)(5z+i)}$$

Here, $\frac{-i}{5}$ and $-5i$ are simple poles of $f(z)$ and the pole $\frac{-i}{5}$ lies inside the unit circle.

$$\therefore \operatorname{Res}\{f(z)\,;\,a\} = \frac{h(a)}{k'(a)} = \lim_{z \to a} \frac{h(z)}{k'(z)}$$

$$\text{Res}\left\{f(z); \frac{-i}{5}\right\} = \lim_{z \to -\frac{i}{5}} \frac{h(z)}{k'(z)}$$

Put $h(z) = 2$ & $k(z) = 5z^2 + i26z - 5$

$$k'(z) = 10z + i26$$

$$\therefore \text{Res}\left\{f(z); \frac{-i}{5}\right\} = \lim_{z \to -\frac{i}{5}} \frac{2}{10z+26i} = \frac{2}{\cancel{10}^{2}\left(-\frac{i}{\cancel{5}}\right) + 26i}$$

$$= \frac{2}{-2i+26i} = \frac{\cancel{2}}{\cancel{24}\, i} = \frac{1}{12i} \;//.$$

$$\therefore \text{Res}\left\{f(z); \frac{-i}{5}\right\} = \frac{1}{12i} \;//.$$

Hence by Cauchy's residue theorem,

$$I = 2\pi i \sum_j \text{Res}\{f(z); z_j\} = \cancel{2}\pi \cancel{i} \times \frac{1}{\cancel{12}\, \cancel{i}} = \frac{\pi}{6} \;//.$$

Type : II

1. Use Contour Integration method to evaluate $\int_0^\infty \frac{dx}{1+x^4}$

Solution:-

Let $f(z) = \frac{1}{1+z^4}$

The roots of the equation $z^4+1=0$ are

$z^4 = -1$

$$\boxed{z = \sqrt[4]{-1}}$$

(ie) Fourth roots of -1.

These are the poles of $f(z)$

By De Moivre's Theorem,

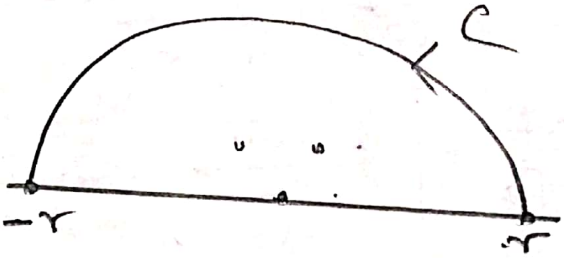
$e^{\frac{i\pi}{4}}, e^{\frac{i3\pi}{4}}, e^{\frac{i5\pi}{4}}, e^{i7\pi/4}$ are the simple poles.

Let us choose the path/contour C consisting of the interval $[-r, r]$ on the real axis and the upper semi-circle $|z| = r$, we denote C_1

$$\therefore \int_C f(z)\,dz = \int_{-r}^{r} f(x)\,dx + \int_{C_1} f(z)\,dz \rightarrow (1)$$

The poles of $f(z)$ lying inside the contour C are $e^{i\pi/4}$ & $e^{i3\pi/4}$ only

we find the residues of $f(z)$ at these points



Consider $e^{i\pi/4}$

$$\text{Res}\{f(z); e^{i\pi/4}\} = \frac{h(e^{i\pi/4})}{k'(e^{i\pi/4})}$$

where,

$h(z) = 1$; $k(z) = z^4 + 1 \Rightarrow k'(z) = 4z^3$

$$\text{Res}\{f(z); e^{i\pi/4}\} = \lim_{z \to e^{i\pi/4}} \frac{1}{4z^3} = \frac{1}{4(e^{i3\pi/4})}$$

$$= \frac{1}{4} e^{-\frac{i3\pi}{4}}$$

//

Consider $e^{i\frac{3\pi}{4}}$.

~~By residue theorem~~, $\text{Res}\left\{ f(z) ; e^{i\frac{3\pi}{4}} \right\} = \lim_{z \to e^{i3\pi/4}} \frac{1}{4z^3}$

$$= \frac{1}{4\left(e^{i3\pi/4}\right)^3} = \frac{1}{4\, e^{i9\pi/4}} = \frac{1}{4} e^{-i9\pi/4}$$

By residue theorem,

$$\int_C f(z)\, dz = 2\pi i \,(\text{sum of the residues at the poles})$$

$$= 2\pi i \left[\frac{e^{-i3\pi/4}}{4} + \frac{e^{-i9\pi/4}}{4} \right]$$

$$= \frac{\pi i}{2} \left[e^{-i\frac{3\pi}{4}} + e^{-i\frac{9\pi}{4}} \right]$$

$$= \frac{\pi i}{2} \left[\cos\frac{3\pi}{4} - i \sin\frac{3\pi}{4} + \cos\frac{9\pi}{4} - i \sin\frac{9\pi}{4} \right]$$

$$= \frac{\pi i}{2} \left[\left(\cancel{\frac{-1}{\sqrt{2}}} - \frac{i}{\sqrt{2}} \right) + \left(\cancel{\frac{1}{\sqrt{2}}} - \frac{i}{\sqrt{2}} \right) \right]$$

$$= \frac{\pi i}{\cancel{2}} \left[\frac{-\cancel{2}i}{\sqrt{2}} \right] = \frac{\pi}{\sqrt{2}} \rightarrow (2)$$

From equn (1),

$$\cancel{\int_{-r}^{r} f(x)\, dx =} \cancel{\int_{-r}^{r} \frac{dx}{1+x^4}}.$$

$$\int_{-r}^{r} f(x)\, dx + \int_{C_1} f(z)\, dz = \frac{\pi}{\sqrt{2}} \rightarrow (3)$$

(3) $\Rightarrow$

As $r \to \infty$, $\int_{C_1} f(z)\, dz \to 0$

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}} \Rightarrow 2\int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}} \Rightarrow \boxed{\int_0^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}}$$