Complex Analysis

Study about complex variables

denoted by w (or) z

$$w = f(z) = u(x,y) + iv(x,y)$$

u, v are real valued function

unit-1

Examples:

i) $f(z) = z^2$

w.k.t $z = x+iy$

$f(z) = (x+iy)^2$

$= x^2 + (iy)^2 + 2xiy$

$= x^2 - y^2 + 2xiy$

$f(z) = x^2 - y^2 + i(2xy)$

$u(x,y) + iv(x,y) = (x^2-y^2) + i(2xy)$

$\therefore u(x,y) = x^2 - y^2$

$v(x,y) = 2xy$

Exercise:

i) $w = z^3$

$f(z) = w$

$f(z) = z^3$

$= (x+iy)^3$

$u+iv = x^3 + (iy)^3 + 3(x^2)(iy) + 3(x)(iy)^2$

$[a+b]^3 = a^3 + 3a^2b + 3ab^2 + b$

$= x^3 + i^3y^3 + 3x^2iy + 3x(i^2)y^2$

$= x^3 + i^3y^3 + 3x^2iy - 3xy^2$

$= x^3 - iy^3 + 3x^2iy - 3xy^2$

$= x^3 - 3xy^2 + i(3x^2y - y^3)$

$\therefore u = x^3 - 3xy^2 \quad u(x,y) = x^3 - 3\overset{x}{y}^2$

$v = 3x^2y - y^3 \quad v(x,y) = 3x^2y - y^3$

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ii) $w = 2\bar{z}^2 + 1$

$f(z) = w$

$f(z) = 2(x - iy)^2 + 1$

$u + iv = 2(x^2 + (iy)^2 - 2(x)(iy)] + 1$

$= 2(x^2 - y^2 - 2xiy) + 1$

$= 2x^2 - 2y^2 - 4xiy + 1$

$= 2x^2 - 2y^2 + 1 - i(4xy)$

$\therefore u = 2x^2 - 2y^2 + 1$ $\quad u(x, y) = 2x^2 - 2y^2 + 1$

$v = 4xy$ $\quad v(x, y) = 4xy$

iii) $w = \frac{1}{z}$

$f(z) = w$

$f(z) = \frac{1}{x + iy}$

$u(x, y) + iv(x, y) = \frac{1}{x + iy} \times \frac{x - iy}{x - iy}$

$= \frac{x - iy}{x^2 - (iy)^2}$

$= \frac{x - iy}{x^2 + y^2}$

$= \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2}$

$\therefore u(x, y) = \frac{x}{x^2 + y^2}$ $\quad v(x, y) = -\frac{y}{x^2 + y^2}$

iv) $w = \frac{z}{1 + z}$

$f(z) = \frac{x + iy}{1 + x + iy}$

$u(x, y) + iv(x, y) = \frac{x + iy}{(x + 1) + iy}$

$= \frac{x + iy}{(x + 1) + iy} \times \frac{(x + 1) - iy}{(x + 1) - iy}$

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$$= \frac{x(x+1) - xiy + iy(x+1) - i^2y^2}{(x+1)^2 - (iy)^2}$$

$$= \frac{x^2 + x - xiy + xiy + iy + y^2}{(x+1)^2 + y^2}$$

$$= \frac{x^2 + x + y^2}{(x+1)^2 + y^2} + \frac{iy}{(x+1)^2 + y^2}$$

$$\therefore u = \frac{x^2 + x + y^2}{(x+1)^2 + y^2} \quad v(x,y) = \frac{y}{(x+1)^2 + y^2}$$

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v) $w = z + \frac{1}{z}$

$f(z) = w$

$$u(x,y) + iv(x,y) = \frac{z^2 + 1}{z}$$

$$= \frac{(x+iy)^2 + 1}{x+iy}$$

$$= \frac{x^2 - y^2 + 2xiy + 1}{x+iy} \times \frac{x-iy}{x-iy}$$

$$= \frac{(x-iy)(x^2 - y^2 + 2xiy + 1)}{x^2 - (iy)^2}$$

$$= \frac{x(x^2 - y^2 + 2xiy + 1) - iy(x^2 - y^2 + 2xiy + 1)}{x^2 + y^2}$$

$$= \frac{x^3 - xy^2 + 2x^2iy + x - ix^2y + iy^3 - 2i^2xy^2 - iy}{x^2 + y^2} \quad -2(-1)xy^2$$

$$= \frac{x^3 - xy^2 + x + 2xy^2 + i(2x^2y - x^2y + y^3 - y)}{x^2 + y^2}$$

$$= \frac{x^3 - xy^2 + x + i(x^2y + y^3 - y)}{x^2 + y^2}$$

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$$= \frac{x(x^2-y^2+1)}{x^2+y^2} + i\frac{y(x^2+y^2-1)}{x^2+y^2}$$

$$\therefore u(x,y) = \frac{x(x^2-y^2+1)}{x^2+y^2} \quad v(x,y) = \frac{y(x^2+y^2-1)}{x^2+y^2}$$

vi) $w = z\bar{z}$

$w = f(z)$

$f(z) = (x+iy)(x-iy)$

$u(x,y) + iv(x,y) = x^2 - iyx + iyx - (i^2y^2)$

$= x^2 - iyx + iyx + y^2$

$= x^2+y^2$

$\therefore u(x,y) = x^2+y^2$

Limits

Definition:

$w = f(z)$

limit l as z tends to z_0

$\varepsilon > 0$ & $\delta > 0$ $\exists$ $0 < |z-z_0| < \delta$

$\Rightarrow |f(z) - l| < \varepsilon$

$$\lim_{z \to z_0} f(z) = l$$

Eg:1

$$f(z) = \begin{cases} z^2 & \text{if } z \neq i \\ 0 & z = i \end{cases}$$

Ans:

z approaches i

$f(z)$ approaches $i^2 = -1$

$$\lim_{z \to i} f(z) = -1$$

To prove $\varepsilon > 0$ & $\delta > 0$

$0 < |z-i| < \delta \Rightarrow |z^2+1| < \varepsilon$

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$|z^2+1| = |(z+i)(z-i)|$

$= |z+i|\,|z-i| \quad \cdots\to(1)$

From that $\delta>0$

choose, Let $\delta \le 1$

$0<|z-i|<1 \Rightarrow |z+i| = |z-i+2i|$

$\le |z-i|+|2i|$

$< 1+2 = 3$

$\therefore |z+i|<3$

From (1)

$0<|z-i|<1$

$\Rightarrow |z^2+1| < 3|z-i|$

$\delta = \min\{1, \varepsilon/3\}$

$0<|z-i|<\delta \Rightarrow |z^2+1|<\varepsilon$

$\therefore \lim_{z\to i} f(z) = -1$

Eg: 2

$\lim_{z\to 2} \frac{z^2-4}{z-2} = 4$

$|z-2|<\delta \Rightarrow |f(z)-4| < \varepsilon$

$f(z) = \frac{z^2-4}{z-2}$

$f(z) = \frac{(z-2)(z+2)}{z-2}$

$= z+2$

$\left|\frac{z^2-4}{z-2} - 4\right|$ i.e. $|f(z)-4|$

$|f(z)-4| = |z+2-4|$

$= |z-2|$ when $z\neq 2$

$= |z-2|<\delta<\varepsilon$

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Eg:3

$f(z) = \frac{\bar{z}}{z}$

$z \to 0$

$= \frac{x - iy}{x + iy}$

put $y = mx$

$= \frac{x - imx}{x + imx}$

$= \frac{x(1 - im)}{x(1 + im)}$

$= \frac{1 - im}{1 + im}$

f(z) does not have a limit as $z \to 0$

Eg:4

$f(z) = \frac{x^2 y^2}{(x + y^2)^3}$

put $y^2 = mx$ (along the parabola)

$f(z) = \frac{x^2 (mx)^2}{(x + mx)^3}$

$= \frac{x^3 m}{x^3 (1 + m)^3}$

$= \frac{m}{(1 + m)^3}$

which depends on m

f(z) does not have a limit as $z \to 0$

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Continuous function:

$z_0 \in D$

f is continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$

Given $\varepsilon > 0 \ \exists \ \delta > 0$ such that
$|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$

$\therefore$ f is continuous at D

Differentiability:
(Def) (note)

Eg:1

$f(z) = z^2$ is differentiable at every point & $f'(z) = 2z$.

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

$$= \lim_{h \to 0} \frac{(z+h)^2 - z^2}{h}$$

$$= \lim_{h \to 0} \frac{z^2 + h^2 + 2zh - z^2}{h}$$

$$= \lim_{h \to 0} \frac{h(h+2z)}{h}$$

$$f'(z) = 2z$$

Eg:2

$f(z) = \bar{z}$ on continuous but not differentiable

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

$$= \lim_{h \to 0} \frac{\overline{(z+h)} - \bar{z}}{h}$$

$$= \lim_{h \to 0} \frac{\bar{z} + \bar{h} - \bar{z}}{h}$$

$$= \lim_{h \to 0} \frac{\bar{h}}{h}$$

$\lim_{h \to 0} \frac{\bar{h}}{h}$ does not exist

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Theorem 2.5

cauchy-Riemann equation (CR eqn)

statement:

Let $f(z) = u(x,y) + iv(x,y)$ be differentiable at a point $z_0 = x_0 + iy_0$. Then $u(x,y)$ and $v(x,y)$ have first order partial derivatives $u_x(x_0,y_0)$, $u_y(x_0,y_0)$, $v_x(x_0,y_0)$ and $v_y(x_0,y_0)$ at (x_0,y_0) and these partial derivatives satisfy the cauchy-Riemann equations (C.R eqn) given by

$$u_x(x_0,y_0) = v_y(x_0,y_0) \ \& \ u_y(x_0,y_0) = -v_x(x_0,y_0)$$

Also $f'(z_0) = u_x(x_0,y_0) + iv_x(x_0,y_0)$

$= v_y(x_0,y_0) - iu_y(x_0,y_0)$

Proof:

$f(z) = u(x,y)$

$f(z) = u(x,y) + iv(x,y)$

$f(z)$ is differentiable at z_0

Neccessary part.

$$\lim_{h \to 0} \frac{f(z_0+h) - f(z_0)}{h} = f'(z_0)$$

$$f'(z_0) = \lim_{h_1+ih_2 \to 0} \frac{u(x_0+h_1, y_0+h_2) + iv(x_0+h_1, y_0+h_2) - u(x_0,y_0) - iv(x_0,y_0)}{h_1+ih_2}$$

$$\lim_{h_1+ih_2 \to 0} \frac{u(x_0+h_1, y_0+h_2) - u(x_0,y_0) + iv(x_0+h_1, y_0+h_2) - v(x_0,y_0)}{h_1+ih_2}$$

Along the real $h \to 0$

$h = h_1$

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$$f'(z_0) = \lim_{h_1 \to 0} \frac{u(x_0+h_1, y_0) - u(x_0,y_0) + i[v(x_0+h_1, y_0) - v(x_0,y_0)]}{h_1}$$

$$= \lim_{h_1 \to 0} \frac{u(x_0+h_1, y_0) - u(x_0,y_0)}{h_1} + i \lim_{h_1 \to 0} \left[\frac{v(x_0+h_1, y_0) - v(x_0, y_0)}{h_1}\right]$$

$f'(z_0) = u_x + i v_x \rightarrow (1)$ $\quad f'(z_0) = u_x(x_0,y_0) + i v_x(x_0,y_0)$

$h \to 0$ along imaginary $h = ih_2$

$$f'(z_0) = \lim_{ih_2 \to 0} \frac{u(x_0+h_1, y_0) - u(x_0,y_0) + i[v(x_0+h_1, y_0) - v(x_0,y_0)]}{ih_2}$$

$$= \lim_{ih_2 \to 0} \frac{u(x_0, y_0+h_2) - u(x_0,y_0)}{i} + i\left[\frac{v_y(x_0, y_0)}{i}\right]$$

$$= \frac{1}{i} u_y(x_0,y_0) + v_y(x_0,y_0)$$

$f'(z_0) = -i u_y(x_0,y_0) + v_y(x_0,y_0) \quad \ldots \rightarrow (2)$

From (1) & (2)

comparing

$$f'(z_0) = u_x(x_0,y_0) + i v_x(x_0,y_0)$$
$$= -i u_y(x_0,y_0) + v_y(x_0,y_0)$$

Equating real and imaginary parts

$$u_x(x_0,y_0) = v_y(x_0,y_0)$$
$$v_x(x_0,y_0) = -u_y(x_0,y_0)$$

Remark 1

$f' = u_x + i v_x$

$\Rightarrow v_y - i u_y$

$|f'(z)|^2 = u_x^2 + v_x^2$

$= u_y^2 + v_y^2$

$\Rightarrow u_x^2 + v_x^2 = u_y^2 + v_y^2$

$= u_x^2 + u_y^2 = v_x^2 + v_y^2$

$|f'(z)|^2 = u_x v_y - u_y v_x$

$|z| = \sqrt{x^2+y^2}$

$[f'(z)]^2 = \begin{vmatrix} u_x & v_y \\ v_x & v_y \end{vmatrix}$

$\frac{\partial(u,v)}{\partial(x,y)}$

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Remark 2

$\bar{z} = x - iy$

$u = x \quad v = -y$

$u_x = 1 \quad v_x = 0$

$u_y = 0 \quad v_y = -1$

It not satisfy the CR equation at any pt z

$f(z) = \bar{z}$ is nowhere differentiable

pg:35 Eg: 1

Let $f(z) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

Sol

$u(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$

$u(x,y) = \dfrac{xy}{x^2+y^2}, \quad v(x,y) = 0$

$u_x(0,0) = \lim_{h \to 0} \left[\dfrac{u(x_0+h, y_0) - u(x_0, y_0)}{h} \right]$

$= \lim_{h \to 0} \dfrac{u(0+h,0) - u(0,0)}{h}$

$= \lim_{h \to 0} \dfrac{u(h,0) - u(0,0)}{h}$

$= \lim_{h \to 0} \dfrac{0-0}{h} \Rightarrow 0$

|||ly

$v_y(0,0) = 0, \quad v_x(0,0) = 0$

$u_y(0,0) = \lim_{h \to 0} \left[\dfrac{u(x_0, y_0+h) - u(x_0, y_0)}{h} \right]$

$= \lim_{h \to 0} \dfrac{u(0, 0+h) - u(0,0)}{h}$

$= \lim_{h \to 0} \dfrac{u(0,h) - u(0,0)}{h}$

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$$= \lim_{h \to 0} \frac{0-0}{h} \Rightarrow 0$$

Along the path $y = mx$

$$f(z) = \frac{xmx}{x^2 + m^2x^2}$$

$$= \frac{x^2 m}{x^2(1+m^2)}$$

$$= \frac{m}{1+m^2} \quad \text{if } x \neq 0$$

$z \to 0$ along $y = mx$, $f(x) = \frac{m^2}{1+m^2}$ is different for different values of m

$f(z)$ does not have a limit $z \to 0$

$f(z)$ does not even ~~function~~ continuous at $z = 0$

$f(z)$ is not differentiable at $z = 0$

Eg: 2

Let $f(z) = \sqrt{|xy|}$

$u(x,y) = \sqrt{|xy|}$ & $v(x,y) = 0$

$$u_x(0,0) = \lim_{h \to 0} \left[\frac{u(x_0+h, y_0) - u(x_0, y_0)}{h} \right]$$

$$= \lim_{h \to 0} \left[\frac{u(0+h, 0) - u(0,0)}{h} \right]$$

$$= \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0-0}{h}$$

$$= 0$$

$$u_y(0,0) = \lim_{h \to 0} \left[\frac{u(x_0, y_0+h) - u(x_0, y_0)}{h} \right]$$

$$= \lim_{h \to 0} \frac{u(0, 0+h) - u(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{u(0,h) - u(0,0)}{h}$$

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$$= \lim_{h \to 0} \frac{0-0}{h}$$

$$= 0$$

$|||^{ly}$ $V_x(0,0) = 0$ & $V_y(0,0) = 0$

$\therefore$ CR equations are satisfied at $z=0$

Let $f(z)$ is not differentiable at $(0,0)$

Along the path $y = mx$

$$\frac{f(z)-f(0)}{z} = \frac{\sqrt{|x m x|}}{x + imx}$$

$$= \frac{x\sqrt{|m|}}{x(1+im)}$$

$$= \frac{\sqrt{m}}{1+im} \text{ if } x \neq 0$$

$z \to 0$ along path $y = mx$,

$$\frac{f(z)-f(0)}{z} \to \frac{\sqrt{|m|}}{1+im}$$ depends on path along $z \to 0$

f is not differentiable at $z=0$

Theorem 2.6

Let $f(z) = u(x,y) + iv(x,y)$ be a function defined in a region D such that u, v and their first order partial derivatives are continuous in D. If the first order partial derivatives of u, v satisfy the Cauchy-Riemann equations at a point $(x,y) \in D$ then f is differentiable at $z = x+iy$.

Proof:

$f(z) = u(x,y) + iv(x,y)$

$u(x,y)$'s first order partial derivatives are continuous at (x,y)

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By mean value theorem

$u(x+h_1, y+h_2) - u(x,y) = h_1 u_x + h_2 u_y(x,y) + h_1 \varepsilon_1 + h_2 \varepsilon_2 \rightarrow (1)$

$\varepsilon_1 \& \varepsilon_2 \to 0 \& h_1 \& h_2 \to 0$

Ill'y

$v(x+h_1, y+h_2) - v(x,y) = h_1 v_x(x,y) + h_2 v_y(x,y) + h_1 \varepsilon_3 + h_2 \varepsilon_4 \rightarrow (2)$

$\varepsilon_3, \varepsilon_4 \to 0 \& h_1, h_2 \to 0$

Let $h = h_1 + ih_2$

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

$$\frac{f(z+h) - f(z)}{h} = \frac{1}{h}[u(x+h_1, y+h_2) + iv(x+h_1, y+h_2) - u(x,y) - iv(x,y)]$$

$$\Rightarrow \lim_{h \to h_1 + ih_2} \frac{u(x+h_1, y+h_2) + iv(x+h_1, y+h_2) - u(x,y) - iv(x,y)}{h_1 + ih_2}$$

$$\Rightarrow \frac{u(x+h_1, y+h_2) - u(x,y) + i[v(x+h_1, y+h_2) - v(x,y)]}{h_1 + ih_2}$$

$$\Rightarrow h_1[u_x(x,y) + h_2 u_y(x,y) + h_1 \varepsilon_1 + h_2 \varepsilon_2) + i(h_1 v_x(x,y) + h_2 v_y(x,y) + h_1 \varepsilon_3 + h_2 \varepsilon_4)] / h_1 + ih_2$$

using (1) & (2)

$$= \frac{h_1 u_x + h_2 v_y + h_1 \varepsilon_1 + h_2 \varepsilon_2 + i[h_1 v_x + h_2 v_y + h_1 \varepsilon_3 + h_2 \varepsilon_4]}{h_1 + ih_2}$$

$$= \frac{h_1[u_x + iv_x] + h_2[u_y + iv_y] + h_1[\varepsilon_1 + i\varepsilon_3] + h_2[\varepsilon_2 + i\varepsilon_4]}{h_1 + ih_2}$$

$= u_x[h_1 + ih_2] - iu_y[ih_2 + h_1]$

$$\frac{h_1[u_x - iv_y) + h_2(u_x + iv_x] + h_1[\varepsilon_1 + i\varepsilon_3] + h_2[\varepsilon_2 + i\varepsilon_4]}{h_1 + ih_2}$$

$$= \frac{u_x[h_1 + ih_2] - iu_y(ih_2 + h_1) + h_1(\varepsilon_1 + i\varepsilon_3) + h_2(\varepsilon_2 + i\ldots}{h_1 + ih_2}$$

$$= u_x(x,y) - iu_y(x,y) + \frac{h_1}{h}(\varepsilon_1 + i\varepsilon_3) + \frac{h_2}{h}(\varepsilon_2 + i\varepsilon_4)$$

since $\left|\frac{h_1}{h}\right| \le 1$, $\frac{h_1}{h}(\varepsilon_1 + i\varepsilon_3) \to 0$ as $h \to 0$

$\frac{h_2}{h}(\varepsilon_2 + i\varepsilon_4) \to 0$ as $h \to 0$

$$\therefore \lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = u_x(x,y) - iu_y(x,y)$$

$\therefore$ f is differentiable

Eg:1

$f(z) = e^{x+iy}$

$$f(z) = e^{x+iy}$$
$$= e^x \cdot e^{iy}$$
$$= e^x \cdot (\cos y + i\sin y)$$
$$= e^x \cos y + ie^x \sin y$$

Here, $u(x,y) = e^x \cos y$ & $v(x,y) = e^x \sin y$

$u_x = e^x \cos y$ $\quad v_x = e^x \sin y$

$u_y = -e^x \sin y$ $\quad v_y = e^x \cos y$

$u_x = v_y$

$u_y = -v_x$

$\therefore$ C.R equation are satisfied

$\therefore$ The function f is differentiable.

Eg:2

(book) $f(z) = |z|^2$

$$f(z) = u(x,y) + iv(x,y)$$
$$= x^2 + y^2$$

$\therefore u(x,y) = x^2 + y^2$ & $v(x,y) = 0$

$u_x = 2x$ $\quad v_x = 0$

$u_y = 2y$ $\quad v_y = 0$

$\therefore$ It is differentiable only if (x,y) is $(0,0)$

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① C·R equation in complex form

Let $f(z) = u(x,y) + iv(x,y)$ be differentiable. Then the C.R equation can be put in complex form as

$f_x = -i f_y$.

Prove (that):

CR equation can be put in the form

$f(z) = u(x,y) + iv(x,y)$

$f_x = -i f_y$ [∵ C.R $f(x) = -i f_y$]

Proof:

$f_x = -i f_y$

$f_x + i f_y = 0$

$f_x = u_x + i v_x$

$f_y = u_y + i v_y$

$f_x = -i f_y$

$u_x + i v_x = -i(u_y + i v_y)$

$u_x + i v_x = \pm v_y - i u_y$

$u_x = v_y$ & $v_x = -u_y$

(or)

$f_x = -i f_y$

$f_x + i f_y = 0$

$f_x = 0,\ f_y = 0$

$u_x + i v_x = 0,\ v_y - i u_y = 0$

$u_x = v_y,\ v_x = -u_y$

C.R eqn

∴ Hence proved

∴ two C.R equation are equivalent to eqn $f_x = -i f_y$.

Theorem 2.8

② C·R equations in polar coordinates

(Book)

Polar form ⇒ $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ → (1)

formula

$\frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$ → (2)

$f(z) = u(r,\theta) + iv(r,\theta)$ at $z = re^{i\theta} \neq 0$

formula

$f' = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$

$x = r\cos\theta$ $y = r\sin\theta$

$\frac{\partial x}{\partial r} = \cos\theta$ $\frac{\partial y}{\partial r} = \sin\theta$

$x = r\cos\theta,\ y = r\sin\theta$

$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial r}$

$= \frac{\partial u}{\partial x}(\cos\theta) + \frac{\partial u}{\partial y}(\sin\theta)$ → (1)

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$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$= \frac{\partial v}{\partial x} \cdot (-r\sin\theta) + \frac{\partial v}{\partial y} (r\cos\theta)$$

[$\because u_x = v_y$

$u_y = -v_x$]

$$= r\left[-\frac{\partial u}{\partial x}(-\sin\theta) + \frac{\partial u}{\partial y}\cos\theta\right]$$

$$\times \frac{1}{r} \cdot \cos\theta = \frac{\partial u}{\partial y} \sin\theta + \frac{\partial u}{\partial x} \cos\theta$$

$$= -r\left[\frac{\partial u}{\partial x}(\sin\theta) - \frac{\partial u}{\partial y}\cos\theta\right]$$

By C.R eqn

$$= -r\left[\frac{\partial u}{\partial y}(\sin\theta) + \frac{\partial u}{\partial x}(\cos\theta)\right]$$

$$= -r \frac{\partial v}{\partial r} \quad \text{From (1)}$$

$$\frac{\partial u}{\partial \theta} = -r \frac{\partial v}{\partial r}$$

$$\boxed{\frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \theta}}$$

IIIly

$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial \theta}$$

$$= \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta)$$

$\because u_x = u_y$

$u_y = -v_x$

$$= r\left[-\frac{\partial v}{\partial x}(-\sin\theta) + \frac{\partial v}{\partial y}\cos\theta\right]$$

$$= r\left[-\frac{\partial v}{\partial x}(-\sin\theta) + \frac{\partial v}{\partial y}\cos\theta\right]$$

$$\frac{1}{r} \cdot \cos\theta = \frac{\partial u}{\partial y}\sin\theta + \frac{\partial u}{\partial x}\cos\theta$$

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r} \cdot \frac{\partial v}{\partial \theta}} \quad \text{--- (2)}$$

$$\frac{\partial u}{\partial r} = \frac{\partial v}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial v}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$= \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta$$

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$f' = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$

$r\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$

$\Rightarrow r\left[\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x} + i\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial r}\right)\right]$

$x = r\cos\theta, \quad y = r\sin\theta$

$\Rightarrow r\left[\frac{\partial u}{\partial x}\cdot\cos\theta + \frac{\partial u}{\partial y}\sin\theta + i\frac{\partial v}{\partial x}\cos\theta + i\frac{\partial v}{\partial y}\sin\theta\right]$

$= r\left[\cos\left(\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial x}\right) + \sin\theta\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right]$

$= r\left[\cos\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + \sin\theta\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\right]$

Here,

$u_x = v_y \qquad u_x + iv_x$

$u_y = -v_x \qquad u_y - iv_y$

$= xf' + iy\left[\frac{\partial u}{\partial y} - i\frac{\partial u}{\partial y}\right]$

$= xf' + iyf'$

$= (x+iy)f'$

$= zf'$

$f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial u}{\partial r}\right)$

Problems:

promblem 1. (cauchy Reimann equations for the functions)

1, $f(z) = z^3$

A. $f(z) = z^3$

$= (x+iy)^3$

$= x^3 + 3x^2 iy + 3x(iy)^2 + (iy)^3$

$= x^3 + 3ix^2y - 3xy^2 - iy^3$

$= (x^3 - 3xy^2) + i(3x^2y - y^3)$

$\therefore u(x,y) = x^3 - 3xy^2 \quad v(x,y) = 3x^2y - y^3$

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$u_x = 3x^2 - 3y^2$ $\quad v_x = 6xy$

$u_y = -6xy$ $\quad v_y = 3x^2 - 3y^2$

Here, $u_x = v_y$ & $u_y = -v_x$

Hence the cauchy reiman eqn are satisfied

Problem 2:

P.T following functions are nowhere differentiab

i) $f(z) = Re z$

Sol $\quad f(z) = Re z \quad , z = x + iy$

$u = x \quad , \quad v = 0 \quad\quad u(x,y) = x, \; v(x,y) = 0$

$u_x = 1 \quad , \quad v_x = 0$

$u_y = 0 \quad , \quad v_y = 0$

since $u_x \neq v_y$.

($\because$ C.R eqn are not satisfied at any pt)

f(z) is nowhere differentiable

ii) $f(z) = e^x(\cos y - i\sin y)$

Sol $\quad = e^x(\cos y) - ie^x \sin y$

$u = e^x \cos y \quad\quad v = -e^x \sin y \quad u(x,y) = e^x \cos y, \; v(x,y) = -e^x \sin$

$u_x = e^x \cos y \quad\quad v_x = -e^x \sin y$

$u_y = -e^x \sin y \quad\quad v_y = -e^x \cos y$

$\therefore$ The C.R eqn are not satisfied at any poi

$\therefore f(x)$ is nowhere differentiable

Problem 3:

P.T $f(z) = \begin{cases} \dfrac{z Re(z)}{|z|} & \text{if } z \neq 0 \\ 0 & \text{if } z = 0 \end{cases}$ is continuous at $z = 0$

but not differentiable at $z = 0$

Sol

To prove that $\lim_{z \to 0} f(z) = 0$

$\Rightarrow |f(z) - 0| = \left| \dfrac{z Re z}{|z|} \right|$

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$= \frac{|z||Re z|}{|z|}$

$= |Re z|$

w.k.T

$|Re z| \leq |z|$

Given $\varepsilon > 0$, choose $\delta = \varepsilon$

$|z| = |z-0| < \delta \Rightarrow |f(z) - 0| < \varepsilon$

Hence f is continuous at $z=0$

To prove that $f(z)$ is not differentiable at $z=0$

$$\frac{f(z)-f(0)}{z-0}$$

$$= \frac{z Re z}{z|z|} = \frac{Re z}{|z|}$$

$z = x+iy$

$$= \frac{x}{\sqrt{x^2+y^2}}$$

Along the path $y = mx$

$$\therefore \frac{f(z)-f(0)}{z-0} = \frac{x}{\sqrt{x^2+m^2x^2}}$$

$$= \frac{x}{x\sqrt{1+m^2}}$$

$$= \frac{1}{\sqrt{1+m^2}}$$

Since the value of the limit depends on m and hence on the path along $z \to 0$

$\therefore \lim_{z \to 0} \frac{f(z)-f(0)}{z-0}$ does not exist

$\therefore f(z)$ is not differentiable at $z=0$

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Problem 4

P.T $f(z)=z\ Im z$ is differentiable only at $z=0$

and find $f'(0)$

Sol.

$f(z)=z Im z$

$=(x+iy)\cdot y$

$=xy+iy^2$

Here $u=xy,\ v=y^2$ $\quad u(x,y)=xy,\ v(x,y)=y^2$

$u_x=y;\ v_x=0$

$u_y=x;\ v_y=2y$

clearly the C.R eqns are satisfied only at $z=0$

further all the first order partial derivatives are

continuous

$\therefore f(z)$ is differentiable at $z=0$

$f'(0)=u_x(0,0)+iv_x(0,0)$

$=0+0$

$=0$

Problem 5

S.T $f(z)=\begin{cases}\dfrac{xy^2(x+iy)}{x^2+y^4} & \text{if } z\neq 0\\ 0 & \text{if } z=0\end{cases}$ is not differentiable

at $z=0$

Sol.

$\dfrac{f(z)-f(0)}{z-0}$

$=\dfrac{xy^2(x+iy)}{x^4+y^4}\cdot\left(\dfrac{1}{x+iy}\right)$ $\quad(z=x+iy)$

$=\dfrac{xy^2}{x^4+y^4}$

Along the path $x=my^2$

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$$\frac{f(z)-f(0)}{z-0} = \frac{x(my^2(y^2))}{(my^2)^2+y^4}$$

$$= \frac{my^4}{m^2y^4+y^4} \Rightarrow \frac{my^4}{(m^2+1)y^4}$$

The value of limit depends on m and hence depends on the path along which $z \to 0$

$\therefore \lim_{z \to 0} \frac{f(z)-f(0)}{z-0}$ does not exist

$f(z)$ is not differentiable at $z=0$

Problem 6:

P.T the function $f(z) = \begin{cases} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & z \neq 0 \\ 0 & z=0 \end{cases}$ satisfies C.R equation at the origin but $f'(0)$ exist (not) (or) does not exist.

Sol.

$$f(z) = \begin{cases} \frac{x^3(1+i)-y^3(1-i)}{x^2+y^2} & \text{if } z \neq 0 \\ 0 & \text{if } z=0 \end{cases}$$

$$f(z) = \frac{x^3+ix^3-y^3+y^3i}{x^2+y^2}$$

$$= \frac{x^3-y^3}{x^2+y^2} + i\left(\frac{x^3+y^3}{x^2+y^2}\right)$$

$\therefore$ here $u(x,y) = \frac{x^3-y^3}{x^2+y^2}$, $v(x,y) = \frac{x^3+y^3}{x^2+y^2}$ if $(x,y) \neq (0,0)$

$u(0,0) = v(0,0) = 0$

$$\lim_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h}$$

$$u_x(0,0) = \lim_{h \to 0} \frac{u(h,0)-u(0,0)}{h}$$

$$= \lim_{h \to 0} \left[\frac{h^3/h^2-0}{h}\right]$$

$$= \lim_{h \to 0} \left[\frac{h^3-0}{h^3}\right] = 1$$

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|||ly $u_y(0,0) = \lim_{h \to 0} \left[\frac{u(x_0, y_0+h) - u(x_0, y_0)}{h} \right]$

$= \lim_{h \to 0} \left[\frac{u(0,h) - u(0,0)}{h} \right]$

$= \lim_{h \to 0} \left[\frac{-h^3/h^2 - 0}{h} \right]$

$u_y(0,0) = -1$

$v_x(0,0) = \lim_{h \to 0} \frac{v(h,0) - v(0,0)}{h}$

$= \lim_{h \to 0} \left[\frac{h^3/h^2 - 0}{h} \right]$

$v_x(0,0) = 1$

$v_y(0,0) = \lim_{h \to 0} \frac{v(0,h) - v(0,0)}{h}$

$= \lim_{h \to 0} \left[\frac{h^3/h^2 - 0}{h} \right]$

$v_y(0,0) = 1$

$\therefore u_x(0,0) = v_y(0,0) = 1$

$u_y(0,0) = -v_x(0,0) = -1$

$\therefore$ C.R eqn are satisfied at $z=0$

$\frac{f(z) - f(0)}{z - 0}$

$f(z) = u(x,y) + i v(x,y)$

$= \frac{x^3 - y^3}{x^2+y^2} + i \frac{x^3+y^3}{x^2+y^2}$

$\frac{f(z) - f(0)}{z - 0} = \frac{x^3 - y^3 + i(x^3 + iy^3)}{(x^2+y^2) \cdot (x+iy)}$

$= \frac{x^3 - y^3}{(x^2+y^2)(x+iy)} + \frac{i x^3 + y^3}{(x^2+y^2) \cdot (x+iy)}$

Along the path $y = mx$

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$$\frac{f(z)-f(0)}{z-0} = \frac{x^3-m^3x^3}{(x^2+m^2x^2)(x+imx)} + i\frac{x^3+m^3x^3}{(x^2+m^2x^2)(x+imx)}$$

$$= \frac{x^3(1-m^3)}{x^2(1+m^2)x(1+im)} + i\frac{x^3(1+m^3)}{x^2(1+m^2)x(1+im)}$$

$$= \frac{1-m^3}{(1+m^2)(1+im)} + i\frac{1+m^3}{(1+m^2)(1+im)}$$

Hence the value of the limit depends on the path along $z \to 0$

$\lim_{z \to 0} \frac{f(z)-f(0)}{z-0}$ does not exist

Hence f is not differentiable at 0.

Problem 7

Prove that $f(z) = \sin x \cosh y + i \cos x \sinh y$ is differentiable at every point

$f(z) = \sin x \cosh y + i \cos x \sinh y$

$u(x,y) = \sin x \cosh y$ $\quad v(x,y) = \cos x \sinh y$

$u_x = \cos x \cosh y$ $\quad v_x = -\sin x \sinh y$

$u_y = \sin x \sinh y$ $\quad v_y = \cos x \cosh y$

Hyperbolic function $\therefore \cosh y = \sinh y, \sinh y = \cosh y$ (differentiate)

$\therefore u_x = v_y$ & $u_y = -v_x \; \forall x, y$

C.R eqn is satisfied at every point

$\therefore$ All first order derivatives are continuous

f(z) is differentiable at every point

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(8) Problem 8

Find constants a and b so that the function $f(z) = a(x^2-y^2)+ibxy+c$ is differentiable at every point

Sol: $u(x,y) = a(x^2-y^2)+c$

$v(x,y) = bxy$

$f(z) = u(x,y)+iv(x,y)$

$u(x,y) = a(x^2-y^2)+c \qquad v(x,y) = bxy$

$u_x = 2ax \qquad v_x = by$

$u_y = -2ay \qquad v_y = bx$

$\therefore u_x = v_y \;\&\; u_y = -v_x$

$2ax = bx \quad -2ay = by$

if and only if $2a = b$

$\therefore$ CR equation is satisfied at all points iff

$2a = b$

The function f(z) is differentiable for all values of a, b with $2a = b$

Problem 9

S.T $f(z) = \sqrt{r}(\cos\theta/2 + i\sin\theta/2)$ where $r>0$ & $0<\theta<2\pi$ is differentiable and find $f'(z)$.

Sol. $f(z) = \sqrt{r}\cos\theta/2 + i\sin\theta/2\sqrt{r}$

$\therefore u = \sqrt{r}\cos(\theta/2)$, $v = \sin\theta/2(\sqrt{r})$

$\frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}}\cos\theta/2$, $\frac{\partial v}{\partial r} = \frac{1}{2\sqrt{r}}\sin\theta/2$ ...(1)

$\frac{\partial u}{\partial \theta} = -\frac{\sqrt{r}}{2}\sin\theta/2$, $\frac{\partial v}{\partial \theta} = \frac{\sqrt{r}}{2}\cos\theta/2$ ...(2)

$u = r^{1/2}\cos\theta/2$

$\frac{1}{2}r^{-1/2}\cos\theta/2$

$\frac{\partial u}{\partial r} = \frac{1}{2\sqrt{r}}\cos\theta/2$

$\frac{\partial u}{\partial \theta} = \frac{\sqrt{r}}{-2}\sin\theta/2$

$(\cos\theta/2 \cdot -\frac{1}{2}\sin$

condition for polar coordinates in polar form

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

$$\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

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i) $\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}\left(\frac{\sqrt{r}}{2}\cos(\theta/2)\right)$ From(2)

$= \frac{1}{\sqrt{r}\cdot\sqrt{r}}\left(\frac{\sqrt{r}}{2}\cos(\theta/2)\right)$

$= \frac{1}{2\sqrt{r}}\cos(\theta/2)$

$= \frac{\partial u}{\partial r}$ From(1)

$\therefore \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$

ii) $\frac{-1}{r}\frac{\partial u}{\partial \theta} = \frac{-1}{r}\left[-\frac{\sqrt{r}}{2}\sin(\theta/2)\right]$ From(2)

$= \frac{-1}{\sqrt{r}\cdot\sqrt{r}}\left[-\frac{\sqrt{r}}{2}\sin(\theta/2)\right]$

$= \frac{1}{2\sqrt{r}}\sin(\theta/2)$

$= \frac{\partial v}{\partial r}$ From(1)

$\therefore \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta}$

Hence proved.

The CR equations (in polar form) are satisfied. Further all first order partial derivatives are continuous.

Here f'(z) exists

$f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$ (By theo 2.8)

$= \frac{r}{z}\left[\frac{1}{2\sqrt{r}}\cos(\theta/2) + \frac{i}{2\sqrt{r}}\sin(\theta/2)\right]$ From(1)

$= \frac{\sqrt{r}\sqrt{r}}{2\sqrt{r}\,z}\left[\cos(\theta/2) + i\sin(\theta/2)\right]$

$= \frac{1}{2z}\left[\sqrt{r}(\cos(\theta/2) + i\sin(\theta/2))\right]$

$= \frac{1}{2z}[\sqrt{z}]$

$= \frac{1}{2\sqrt{z}}$

$z = r(\cos\theta + i\sin\theta)$

$\sqrt{z} = \sqrt{r(\cos\theta + i\sin\theta)}$

$\sqrt{z} = \sqrt{r}(\cos\theta + i\sin\theta)^{1/2}$

$\sqrt{z} = \sqrt{r}(\cos\theta/2 + i\sin\theta/2)$

$\therefore f'(z) = \frac{1}{2\sqrt{z}}$

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Analytic functions: (imp)(book)

Analytic is differentiable but the converse not true.

Theorem 2.10

An analytic function in a region D with its derivative zero at every point of the domain is a constant

$f(z) = u(x,y) + iv(x,y)$

$f'(z) = 0$

$f'(z) = u_x + iv_x = v_y - iu_y$

$0 = u_x + iv_x = v_y - iu_y$

$\therefore u_x = v_y = u_y = v_x$ (or) $u_x = u_y = v_x = v_y$

$= u(x,y)$ and $v(x,y)$ are constant function

Hence $f(z)$ is constant

Problems

Problem 1.

An analytic function in a region with constant modulus is constant.

Let $f(z) = u(x,y) + iv(x,y)$ be a analytic in domain D

$|f(z)|$ is constant,

$u^2 + v^2 = c^2$ where c is constant $\to (1)$

Diff (1) w.r.to x

$2uu_x + 2vv_x = 0$

i.e) $uu_x + vv_x = 0 \to (1)$

Similarly diff partially with respect to y

$2uu_y + 2vv_y = 0$

i.e) $uu_y + vv_y = 0$

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using CR eqns in (1) & (2)

$$u u_x - v u_y = 0 \to (3) \quad (u_x = v_y,\ u_y = -v_x)$$

$$u u_y + v u_x = 0 \to (4)$$

Eliminating u_y from (3) & (4)

$(3) \times u \Rightarrow u^2 u_x - v u u_y = 0$

$(4) \times v \Rightarrow v^2 u_x + u v u_y = 0$

$$u^2 u_x + v^2 u_x = 0$$

$\Rightarrow u_x (u^2 + v^2) = 0 \Rightarrow \boxed{u_x = 0}$

Eliminating u_x from (3) & (4)

$(3) \times v \Rightarrow$ since $u^2 + v^2 = c$, $u_x = 0$

similarly, $v_x = 0$

$$f'(z) = u_x + i v_x = 0$$

Hence f is constant

Problem 2.

Any analytic function $f(z) = u + iv$ with $\arg f(z)$ constant is itself a constant function.

Sol: $\arg f(z) = \tan^{-1}(v/u) = c$, where c is a constant

$\frac{v}{u} = k$, where k is constant

$v = ku$

Hence, $v_x = k u_x$ & $v_y = k u_y$

Eliminating k from above equations

$$u_x v_y = v_x u_y \qquad u_x^2 + u_y^2 = 0$$

$\therefore u_x v_y - u_y v_x = 0$

$\therefore u_x^2 + u_y^2 = 0$ (using C.R)

$\therefore u_x = 0$ & $u_y = 0$

hence u is constant

to prove v is constant

$\therefore f = u + iv$ is constant

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Problem 3:

If $f(z)$ and $f(\bar{z})$ are analytic in a region D S.T $f(z)$ is constant in region.

Sol.

Let $f(z) = u(x,y) + iv(x,y)$

$\therefore f(\bar{z}) = u(x,y) - iv(x,y)$

$= u(x,y) + i[-v(x,y)]$

Since $f(z)$ is analytic in D, $u_x = v_y$ & $u_y = -v_x$ (By using CF)

$f(\bar{z})$ is analytic in D

$u_x = -v_y$ & $u_y = v_x$

By adding $u_x = 0$ & $u_y = 0$

$f(z) = u + iv$ $\qquad$ $f(\bar{z}) = u - iv$

$u_x = v_y, u_y = -v_x \rightarrow (1)$ $\qquad$ $u_x = -v_y, u_y = v_x \rightarrow (2)$

Add (1) & (2)

$2u_x = 0, 2u_y = 0$

$u_x = 0, u_y = 0$

Hence, $u_x = 0 = v_x$

$f'(z) = u_x + iv_x = 0$

$f(z)$ is constant in D.

Problem 4:

Prove that the functions $f(z)$ and $f(\bar{z})$ are simultaneously analytic.

$f(z) = u(x,y) + iv(x,y)$ is analytic in region D.

First order partial derivatives of u and v are continuous and satisfy CR eqn

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \rightarrow (1) \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \rightarrow (2)$$

$f(\bar{z}) = u(x,-y) - iv(x,-y)$

$= u_1(x,y) + iv_1(x,y)$ where $u_1(x,y) = u(x,-y)$ & $v_1(x,y) = -v(x,-y)$

$$\frac{\partial u_1}{\partial x} = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial v_1}{\partial y} \text{ using (1)}$$

$$\frac{\partial u_1}{\partial y} = -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = -\frac{\partial v_1}{\partial x}$$

u_1 & v_1 are continuous

$\therefore f(\bar{z})$ is analytic in D then $f(z)$ is also analytic in D.

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Harmonic function:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$ this eqn is called Laplace's equation.

u(x, y) function of two real variables x & y

Theorem 2.11

The real and imaginary parts of an analytic fn are harmonic function

Pf: $f(z) = u(x, y) + iv(x, y)$ be an analytic function

u and v have continuous partial derivatives of first order satisfy the CR eqn given by

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \& \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \& \quad \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$

$$= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y}$$

$$= 0$$

Then u is harmonic function

IIIly

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial y}\right)$$

$$= \frac{\partial}{\partial x}\left(-\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= -\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial x \partial y}$$

$$= 0$$

Then v is also harmonic function

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Remark

The cauchy Riemann equations can be used to obtain a harmonic conjugate of a given harmonic function.

Find the analytic function $f(z) = u+iv$

Given

Let $u(x,y) = x^2 - y^2$

$u_x = 2x$, $u_{xx} = 2$

$u_y = -2y$, $u_{yy} = -2$

$u_{xx} + u_{yy} = 2-2 = 0$

Hence u is harmonic

$u = 0$

Let v be the conjugate (harmonic) of u

Here, $u_x = 2x$

using C.R eqn $\boxed{v_y = 2x} \rightarrow (1)$

In (1) diff w.r.to y

$\boxed{v = 2xy + \phi(x)} \rightarrow (2)$

Diff w.r.to x

$v_x = 2y + \phi'(x)$

$-u_y = 2y + \phi'(x)$

$-2y = 2y + \phi'(x) \Rightarrow \phi'(x) = 0$

$\Rightarrow \boxed{\phi(x) = c} \rightarrow (3)$

Sub (3) in (2)

$v = 2xy + c$

Hence an analytic function is

$f(z) = u + iv$

$= x^2 - y^2 + i(2xy + c)$

$= (x+iy)^2 + ic$

$\boxed{f(z) = z^2 + ic}$

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Problem 1

PT $u = 2x - x^3 + 3xy^2$ is harmonic and find its harmonic conjugate. Also find the corresponding analytic function

Sol

$u = 2x - x^3 + 3xy^2$

$u_x = 2 - 3x^2 + 3y^2$ $\quad u_{xx} = -6x$

$u_y = 6xy$ $\quad u_{yy} = 6x$

$u_{xx} + u_{yy} = -6x + 6x = 0$

$\therefore u$ is harmonic

$u = 0$

Let v be the harmonic conjugate of u

Here,

$u_x = 2 - 3x^2 + 3y^2$

using CR eqn $v_y = 2 - 3x^2 + 3y^2 \rightarrow (1)$

In (1) diff w.r to y (integrate)

~~$v = 2 3x - x^3 + y^3$~~

$v = 2y - 3x^2y + y^3 + \phi(x) \rightarrow (2)$

Diff (2) w r to x

$v_x = -6xy + \phi'(x)$

$-u_y = -6xy + \phi'(x) \Rightarrow -6xy = -6xy + \phi'(x)$

$\Rightarrow \phi'(x) = 0$

$\phi(x) = c \rightarrow (3)$

Sub (3) in (2)

$v = 2y - 3x^2y + y^3 + c$

Hence an analytic function is

$f(z) = u + iv$

$= 2x - x^3 + 3xy^2 + i(2y - 3x^2y + y^3 + c)$

$= 2x - x^3 + 3xy^2 + i3x^2y + i^3y^3) + ic$

$= 2(x+iy) - (x^3 + i^2 3xy^2 + i3x^2y + i^3y^3) + ic$

$f(z) = 2z - z^3 + ic$

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Problem 2

10m S.T $u = \log\sqrt{x^2+y^2}$ is harmonic and determine its conjugate and hence find the corresponding analytic function $f(z)$

Sol.

$u = \log\sqrt{x^2+y^2}$

$u = \log(x^2+y^2)^{1/2} \Rightarrow \frac{1}{2}\log(x^2+y^2)$

$u_x = \frac{1}{2}\cdot\frac{1}{x^2+y^2}(2x)$ $\quad u_y = \frac{1}{2}\frac{1}{x^2+y^2}(2y)$

$u_x = \frac{x}{x^2+y^2}$ $\quad u_y = \frac{y}{x^2+y^2}$

$u_{xx} = \frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2}$ $\quad u_{yy} = \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2}$

$= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2}$ $\quad = \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}$

$u_{xx} = \frac{y^2-x^2}{(x^2+y^2)^2}$ $\quad u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$

Here $u_x = \frac{x}{x^2+y^2}$, $u_y = \frac{y}{x^2+y^2}$, $u_{xx} = \frac{y^2-x^2}{(x^2+y^2)^2}$,

$u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2}$

$\therefore u_{xx}+u_{yy} = \frac{x^2-y^2}{(x^2+y^2)^2} + \frac{y^2-x^2}{(x^2+y^2)^2}$

$u = 0$

$\therefore u$ is harmonic

Let v be a conjugate of u

Here, $u_x = \frac{x}{x^2+y^2}$

using C.R equation

$v_y = \frac{x}{x^2+y^2} \rightarrow (1)$

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Integrate (1) w r to y

$$v = \int \frac{x}{x^2+y^2}$$

$$v = \tan^{-1}(y/x) + \phi(x) \quad \to (2)$$

Diff (2) w r to x

$$v_x = \frac{1}{1+(y/x)^2}\left(\frac{-y}{x^2}\right) + \phi'(x)$$

$$= \frac{1}{\frac{x^2+y^2}{x^2}}\left(\frac{-y}{x^2}\right) + \phi'(x)$$

$$-u_y = \frac{x^2}{x^2+y^2} \times \frac{-y}{x^2} + \phi'(x)$$

$$-u_y = \frac{-y}{x^2+y^2} + \phi'(x) \quad (\because \text{By C.R eqn})$$

$$\frac{-y}{x^2+y^2} = \frac{-y}{x^2+y^2} + \phi'(x)$$

$$\phi'(x) = 0$$

$$\phi(x) = c \to (3)$$

sub (3) in (2)

$$v = \tan^{-1}(y/x) + c$$

Hence an analytic function is

$$f(z) = u + iv$$

$$f(z) = \log\sqrt{x^2+y^2} + i(\tan^{-1}(y/x) + c)$$

Integrate $\int \frac{x}{x^2+y^2}$ · $\tan^{-1}y$

Diff $\tan^{-1}(y/x)$ $\frac{1}{1+x^2}$

yx^{-1}

$-1yx^{-2}$

$-y/x^2$

Problem 3 (Pg: 55) [M.T method]

ST $u(x,y)$: $\sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$ is harmon

Find an analytic function $f(z)$ in terms of z with the

Given u from its real part

$$u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$$

$$u_x = \cos x \cosh y - 2\sin x \sinh y + 2x + 4y$$

$$u_x = \cos x \cosh y - 2\sin x \sinh y + 2x + 4y$$

$$u_{xx} = -\sin x \cosh y - 2\cos x \sinh y + 2 - \ldots$$

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$u_y = \sin x \sinh y + 2\cos x \cosh y - 2y + 4x$

$u_{yy} = \sin x \cosh y + 2\cos x \sinh y - 2$

$u_{xx} + u_{yy} = -\sin x \cosh y - 2\cos x \sinh y + 2 + \sin x \cosh y +$

$2\cos x \sinh y - 2$

$u_{xx} + u_{yy} = 0$

$\therefore u$ is harmonic

Let v be the harmonic conjugate of u.

W.K.T

$\phi_1(x,y) = u_x$

$= \cos x \cosh y - 2\sin x \sinh y + 2x + 4y$

$\phi_1(z,0) = \cos z \cosh 0 - 2\sin z \sinh 0 + 2z + 4(0)$

$= \cos z + 2z$

Then,

$\phi_2(x,y) = u_y$

$= \sin x \sinh y + 2\cos x \cosh y - 2y + 4x$

$\phi_2(z,0) = \sin z \sinh(0) + 2\cos z \cosh(0) - 2(0) + 4z$

$= 2\cos z + 4z$

[By Milne thompson method]

$f(z) = \int [\phi(z,0) - i\phi_2(z,0)]\,dz + c$

$= \int [\cos z + 2z - i(2\cos z + 4z)]\,dz$

$f(z) = \sin z + z^2 - 2i\sin z - 2iz^2 + c$

Problem 4

If $f(z) = u(x,y) + iv(x,y)$ is an analytic function

$u(x,y) = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$ find $f(z)$

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$u(x,y) = \dfrac{\sin 2x}{\cosh 2y + \cos 2x}$ u v

$u_x = \dfrac{(\cosh 2y + \cos 2x)\cos 2x \cdot 2 - \sin 2x(-\sin 2x)\cdot 2}{(\cosh 2y + \cos 2x)^2}$

$u_x = \dfrac{(\cosh 2y + \cos 2x)\cdot 2\cos 2x + 2\sin^2 2x}{(\cosh 2y + \cos 2x)^2}$

$u_x = \dfrac{2\cosh 2y \cos 2x + 2\cos^2 2x + 2\sin^2 2x}{(\cosh 2y + \cos 2x)^2}$

$= \dfrac{2\cosh 2y \cos 2x + 2(\cos^2 2x + \sin^2 2x)}{(\cosh 2y + \cos 2x)^2}$

$u_x = \dfrac{2\cosh 2y \cos 2x + 2}{(\cosh 2y + \cos 2x)^2}$

$u_y = \dfrac{(\cosh 2y + \cos 2x)(0) - (\sin 2x)(\sinh 2y)(2)}{(\cosh 2y + \cos 2x)^2}$

$= \dfrac{-2\sin 2x \sinh 2y}{(\cosh 2y + \cos 2x)^2}$ u

$u_{yy} = (\cosh 2y + \cos 2x)^2 ($

$\phi_1(x,y) = u_x$ & $\phi_2(x,y) = u_y$

$\therefore \phi_1(z,0) = \dfrac{2\cos 2z \cosh 0 + 2}{(\cosh 0 + \cos 2z)^2}$

$= \dfrac{2\cos 2z + 2}{(1 + \cos 2z)^2}$

$= \dfrac{2(\cos 2z + 1)}{(1 + \cos 2z)(1 + \cos 2z)}$

$= \dfrac{2}{1 + \cos 2z}$

$= \dfrac{2}{2\cos^2 z} = \dfrac{1}{\cos^2 z} = \sec^2 z$

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$\phi_2(z,0)=0$

$\phi_2(x,y)= u_y$

$$= \frac{-2\sin 2x \sinh 2y}{(\cosh 2y+\cos 2x)^2}$$

$$\phi_2(z,0) = \frac{-2\sin 2z \sinh 2(0)}{(\cosh 2(0)+\cos 2z)^2}$$

$$= \frac{-2\sin 2z \sinh 0}{(\cosh 0+\cos 2z)^2}$$

$$= \frac{-2\sin 2z(0)}{(1+\cos 2z)^2}$$

$$= \frac{-0}{(1+\cos 2z)^2}$$

$\phi(z,0) = 0$

Milne-Thompson method

$$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]dz + c$$

$$= \int(\sec^2 z - i(0))dz + c$$

$$= \int \sec^2 z\, dz + c$$

$f(z) = \tan z + c$

5) Find the analytic function $f(z)=u+iv$ if

$$u+v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

Sol

Given

$$u+v = \frac{\sin 2x}{\cosh 2y - \cos 2x}$$

$$u_x + v_x = \frac{(\cosh 2y - \cos 2x)2\cos 2x - \sin 2x(-2\sin 2x)}{(\cosh 2y - \cos 2x)^2}$$

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$$u_x + v_x = \frac{2(\cosh 2y - \cos 2x)\cos 2x - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2} \rightarrow (1)$$

$$u_y + v_y = \frac{(\cosh 2y - \cos 2x)(0) - \sin 2x(2\sinh 2y)}{(\cosh 2y - \cos 2x)^2}$$

$$u_y + v_y = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2} \rightarrow (2)$$

since the required function $f(z) = u + iv$ is to be analytic u and v satisfy the CR eqn

$$u_x = v_y$$

$$u_y = -v_x$$

$$u_x - u_y = \frac{2(\cosh 2y - \cos 2x)\cos 2x - 2\sin^2 2x}{(\cosh 2y - \cos 2x)^2}$$

$$u_x(z,0) - u_y(z,0) = \frac{2(\cosh 2(0) - \cos 2z)\cos 2z - 2\sin^2 2z}{(\cosh 2(0) - \cos 2z)^2}$$

$$= \frac{2(\cosh 0 - \cos 2z)\cos 2z - 2\sin^2 2z}{(\cosh 0 - \cos 2z)^2}$$

$$= \frac{2(1 - \cos 2z)\cos 2z - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2\cos^2 2z - 2\sin^2 2z}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2(\cos^2 2z + \sin^2 2z)}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2(1)}{(1 - \cos 2z)^2}$$

$$= \frac{2\cos 2z - 2}{(1 - \cos 2z)^2}$$

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$$= \frac{-2(1-\cos 2z)}{(1-\cos 2z)^2}$$

$$= \frac{-2}{(1-\cos 2z)}$$

$$= \frac{-2}{2\sin^2 z}$$

$$= \frac{-1}{\sin^2 z}$$

$u_x(z,0) - u_y(z,0) = -\operatorname{cosec}^2 z \rightarrow (4)$

using CR eqn in (3)

$$u_y + u_x = \frac{-2\sin 2x \sinh 2y}{(\cosh 2y - \cos 2x)^2}$$

$$u_y(z,0) + u_x(z,0) = \frac{-2\sin 2z \sinh 2(0)}{(\cosh 2(0) - \cos 2z)^2}$$

$$= \frac{-2\sin 2z \sinh 0}{(\cosh 0 - \cos 2z)^2}$$

$$= \frac{0}{(1-\cos 2z)^2}$$

$u_y z(0,0) + u_x(z,0) = 0 \rightarrow (5)$

Now adding (4) & (5)

$u_x(z,0) - u_y(z,0) + u_y(z,0) + u_x(z,0) = -\operatorname{cosec}^2 z + 0$

$2u_x(z,0) = -\operatorname{cosec}^2 z$

$u_x(z,0) = \frac{-1}{2}\operatorname{cosec}^2 z \rightarrow (6)$

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Subtracting (4) & (5)

$u_x(z,0) - u_y(z,0) - u_y(z,0) - u_x(z,0) = -\mathrm{cosec}^2 z - 0$

$-2u_y(z,0) = -\mathrm{cosec}^2 z$

$u_y(z,0) = \frac{1}{2}\mathrm{cosec}^2 z \rightarrow (7)$

Now

$f(z) = u(z,0) + iv(z,0)$

$f'(z) = u_x(z,0) - iv_y(z,0)$

$= -\frac{1}{2}\mathrm{cosec}^2 z - i\frac{1}{2}\mathrm{cosec}^2 z$

$f'(z) = -\frac{1}{2}(1+i)\mathrm{cosec}^2 z$

Integrating with respect to z

$f(z) = \int -\frac{1}{2}(1+i)\mathrm{cosec}^2 z \, dz$

$f(z) = \left(\frac{1+i}{2}\right)\cot z + c$

Problem 6

Given $v(x,y) = x^4 - 6x^2y^2 + y^4$ find $f(z) = u(x,y) + iv(x,y)$ such that $f(z)$ is analytic

1) $v(x,y) = x^4 - 6x^2y^2 + y^4$

$v_x = 4x^3 - 12xy^2$

$v_y = -12x^2y + 4y^3$

Let $f(z) = u + iv$ be the required analytic function

By CR equation $u_x = v_y$

$u_x = -12x^2y + 4y^3$

∴ Integrating with respect to x

$u = \frac{-12x^3}{3}y + 4y^3x + \lambda(y)$

$= -4x^3y + 4y^3x + \lambda(y)$

where $\lambda(y)$ is an arbitary function of y

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$u_y = -4x^3 + 12xy^2 + \lambda'(y) = -v_x$

$u_y = -4x^3 + 12xy^2 + \lambda'(y)$

$-(4x^3 - 12xy^2) = -4x^3 + 12xy^2 + \lambda'(y)$

$\therefore \lambda'(y) = 0$

$\lambda'(y) = c$

where c is constant

$u = -4x^3y + 4xy^3 + c$

$\therefore f(z) = u + iv$

$= (-4x^3y + 4xy^3 + c) + i(x^4 - 6x^2y^2 + y^4)$

$= i[(x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3)] + c$

$[\because -1[4x^3y - 4xy^3 + c]]$

$i^2[4x^3y - 4xy^3 + c)$

$= i[x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3)] + c$

$(x+iy)^4 = 4c_0x^4 + 4c_1x^3(iy) + 4c_2x^2(iy)^2 + 4c_3x(iy)^3 + 4c_4(iy)^4$

$= x^4 + i4x^3y - 6x^2y^2 - i4xy^3 + y^4$

$= (x^4 - 6x^2y^2 + y^4) + i(4x^3y - 4xy^3).$

$\therefore f(z) = i(x+iy)^4 + c$

$f(z) = iz^4 + c \qquad (\because z = x + iy)$

By Milne thompson method

(i) $\phi_1(x,y) = v_y$

$\phi_2(x,y) = v_x$

$\therefore \phi_1(x,y) = -12x^2y + 4y^3$

$\phi_2(x,y) = 4x^3 - 12xy^2$

$\phi_1(z,0) = -12z^2(0) + 4(0)$

$\phi_1(z,0) = 0$

$\phi_2(z,0) = 4z^3 - 12z(0)$

$\phi_2(z,0) = 4z^3$

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$\therefore f(z) = \int [\phi_1(z,0) + i\phi_2(z,0)]\, dz$

$= \int 0 + i4z^3\, dz$

$= i\int 4z^3\, dz$

$f(z) = iz^4 + c$

Problem 7

Find the analytic function $f(z) = u + iv$ given that $u - v = e^x(\cos y - \sin y)$.

Sol

$u - v = e^x(\cos y - \sin y) \rightarrow (1)$

$u_x - v_x = e^x(\cos y - \sin y) \rightarrow (2)$

$u_y - v_y = e^x[-\sin y - \cos y]$

$u_y - v_y = -e^x[\sin y + \cos y] \rightarrow (3)$

since the required function is to be analytic it has satisfy the C.R eqns

using C.R eqn in (3)

$u_y = -v_x, \quad u_x = v_y$

$-v_x - u_x = -e^x(\sin y + \cos y) \rightarrow (4)$

solving (2) & (4)

(2) $\Rightarrow$ $u_x - v_x = e^x(\cos y) - e^x \sin y$

(4) $\times -1 \Rightarrow$ $u_x + v_x = e^x \sin y + e^x \cos y$

$2u_x = 2e^x \cos y$

$u_x = e^x \cos y \rightarrow (5)$

$u_x + v_x = -e^x \cos y + e^x \sin y$

$\Rightarrow -u_x - v_x$

sub $u_x = e^x \cos y$ in (2)

$e^x \cos y - v_x = e^x \cos y - e^x \sin y$

$v_x = e^x \sin y \rightarrow (6)$

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Integrating (6) with respect to x

$$V = e^x \sin y + f(y)$$

$$\therefore V_y = e^x \cos y + f'(y) \rightarrow (7)$$

using CR eqn in (5) & (7)

$$u_x = -V_y$$

$$e^x \cos y = -e^x \cos y + f'(y)$$

$$f'(y) = 0$$

Hence $f(y) = c_1$

where c_1 is constant

$$\therefore V = e^x \sin y + c_1$$

From (1)

$$u = e^x \cos y + c_2$$

then, $f(z) = u + iv \Rightarrow e^x \cos y + c_2 + e^x \sin y + c_1$

$$= e^x(\cos y + i e^x \sin y) + c_1 + i c_2$$

$$= e^x(\cos y + i \sin y) + (c_1 + i c_2)$$

$$= e^x e^{iy} + \alpha \quad \text{(where } \alpha \text{ is complex constant)}$$

$$= e^{x+iy} + \alpha$$

$$f(z) = e^z + \alpha \qquad (z = x + iy)$$

Problem 8

If $u + v = (x - y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ find the analytic function $f(z)$ in terms of z.

Sol.

$$u + v = (x - y)(x^2 + 4xy + y^2) \rightarrow (1)$$

Differentiating (1) partially w.r.t x

$$u_x + v_x = (x^2 + 4xy + y^2)(1) + (x - y)(2x + 4y) \rightarrow (2)$$

Differentiating (1) partially w.r.t y

$$u_y + v_y = (-1)(x^2 + 4xy + y^2) + (x - y)(4x + 2y) \rightarrow (3)$$

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$f = u + iv$ is analytic,

u and v satisfy the CR eqn

$u_x = v_y$ & $u_y = -v_x$

using CR eqn in (3)

$$-v_x + u_x = -(x^2+4xy+y^2) + (x-y)(4x+2y) \rightarrow (4)$$

Adding (2) & (4)

~~$2u_x - (x-y)$~~

$$u_x + v_x - v_x + u_x = x^2+4xy+y^2 - (x^2+4xy+y^2) + (x-y)(2x+4y) + (x-y)(4x+2y)$$

$$2u_x = (x-y)[6x+6y]$$

$$u_x = \frac{(x-y)6(x+y)}{2}$$

$$u_x = 3(x-y)(x+y)$$

$$u_x = 3(x^2-y^2) \rightarrow (5)$$

Subtracting (4) from (2)

$$-v_x + u_x - u_x - v_x = -(x^2+4xy+y^2) + (x-y)(4x+2y) - (x^2+4xy+y^2) - (x-y)(2x+4y)$$

$$-2v_x = -x^2-4xy-y^2+4x^2+8xy-4xy-2y^2-x^2-4xy-y^2-[2x^2+4xy-2xy-4y^2]$$

$$= -x^2-4xy-y^2+4x^2+2xy-4xy-2y^2-x^2-4xy+y^2-2x^2-4xy+2xy+4y^2$$

$$= -4x^2 + -16xy + 4xy + 4x^2 - 4y^2 + 4y^2$$

$$-2v_x = -12xy$$

$$v_x = 12xy/2 \rightarrow (6)$$

$$v_x = 6xy$$

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using C.R eqn in (6)

$u_y = -6xy$

$\phi(x,y) = u_x, \phi_2(x,y) = u_y$

$\therefore \phi_1(z,0) = 3(z^2 - 0)$

$\phi_1(z,0) = 3z^2$

$\therefore \phi_2(z,0) = -6z(0)$

$\phi_2(z,0) = 0$

By Milne-Thompson method

$f(z) = \int[\phi_1(z,0) - i\phi_2(z,0)]dz$

$= \int 3z^2 dz$

$f(z) = z^3 + c$

Problem 9

Find the real part of the analytic function whose imaginary part is $e^{-x}(2xy\cos y + (y^2 - x^2)\sin y)$ construct the analytic function.

Sol.

Let $v = e^{-x}[2xy\cos y + (y^2 - x^2)\sin y]$

$f(z) = u + iv$ be required analytic function

To prove that v is harmonic.

Let $\phi_1(x,y) = v_y, \phi_2(x,y) = v_x$

$v_y = e^{-x}[(2x\cos y - 2xy\sin y + (y^2 - x^2)\cos y + 2y\sin y)]$

$v_y = e^{-x}(2x\cos y - 2xy\sin y + 2y\sin y + (y^2 - x^2)\cos y)$

$v_x = e^{-x}[-2y\cos y + 2xy\cos y + (y^2 - x^2)\sin y + 2x\sin y)](-1)$

$v_x = e^{-x}[-2y\cos y - (y^2 - x^2)\sin y + 2x\sin y - 2xy\cos y]$

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$\therefore \phi_1(z,0) = e^{-z}(2z\cos(0) - 2z(0)\sin(0) + 2(0)\sin(0) + (0-z^2)\cos(0)]$

$= e^{-z}(2z(1) - 0 + 0 + -z^2(1))$

$\phi_1(z,0) = e^{-z}(2z - z^2)$

$\phi_2(z,0) = e^{-z}[-2(0)\cos(0) - (0-z^2)\sin(0) + 2(z)\sin(0) - (2(z)(0)\cos(0)]$

$= e^{-z}[(0) + z^2(0) - (0) - (0)]$

$\phi_2(z,0) = 0$

By Milne thompson method

$f(z) = \int [\phi_1(z,0) - i\phi_2(z,0)]\,dz$

$= \int e^{-z}(2z - z^2)\,dz$

$= \int e^{-z} 2z\,dz - \int e^{-z} z^2\,dz$

$u = z^2 \quad dv = e^{-z}dz$

$\frac{du}{dz} = 2z \quad v = \frac{e^{-z}}{-1}$

$= \int 2z e^{-z} dz - [-z^2 e^{-z} - \int -e^{-z} 2z\,dz]$

$= \int 2z e^{-z} dx + z^2 e^{-z} - \int 2z e^{-z} dx$

$f(z) = z^2 e^{-z}$

$e^{iy} = \cos y + i\sin y$

$= (x+iy)^2 e^{-z}$

$= (x+iy)^2 e^{-(x+iy)} \quad (z = x+iy)$

$= [(x^2) + (iy)^2 + 2xiy]\, e^{-x} e^{-iy}$

$= (x^2 - y^2 + 2xiy)\, e^{-x}(\cos y - i\sin y)$

$f(z) = e^{-x}[(x^2-y^2) + 2xiy](\cos y - i\sin y)$

Real part of f(z):

$f(z) = e^{-x}[(x^2-y^2)\cos y + 2xy\sin y)]$

$\therefore u = e^{-x}[(x^2-y^2)\cos y + 2xy\sin y)$

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Bilinear Transformations:

1) Translation: $w = z+b$

∞ is only fixed point of transformation $b \neq 0$

2) Rotation: $w = az$ where $|a| = 1$

0 & ∞ is fixed point of transformation

3) Magnification or contraction: $w = bz$ ($b>0$, real) ($b>1$) ($b<1$)

0 ∞ is fixed point of this transformation

4) Inversion $w = 1/z$

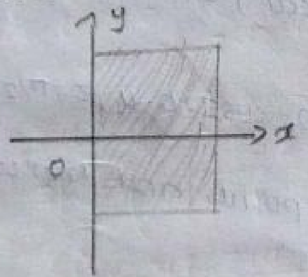
Prove that

The transformation $w = 1/z$ is the inversion w.r.t the unit circle followed by reflection about the real axis (Refer book)

Solved problems:

Problem 1:

Under the transformation $w = iz+i$ show that the half plane $x>0$ maps onto the half plane $v>1$.



Sol Let $z = x+iy$ &

$w = u+iv$

$w = iz+i \Rightarrow w = i(x+iy)+i$

$= xi + (i)^2 y + i$

$= xi - y + i$

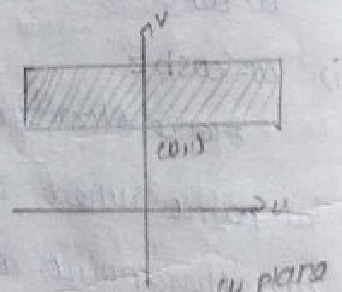
$= -y + i(x+1)$

$\therefore u+iv = -y+i(x+1)$

$\therefore u = -y$ & $v = x+1$

$x>0 \Leftrightarrow v>1$

$\therefore$ The half plane $x>0$ is mapped into half plane $v>1$.



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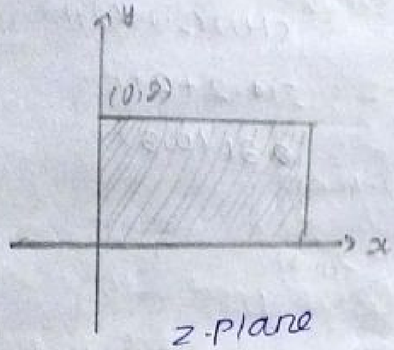
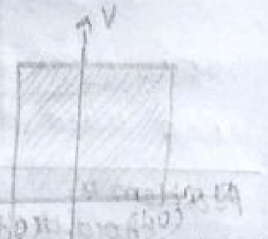
Problem 2

Show that the given region in z-plane given by $x>0$ and $0<y<2$ is mapped into the region in the w plane given by $-1<u<1$ and $v>0$ under the transformation $w = iz+1$





A. Let $z = x+iy$ & $w = u+iv$

$w = iz+1$

$\Rightarrow i(x+iy)+1$

$\Rightarrow xi - y + 1$

$\Rightarrow (-y+1) + ix$

$u = 1-y$, $v = x$

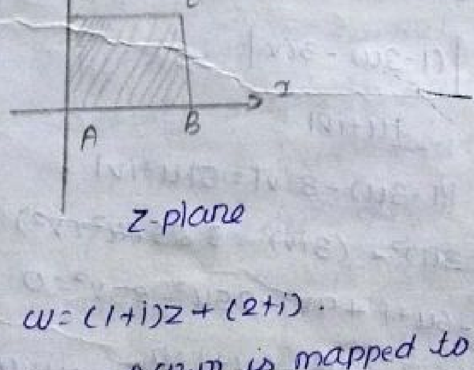
$\therefore x>0, 0<y<2 \Leftrightarrow v>0$ & $-1<u<1$

Hence the given region is mapped into region $v>0$ and $-1<u<1$.

Problem 3

Find the image of the square region with vertices $(0,0), (2,0), (2,2), (0,2)$ under the transformation.

$w = (1+i)z+(2+i)$



w-plane

A: $w = (1+i)z+(2+i)$.

$A(0,0)$ is mapped to $A' = (1+i)(0+0i)+(2+i)$

$= (1+i)(0)+(2+i)$

$= (2+i)\ (2,1)$

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B(2,0) is mapped into $B' = (1+i)(2+0i)+(2+i)$

$= 4+3i \; (4,3)$

C(2,2) is mapped into $C' = (1+i)(2+2i)+(2+i)$

$= 2+2i+2i-2+2+i$

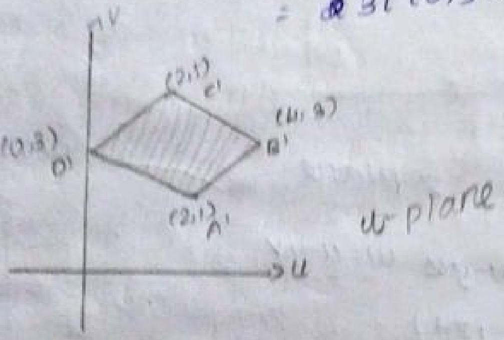
$= 2+5i \; (2,5)$

D(0,2) is mapped into $D' = (1+i)(0+2i)+(2+i)$

$= (1+i)(2i)+(2+i)$

$= 2i-2+2+i$

$= 3i \; (0,3)$



Problem 4

Show that by means of inversion $w = 1/z$ the circle given by $|z-3| = 5$ is mapped into circle $|w + 3/16| = 5/16$.

Sol. The circle $|z-3| = 5$ is mapped into

$|1/w - 3| = 5$ where $z = 1/w$

$$\left|\frac{1}{w} - 3\right| = 5 \Rightarrow \left|\frac{1}{u+iv} - 3\right| = 5$$

$$\left|\frac{1-3(u+iv)}{u+iv}\right| = 5$$

$$\left|\frac{1-3u-3iv}{u+iv}\right| = 5$$

$$\frac{|(1-3u)-3iv|}{|u+iv|} = 5$$

$$|(1-3u)-3iv| = 5|u+iv|$$

$$(1-3u)^2 - (3iv)^2 = 25(u^2+v^2)$$

$$9u^2-6u+1+9v^2-25u^2-25v^2 = 0$$

$$-16u^2-16v^2-6u+1 = 0 \quad \text{(Mult (-) on b/s)}$$

$$16u^2+16v^2+6u-1 = 0 \quad \text{(Divide by 16)}$$

$$u^2+v^2+\frac{6}{16}u-\frac{1}{16} = 0$$

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∴ This is a circle with centre

centre $(-u,-v)$

$2u = 6/16$ $2v = 0$ ∴ centre $(-3/16, 0)$

$u = 6/32$ $v = 0$

$u = 3/16$ $v = 0$

Radius: $\sqrt{u^2+v^2-c}$

$= \sqrt{9/256 + 0 + 1/16}$

$= \sqrt{\frac{9+16}{256}}$

$= \sqrt{+25/256}$

∴ Radius $= 5/16$

Hence the image circle in the w-plane is given by eqn

$|w + 3/16| = 5/16$

Problems:

Find the image of the circle $|z-3i| = 3$ under the map $w = 1/z$.

Sol. The image of the circle $|z-3i| = 3$ under the transformation $w = 1/z$ is by eqn $\left|\frac{1}{w} - 3i\right| = 3$ $(z = 1/w)$

$$\left|\frac{1}{w} - 3i\right| = 3$$

$$\left|\frac{1}{u+iv} - 3i\right| = 3$$

$$\left|\frac{1-3i(u+iv)}{u+iv}\right| = 3$$

$$|1-3i(u+iv)| = 3|u+iv|$$

$$|1-3iu+3v| = 3|u+iv|$$

$$|(1+3v)-3iu| = 3|u+iv|$$

$$(1+3v)^2 + 9u^2 = 9(u^2+v^2)$$

$$1+6v+9v^2+9u^2 = 9u^2+9v^2$$

$\Rightarrow 6v+1 = 0$ (st. line eqn)

Hence the image of circle $|z-3i| = 3$ under $w = 1/z$ in z plane is st. line $6v+1 = 0$ in w plane

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Problem 6

Find the image of the strip $2<x<3$ under $w=1/z$

Sol.

The transformation $w=1/z$ written in cartesian coordinates as

$$x=\frac{u}{u^2+v^2} \quad \& \quad y=\frac{-v}{u^2+v^2}$$

$$x>2 \Rightarrow \frac{u}{u^2+v^2}>2$$

$$2(u^2+v^2)-u<0$$

$$u^2+v^2-u/2<0$$

∴ The region $x>2$ is mapped into a region represented by $u^2+v^2-u/2<0$, which is interior of circle with centre $(1/4, 0)$ & radius $1/4$

centre $(-u,-v)$ radius $=\sqrt{u^2+v^2-d}$

$2u=1/2$ $2v=0$ $=\sqrt{1/16+0-0}$

$u=1/4$ $v=0$ radius $=1/4$

Now, $x<3 \Rightarrow \frac{u}{u^2+v^2}<3$

$$3(u^2+v^2)-u>0$$

$$u^2+v^2-u/3>0$$

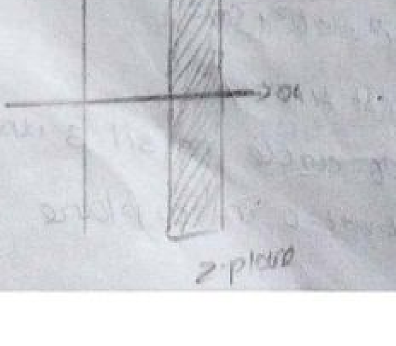
centre $(-u,-v)$ radius $=\sqrt{u^2+v^2-d}$

$2u=1/3$ $2v=0$ $=\sqrt{1/36+0-0}$

$u=1/6$ $v=0$ radius $=1/6$

The region $x<3$ is mapped onto the exterior of the circle with centre $(1/6, 0)$ & radius $1/6$.

∴ The strip $2<x<3$ is mapped onto region bounded by the circles $u^2+v^2=u/2$ and $u^2+v^2=u/3$ in w-plane



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Bilinear transformations:

$$w = T(z) = \frac{az+b}{cz+d}$$

a, b, c, d are complex constants & $ad-bc \neq 0$

$T(\infty) = a/c$ & $T(-d/c) = \infty$

T become 1-1 onto map of extended complex plane onto itself.

$$z = T^{-1}(w) = \frac{-dw+b}{cw-a}$$

Theorem 3.1

Any bilinear transformation can be express as a product of translation, rotation, magnification or contraction and inversion.

Proof: Let $w = T(z) = \frac{az+b}{cz+d}$ where $ad-bc \neq 0$ by bilinear transformation → (1)

case i) $c=0$

Hence $d \neq 0$ (since $ad-bc \neq 0$)

$(1) \Rightarrow w = \frac{az+b}{(0)z+d}$

$w = az+b/d$

$= (a/d)z + (b/d)$

Now, let $T_1(z) = (a/d)z$ & $T_2(z) = z + (b/d)$

T_1 and T_2 are elementary transformations

$(T_2 \circ T_1)(z) = T_2[(a/d)z]$

$= (a/d)z + (b/d)$

$= T(z)$

case ii) $c \neq 0$

$w = \frac{az+b}{cz+d}$

$= \frac{az + b - ad/c + ad/c}{c(z+d/c)}$

$= \frac{a(z+d/c)] + b - (ad/c]}{c(z+d/c)}$

$w = a/c + \frac{b - ad/c}{c(z+d/c)}$

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Let $T_1(z) = cz+d$

$T_2(z) = 1/z$

$T_3(z) = (b - ad/c) z$

$T_4(z) = z + (a/c)$

Then $T(z) = (T_4 \circ T_3 \circ T_2 \circ T_1)(z)$

$T(z) = (T_4 \circ T_3 \circ T_2)(cz+d)$

$= (T_4 \circ T_3) T_2(cz+d)$

$= (T_4 \circ T_3)(1/cz+d)$

$= T_4(T_3(1/cz+d))$

$= T_4 \left(b - \frac{ad}{c}\right)\left(\frac{1}{cz+d}\right)$

$$T(z) = \frac{b - \frac{ad}{c}}{cz+d} + a/c$$

Hence the theorem

Solved problems

Problem 1

S.T the transformation $w = \frac{5-4z}{4z-2}$ maps the unit circle $|z| = 1$ into a circle of radius unity and centre $-1/2$

Sol.

$$w = \frac{5-4z}{4z-2}$$

$4wz - 2w = 5 - 4z$

$(4wz + 4z) = 5 + 2w$

$(4w+4)z = 5 + 2w$

$$z = \frac{5+2w}{4w+4}$$

$|z| = 1 \Rightarrow z\bar{z} = 1$

$$\left[\frac{5+2w}{4w+4}\right]\left[\frac{5+2\bar{w}}{4\bar{w}+4}\right] = 1$$

$$\frac{25 + 4w\bar{w} + 10w + 10\bar{w}}{16w\bar{w} + 16w + 16\bar{w} + 16} = 1$$

$25 + 4w\bar{w} + 10w + 10\bar{w} = 16w\bar{w} + 16(w + \bar{w}) + 16$

$\Rightarrow 25 + 4w\bar{w} + 10w + 10\bar{w} - 16w\bar{w} - 16w - 16\bar{w} - 16 = 0$

$-12w\bar{w} - 6\bar{w} - 6w + 9 = 0$

$\Rightarrow 12w\bar{w} + 6\bar{w} + 6w - 9 = 0$

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$\Rightarrow w\bar{w} + \frac{1}{2}\bar{w} + \frac{1}{2}w - 3/4 = 0$

The eqn of circle is

$z\bar{z} + \bar{\alpha} z + \alpha \bar{z} + \beta = 0$

where β is real no

centre is $-\alpha$ & radius $\sqrt{\alpha\bar{\alpha} - \beta}$

centre $\Rightarrow -\alpha = -1/2$

radius: $\sqrt{\alpha\bar{\alpha} - \beta}$

$= \sqrt{1/4 + 3/4}$

$= \sqrt{1}$

$= 1$

This represents the equation of the circle with centre $-1/2$ & radius is 1.

Hence the result.

Problem 2.

Show that the transformation $w = \frac{2z+3}{z-4}$ maps the circle $z\bar{z} - 2(z+\bar{z}) = 0$ into a st. Line given by $2(w+\bar{w}) + 3 = 0$.

Sol.

$w = \frac{2z+3}{z-4}$

$w(z-4) = 2z+3$

$wz - 4w - 2z - 3 = 0$

$z(w-2) = 3 + 4w$

$\therefore z = \frac{3+4w}{w-2}$

The image of circle $z\bar{z} - 2(z+\bar{z}) = 0$ is

$$\left[\frac{3+4w}{w-2}\right]\left[\frac{3+4\bar{w}}{\bar{w}-2}\right] - 2\left[\frac{3+4w}{w-2} + \frac{3+4\bar{w}}{\bar{w}-2}\right] = 0$$

$$\frac{(3+4w)(3+4\bar{w}) - 2(3+4w)(\bar{w}-2) - 2(3+4\bar{w})(w-2)}{(w-2)(\bar{w}-2)} =$$

$\Rightarrow 9 + 12\bar{w} + 12w + 16w\bar{w} - 6\bar{w} + 12 - 8w\bar{w} + 16w - 6w + 12 - 8w\bar{w} + 16\bar{w} =$

$\Rightarrow 22\bar{w} + 22w + 33 = 0$

$(\div) 11 \quad 2\bar{w} + 2w + 3 = 0$ which is obviously a st Line

$2(\bar{w} + w) + 3 = 0$

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Problem 3

Show that $w = \frac{z-1}{z+1}$ maps the imaginary axis in the z-plane onto the circle $|w|=1$. What portion of the z-plane corresponds to the interior of the circle $|w|$

Sol:

$|w| = 1$

$\left|\frac{z-1}{z+1}\right| = 1$

$|z-1| = |z+1|$

$z = x+iy$

$|x+iy-1| = |x+iy+1|$

$(x-1)^2+y^2 = (x+1)^2+y^2$

$x^2-2x+1 = x^2+2x+1$

$\Rightarrow 4x = 0$

$x = 0$

Hence the transformation $w = \frac{z-1}{z+1}$ maps the imaginary axis $x=0$ onto the unit circle $|w|=1$.

Also since the point $z=1$ is mapped to $w=0$ it follows that the half plane $x>0$ is mapped onto the interior of the circle $|w|=1$

Cross ratio:

$$(z_1, z_2, z_3, z_4) = \begin{cases} \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)} & \text{if none of } z_1, z_2, z_4 \text{ is } \infty \\ \frac{z_1-z_3}{z_1-z_4} & \text{if } z_2 \text{ is } \infty \\ \frac{z_2-z_4}{z_1-z_4} & \text{if } z_3 \text{ is } \infty \\ \frac{z_1-z_3}{z_2-z_3} & \text{if } z_4 \text{ is } \infty \\ \frac{z_2-z_4}{z_2-z_3} & \text{if } z_1 \text{ is } \infty \end{cases}$$

Theorem 3.2

Any bilinear transformation preserves cross ratio:

Proof. Let $w = \frac{az+d}{cz+d}$, $ad-bc \neq 0$ be the bilinear transformation.

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Let z_1, z_2, z_3, z_4 be four distinct pts. Let their images under this transformation be w_1, w_2, w_3, w_4 respectively

We assume that all z_i & w_i are diff from ∞

We claim that $(z_1, z_2, z_3, z_4) = (w_1, w_2, w_3, w_4)$

$$w_i = \frac{az_i+b}{cz_i+d} \quad (i = 1,2,3,4)$$

$$w_1 - w_3 = \frac{az_1+b}{cz_1+d} - \frac{az_3+b}{cz_3+d}$$

$$= \frac{(az_1+b)(cz_3+d) - (az_3+b)(cz_1+d)}{(cz_1+d)(cz_3+d)}$$

$$= \frac{az_1cz_3 + az_1d + bcz_3 + bd - cz_1az_3 - cz_1b - daz_3 - db}{(cz_1+d)(cz_3+d)}$$

$$= \frac{az_1d - bcz_1 + bcz_3 - daz_3}{(cz_1+d)(cz_3+d)}$$

$$= \frac{(ad-bc)z_1 + (bc-da)z_3}{(cz_1+d)(cz_3+d)}$$

$$= k_1(z_1 - z_3)$$

IIIly

$$w_2 - w_4 = \frac{az_2+b}{cz_2+d} - \frac{az_4+b}{cz_4+d}$$

$$= \frac{(az_2+b)(cz_4+d) - (az_4+b)(cz_2+d)}{(cz_2+d)(cz_4+d)}$$

$$= \frac{az_2cz_4 + az_2d + bcz_4 + bd - az_4cz_2 - az_4d - bcz_2 - bd}{(cz_2+d)(cz_4+d)}$$

$$= \frac{az_2d - bcz_2 + bcz_4 - az_4d}{(cz_2+d)(cz_4+d)}$$

$$= \frac{(ad-bc)z_2 + (bc-ad)z_4}{(cz_2+d)(cz_4+d)}$$

$$= k_2(z_2 - z_4)$$

$$\therefore (w_1 - w_3)(w_2 - w_4) = k_1k_2(z_1 - z_3)(z_2 - z_4)$$

$$= k(z_1 - z_3)(z_2 - z_4)$$

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$$\therefore \frac{(w_1-w_3)(w_2-w_4)}{(w_1-w_4)(w_2-w_3)} = \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)}$$

It is similar if one of the z_i or w_i is ∞.

Solved problems

Problem 1:

Find the bilinear transformation which maps the points $z_1=2$, $z_2=i$, $z_3=-2$ onto $w_1=1$, $w_2=i$, $w_3=-1$ respectively.

Sol. Let the image of any point z under required bilinear transformation be w. The required bilinear transformation is given by eqn

$$(w,1,i,-1) = (z,2,i,-2)$$

$$\frac{(w-i)(1+1)}{(w+1)(1-i)} = \frac{(z-i)(2+2)}{(z+2)(2-i)}$$

$$\frac{2(w-i)}{(w+1)(1-i)} = \frac{4(z-i)}{(z+2)(2-i)}$$

$$\frac{(w-i)}{w-iw+1-i} = \frac{2(z-i)}{2z-iz+4-2i}$$

$$(w-i)(2z-iz+4-2i) = 2(z-i)(w-iw+1-i)$$

$$2zw-izw+4w-2iw-2iz+z-4i+2 = 2[zw-ziw+z-zi-wi+w-i+1]$$

$$2zw-izw+4w-2iw-2iz+z-4i+2-2zw+2ziw-2z+2zi+2wi-2w+2i-2$$

$$+iwz+6w-3z-2i=0 \text{ (verify)} \Rightarrow iwz+6w-3z-2i=0$$

$$\therefore w(iz+6) = 3z+2i$$

$$\therefore w = \frac{3z+2i}{iz+6}$$

This is the required bilinear transformation

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Find the bilinear transformation which maps the points $z=-1,1,\infty$ respectively on $w=-i,-1,i$.

Sol. Let the image of any point z under the required bilinear transformation. since bilinear transformation preserves cross ratio,

$$(z,-1,1,\infty)=(w,-i,-1,i)$$

$$\text{i.e.}\ \frac{z-1}{-1-1}=\frac{(w+i)(-i-i)}{(w-i)(-i+1)}$$

$$\therefore (z-1)(w-iw-i-1)=4iw+4i$$

$$wz-iwz-zi-z-w+iw+i+1=4iw+4i$$

$$wz-iwz-w+iw-4iw=4i+zi+z-i-1$$

$$w(z-iz-1+i-4i)=4i+z(i+1)(i-1)$$

$$w(z-1-i(z-1)-4i)=4i+(i+1)(z-1)$$

$$\therefore w=\frac{(i+1)z+3i-1}{(1-i)z-3i-1}$$

Problem 3.

Find the bilinear transformation which maps the points $z_1=0,\ z_2=-i,\ z_3=-1$ into $w_1=i,\ w_2=1$ & $w_3=0$ respectively.

Sol. Let the image of any point z under the required bilinear transformation, since bilinear transformation preserves cross ratio

$$(z,0,-i,-1)=(w,i,1,0)$$

$$\frac{(z+i)(0+1)}{(z+1)(0+i)}=\frac{(w-1)(i-0)}{(w-0)(i-1)}$$

$$\therefore (z+i)w(i-1)=-(w-1)(z+1)$$

$$(z+i)(wi-w)=-zw-w+z+1$$

$$zwi-zw-w+iw+zw+w=z+1$$

$$w(zi-i)=z+1$$

$$\therefore w=-i\left(\frac{z+1}{z-1}\right)$$ which is required bilinear transformation

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Problem 4

Determine the bilinear transformation which maps 0, 1, ∞ into i, -1, -i respectively.

Sol:

The required bilinear transformation is

$$(w, i, -1, -i) = (z, 0, 1, \infty)$$

$$\frac{z-1}{-1-0} = \frac{(w+1)(+i+i)}{(w+i)(-1+i)}$$

$$(z-1)(w+i)(-1+i) = -1(w+1)(i+1)$$

$$-(z-1)w + (z-1)wi - (z-1)i - 1(z-1) = -1(wi + w + i + 1)$$

$$-wz + w + zwi - wi - zi + i - z + i = -wi + w - i + 1$$

$$\therefore 1 - z = \frac{2i(w+1)}{iw + w - 1 + i}$$

$$\therefore 2iw + 2i = iw + w - 1 + 1 - iwz - wz + z - iz$$

$$\therefore w(2i - i - 1 + iz + z) = z - iz - 2i + 1 + i$$

$$w[(i-1) + (1+i)z] = z(1-i) - (1+i)$$

$$\therefore w = \frac{z(1-i) - (1+i)}{z(1+i) - (1-i)}$$

$$w = \frac{z - \left(\frac{1+i}{1-i}\right)}{z - \left(\frac{1-i}{1+i}\right)}$$

$$\therefore w = \frac{z - \left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)}{z - \left(\frac{1-i}{1+i} \times \frac{1-i}{1-i}\right)}$$

$$w = \frac{z - i}{z + 1}$$

∴ The required bilinear transformation is

$$w = \frac{z-i}{z+1}$$

The equation of the left half of w-plane and the interior of the unit circle in z plane

$Re\,w < 0$ & $|z| < 1$

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$$\text{Re}\, w < 0 \iff \text{Re}\left(\frac{z-i}{z+i}\right) < 0$$

$$\iff \text{Re}\left[\frac{(z-i)(\bar{z}+\bar{i})}{|z+i|^2}\right] < 0$$

$$\iff \text{Re}(z-i)(\bar{z}-i) < 0$$

$$\iff \text{Re}(z\bar{z} - i(z+\bar{z}) - 1) < 0$$

$$\iff \text{Re}(z\bar{z}) - 1 < 0$$

$$\iff |z|^2 < 1$$

$$\iff |z| < 1$$

The left half plane is mapped into interior of unit circle.

Problem 5

Find the bilinear transformation which maps $-1, 0, 1$ of the z plane onto $-1, -i, 1$ of w plane. Show the under this transformation upper half of z plane maps onto interior of unit circle $|w| = 1$

Sol.

The required bilinear transformation is

$$(w, -1, -i, 1) = (z, -1, 0, 1)$$

$$\therefore \frac{(w+i)(-1-1)}{(w-1)(-1+i)} = \frac{(z-0)(-1-1)}{(z-1)(-1-0)}$$

$$\frac{-2(w+i)}{(w-1)(-1+i)} = \frac{-2z}{1-z}$$

$$(1-z)(w+i) = z(w-1)(i-1)$$

$$w+i = iwz + z$$

$$w(1-iz) = z-i$$

$$w = \frac{z-i}{1-iz} = \frac{i(z-i)}{i(1-iz)}$$

$$= i\left(\frac{z-i}{z+i}\right)$$

$\therefore$ The required bilinear transformation $w = i\left(\frac{z-i}{z+i}\right)$

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The equation of upper half of w plane and interior of unit circle in z plane are

$Im\, w > 0$ & $|z| < 1$

$\therefore Im\, w > 0 \Leftrightarrow Im\left(i\left(\frac{z-i}{z+i}\right)\right) > 0$

$\Leftrightarrow Re\left(\frac{z-i}{z+i}\right) < 0$

$|z| < 1$

Hence the upper half plane is mapped into interior of unit circle.

Fixed point of Bilinear transformation

Problem 1

Find the invariant points of transformation

i) $w = \frac{1+z}{1-z}$

Sol

i) The invariant points of $w = f(z)$ are $f(z) = z$

$\therefore f(z) = z \Rightarrow z = \frac{1+z}{1-z}$

$\Rightarrow z - z^2 = 1 + z$

$\Rightarrow 1 + z^2 = 0$

$\therefore z = \pm i$

$\therefore$ i & $-i$ are two fixed points of transformation

ii) $w = \frac{1}{z-2i}$

$f(z) = z$ & $w = f(z)$

$\frac{1}{z-2i} = z$

$1 = z^2 - 2iz$

$\Rightarrow z^2 - 2iz - 1 = 0$

$(z-i)^2 = 0$

Hence i is the fixed point

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Problem 2.

Prove that the transformation $w = \bar{z}$ is not a bilinear transformation.

Sol.

Any bilinear transformation, other than the identity transformation has two fixed points. However the transformation $w = \bar{z}$ has infinitely many fixed points, all real numbers.

Hence it is not a bilinear transformation.

Some special bilinear transformation:

Problem 1:

Find the general bilinear transformation which maps the unit circle $|z|=1$ onto $|w|=1$ and the points $z=1$ to $w=1$ & $z=-1$ to $w=-1$

Sol.

Any bilinear transformation maps $|z|=1$ onto $|w|=1$ is $w = e^{i\lambda}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)$ where λ is real.

Since 1 and -1 are again mapped to 1, -1

$$1 = e^{i\lambda}\left(\frac{1-\alpha}{\bar{\alpha}-1}\right) \rightarrow (1)$$

$$-1 = e^{i\lambda}\left(\frac{-1-\alpha}{-\bar{\alpha}-1}\right) = e^{i\lambda}\left(\frac{1+\alpha}{1+\bar{\alpha}}\right) \rightarrow (2)$$

Dividing (1) by (2)

$$-1 = \frac{(1-\alpha)(1+\bar{\alpha})}{(\bar{\alpha}-1)(1+\alpha)}$$

$$-1 = \left(\frac{1-\alpha}{\bar{\alpha}-1}\right)\left(\frac{1+\bar{\alpha}}{1+\alpha}\right)$$

$$\therefore -\bar{\alpha} + \alpha\bar{\alpha} + 1 + \alpha = 1 + \bar{\alpha} - \alpha - \alpha\bar{\alpha}$$

$$\therefore -2\bar{\alpha} + 2\alpha = 0$$

$$\therefore \alpha = \bar{\alpha} \rightarrow (3)$$

using (3) in (1)

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$\therefore 1 = e^{i\lambda}\left(\frac{1-\alpha}{\alpha-1}\right)$

$1 = -e^{+i\lambda}$

$\therefore e^{+i\lambda} = -1$

$\boxed{\therefore e^{i\lambda} = -1}$

$\therefore$ The required transformation is

$$w = \frac{\alpha - z}{\alpha z - 1}$$

Problem 2:

Prove that the transformation given by $\bar{a}wz - bw - \bar{b}z + a = 0$ maps the unit circle $|z|=1$ onto unit circle $|w|=1$ if $|b| \neq |a|$

Sol.

$\bar{a}wz - bw - \bar{b}z + a = 0$

$\therefore w(\bar{a}z - b) = \bar{b}z - a$

$$w = \frac{\bar{b}z - a}{\bar{a}z - b}$$

using $w\bar{w} - 1$

$$\Rightarrow \left(\frac{\bar{b}z - a}{\bar{a}z - b}\right)\left(\frac{b\bar{z} - \bar{a}}{a\bar{z} - \bar{b}}\right) - 1$$

$= b\bar{b}z\bar{z} - \bar{b}z\bar{a} - ab\bar{z} + a\bar{a} - a\bar{a}z\bar{z} + \bar{a}z\bar{b} + ab\bar{z} - b\bar{b} / |\bar{a}z - b|^2$

$= b\bar{b}(z\bar{z} - 1) - a\bar{a}(z\bar{z} - 1) / |\bar{a}z - b|^2$

$= (z\bar{z} - 1)(b\bar{b} - a\bar{a}) / |\bar{a}z - b|^2$

$$= \frac{(z\bar{z} - 1)(|b|^2 - |a|^2)}{|\bar{a}z - b|^2}$$

If $|b| \neq |a|$ then

$w\bar{w} - 1 = 0$

$\Leftrightarrow z\bar{z} - 1 = 0$

$\therefore$ The unit circle $|z| = 1$ is mapped onto unit circle $|w| = 1$ if $|b| \neq |a|$

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Problem 3.

Show that bilinear transformation which maps the unit circle $|z|=1$ onto unit circle $|w|=1$ can in form

$w = e^{i\lambda}\left(\frac{az+b}{\bar{b}z+\bar{a}}\right)$ where λ is real

Further this transformation $|z|\leq 1$ onto circular disc $|w|\leq 1$ iff $|a|>|b|$

Also the point $z=1$ show that $\frac{1}{w-1} = \frac{1}{z-1} + \frac{1}{K}$ where $K = 1+\frac{\bar{a}}{b}$.

Sol.

The any bilinear transformation maps $|z|=1$ onto $|w|=1$ in the form $w = e^{i\mu}\left(\frac{z-\alpha}{\bar{\alpha}z-1}\right)$ where μ is real and maps $|z|\leq 1$ onto $|w|\leq 1$ iff $|\alpha|<1$

choose $a=1$ & $b=-a$

$$\therefore w = e^{i\mu}\left(\frac{z-a}{\bar{\alpha}z-1}\right)$$

$$= e^{i\mu}\left(\frac{az+b}{-\bar{b}z-\bar{a}}\right)$$

$$= e^{i\mu}\left(\frac{az+b}{\bar{b}z+\bar{a}}\right)$$

where $e^{i\lambda} = -e^{i\mu}$ and λ is real $\rightarrow (1)$

Further $|\alpha|<1 \Leftrightarrow |-b|<a$ (since $a=1$)

$\Leftrightarrow |b|<|a|$

$\therefore$ The transformation (1) maps $|z|\leq 1$ onto $|w|\leq 1$ iff $|a|>|b|$

suppose $z=1$ is only fixed point of (1)

$\therefore z=1$ is only root of eqn $z = e^{i\lambda}\left(\frac{az+b}{\bar{b}z+\bar{a}}\right)$

i.e) $\bar{b}z^2 + (\bar{a}-e^{i\lambda})z - be^{i\lambda} = \bar{b}(z-1)^2$

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$z^2\bar{b} + z\bar{a} - e^{i\lambda} az - bi\lambda = 0$

$\bar{b}z^2 + z(\bar{a} - e^{i\lambda}a) - be^{i\lambda} = \bar{b}(z-1)$ (assumption)

Equating the corresponding coefficients

$\bar{a} - ae^{i\lambda} = -2\bar{b} \quad \to (2)$

$be^{i\lambda} = -\bar{b} \quad \to (3)$

(2) ⇒

$\bar{a} + \bar{b} = -\bar{b} + ae^{i\lambda}$

using (3)

$\bar{a} - be^{i\lambda} = ae^{i\lambda} - \bar{b}$

$$w - 1 = e^{i\lambda}\left(\frac{az+b}{\bar{b}z+\bar{a}}\right) - 1$$

$$= \frac{e^{i\lambda}az + be^{i\lambda} - \bar{b}z - \bar{a}}{\bar{b}z+\bar{a}}$$

$$= \frac{(ae^{i\lambda} - \bar{b})z + (be^{i\lambda} - \bar{a})}{\bar{b}z+\bar{a}}$$

$$= \frac{(ae^{i\lambda} - \bar{b})z - (\bar{a} - be^{i\lambda})}{\bar{b}z+\bar{a}}$$

$$w - 1 = \frac{(z-1)(ae^{i\lambda} - \bar{b})}{\bar{b}z + \bar{a}} \quad \text{using (5)}$$

$$= \frac{(z-1)(ae^{i\lambda} - \bar{b})}{\bar{b} + \bar{a} + (z-1)\bar{b}}$$

$$= \frac{(z-1)(\bar{a} + \bar{b})}{(\bar{a}+\bar{b}) + (z-1)\bar{b}} \quad \text{using (4)}$$

$$\therefore \frac{1}{w-1} = \frac{(\bar{a}+\bar{b}) + (z-1)\bar{b}}{(z-1)(\bar{a}+\bar{b})}$$

$$= \frac{1}{z-1} + \frac{\bar{b}}{\bar{a}+\bar{b}}$$

$$= \frac{1}{z-1} + \frac{1}{1 + (\bar{a}/\bar{b})}$$

$$= \frac{1}{z-1} + 1/K \quad \text{where } K = 1 + \bar{a}/\bar{b}$$

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