complex integration

Definite Integral:

Let $f(t) = u(t) + iv(t)$ be a continuous complex valued function defined on $[a,b]$

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

Lemma:

$$\left|\int_a^b f(t)dt\right| \le \int_a^b |f(t)|dt$$

Pf.

Let $\int_a^b f(t)dt = re^{i\theta}$

$$\therefore \left|\int_a^b f(t)dt\right| = r$$

$$= e^{-i\theta}\int_a^b f(t)dt$$

$$= Re\left(e^{-i\theta}\int_a^b f(t)dt\right) \quad \text{since } r \text{ is real}$$

$$= Re\left(\int_a^b e^{-i\theta} f(t)dt\right)$$

$$= \int_a^b Re\left(e^{-i\theta} f(t)dt\right)$$

$$\le \int_a^b |e^{-i\theta} f(t)|dt$$

$$= \int_a^b |e^{-i\theta}||f(t)|dt$$

$$= \int_a^b |f(t)|dt$$

$$\boxed{\therefore \left|\int_a^b f(t)dt\right| \le \int_a^b |f(t)|dt}$$

Length:

Let C be a piecewise differentiable curve by equation $z = z(t)$ where $a \le t \le b$. The length l of C is defined by

$$\boxed{l = \int_a^b |z'(t)|dt}$$

Problem 1

Evaluate $\int_C f(z)dz$ where $f(z) = y - x - i3x^2$ and C is

line segment from $z=0$ to $z=1+i$.

A. The parametric eqn of C as $x=t$ and $y=t$

where $0 \le t \le 1$

$\therefore z(t) = x(t) + iy(t)$

$= t + it$

$\therefore z'(t) = 1 + i$ (Diff w.r.to t)

$f(z(t)) = t - t - i3t^2$

$= -i3t^2$

$\therefore \int_C f(z)dz = \int_0^1 f(z(t))z'(t)dt$

$= \int_0^1 -i3t^2(1+i)dt$

$= -3i(1+i)\left[\frac{t^3}{3}\right]_0^1$

$\int_C f(z)\,dz = 1 - i$

Problem 2

P.T $\int_C \frac{dz}{(z-a)^n} = \begin{cases} 0 & \text{if } n \neq 1 \\ 2\pi i & \text{if } n = 1 \end{cases}$ where C is the circle

with centre a and radius r, $n \in Z$

A. The parameters eqn of circle C is given by

$z - a = re^{it}$, $0 \le t \le 2\pi$

$\therefore z'(t) = ire^{it}$

$\int_C \frac{dz}{(z-a)^n} = \int_0^{2\pi} \frac{ire^{it}}{(re^{it})^n}dt$

$= \frac{i}{r^{n-1}}\int_0^{2\pi} e^{i(1-n)t}dt$

$= \frac{i}{r^{n-1}}\left[\frac{e^{i(1-n)t}}{i(1-n)}\right]_0^{2\pi}$ $[n \neq 1]$

$= \frac{1}{(1-n)r^{n-1}}[e^{i(1-n)2\pi} - 1]$

$= \frac{1}{(1-n)r^{n-1}}[1-1] \Rightarrow 0.$

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If $n=1$,

$$\int_C \frac{dz}{z-a} = 2\pi i$$

Hence result

Problem 3

Evaluate $\int_C \bar{z}\,dz$ from $z=0$ to $z=4+2i$ along the curve C consisting of line segment from $z=0$ to $2i$ followed by line segment from $z=2i$ to $z=4+2i$.

Sol.

Let C_1 denote line segment joining 0 to $2i$ & C_2 denote line segment joining $2i$ to $4+2i$

Then, $C = C_1 + C_2$

The parametric eqn of C_1 is given by $x(t)=0$ & $y(t)=t$

$0 \le t \le 4$.

$z(t) = t+2i$ & $z'(t)=1$

$$\therefore \int_{C_2} \bar{z}\,dz = \int_0^4 (t-2i)\,dt$$

$$= \left[\frac{t^2}{2} - 2it\right]_0^4$$

$$= \frac{16}{2} - 8i$$

$$= 8-8i$$

$$\therefore \int_C \bar{z}\,dz = \int_{C_1} \bar{z}\,dz + \int_{C_2} \bar{z}\,dz$$

$$= 2+8-8i$$

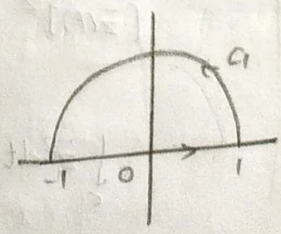
$$\int_C \bar{z}\,dz = 10-8i$$

Problem 4

Evaluate $\int_C |z|\,\bar{z}\,dz$ where C is closed curve consisting of the upper semicircle $|z|=1$ & segment $-1 \le x \le 1$.

Sol.

Let $f(z) = |z|\,\bar{z}$



$$\therefore \int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz$$

where C_1 is upper semicircle $|z|=1$

& C_2 is line segment $-1 \le x \le 1$.

The parametric eqn of C_1 is

$$z = e^{it}, 0 \le t \le \pi$$

$$z'(t) = ie^{it}$$

$$\therefore \int_{C_1} f(z)dz = \int_0^{\pi} e^{-it}e^{it}dt$$

$$= \pi i.$$

The parametric eqn of C_2 is given $y=0, x=t$ where $-1 \le t \le 1$

$$z(t) = t \ \& \ z'(t) = 1$$

$$|z(t)| = \begin{cases} -t & \text{if } 0-1 \le t < 0 \\ t & \text{if } 0 < t \le 1 \end{cases}$$

$$\text{Hence } \int_{C_2} |z|\bar{z}dz = \int_{-1}^{0} -t\,dt + \int_0^1 t\,t\,dt$$

$$= \left(\frac{-t^3}{3}\right)_{-1}^{0} + \left(\frac{t^3}{3}\right)_0^1$$

$$= -1/3 + 1/3$$

$$= 0$$

$$\int_C |z|\bar{z}dz = \int_{C_1} |z|\bar{z}dz + \int_{C_2} |z|\bar{z}dz$$

$$= \pi i.$$

Problem 5

$$\text{P.T} \int_C \bar{z}^2 dz = \begin{cases} 0 & \text{if } C \text{ is unit circle } |z| = 1 \\ 4\pi i & \text{if } C \text{ is circle } |z-1| = 1 \end{cases}$$

S.

Let C be unit circle $|z| = 1$

The parametric eqn of C is given by $z(t) = e^{it}$ where $0 \le t \le 2\pi$

$$z'(t) = ie^{it}$$

$$\bar{z}(t) = -ie^{it}$$

$$[\bar{z}(t)]^2 = -(i)^2(e^{-it})^2$$

$$= (e^{-2it})$$

$$\therefore \int_C \bar{z}^2 dt = \int_0^{2\pi} [\bar{z}(t)]^2 z'(t)dt$$

$$= i\int_0^{2\pi} e^{-it}dt \Rightarrow -[e^{-it}]_0^{2\pi}$$

$$\Rightarrow 0$$

Let C be circle $|z-1|=1$.

The parametric eqn of C is $z(t)=1+e^{it}$ where

$0 \le t \le 2\pi$

$$z'(t)= ie^{it}$$

$$\int_C z^2 dz = \int_0^{2\pi} (1+e^{-it})^2 ie^{it} dt$$

$$= i\int_0^{2\pi} (e^{it}+e^{-it}+2)\, dt$$

$$= i\left[\frac{e^{it}}{i} - \frac{e^{-it}}{i} + 2t\right]_0^{2\pi}$$

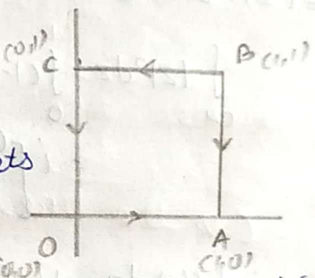
$$= \left[e^{it}-e^{-it}+2it\right]_0^{2\pi}$$

$$= 4\pi i$$

Problem 6:

S.T $\int_C |z|^2 dz = -1+i$, where C is square with vertices

O(0,0), A(1,0), B(1,1) & C(0,1).



Sol

$C = C_1+C_2+C_3+C_4$ where

C_1, C_2, C_3 & C_4 are linesegments

OA, AB, BC & CO.

The parametric eqn of C_1 is given by $x=t$ & $y=0$,

$0 \le t \le 1$

$$z(t)=t \ \& \ z'(t)=1.$$

$$\therefore \int_{C_1} |z|^2 dz = \int_0^1 t^2 dt = 1/3.$$

The parametric eqn of C_2 is given by $y=t$ & $x=1$

where $0 \le t \le 1$.

$$z(t)=1+it, \ z'(t)=i$$

$$\int_{C_2} |z|^2 dz = \int_0^1 |1+it|^2 dt$$

$$= i\int_0^1 (1+t^2)dt$$

$$= i\left[t+t^3/3\right]_0^1$$

$$= i\left[1+1/3\right]$$

$$= 4i/3$$

The parametric eqn of C_3 is given by $y=1$ & $x=1-t$,

$0 \le t \le 1$

Hence $z(t) = (1-t)+i$ & $z'(t) = -1$

$$\therefore \int_{C_3} |z|^2 dz = \int_0^1 [(1-t)^2+1](-1)dt$$

$$= -\int_0^1 (t^2-2t+2)dt$$

$$= -\left[\frac{t^3}{3} - \frac{2t^2}{2} + 2t\right]_0^1$$

$$= -\frac{1}{3} + 1 - 2$$

$$= -\frac{1}{3} - 1 \Rightarrow -\frac{1+3}{3}$$

$$= -\frac{4}{3}$$

The parametric eqn of C_4 is given by $x=0, y=1-t$,

$0 \le t \le 1$. Hence $z(t) = i(1-t)$ and $z'(t) = -i$

$$\int_{C_4} |z|^2 dz = \int_0^1 (1-t)^2(-i)dt$$

$$= i\left[\frac{(1-t)^3}{3}\right]_0^1 \Rightarrow -i/3$$

$$\therefore \int_C f(z)\,dz = \frac{1}{3} + \frac{4i}{3} - \frac{4}{3} - \frac{i}{3}$$

$$= -1+i$$

Problem 7:

Evaluate the integral $\int_C (x^2 - iy^2)\,dz$ where c is parabola $y = 2x^2$ from $(1,2)$ to $(2,8)$.

A. Let $f(z) = x^2 - iy^2$. The parametric eqn of c is $x = t$ and $y = 2t^2$, where $1 \le t \le 2$.

$$\therefore z(t) = x(t) + iy(t)$$

$$= t + i2t^2$$

$$z'(t) = 1 + 4it$$

$$\therefore \int_C (x^2 - iy^2)\,dz = \int_1^2 (t^2 - 4it^4)(1+4it)\,dt$$

$$= \int_1 [(t^2+16t^5) + i(4t^3 - 4t^4)]dt$$

$$= \left[\left(\frac{t^3}{3} + \frac{16t^6}{6}\right) + i\left(t^4 - \frac{4t^5}{5}\right)\right]_1^2$$

$$= \frac{8}{3} + \frac{8}{3}\times 64 + i\left[16 - \frac{4(32)}{5}\right) - \left(\frac{1}{3} + 8 + i + 1 - 4/5\right)$$

$$= \frac{511}{3} - \frac{49}{5}i$$

Problem 8

Evaluate $\int_C \frac{z+2}{z} dz$ where C is semi circle $z = 2e^{i\theta}$, where $0 \le \theta \le \pi$.

Sol.

$z'(\theta) = 2ie^{i\theta}$

$dz = 2ie^{i\theta} d\theta$

$$\int_C \frac{z+2}{z} dz = \int_0^{\pi} \left(\frac{2e^{i\theta}+2}{2e^{i\theta}}\right)(2ie^{i\theta}d\theta)$$

$$= 2i\int_0^{\pi} (1+e^{i\theta})d\theta$$

$$= 2i\left[\theta + \frac{e^{i\theta}}{i}\right]_0^{\pi}$$

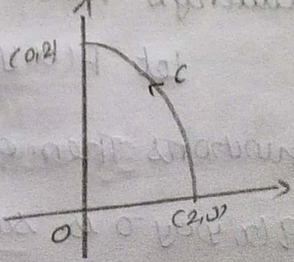
$$= 2i[(\pi - 1/i) - (1/i)]$$

$$= 2i\left[\frac{\pi i - 2}{i}\right]$$

$$\int_C \frac{z+2}{z} dz = -4 + 2\pi i$$

Problem 9.

Let C be arc of the circle $|z| = 2$ from $z = 2$ to $z = 2i$ that lies in the first quadrant. Without actually evaluating the integral S.T $\left|\int_C \frac{dz}{z^2+1}\right| \le \pi/3$



Sol.

Let $f(z) = \frac{1}{z^2+1}$

Since C is circular arc of radius 2 lying in first quadrant the length l of c is given by

$$l = \frac{1}{4}(2\pi \times 2)$$
$$= \pi$$

Also, on C, $|z^2+1| = |z^2-(-1)| \geq |z^2|-|-1|$
$$= |z^2|-1$$
$$= 3$$

$$|z^2+1| \geq 3$$
$$\therefore \left|\frac{1}{z^2+1}\right| \leq 1/3$$

Hence by $\left|\int_C \frac{dz}{z^2+1}\right| \leq \pi/3$

Problem 10.

Without evaluating the integral show that $\left|\int_C \frac{dz}{z^4}\right| \leq 4\sqrt{2}$ where c is the line segment from $z=i$ to $z=1$

Ans. C is the line segment joining (0,1) to (1,0) and its length is obviously $\sqrt{2}$. As z varies on C, the minimum value of $|z|$ is line perpendicular distance from the origin to line segment C.

$C, |z| \geq 1/\sqrt{2}$, so that $|z^4| \geq 1/4$

$$\therefore \left|\frac{1}{z^4}\right| \leq 4.$$

$$\therefore \left|\int_C \frac{dz}{z^4}\right| \leq 4\sqrt{2}$$

Cauchy's theorem:

Let $p(x,y)$ and $q(x,y)$ be two real valued functions. Then differential equation $p(x,y)dx + q(x,y)dy = 0$ is said to be exact if there exists a function $u(x,y)$ such that $\partial u/\partial x = p$ & $\partial u/\partial y = q$.

cauchy integral theorem:

Problems:

Problem 1:

Evaluate using cauchy's integral formula.

$$\frac{1}{2\pi i}\int_c \frac{z^2+5}{z-3}dz$$ where c is $|z|=4$

Sol.

$f(z)=z^2+5$ is analytic inside on $|z|=4$ and $z=3$ lies inside it.

$\therefore$ By cauchy's integral formula

$$f(z)=\frac{1}{2\pi i}\int_c \frac{f(z)\,dz}{z-z_0}$$

$$f(z)=\frac{1}{2\pi i}\int_c \frac{z^2+5}{z-3}dz$$

$$f(3)=3^2+5$$
$$=14$$

Problem 2

Evaluate $\int_c \frac{z\,dz}{z^2-1}$ where C is positively oriented circle $|z|=2$.

Sol.

$$\frac{1}{z^2-1}=\frac{1}{(z+1)(z-1)}$$
$$=\frac{1}{2}\left(\frac{1}{z-1}-\frac{1}{z+1}\right)$$ By Partial fraction

$$\frac{A}{z+1}+\frac{B}{z-1}=1$$

$$A(z-1)+B(z+1)=1$$

$z=\cancel{2}\ 1\ \cancel{0}\ -1$

$2B=1$

$A(-2)=1$

$B=1/2$

$A=-1/2$

$$\therefore\int_c \frac{z}{z^2-1}dz=\frac{1}{2}\int_c\frac{z}{z-1}dz-\frac{1}{2}\int_c\frac{z\,dz}{z+1}$$

$f(z)=z$ is analytic & 1, -1 lies interior of C.

∴ By cauchy integral formula $\int_C \frac{z\,dz}{z-1} = 2\pi i f(1)$
$= 2\pi i$

$\int_C \frac{z\,dz}{z+1} = 2\pi i f(-1)$
$= -2\pi i$

$\therefore \int_C \frac{z\,dz}{z^2-1} = \frac{1}{2}(2\pi i) - \frac{1}{2}(-2\pi i)$
$= 2\pi i$

Problem 3

Evaluate $\int_C \frac{e^z}{z^2+4}\,dz$ where C is positively oriented circle $|z-i| = 2$.

Sol.

$$\frac{1}{z^2+4} = \frac{1}{(z+2i)(z-2i)}$$
$$= \frac{1}{4i}\left(\frac{1}{z-2i} - \frac{1}{z+2i}\right) \quad \text{(By partial fraction)}$$

$2i$ lies inside C and by cauchy's integral formula

$$\int_C \frac{e^z}{z-2i}\,dz = 2\pi i e^{2i}.$$

$-2i$ lies outside C and hence $\frac{e^z}{z+2i}$ analytic inside and on C

By cauchy's theorem $\int_C \frac{e^z}{z+2i}\,dz = 0$

$$\therefore \int_C \frac{e^z}{z^2+4}\,dz = \frac{1}{4i}(2\pi i e^{2i} - 0)$$
$$= \pi/2\, e^{2i}$$

Problem 4

Evaluate $\int_C \left(\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}\right) dz$ where C is circle $|z| = 3$.

Sol.

By partial fractions

$$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$A(z-2)+B(z-1)=1$

$z=2$ $B(1)=1$

$z=1$ $A=-1$

$\therefore \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$

Let $f(z) = \sin\pi z^2 + \cos\pi z^2$, $f(z)$ is analytic inside.

Then $f(z)$ is analytic inside and on C and the Points 1 and 2 lie inside C.

Hence by cauchy's integral formula

$$\int_C \frac{f(z)}{z-1}\,dz = 2\pi i f(1)$$

$$= 2\pi i(\sin\pi + \cos\pi)$$

$$= -2\pi i$$

$$\therefore \int_C \frac{f(z)}{z-2}\,dz = 2\pi i f(2)$$

$$= 2\pi i(\cos 4\pi + \sin 4\pi)$$

$$= 2\pi i$$

Hence $\int_C \frac{f(z)}{(z-1)(z-2)}\,dz = 2\pi i - (-2\pi i)$

$= 4\pi i$

Problem 5

Let c denote the boundary of square whose sides lie along the lines $x=\pm 2$ and $y=\pm 2$ where c is described in positive sense

Evaluate i) $\int \frac{z\,dz}{2z+1}$

A. $\int_C \frac{z\,dz}{2z+1} = \frac{1}{2}\int_C \frac{z\,dz}{z+1/2}$

$= \frac{1}{2}(2\pi i)(-\frac{1}{2})$ [By cauchy's integral formula]

$= -\pi i/2$

ii) $\int_C \frac{\cos z}{z(z^2+8)}$

Let $f(z) = \int_C \frac{\cos z}{z(z^2+8)}$

$f(z) = \frac{\cos z}{z(z^2+8)}$

The points where $f(z)$ is not analytic are $\pm i 2\sqrt{2}$ and these points lie outside C.

Hence $f(z)$ is analytic inside and on C.

By cauchy's integral formula

$$\int_C \frac{\cos z}{z(z^2+8)} dz = \int_C \frac{f(z) dz}{z}$$

$$= 2\pi i f(0)$$

$$= 2\pi i (1/8) \quad \left[\frac{\cos 0}{0+8} = 1/8 \right]$$

$$\int_C \frac{\cos z}{z(z^2+8)} dz = \pi i/4$$

Problem 6

Evaluate $\frac{z dz}{(9-z^2)(z+i)}$ where c is circle $|z|=2$ taken in positive sense.

$$\int_C \frac{z dz}{(9-z^2)(z+i)} = \int_C \frac{z dz}{(3-z)(3+z)(z+i)}$$

But $|z|=2$

$z=3, z=-3$ not exist

$z+i=0$

$z=-i$

$|z| = |-i|$

$|z| = 1$

Lies b/w $|z|=2$

$-2<z<2$

$$\therefore \int_C \frac{z dz}{(9-z^2)(z+i)} = \int_C \frac{f(z)}{z+i} dz$$

$$= 2\pi i f(-i)$$

$$= 2\pi i \left(-1/10\right)$$

$$= \pi/5$$

Higher derivatives

Problem 1:

Evaluate $\int_C \frac{\sin z}{(z-\pi/2)^2}\,dz$ where C is the circle $|z|=2$

Sol. Let $f(z)=\sin z$

$f'(z)=\cos z$

$\pi/2$ lies inside $|z|=2$

$$\int_C \frac{\sin z\,dz}{(z-\pi/2)^2} = 2\pi i f'(\pi/2)$$

$$= 2\pi i(\cos \pi/2)$$

$$= 0$$

Problem 2:

Evaluate $\int_C \frac{z^3 dz}{(2z+i)^3}$ where C is unit circle.

Sol.

$$\int_C \frac{z^3 dz}{(2z+i)^3} = \frac{1}{8}\int_C \frac{z^3 dz}{(z+i/2)^3}$$

Let $f(z)=z^3$

$f'(z)=3z^2,$

$f''(z)=6z$

Also $-i/2$ lies inside C.

$$\int_C \frac{z^3 dz}{(2z+i)^3} = \frac{1}{8}\left(\frac{2\pi i}{2!}\right) f''(-i/2)$$

$$= \frac{2\pi i}{16}(-3i)$$

$$= 3\pi/8$$

Problem 3

Evaluate $\int_C \frac{(e^z+z\sinh z)\,dz}{(z-\pi i)^2}$ where C is circle $|z|=4$.

Sol Let $f(z)=e^z+z\sinh z$

$f'(z)=e^z+z\cosh z+\sinh z$

Also πi lies inside C,

$$\int_C \frac{f(z)}{(z-\pi i)^2}\,dz = 2\pi i f'(\pi i)$$

$$= 2\pi i[e^{\pi i} + \pi i \cosh \pi i + \sinh \pi i]$$

$$= 2\pi i(-1-\pi i)$$

$$= -2\pi i(1+\pi i)$$

Problem 4.

S.T when f is analytic within and on a simple closed curve c and z_0 is not on c then $\int_C \frac{f'(z)\,dz}{z-z_0} = \int_C \frac{f(z)\,dz}{(z-z_0)^2}$

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case(i)

Suppose z_0 is in exterior of C.

$\frac{f(z)}{(z-z_0)^2}$ & $\frac{f'(z)}{z-z_0}$ are analytic inside and on C

$\therefore$ By cauchy's theorem $\int_C \frac{f'(z)\,dz}{z-z_0} = \int_C \frac{f(z)\,dz}{(z-z_0)^2}$

$= 0$

case(ii)

z_0 lies in interior of C

By cauchy's theorem, $\int_C \frac{f'(z)}{(z-z_0)}\,dz$

$= 2\pi i f'(z_0)$

By using higher derivatives formula

$$\int_C \frac{f(z)}{(z-z_0)^2}\,dz = 2\pi i f'(z_0)$$

$$\therefore \int_C \frac{f'(z)}{z-z_0}\,dz = \int_C \frac{f(z)}{(z-z_0)^2}\,dz$$

Problem 5:

Let the function $f(z) = u(x,y) + iv(x,y)$ be continuous in closed bounded region D and let it analytic and not constant in interior of D. S.T function $u(x,y)$ reaches this maximum value on boundary of D and never in interior of D.

A. Consider $e^{f(z)}$. Since $f(z)$ is continuous in closed bounded region D and nonconstant in interior of D. $e^{f(z)}$ is also continuous in closed bounded region D and analytic & non-constant in interior of D.

The maximum value of $|e^{f(z)}|$ is attained only boundary point of D.

$\therefore |e^{f(z)}| = e^{u(x,y)}$

$\therefore$ Maximum value $e^{u(x,y)}$ is attained at boundary point of D.

$\therefore$ Maximum value of $u(x,y)$ is attained only at a boundary point of D.

Problem 6.

Evaluate $\int_C \frac{\sin 2z}{(z-i\pi/4)^4} dz$ where C is $|z|=1$

A. Let $f(z) = \sin 2z$. Since $f(z)$ is analytic & $\pi i/4$ lies inside C.

$$\therefore \int_C \frac{\sin 2z}{(z-\pi i)^4} = \frac{2\pi i}{3!} f'''(\pi i/4)$$

$f'(z) = 2\cos 2z$, $f''(z) = -4\sin 2z$

$f'''(z) = -8\cos 2z$

$f'''(\pi i/4) = -8\cos(\pi i/2)$

$= -8\cosh(\pi/2)$

$$\therefore \int_C \frac{\sin z}{(z-\pi i)^4} dz = \frac{-8\pi i \cosh(\pi/2)}{3}$$

Problem 7.

Evaluate $\int_C \frac{e^{2z}}{(z+1)^4} dz$ where c is circle $|z|=2$

Sol

Let $f(z) = e^{2z}$

$f(z)$ is analytic &

$f'(z) = 2e^{2z}$

$f''(z) = 4e^{2z}$

$f'''(z) = 8e^{2z}$

$$\int_C \frac{e^{2z}}{(z+1)^4} dz = \left(\frac{2\pi i}{3!}\right) f'''(-1)$$

$$= \left(\frac{2\pi i}{6}\right)(8e^{-2})$$

$$= \frac{i8\pi e^{-2}}{3}$$

Problem 8:

Evaluate $\int_C \frac{e^z}{(z+2)(z+1)^2}$ dz where c is $|z|=3$

Sol.

$$\frac{1}{(z+2)(z+1)^2} = \frac{(z+2)-(z+1)}{(z+2)(z+1)^2}$$

$$= \frac{1}{(z+1)^2} - \frac{1}{(z+2)(z+1)}$$

$$= \frac{1}{(z+1)^2} - \frac{1}{z+1} + \frac{1}{z+2}$$

$$\therefore \int_C \frac{e^z}{(z+2)(z+1)^2} dz = \int_C \frac{e^z}{z+2} dz - \int_C \frac{e^z}{z+1} dz + \int_C \frac{e^z}{(z+1)^2} dz \rightarrow (1)$$

$z=-2,-1$ lie in interior of c.

$f(z) = e^z$, It is analytic in c.

$f'(z) = e^z$

By cauchy's integral formula

$$\int_C \frac{e^z}{z+2} dz = 2\pi i f(-2)$$

$$= 2\pi i e^{-2}$$

$$\int_C \frac{e^z}{z+1}\,dz = 2\pi i f(-1)$$

$$= 2\pi i e^{-1}$$

$$\int_C \frac{e^z}{(z+1)^2}\,dz = \left(\frac{2\pi i}{1!}\right) f'(-1)$$

$$= 2\pi i e^{-1}$$

From (1)

$$\int_C \frac{e^z}{(z+2)(z+1)^2}\,dz = 2\pi i\,[e^{-2} - \cancel{e^{-1}} + \cancel{e^{-1}}]$$

$$= 2\pi i e^{-2}$$