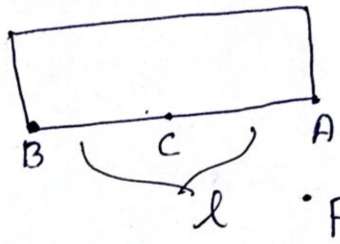
1). The two ends of a train moving with constant acceleration pass a certain point with velocities u and v, show that the velocity with which the middle point of the train passes the same point is $\sqrt{\frac{u^2+v^2}{2}}$.

Soln: Let A and B be the two ends of the train and $AB = l$ is the mid-point of BA.



When the end A crosses the point P, velocity of every point of the train = velocity of A = u.

Hence at this instant, velocity of C = velocity of B = u.

When B comes to P, its velocity = v. So if f is the constant acceleration considering the motion of B, we have

$$v^2 = u^2 + 2f \cdot BA = u^2 + 2f \cdot l \quad \text{—(1)}$$

When C comes to P, its velocity v_1 will be given by

$$v_1^2 = u^2 + 2f \cdot CA = u^2 + \cancel{2}f \cdot \tfrac{1}{\cancel{2}}l = u^2 + fl \quad \text{—(2)}$$

from (1), $fl = \dfrac{v^2 - u^2}{2}$

Substituting in (2)

$$v_1^2 = u^2 + \frac{v^2 - u^2}{2} = \frac{2u^2 + v^2 - u^2}{2} = \frac{u^2 + v^2}{2}$$

$$\therefore v_1 = \sqrt{\frac{u^2 + v^2}{2}}$$

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(2). A train goes from rest at one station to rest another station 3 km. off, being uniformly accelerated for the first two-thirds of the journey and uniformly retarded for the remainder and takes 3 minutes to describe the whole distance. Find the acceleration, the retardation and the maximum velocity.

Soln: (2)

Let AC be the distance described under Constant acceleration. $AC = \frac{2}{3} \times 3000 = 2000\,m$.

Let f_1 be the acceleration and t_1 the time taken for the portion AC. The initial velocity for this motion is 0 and let v be the velocity at C.

Then $v = f_1 t_1$ $\quad [v = u + ft]$ —— (1)

$2000 = \frac{1}{2} f_1 t_1^2$ $\quad [s = ut + \frac{1}{2}ft^2]$ —— (2)

s distance

$v^2 = 2f_1 \times 2000$ $\quad [v^2 = u^2 + 2fs]$ —— (3)

$v^2 = 4000 f_1$

The portion CB is the distance described under Constant retardation. Let this retardation $= f_2$ and time taken for this part be t_2. CB = 1000m

The initial velocity for this motion is v and the final velocity is 0.

opposite

$\therefore 0 = v - f_2 t_2$ (ie) $v = f_2 t_2$ $\quad [v = u - ft]$ —— (4)

$1000 = vt_2 - \frac{1}{2} f_2 t_2^2$ $\quad [s = ut - \frac{1}{2}ft^2]$

$= f_2 t_2^2 - \frac{1}{2} f_2 t_2^2$ using (4)

$1000 = \frac{f_2 t_2^2}{2}$ —— (5)

$0 = v^2 - 2f_2 \times 1000$ (ie) $v^2 = 2000 f_2$ $\quad [v^2 = u^2 - 2fs]$ —— (6)

s distance

Also total time

minutes second

$t_1 + t_2 = 3 \times 60 = 180$ —— (7)

Equating (3) & (6)

$\cancel{4}000 f_1 = \cancel{2}000 f_2$

(ie) $2f_1 = f_2$ —— (8)

Equating 1 & 4

$f_1 t_1 = f_2 t_2$

(ii) $\cancel{f_1} t_1 = 2 \cancel{f_1} t_2$ using 8

(or) $t_1 = 2t_2$ —— 9

Solving 7 & 9

$$\cancel{t_1} + t_2 = 180$$
$$\underline{-\cancel{t_1} \mp 2t_2 = -0}$$
$$3t_2 = 180$$

$\boxed{t_2 = 60}$ sec

$\boxed{t_1 = 120}$ sec

From (2) $2000 = \frac{1}{2} f_1 \times 120^2$

$$\frac{\cancel{2000} \times \cancel{2}}{\cancel{120} \times \cancel{120}} = f_1$$

$$\frac{5}{18} = f_1$$

$\therefore f_1 = 5/18 \ m/sec^2$

$f_2 = 2f_1 = \cancel{2} \times 5/\cancel{18}_9 \ m/sec^2$

$f_2 = 5/9 \ m/sec^2$

The maximum velocity = velocity at C =

$v = f_2 t_2 = 5/\cancel{9}_3 \times \cancel{60}^{20} = \frac{100}{3} \ m/sec.$

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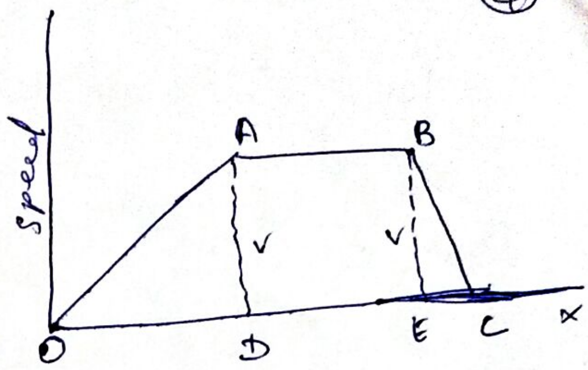
(3). The Speed of a train increases at a constant rate a from 0 to v, and then remains constant for an interval and finally decrease to 0 at a constant rate 3. If l be the total distance described, prove that the total time occupied is $\frac{l}{v} + \frac{v}{2l}\left[\frac{1}{\alpha} + \frac{1}{\beta}\right]$.

Soln:

Let OABC represent the velocity - time graph. The line OA represents the motion under Constant acceleration. The line AB which is parallel to OX represents the motion under Constant velocity and BC represents the motion under retardation.



Draw AD, BE perpendicular to OX.

$AD = BE = v$

α = Constant acceleration

α = slope of OA = $\tan \angle AOD = \frac{AD}{OD} = \frac{v}{OD}$

$\therefore OD = v/\alpha$ ——— (1)

β = Constant retardation

β = slope of BC = $\frac{BE}{EC} = \frac{v}{EC}$

$\therefore EC = v/\beta$ ——— (2)

l = Total distance

= area of trap. OABC

$l = \frac{1}{2}[AB + OC] . AD$

$l = \frac{1}{2}[\underbrace{AB}_{DE} + OD + DE + EC] . AD$

$l = \frac{1}{2}[2DE + OD + EC] . v$

$l = \frac{1}{2}(2DE + v/\alpha + v/\beta) . v$

$\frac{2l}{v} = 2DE + v/\alpha + v/\beta$

$\frac{2l}{v} - v/\alpha - v/\beta = 2DE$

$\left[\frac{2l}{v} - v/\alpha - v/\beta\right] \cdot \frac{1}{2} = DE$ ——— (3)

Adding (1), (2) & (3) we get total time

$$= \frac{v}{\alpha} + \frac{v}{\beta} + \frac{\frac{2l}{v} - \frac{v}{\alpha} - \frac{v}{\beta}}{2}$$

$$= \frac{2v}{2\alpha} + \frac{2v}{2\beta} + \frac{\cancel{2}l}{\cancel{2}v} - \frac{v}{2\alpha} - \frac{v}{2\beta}$$

$$= \frac{v}{2\alpha} + \frac{v}{2\beta} + \frac{l}{v}$$

$$= \frac{v}{2}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + l/v$$

$$= \frac{l}{v} + \frac{v}{2}\left(\frac{1}{\alpha} + 1/\beta\right).$$

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(4). Show that the greatest height reached by a projectile whose initial velocity is v and angle of projection is α is unaltered if v is increased to kV and α is decreased by λ where $\operatorname{cosec}\lambda = k(\cot\lambda - \cot\alpha)$.

Soln:

Initial velocity = V

If v increased by $\boxed{kv}$

α decreased by $\boxed{\alpha - \lambda}$

$$\text{greatest height} = \frac{v^2 \sin^2\alpha}{2g} \quad \text{— (1)}$$

$$= \frac{k^2 v^2 \sin^2(\alpha - \lambda)}{2g} \quad \text{— (2)}$$

equation (1) equal to (2).

$$\frac{\cancel{v^2} \sin^2\alpha}{\cancel{2g}} = \frac{k^2 \cancel{v^2} \sin^2(\alpha - \lambda)}{\cancel{2g}}$$

$$\sin^2\alpha = k^2 \sin^2(\alpha - \lambda)$$

Square root both sides,

$$\sin\alpha = k \sin(\alpha - \lambda)$$

$\sin(A-B) = \sin A \cos B - \cos A \sin B$

($\div \sin\alpha \sin\lambda$ both sides)

$$\frac{\sin\alpha}{\sin\alpha \sin\lambda} = \frac{k}{\sin\alpha \sin\lambda}\left[\sin\alpha \cos\lambda - \cos\alpha \sin\lambda\right]$$

It will $\frac{\sin\alpha}{\sin\alpha \sin\lambda} = k\left[\frac{\sin\alpha\cos\lambda}{\sin\alpha\sin\lambda} - \frac{\cos\alpha\sin\lambda}{\sin\alpha\sin\lambda}\right]$

$$\frac{1}{\sin\lambda} = k\left[\frac{\cos\lambda}{\sin\lambda} - \frac{\cos\alpha}{\sin\alpha}\right]$$

$$\operatorname{cosec}\lambda = k\left[\cot\lambda - \cot\alpha\right]$$

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(5). A particle is projected under gravity in a vertical plane with a velocity u at an angle α to the horizontal. If the range on the horizontal be R and the greatest height attained by h, show that $\frac{u^2}{2g} = h + \frac{R^2}{16h}$ and $\tan\alpha = \frac{4h}{R}$.

Soln:

velocity $= u$, angle $= \alpha$

greatest height $(h) = \frac{u^2 \sin^2\alpha}{2g}$ — (1)

horizontal range $(R) = \frac{2u^2 \sin\alpha\cos\alpha}{g}$ — (2)

equation (2) ÷ (1) $\frac{R}{h} = \frac{2\cancel{u^2}\sin\alpha\cos\alpha}{\cancel{g}} \times \frac{2\cancel{g}}{\cancel{u^2}\sin^2\alpha} = \frac{4\cos\alpha}{\sin\alpha}$

$\boxed{\frac{R}{h} = 4\cot\alpha}$ (or) $\boxed{\tan\alpha = \frac{4h}{R}}$

$$\operatorname{cosec}^2\alpha = 1 + \cot^2\alpha$$

$$\operatorname{cosec}^2\alpha = 1 + \frac{R^2}{16h^2} \quad \text{— (3)}$$

from (1) $\sin^2\alpha = \frac{2gh}{u^2}$

$\operatorname{cosec}^2\alpha = \frac{u^2}{2gh}$ (Reciprocal) — (4)

eqn (3) $\frac{u^2}{2gh} = 1 + \frac{R^2}{16h^2}$

$$\frac{u^2}{2gh} = \frac{h}{h} + \frac{R^2}{16h^2} \Rightarrow \frac{u^2}{(2g)\cancel{h}} = \frac{1}{\cancel{h}}\left[h + \frac{R}{16h}\right]$$

$$\boxed{\frac{u^2}{2g} = h + \frac{R}{16h}}$$