Linear Differential Equations with constant coefficients:

The General linear differential equation of order n is of form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_n y = f(x)$$

where $a_1, a_2, \dots a_n$ are real constant. This equation can also be written in operator form as

$$(D^n y + a_1 D^{n-1} y + \dots + a_n) y = f(x) \rightarrow (1)$$

The soln of (1) consists of two parts viz., (i) complementary function, (ii) Particular integral (ie) $y = y_c + y_p$ where y_c is complementary function, y_p is a particular integral.

To find Complementary function:

To find complementary function, we have to form the auxilary equation which is obtained by putting $D = m$ and $f(x) = 0$. Therefore, the auxilary equa (1) is

$$(m^n + a_1 m^{n-1} + a_2 m^{n-2} + \dots + a_n) = 0$$

case (i):

If all roots $m_1, m_2 \dots m_n$ are real & different then the complementary function (C.F), $y_c = Ae^{m_1 x} + Be^{m_2 x} + ce^{m_3} \dots$

case (ii):

If any two roots are equal say $m_1 = m_2 = m$, the C.F given by, $y = (Ax + B)e^{mx}$.

case (iii):

If any three roots are equal say $m_1 = m_2 = m_3 = m$, then CF is $y = (Ax^2 + Bx + c)e^{mx}$.

case (iv):

If the roots are imaginary say $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$ then C.F is $y_c = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

To find particular Integral (P.I):

Type I:

If $f(x) = e^{ax}$ then particular integral is given by,

$$P.I = \frac{1}{\phi(D)} \cdot e^{ax}$$

$$= \frac{1}{\phi(a)} \cdot e^{ax} \quad \text{provided } \phi(a) \neq 0$$

If $\phi(a) = 0$, then $P.I = \frac{1}{\phi(D)} e^{ax}$

$$= x \frac{1}{\phi'(D)} e^{ax}$$

where $\phi'(D)$ means derivatives of $\phi(D)$.

8. Solve: $(D^2 - 6D + 9)y = e^{3x}$

put $D = m$

$m^2 - 6m + 9 = 0$

$(m-3)(m-3) = 0$

$m = 3,\ m = 3$

$\therefore$ The roots are equal.

$C.F = (Ax+B)e^{mx}$

$= (Ax+B)e^{3x}$

$P.I = \frac{1}{D^2 - 6D + 9} \cdot e^{3x}$

$= \frac{1}{9 - 18 + 9} \cdot e^{3x} = 0$

$P.I = \frac{x}{2D - 6} \cdot e^{3x}$

$= \frac{x}{6-6} \cdot e^{3x} = \frac{x^2}{2} e^{3x}$

$\therefore$ The complete soln, $y = C.F + P.I$

$= (Ax+B)e^{3x} + \frac{x^2}{2} e^{3x}$

$y = e^{3x}\left[(Ax+B) + \frac{x^2}{2}\right]$

Type II :

If $f(x) = \sin ax$ (or) $\cos ax$, then P.I is given by,

$P.I. = \frac{1}{\phi(D)} \cdot \sin ax$ (or) $\cos ax$

In $\phi(D)$ replace D^2 by $-a^2$, provided $\phi(D) \neq 0$. If $\phi(D) = 0$, when we replace D^2 by $-a^2$ then $P.I = x \cdot \frac{1}{\phi'(D)} \sin ax$ (or) $\cos ax$.

NOTE:

If $f(x) = \sin(ax+b)$ (or) $\cos(ax+b)$ then the method of finding particular integral is the same as explained earlier.

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. Solve : $(D^2-4D+3)y = \cos 2x$

$m^2-4m+3=0$

$(m-3)(m-1)=0$

$m=3,1$

$\therefore$ The roots are different.

$C.F = Ae^{mx}+Be^{mx}$

$= Ae^{3x}+Be^{x}$

$P.I = \frac{1}{D^2-4D+3} \cdot \cos 2x$

Replace D^2 by $-(2)^2$

$P.I = \frac{1}{-4-4D+3} \cdot \cos 2x = \frac{1}{-4D-1} \cdot \cos 2x$

$= \frac{4D-1}{-(4D+1)(4D-1)} \cos 2x$

$$= -\frac{4D-1}{16D^2-1} \cdot \cos 2x$$

$$= -\frac{4D-1}{16(-4)-1} \cdot \cos 2x$$

$$= -\frac{4D-1}{-65} \cdot \cos 2x$$

$$= \frac{4D(\cos 2x)}{65} - \frac{\cos 2x}{65}$$

$$P.I = \frac{-8\sin 2x}{65} - \frac{\cos 2x}{65} = \frac{-1}{65}\left[8\sin 2x + \cos 2x\right]$$

$\therefore$ The complete soln, $y = C.F + P.I$

$$y = Ae^{3x} + Be^{x} - \frac{1}{65}\left[8\sin 2x + \cos 2x\right]$$

Type III:

If $f(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_n$ where $a_0 x^n + a_1 x^{n-1} + \dots + a_n$ is a pure algebraic function.

$$P.I = \frac{1}{\phi(D)} (a_0 x^n + a_1 x^{n-1} + \dots + a_n)$$

$$= [\phi(D)]^{-1} (a_0 x^n + a_1 x^{n-1} + \dots + a_n)$$

Expand $[\phi(D)]^{-1}$ by using binomial thm in ascending powers of D and then operate on $a_0 x^n + a_1 x^{n-1} + \dots + a_n$

NOTE: 1

* $(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$
* $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$
* $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$
* $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

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Solve: $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2+3$

$$m^2 - 5m + 6 = 0$$

$$(m-3)(m-2) = 0$$

$$m = 3, 2$$

∴ The roots are different.

$$C.F = Ae^{mx} + Be^{mx}$$

$$= Ae^{3x} + Be^{2x}$$

$$P.I = \frac{1}{D^2-5D+6} \cdot (x^2+3)$$

$$= \frac{1}{6}\left[1 + \frac{D^2-5D}{6}\right]^{-1}(x^2+3)$$

$$= \frac{1}{6}\left[1 - \left(\frac{D^2-5D}{6}\right) + \left(\frac{D^2-5D}{6}\right)^2 + \cdots\right](x^2+3)$$

$$= \frac{1}{6}\left[1 - \frac{D^2}{6} + \frac{5D}{6} + \frac{D^4}{36} + \frac{25D^2}{36} - \frac{10D^3}{36}\right](x^2+3)$$

$$= \frac{1}{6}\left[1 - \frac{D^2}{6} + \frac{5D}{6} + \frac{25D^2}{36}\right](x^2+3)$$

$$= \frac{1}{6}\left[x^2+3 - \frac{D^2}{6}(x^2+3) + \frac{5D}{6}(x^2+3) + \frac{25}{36}D^2(x^2+3)\right]$$

$$= \frac{1}{6}\left[x^2+3 - \frac{1}{3} + \frac{5}{3}x + \frac{25}{18}\right]$$

$$P.I = \frac{1}{6}\left[x^2 + \frac{5x}{3} + \frac{73}{18}\right] = \frac{1}{6}\left[\frac{18x^2+30x+73}{18}\right]$$

∴ The complete soln is, $y = C.F + P.I$

$$y = Ae^{3x} + Be^{2x} + \frac{1}{108}\left[18x^2 + 30x + 73\right]$$

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