2 mark:

1) Find (i) ∇r (ii) $\nabla(1/r)$

(i) $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\nabla \cdot \vec{r} = \left(\vec{i}\,\partial/\partial x + \vec{j}\,\partial/\partial y + \vec{k}\,\partial/\partial z\right) \cdot (x\vec{i} + y\vec{j} + z\vec{k})$

$= \partial/\partial x\,(x) + \partial/\partial y\,(y) + \partial/\partial z\,(z)$

$= 1 + 1 + 1 = 3.$

(ii) $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} = \Sigma x\vec{i}$

$r^2 = x^2 + y^2 + z^2 = \Sigma x^2$

partially diff w.r.to x

$2r \frac{\partial r}{\partial x} = 2x.$

$\frac{\partial r}{\partial x} = x/r$

III^{ly}, $\frac{\partial r}{\partial y} = \frac{y}{r}$

$\frac{\partial r}{\partial z} = \frac{z}{r}$

$\nabla(1/r) = \Sigma \vec{i}\,\partial/\partial x\,(1/r)$

$= \Sigma \vec{i}\,(-1/r^2)\,\frac{\partial r}{\partial x}$

$= \Sigma \vec{i}\,(-1/r^2)(x/r)$

$= \Sigma \vec{i}\,(-x/r^3)$

$= -1/r^3\ \Sigma x\vec{i}.$

$= -1/r^3\,[x\vec{i} + y\vec{j} + z\vec{k}]$

$= -1/r^3 \cdot \vec{r}$

2). If $\phi = x^2+y^2+z^2-8$ then find grad ϕ at $[2,0,2]$

$$\phi = x^2+y^2+z^2-8$$

$$\nabla\phi = \vec{i}\,\partial/\partial x\,(x^2+y^2+z^2-8) + \vec{j}\,\partial/\partial y\,(x^2+y^2+z^2-8) + \vec{k}\,\partial/\partial z\,(x^2+y^2+z^2-8)$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$\nabla\phi$ at $(2,0,2)$

$$= 4\vec{i} + 0\vec{j} + 4\vec{k}$$

5) $x^2+y^2-z = 10$ at $(1,1,1)$. find the unit normal

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\phi = x^2+y^2-z^2 = 10.$$

$$\nabla\phi = \vec{i}\,\partial/\partial x\,(x^2+y^2-z-10) + \vec{j}\,\partial/\partial y\,(x^2+y^2-z-10) + \vec{k}\,\partial/\partial z\,(x^2+y^2-z-10).$$

$$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(-1)$$

$$= 2x\vec{i} + 2y\vec{j} - \vec{k}.$$

at $(1,1,1)$

$$\nabla\phi = 2\vec{i} + 2\vec{j}$$

$$|\nabla\phi| = \sqrt{4+4}$$

$$= \sqrt{8}$$

$$\therefore \hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2\vec{i} + 2\vec{j}}{\sqrt{8}}$$

4) Define line Integral.

An Integral which is evaluated along a curve is called line Integral.

5) Define surface Integral:

- An Integral which is evaluated along a surface is called a surface Integral.

6) State green's theorem:

If $M(x,y)$ and $N(x,y)$ are continuous functions with continuous partial derivatives in a region R in the xy plane and on its boundary C which is a simple closed curve

then $\oint_C (Mdx + Ndy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy.$

7) State stokes theorem.

If S is an open surface bounded by a simple closed curve C and if $\vec{F}$ is continous having continuous partial derivatives in S and on C

$$\oint_C \vec{F}.d\vec{r} = \iint_S curl\, \vec{F}.\hat{n}\, ds.$$

8) Prove that $\int_C \vec{r}.d\vec{r} = 0.$

$$\oint_C \vec{r}.d\vec{r} = \iint_S curl\, \vec{F}.\hat{n}\, ds.$$

To find, $\iint_S curl\, \vec{F}.\hat{n}\, ds$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$curl\, \vec{F} = \nabla \times \vec{F}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(z) - \frac{\partial}{\partial z}(y)\right] - \vec{j}\left[\frac{\partial}{\partial x}(z) - \frac{\partial}{\partial z}(x)\right] + \vec{k}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial y}(x)\right]$$

$$= \vec{i}(0) + \vec{j}(0) + \vec{k}(0).$$

$$= 0.$$

$$\therefore \iint_S \text{curl } \vec{F}\cdot\hat{n}\, ds = 0.$$

---

9. Define fourier series.

The fourier series for the function $f(x)$ defined in the interval $(c, c+2\pi)$ with period 2π is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \text{--- (1)}.$$

where,

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\, dx \quad \text{--- (2)}$$

$$a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \cos nx\, dx \quad \text{--- (3)}.$$

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x) \sin nx\, dx \quad \text{--- (4)}$$

The values of a_0, a_n and b_n are known as fourier - Eulers formula.

---

10. Find a sine series for $f(x) = c$ in the range 0 to π.

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx.$

$$= \frac{2}{\pi}\int_0^{\pi} c \sin nx\, dx.$$

Scanned with CamScanner

$$= \frac{2c}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{2c}{\pi n}(1-\cos n\pi)$$

$$= \frac{2c}{\pi n}(1-(-1)^n)$$

when n is even $b_n = 0$.

when n is odd $b_n = \frac{4c}{n\pi}$

Hence $c = 4c/\pi \left(\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \dots\right)$

putting $x = \pi/2, \pi/4, \dots = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

---

5 mark:

11) prove that $\text{grad}\ \phi/\psi = \frac{\psi\, \text{grad}\, \phi - \phi\, \text{grad}\, \psi}{\psi^2}$

$$\text{grad}\ \phi/\psi = \vec{i}\,\frac{\partial}{\partial x}\left(\frac{\phi}{\psi}\right) + \vec{j}\,\frac{\partial}{\partial y}\left(\frac{\phi}{\psi}\right) + \vec{k}\,\frac{\partial}{\partial z}\left(\frac{\phi}{\psi}\right)$$

$$= \vec{i}\left(\frac{\psi\frac{\partial\phi}{\partial x} - \phi\frac{\partial\psi}{\partial x}}{\psi^2}\right) + \vec{j}\left(\frac{\psi\frac{\partial\phi}{\partial y} - \phi\frac{\partial\psi}{\partial y}}{\psi^2}\right)$$

$$\vec{k}\left(\frac{\psi\frac{\partial\phi}{\partial z} - \phi\frac{\partial\psi}{\partial z}}{\psi^2}\right)$$

$$= \frac{\psi\left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right) - \phi\left(\vec{i}\frac{\partial\psi}{\partial x} + \vec{j}\frac{\partial\psi}{\partial y} + \vec{k}\frac{\partial\psi}{\partial z}\right)}{\psi^2}$$

$$\text{grad}\left(\frac{\phi}{\psi}\right) = \frac{\psi\, \text{grad}\, \phi - \phi\, \text{grad}\, \psi}{\psi^2}$$

---

12) Evaluate $\iint_S \vec{F}\cdot\hat{n}\, ds$ where $\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$ and s is the part of the surface of the sphere $x^2+y^2+z^2=1$ which lies in the first octant.

$$\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$$

Let $\phi = x^2+y^2+z^2-1$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2x\vec{i}+2y\vec{j}+2z\vec{k}}{\sqrt{4x^2+4y^2+4z^2}} = \frac{2(x\vec{i}+y\vec{j}+z\vec{k})}{2\sqrt{x^2+y^2+z^2}}$$

$$= \frac{(x\vec{i}+y\vec{j}+z\vec{k})}{\sqrt{x^2+y^2+z^2}} = \frac{x\vec{i}+y\vec{j}+z\vec{k}}{1} \quad [\because x^2+y^2+z^2=1]$$

$$= x\vec{i}+y\vec{j}+z\vec{k}$$

$$\vec{F}\cdot\hat{n} = xyz + xyz + xyz = 3xyz.$$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_R \vec{F}\cdot\hat{n}\frac{dx\,dy}{|\hat{n}\cdot\vec{k}|}$$

where R is the projection of s on the xy plane. Clearly, the projection R is bounded by the lines x axis ($y=0$) & y axis ($x=0$), and the circle $x^2+y^2=1$, $z=0$.

$$|\hat{n}\cdot\vec{k}| = (x\vec{i}+y\vec{j}+z\vec{k})\cdot\vec{k} = z$$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_R 3xyz\frac{dx\,dy}{z}$$

$$= 3\iint_R xy\, dx\, dy$$

Scanned with CamScanner

$$= 3\int_0^1\int_0^{\sqrt{1-x^2}} xy \, dy \, dx$$

$$= 3\int_0^1 \left(x.\frac{y^2}{2}\right)_0^{\sqrt{1-x^2}} dx$$

$$= \frac{3}{2}\int_0^1 x(1-x^2)\,dx$$

$$= \frac{3}{2}\int_0^1 x - x^3 \, dx$$

$$= \frac{3}{2}\left(\frac{x^2}{2} - \frac{x^4}{4}\right)_0^1 dx$$

$$= \frac{3}{2}\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$= \frac{3}{2}\left(\frac{1}{4}\right)$$

$$= \frac{3}{8}.$$

18) Evaluate $\iint_S \vec{F}.\hat{n}\,ds$ where $\vec{F} = 4x\vec{i} - 2y\vec{j} + z^2\vec{k}$ and S is the surface boundary of $x^2+y^2=4$; $z=0$ and $z=3$

$$\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$$

$$\iint_S \vec{F}.\hat{n}\,ds = \iiint_V \nabla.\vec{F}\,dV$$

To find the limit $z=0$ & $z=3$.

$$x^2+y^2 = 4$$

$$y^2 = 4 - x^2$$

$$y = \pm\sqrt{4-x^2}$$

y varies from $y = -\sqrt{4-x^2}$ & $y = \sqrt{4-x^2}$

put $y=0$ in ①,

$$x^2+y^2 = 4.$$

$$x^2+0 = 4$$

$$x^2 = 4.$$

$$x = \pm 2$$

x varies from -2 to 2.

$$\operatorname{div} \vec{F} = \nabla \cdot \vec{F}$$

$$= \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}(4x\vec{i} - 2y^2\vec{j} + z^2\vec{k})$$

$$= \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)$$

$$= 4 - 4y + 2z.$$

$$\iiint_V \nabla \cdot \vec{F}\, dv = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^3 (4-4y+2z)\, dz\, dy\, dx.$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left(4z - 4yz + 2\frac{z^2}{2}\right)_0^3 dy\, dx.$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9)\, dy\, dx.$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y)\, dy\, dx.$$

$$= \int_{-2}^{2} \left(21 - 12\frac{y^2}{2}\right)_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx.$$

$$= \int_{-2}^{2} (21y - 6y^2)_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx.$$

$$= \int_{-2}^{2} \left[21(\sqrt{4-x^2}) - 6(4-x^2)\right] - \left[-21(\sqrt{4-x^2}) - 6((4-x^2)\right] dx$$

$$= \int_{-2}^{2} \left[21(\sqrt{4-x^2}) - 6(4-x^2) + 21\sqrt{4-x^2} + 6(4-x^2)\right] dx$$

$$= \int_{-2}^{2} (42(\sqrt{4-x^2}))\, dx.$$

$$= 42\int_{-2}^{2} \sqrt{4-x^2}\, dx.$$

$$\left[\int\sqrt{a^2-x^2} = \frac{x}{a}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)\right]$$

$$= 42\left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_{-2}^{2}$$

$$= 42\left[\frac{2}{2}\sqrt{4-4} + \frac{4}{2}\sin^{-1}\left(\frac{2}{2}\right)\right]$$

$$-\left[\frac{-2}{2}\sqrt{4-(-2)^2} + \frac{4}{2}\sin^{-1}\left(-\frac{2}{2}\right)\right]$$

$$= 42\left[2\sin^{-1}(1) - (0 + 2\sin^{-1}(-1))\right]$$

$$= 42\left[2\times\frac{\pi}{2} - \left(-2\frac{\pi}{2}\right)\right]$$

$$= 42\left(2\frac{\pi}{2} + 2\frac{\pi}{2}\right)$$

$$= 42(2\pi)$$

$$= 84\pi$$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\, ds = 84\pi$$

---

14. Obtain the fourier series for the function

$$f(x) = \begin{cases} 1, & 0 < x < \pi \\ 0, & \pi < x < 2\pi \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx\, dx.$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} 1\, dx + \int_{\pi}^{2\pi} 0\, dx\right]$$

$$= \frac{1}{\pi}(x)_0^{\pi}$$

$$= \frac{1}{\pi}(\pi - 0)$$

$$= 1$$

Scanned with CamScanner

$a_0 = 1$

$$a_n = \frac{1}{\pi}\left[\int_0^{\pi} \cos nx\,dx + \int_{\pi}^{2\pi} 0 \cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi} + 0.$$

$$= \frac{1}{\pi}(0-0).$$

$$= 0.$$

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} \sin nx\,dx + \int_{\pi}^{2\pi} 0 \sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi} + 0.$$

$$= -\frac{1}{n\pi}\left[(-1)^n - 1\right]$$

$$= \frac{-1}{n\pi}\left[(-1)^n - 1\right]$$

$$= \frac{1}{n\pi}\left[1 - (-1)^n\right]$$

$$b_n = \begin{cases} \frac{2}{n\pi} & ; n = 1, 3, 5, \ldots \\ 0 & ; n = 2, 4, 6, \ldots \end{cases}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$= \frac{1}{2} + \sum_{n=1,3,5\ldots}^{\infty} \frac{2}{n\pi} \sin nx.$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(2n-1)\pi} \sin(2n-1)x.$$

$$= \frac{1}{2} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{1}{2n-1} \sin(2n-1)x.$$

16. If $f(x) = x$ when $0 < x < \pi/2$; $f(x) = \pi - x$ when $x > \pi/2$

Expand $f(x)$ as a sine series in the interval $(0, \pi)$

Let $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$.

where $b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$.

$$= \frac{2}{\pi} \int_0^{\pi/2} x \sin nx \, dx + \frac{2}{\pi} \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx.$$

$$= \frac{2}{\pi} \left\{ \left[-x \frac{\cos nx}{n} \right]_0^{\pi/2} + \int_0^{\pi/2} \frac{\cos nx}{n} dx \right\}$$

$$+ \frac{2}{\pi} \left\{ \left[\frac{-(\pi - x) \cos nx}{n} \right]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi} \frac{\cos nx}{n} dx \right\}$$

$$= \frac{2}{n^2 \pi} \left[\sin nx \right]_0^{\pi/2} - \frac{2}{n^2 \pi} \left[\sin nx \right]_{\pi/2}^{\pi}$$

$$= \frac{2}{n^2 \pi} \sin n\pi/2 + \frac{2}{n^2 \pi} \sin n\pi/2$$

$$= \frac{4}{n^2 \pi} \sin n\pi/2$$

When n is even $b_n = 0$.

When n is odd and is of the form $4p+1$, $b_n = 4/n^2\pi$.

when n is odd is of the form $4p-1$

$b_n = -4/n^2\pi$

$b_2 = b_4 = b_6 = \dots = 0$.

$b_1 = 4/1^2\pi \quad b_5 = 4/5^2\pi \quad b_9 = 4/9^2\pi$

$b_3 = -4/3^2\pi \quad b_7 = -4/7^2\pi$

$$f(x) = 4/\pi \left\{ \frac{\sin x}{1^2} - \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} - \dots \right\}$$

10 mark

16. (i) Prove that $\nabla^2(r^n) = n(n+1)r^{n-2}$ where $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$ and hence deduce $\nabla^2(1/r)$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$r^2 = x^2 + y^2 + z^2$$

$$\frac{\partial r}{\partial x} = x/r \; ; \quad \frac{\partial r}{\partial y} = y/r \; ; \quad \frac{\partial r}{\partial z} = z/r$$

$$\nabla^2(r^n) = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right] r^n$$

$$= \frac{\partial^2 r^n}{\partial x^2} + \frac{\partial^2 r^n}{\partial y^2} + \frac{\partial^2 r^n}{\partial z^2}$$

$$= \sum \frac{\partial^2}{\partial x^2}(r^n)$$

$$= \sum \partial/\partial x \left[\partial/\partial x \, r^n\right]$$

$$= \sum \partial/\partial x \left[n r^{n-1} \frac{\partial r}{\partial x}\right]$$

$$= \sum \partial/\partial x \left[n r^{n-1} x/r\right]$$

$$= n \sum \partial/\partial x \left[r^{n-2} x\right]$$

$$= n \sum \left[(n-2) r^{n-3} \frac{\partial r}{\partial x} (x) + 1 (r^{n-2})\right]$$

$$= n \sum \left[(n-2) r^{n-3} (x/r) x + r^{n-2}\right]$$

$$= n \sum \left[(n-2) r^{n-4} . x^2 + r^{n-2}\right]$$

$$= n \left[\sum (n-2) r^{n-4} x^2 + \sum r^{n-2}\right]$$

$$= n \left[(n-2) r^{n-4} \sum x^2 + r^{n-2} \sum (1)\right]$$

$$= n \left[(n-2) r^{n-4} (x^2 + y^2 + z^2) + r^{n-2} \sum (1+1+1)\right]$$

$$= n(n-2) r^{n-4} . r^2 + n r^{n-2} . 3$$

$= n(n-2)\, r^{n-2} + 3n\, r^{n-2}$

$= n r^{n-2}[(n-2)+3]$

$= n r^{n-2}[n+1]$

$\nabla^2 r^n = n(n+1)\, r^{n-2} \rightarrow (1)$

put $n = (-1)$ in equ (1)

$\nabla^2 (r^{-1}) = (-1)[-1+1] r^{-1-2}$

$= 0.$

$\therefore \nabla^2 (1/r) = 0.$

(ii) S.T $\nabla^2 (e^r) = e^r + 2/r\, e^r$

$\nabla^2 e^r = \sum \frac{\partial^2}{\partial x^2}(e^r)$

$= \sum \partial/\partial x \left[e^r \frac{\partial r}{\partial x} \right]$

$= \sum \partial/\partial x \left[e^r x/r \right]$

$= \sum \partial/\partial r \left[e^r . x (1/r) \right]$

$= \sum \left[e^r x (-1/r^2) \frac{\partial r}{\partial x} + e^r (1).(1/r) + (e^r \frac{\partial r}{\partial x}) x (1/r) \right]$

$= \sum \left[-e^r \frac{x}{r^2} \cdot \frac{x}{r} + \frac{e^r}{r} + e^r \frac{x}{r} \cdot \frac{x}{r} \right]$

$= \sum \left[\frac{-x^2}{r^3} e^r + \frac{e^r}{r} + \frac{x^2}{r^2} e^r \right]$

$= -\left(\frac{x^2+y^2+z^2}{r^3}\right) e^r + \frac{3e^r}{r} + \left(\frac{x^2+y^2+z^2}{r^2} e^r\right)$

$= \frac{-r^2 e^r}{r^3} + \frac{3e^r}{r} + \frac{r^2}{r^2} e^r$

$$= -\frac{e^r}{r} + \frac{3e^r}{r} + e^r$$

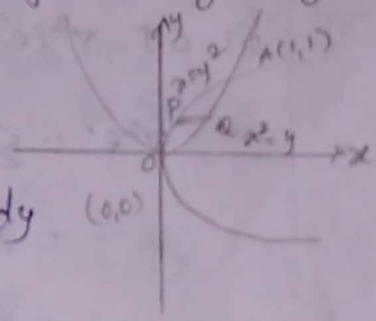
$$= \frac{2e^r}{r} + e^r$$

$$\nabla^2(e^r) = e^r + \frac{2}{r} e^r.$$

Hence proved.

---

17) Verify Green's theorem in the plane for $\int_C (3x^2-8y^2)\,dx + (4y-6xy)\,dy$, where C is the boundary of the region defined by $x=y^2$, $y=x^2$.



$$\oint_C M\,dx + N\,dy = \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy$$

To evaluate:

$$\iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dx\,dy.$$

$M = 3x^2 - 8y^2 \qquad N = 4y - 6xy$

$\frac{\partial M}{\partial y} = -16y \qquad \frac{\partial N}{\partial x} = -6y$

$x = y^2$ (or) $y^2 = x$ is a parabola about x axis

$y = x^2$ (or) $x^2 = y$ is a parabola about y axis.

At P, $x = y^2$ At Q, $x = \sqrt{y}$

At O, $y = 0$. At A, $y = 1$.

$$= \iint_R \left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] dy\,dx.$$

$$= \int_0^1 \int_{y^2}^{\sqrt{y}} (-6y + 16y)\, dy\, dx.$$

$$= 10 \int_0^1 \int_{y^2}^{\sqrt{y}} y\, dx . dy.$$

$$= 10 \int_0^1 (xy)_{y^2}^{\sqrt{y}}\, dy.$$

$$= 10 \int_0^1 (\sqrt{y}.y - y^2.y)\, dy.$$

$$= 10 \int_0^1 (y^{1/2}.y - y^3)\, dy$$

$$= 10 \int_0^1 (y^{3/2} - y^3) dy$$

$$= 10 \left[\frac{y^{5/2}}{5/2} - \frac{y^4}{4} \right]_0^1$$

$$= 10 \left[\frac{1}{5/2} - \frac{1}{4} \right]$$

$$= 10 \left[\frac{8-5}{20} \right]$$

$$= 3/2.$$

To find $\oint_C Mdx + Ndy$.

Here the line integral over simple closed curve C bounding the surface OAO, consisting the curves OA and AO.

$$\oint_C Mdx + Ndy = \int_{OAO} Mdx + Ndy$$

$$= \int_{\substack{OA \\ x^2=y}} Mdx + Ndy + \int_{\substack{AO \\ y^2=x}} Mdx + Ndy \rightarrow (2).$$

$$Mdx + Ndy = (3x^2-8y^2)\,dx + (4y-6xy)\,dy \quad \text{---}(3)$$

On OA

$x^2 = y$

$2x\,dx = dy$ in equ (3),

$$Mdx + Ndy = [3x^2 - 8x^4]\,dx + [4x^2 - 6x.x^2]\,2x\,dx.$$

$$= (3x^2-8x^4)\,dx + (8x^3 - 12x^4)\,dx.$$

$$= (3x^2 - 8x^4 + 8x^3 - 12x^4)\,dx.$$

$$= (3x^2 + 8x^3 - 20x^4)\,dx$$

$$\int_{OA} Mdx + Ndy = \int_0^1 (3x^2 + 8x^3 - 20x^4)\,dx$$

$$= \left[\frac{3x^3}{3} + \frac{8x^4}{4} - \frac{20x^5}{5}\right]_0^1$$

$$= (x^3 + 2x^4 - 4x^5)_0^1$$

$$= [(1+2-4) - 0]$$

$$= -1$$

On AO.

$$\oint_{AO} Mdx + Ndy = \oint_{AO} (3x^2-8y^2)\,dx + (4y-6xy)\,dy$$

$y^2 = x$ $\quad$ $2y = \frac{dx}{dy}$

$x = y^2$ $\quad$ $dx = 2y\,dy$.

$$\oint_1^0 (3(y^2)^2 - 8y^2)\,dy \cdot 2y + (4y - 6y^3)\,dy$$

$$\oint_1^0 (3y^4 - 8y^2)\,2y\,dy + (4y-6y^3)\,dy.$$

$$\Rightarrow \oint_1^0 (6y^5 - 16y^3 + 4y - 6y^3)\,dy$$

$= \oint_0^1 [6y^5 + 4y - 22y^3]\, dy$

$= \left[6y^6/6 + 4y^2/2 - 22y^4/4\right]_0^1$

$= [0 - (1 + 2 - 11/2)]$

$= [3 - 11/2]$

$= \left[\frac{6-11}{2}\right]$

$= 5/2$

$\oint_C M dx + N dy = \int_{OA} M dx + N dy + \int_{AO} M dx + N dy$

$= -1 + 5/2$

$= 3/2$

$\oint_C M dx + N dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx . dy$

Hence verified.

---

18) If $f(x) = 1/2 (\pi - x)$ as a fourier series with period 2π to be valid in the interval 0 to 2π

$f(x)$ is defined $(0, 2\pi)$

$f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \rightarrow (1)$.

gn: $f(x) = \frac{\pi - x}{2}$

$a_0 = 1/\pi \int_0^{2\pi} f(x)\, dx$

$= 1/\pi \int_0^{2\pi} \frac{\pi - x}{2}\, dx$

$= 1/2\pi \int_0^{2\pi} (\pi - x)\, dx$

$= 1/2\pi \left[\pi x - x^2/2\right]_0^{2\pi}$

Scanned with CamScanner

$= \frac{1}{2\pi}\left[2\pi^2 - \frac{4\pi^2}{2}\right]$

$= \frac{1}{2\pi}\left[2\pi^2 - 2\pi^2\right]$

$= 0$

$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx.$

$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)\cos nx\,dx.$

$= \frac{1}{2\pi}\int_0^{2\pi}(\pi - x)\cos nx\,dx.$

$= \frac{1}{2\pi}\left[\left[(\pi - x)\frac{\sin nx}{n}\right]_0^{2\pi} - \int_0^{2\pi}\frac{\sin nx}{n}(-dx)\right]$

$= \frac{1}{2\pi}\left[(\pi - 2\pi)\frac{\sin 2\pi n}{n} - \left[(\pi - 0)\frac{\sin n(0)}{n}\right]\right] + \int_0^{2\pi}\frac{\sin nx}{n}\,dx.$

$= \frac{1}{2\pi}(0) + \left(\frac{-\cos nx}{n^2}\right)_0^{2\pi}$

$= \frac{1}{2\pi}\left[\frac{-\cos 2n\pi}{n^2} + \frac{\cos n(0)}{n^2}\right]$

$= \frac{1}{2\pi}\left[\frac{-1}{n^2} + \frac{1}{n^2}\right]$

$= 0$

$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx.$

$= \frac{1}{\pi}\int_0^{2\pi}\left(\frac{\pi - x}{2}\right)\sin nx\,dx.$

$= \frac{1}{2\pi}\int_0^{2\pi}(\pi - x)\sin nx\,dx.$

$= \frac{1}{2\pi}\left[\left[(\pi - x)\left(\frac{-\cos nx}{n}\right)\right]_0^{2\pi} - \int_0^{2\pi} -\frac{\cos nx}{n}(-dx)\right]$

$$= \frac{1}{2\pi}\left[\left\{\left[-(\pi-2\pi)\frac{\cos n2\pi}{n}\right] - \left[-(\pi-0)\frac{\cos n0}{n}\right]\right\} - \int_0^{2\pi}\frac{\cos nx}{n}\,dx\right]$$

$$= \frac{1}{2\pi}\left[\left[-(-\pi)\frac{1}{n} - (\pi)\left(-\frac{1}{n}\right)\right] - \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{2\pi}\right]$$

$$= \frac{1}{2\pi}\left[\frac{2\pi}{n}\right] - \frac{1}{n^2}\left[\sin 2\pi n - \sin(0)\right]$$

$$= \frac{1}{2\pi}\left[\frac{2\pi}{n}\right] - 0.$$

$$b_n = \frac{1}{n}.$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$= \sum_{n=1}^{\infty} \frac{1}{n} \sin nx \rightarrow ②$$