VECTOR CALCULUS AND FOURIER SERIES...

UNIT - I

5 MARKS :

1) Prove that grad $(\phi \pm \psi)$ = grad ϕ $\pm$ grad ψ

Soln:

$$\text{grad}(\phi+\psi) = \vec{i}\frac{\partial}{\partial x}(\phi+\psi) + \vec{j}\frac{\partial}{\partial y}(\phi+\psi) + \vec{k}\frac{\partial}{\partial z}(\phi+\psi)$$

$$= \vec{i}\frac{\partial\phi}{\partial x} + \vec{i}\frac{\partial\psi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{j}\frac{\partial\psi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} + \vec{k}\frac{\partial\psi}{\partial z}$$

$$= \left[\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right] + \left[\vec{i}\frac{\partial\psi}{\partial x} + \vec{j}\frac{\partial\psi}{\partial y} + \vec{k}\frac{\partial\psi}{\partial z}\right]$$

$$\text{grad}(\phi+\psi) = \text{grad}\,\phi + \text{grad}\,\psi$$

Hence proved

$$\text{grad}(\phi-\psi) = \vec{i}\frac{\partial}{\partial x}(\phi-\psi) + \vec{j}\frac{\partial}{\partial y}(\phi-\psi) + \vec{k}\frac{\partial}{\partial z}(\phi-\psi)$$

$$= \vec{i}\frac{\partial\phi}{\partial x} - \vec{i}\frac{\partial\psi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} - \vec{j}\frac{\partial\psi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} - \vec{k}\frac{\partial\psi}{\partial z}$$

$$= \left[\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right] - \left[\vec{i}\frac{\partial\psi}{\partial x} + \vec{j}\frac{\partial\psi}{\partial y} + \vec{k}\frac{\partial\psi}{\partial z}\right]$$

$$\text{grad}(\phi - \psi) = \text{grad}\,\phi - \text{grad}\,\psi$$

Hence proved//

2) Find the directional derivative of $\phi = x^2yz + 4xz^2 + xyz$ at $(1,2,3)$ in the direction of $2\vec{i} + \vec{j} - \vec{k}$.

Soln:

$$\phi = x^2yz + 4xz^2 + xyz$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= (2xyz + 4z^2 + yz)\vec{i} + (x^2z + xz)\vec{j} + (x^2y + 8xz + xy)\vec{k}$$

$$\nabla\phi_{(1,2,3)} = 54\vec{i} + 6\vec{j} + 28\vec{k}$$

Given:

$$\vec{a} = 2\vec{i} + \vec{j} - \vec{k}$$

$$|\vec{a}| = \sqrt{4+1+1} = \sqrt{6}$$

Directional directive $= \nabla\phi \dfrac{\vec{a}}{|\vec{a}|}$

$$= (54\vec{i} + 6\vec{j} + 28\vec{k}) \cdot \frac{2\vec{i} + \vec{j} - \vec{k}}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}}\left[(54\vec{i} + 6\vec{j} + 28\vec{k}) \cdot (2\vec{i} + \vec{j} - \vec{k})\right]$$

$$= \frac{1}{\sqrt{6}}[108 + 6 - 28]$$

$$= \frac{1}{\sqrt{6}}[114 - 28]$$

$$= \frac{1}{\sqrt{6}}(86)$$

$$= \frac{86}{\sqrt{6}}$$

3) Find the angle b/w the surface $z = x^2+y^2-3$ and $x^2+y^2+z^2=9$ at $(2,-1,2)$.

soln

$$\phi_1 = x^2+y^2-z-3$$

$$\nabla\phi_1 = 2x\vec{i} + 2y\vec{j} - \vec{k}$$

$$(\nabla\phi_1)_{(2,-1,2)} = 4\vec{i} - 2\vec{j} - \vec{k}$$

$$|\nabla\phi_1| = \sqrt{16+4+1} = \sqrt{21}$$

$$\phi_2 = x^2+y^2+z^2-9$$

$$\nabla\phi_2 = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$(\nabla\phi_2)_{(2,-1,2)} = 4\vec{i} - 2\vec{j} + 4\vec{k}$$

$$|\nabla\phi_2| = \sqrt{16+4+16} = \sqrt{36} = 6$$

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$\cos\theta = \frac{(4\vec{i}-2\vec{j}-\vec{k})\cdot(4\vec{i}-2\vec{j}+4\vec{k})}{\sqrt{21}\,\sqrt{36}}$$

$$\theta = \cos^{-1}\left(\frac{8}{3\sqrt{21}}\right)$$

4) If $\vec{F} = x^2y\vec{i} + y^2z\vec{j} + z^2x\vec{k}$. Find curl (curl $\vec{F}$)

soln

$$\vec{F} = x^2y\vec{i} + y^2z\vec{j} + z^2x\vec{k}$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^2z & z^2x \end{vmatrix}$$

$$= \vec{i}\,[0-y^2] - \vec{j}\,[z^2-0] + \vec{k}\,[0-x^2]$$

$$= -y^2\vec{i} - z^2\vec{j} - x^2\vec{k}$$

$$\text{curl}\,(\text{curl}\,F) = \nabla\times(\nabla\times F) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & -z^2 & -x^2 \end{vmatrix}$$

$$= \vec{i}\,[0+2z] - \vec{j}\,[-2x-0] + \vec{k}\,[0+2y]$$

$$= 2z\vec{i} + 2x\vec{j} + 2y\vec{k}$$

$$= 2\left[z\vec{i} + x\vec{j} + y\vec{k}\right]$$

5, Prove that $\nabla\cdot\nabla\phi = \nabla^2\phi$

Soln

$$\nabla\cdot\nabla\phi = \left[\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right]\cdot\left[\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right]$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= \nabla^2\phi$$

$$\nabla\cdot\nabla\phi = \nabla^2\phi$$

Hence proved //

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10 MARKS :

1) A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$ where t is the time, find the component of its velocity and acceleration at time $t=1$ in the direction $\vec{i} - 3\vec{j} + 2\vec{k}$.

Soln:

Hence the component of its velocity vector in the direction $\vec{i} - 3\vec{j} + 2\vec{k}$.

Let,

$$r = 2t^2\vec{i} + (t^2 - 4t)\vec{j} + (3t-5)\vec{k}$$

Components of its velocity :

$$V = \frac{dr}{dt}$$

$$= 2(2t)\vec{i} + (2t-4)\vec{j} + (3-0)\vec{k}$$

$$= 4t\vec{i} + (2t-4)\vec{j} + 3\vec{k}$$

$$V_{t=1} = 4(1)\vec{i} + [2(1)-4]\vec{j} + 3\vec{k}$$

$$= 4\vec{i} + (2-4)\vec{j} + 3\vec{k} \Rightarrow 4\vec{i} - 2\vec{j} + 3\vec{k}$$

$$\frac{\vec{i} - 3\vec{j} + 2\vec{k}}{|\vec{i} - 3\vec{j} + 2\vec{k}|} \cdot V$$

$$= \frac{(\vec{i} - 3\vec{j} + 2\vec{k}) \cdot (4\vec{i} - 2\vec{j} + 3\vec{k})}{\sqrt{1+9+4}}$$

$$= \frac{(4 \times 1) + (-3 \times -2) + (2 \times 3)}{\sqrt{14}}$$

$$= \frac{4+6+6}{\sqrt{14}} \Rightarrow \frac{16}{\sqrt{14}}$$

Components of Acceleration Vector :

$$a = \frac{dv}{dt}$$

$$v = 4t\vec{i} + (2t-4)\vec{j} + 3\vec{k}$$

$$a = 4\vec{i} + [2(1) - 0]\vec{j} + 0$$

$$= 4\vec{i} + 2\vec{j}$$

$$\frac{\vec{i} - 3\vec{j} + 2\vec{k}}{|\vec{i} - 3\vec{j} + 2\vec{k}|} \cdot a$$

$$= \frac{(\vec{i} - 3\vec{j} + 2\vec{k}) \cdot (4\vec{i} + 2\vec{j})}{\sqrt{1+9+4}}$$

$$= \frac{(4\times 1) + (-3\times 2) + (2\times 0)}{\sqrt{14}}$$

$$= \frac{4-6}{\sqrt{14}} \Rightarrow \frac{-2}{\sqrt{14}}$$

2, Prove that $\vec{F} = (2x+yz)\vec{i} + (4y+zx)\vec{j} - (6z-xy)\vec{k}$ is solenoidal as well as irrotational and also find scalar potential of $\vec{F}$.

soln: Gvn:

$$\vec{F} = (2x+yz)\vec{i} + (4y+zx)\vec{j} + (6z-xy)\vec{k}$$

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(2x+yz) + \frac{\partial}{\partial y}(4y+zx) - \frac{\partial}{\partial z}(6z-xy)$$

$$= 2+4-6$$

$$\nabla \cdot \vec{F} = 0$$

The given vector is solenoidal.

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x+yz & 4y+zx & -6z+xy \end{vmatrix}$$

$$= \vec{i}(x-x) - \vec{j}(y-y) + \vec{k}(z-z) = \vec{0}$$

The given vector is irrotational.

$$\phi = (2x+yz)\vec{i} + (4y+zx)\vec{j} - (6z-xy)\vec{k}$$

$$\frac{\partial\phi}{\partial x} = 2x+yz \qquad \frac{\partial\phi}{\partial y} = 4y-zx \qquad \frac{\partial\phi}{\partial z} = -6z+xy$$

$$\int \partial\phi = \int 2x+yz\ \partial x \qquad \int \partial\phi = \int 4y-zx\ \partial x \qquad \int\partial\phi = \int -6z+xy\ \partial z$$

$$\phi_1 = \frac{2x^2}{2} + yzx + c \qquad \phi_2 = \frac{4y^2}{2} + zxy + C \qquad \phi_3 = \frac{-6z^2}{2} + xyz + c$$

$$\phi = \phi_1 + \phi_2 + \phi_3 + C$$

$$\therefore \phi = x^2 + 2y^2 - 3z^2 + xyz + C$$

3) Find a and b such that the surface $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at $(1,-1,2)$

Soln:

Let, $\phi_1 = ax^2 - byz - (a+2)x$

$$\nabla\phi_1 = \vec{i}\left[2ax - (a+2)\right] + \vec{j}(-bz) + \vec{k}(-by)$$

$$\nabla\phi_{(1,-1,2)} = \vec{i}(2a-a-2) + \vec{j}(-2b) + b\vec{k}$$

$$= \vec{i}[a-2] - 2b\vec{j} + b\vec{k}$$

$$|\nabla\phi_1| = \sqrt{(a-2)^2 + (2b)^2 + b^2}$$

$= \sqrt{(a-2)^2 + 4b^2 + b^2}$

$= \sqrt{(a-2)^2 + 5b^2}$

Let, $\phi_2 = 4x^2y + z^3 - 4$

$\nabla\phi_2 = 8xy\,\vec{i} + 4x^2\vec{j} + 3z^2\,\vec{k}$

$\nabla\phi_2\,(1,-1,2) = -8\vec{i} + 4\vec{j} + 12\,\vec{k}$

$|\nabla\phi_2| = \sqrt{64+16+144}$

$= \sqrt{224}$

Given orthogonally $(\theta = 90^\circ)$

$\cos 90^\circ = 0$

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\,|\nabla\phi_2|}$$

$$= \frac{-8(a-2)^2 - 8b + 12b}{\sqrt{(a-2)^2 + 5b^2}\sqrt{224}}$$

$$\therefore 0 = \frac{-8a+16-8b+12b}{\sqrt{(a-2)^2+5b^2}\ \sqrt{224}}$$

$\Rightarrow -8a+16-8b+12b = 0$

$-8a+16+4b = 0$

$\div 4$

$-2a+b+4 = 0$

$2a-b-4 = 0$

Since the points $(1,-1,2)$ lies on the

surface $\phi_1(x, y, z) = 0$.

Sub in $ax^2 - byz = (a+2)x$

$$\Rightarrow a + 2b - (a+2) = 0$$

$$a + 2b - a - 2 = 0$$

$$2b - 2 = 0 \quad \Rightarrow b = 2/2 \quad \Rightarrow \boxed{b = 1}$$

$$2a - (1) - 4 = 0$$

$$2a - 5 = 0$$

$$2a = 5 \quad \Rightarrow \boxed{a = 5/2}$$

4) Prove div (u grad v) = $u\nabla^2 v$ + (grad u) · (grad v)

Soln:

$$\nabla \cdot (u \nabla v) = u \nabla^2 v + \nabla u \cdot \nabla v$$

$$u \nabla v = u \left[\vec{i} \frac{\partial v}{\partial x} + \vec{j} \frac{\partial v}{\partial y} + \vec{k} \frac{\partial v}{\partial z} \right]$$

$$= \vec{i}\, u \frac{\partial v}{\partial x} + \vec{j}\, u \frac{\partial v}{\partial y} + \vec{k}\, u \frac{\partial v}{\partial z}$$

$$\nabla \cdot (u \nabla v) = \left[\vec{i} \frac{\partial}{\partial x} + \vec{j} \frac{\partial}{\partial y} + \vec{k} \frac{\partial}{\partial z} \right] \cdot \left[\vec{i}\, u \frac{\partial v}{\partial x} + \vec{j}\, u \frac{\partial v}{\partial y} + \vec{k}\, u \frac{\partial v}{\partial z} \right]$$

$$= \frac{\partial}{\partial x}\left[u \frac{\partial v}{\partial x} \right] + \frac{\partial}{\partial y}\left[u \frac{\partial v}{\partial y} \right] + \frac{\partial}{\partial z}\left[u \frac{\partial v}{\partial z} \right]$$

$$= u \frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} + u \frac{\partial^2 v}{\partial y^2} + \frac{\partial u}{\partial y} \frac{\partial v}{\partial y}$$

$$+ u \frac{\partial^2 v}{\partial z^2} + \frac{\partial u}{\partial z} \frac{\partial v}{\partial z}$$

$$= u\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right] + \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial z}$$

$$= u\nabla^2 v + \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial z} \rightarrow (1)$$

$$\nabla u = \vec{i}\frac{\partial u}{\partial x} + \vec{j}\frac{\partial u}{\partial y} + \vec{k}\frac{\partial u}{\partial z}$$

$$\nabla v = \vec{i}\frac{\partial v}{\partial x} + \vec{j}\frac{\partial v}{\partial y} + \vec{k}\frac{\partial v}{\partial z}$$

$$\nabla u\cdot\nabla v = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y} + \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial z} \rightarrow (2)$$

From (1) and (2),

$$\nabla\cdot(u\nabla v) = u\nabla^2 v + \nabla u\cdot\nabla v$$

Hence proved//

5) A particle moves along the curve $x=e^{-t}$, $y=2\cos 3t$, $z=2\sin 3t$ determine the velocity and acceleration at any time t and there magnitude at $t=0$.

Soln:

Let,

$$r = e^{-t}\,\vec{i} + 2\cos 3t\,\vec{j} + 2\sin 3t\,\vec{k}$$

Velocity $\Rightarrow V = \frac{dr}{dt}$

$$= -e^{-t}\,\vec{i} - 2\sin 3t\cdot(3)\,\vec{j} + 2\cos 3t\,(3)\,\vec{k}$$

$$= -e^{-t}\,\vec{i} - 6\sin 3t\,\vec{j} + 6\cos 3t\,\vec{k}$$

$$V_{t=0} = -e^{-0}\,\vec{i} - 6\sin 3(0)\,\vec{j} + 6\cos 3(0)\,\vec{k}$$

$$= -\vec{i} - 0\vec{j} + 6\vec{k}$$

$$= -\vec{i} + 6\vec{k}$$

$$|v|_{t=0} = \sqrt{(-1)^2 + (6)^2}$$

$$= \sqrt{1+36} = \sqrt{37}$$

Acceleration :

$$a = \frac{dv}{dt}$$

$$v = -e^{-t}\vec{i} - 6\sin 3t\,\vec{j} + 6\cos 3t\,\vec{k}$$

$$\frac{dv}{dt} = -(-e^{-t})\vec{i} - 6\cos 3t(3)\vec{j} - 6\sin 3t(3)\vec{k}$$

$$= e^{-t}\vec{i} - 18\cos 3t\,\vec{j} - 18\sin 3t\,\vec{k}$$

$$a_{t=0} = \vec{i} - 18\vec{j} - 0\vec{k}$$

$$= \vec{i} - 18\vec{j}$$

$$|a|_{t=0} = \sqrt{(1)^2 + (-18)^2}$$

$$= \sqrt{1+324}$$

$$= \sqrt{325}$$