UNIT- I

VECTOR CALCULUS AND FOURIER SERIES

5 mark :-

1. P.T $\text{grad}(\phi \pm \psi) = \text{grad}\,\phi \pm \text{grad}\,\psi$

Sol:-

$$\text{grad}(\phi \pm \psi) = \text{grad}\,\phi \pm \text{grad}\,\psi$$

$$\text{grad}(\phi+\psi) = \vec{i}\frac{\partial}{\partial x}(\phi+\psi) + \vec{j}\frac{\partial}{\partial y}(\phi+\psi) + \vec{k}\frac{\partial}{\partial z}(\phi+\psi)$$

$$= \vec{i}\frac{\partial\phi}{\partial x} + \vec{i}\frac{\partial\psi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{j}\frac{\partial\psi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} + \vec{k}\frac{\partial\psi}{\partial z}$$

$$= \left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right) + \left(\vec{i}\frac{\partial\psi}{\partial x} + \vec{j}\frac{\partial\psi}{\partial y} + \vec{k}\frac{\partial\psi}{\partial z}\right)$$

$$= \text{grad}\,\phi + \text{grad}\,\psi$$

$$\text{grad}(\phi-\psi) = \vec{i}\frac{\partial}{\partial x}(\phi-\psi) + \vec{j}\frac{\partial}{\partial y}(\phi-\psi) + \vec{k}\frac{\partial}{\partial z}(\phi-\psi)$$

$$= \vec{i}\frac{\partial\phi}{\partial x} - \vec{i}\frac{\partial\psi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} - \vec{j}\frac{\partial\psi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} - \vec{k}\frac{\partial\psi}{\partial z}$$

$$= \left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right) - \left(\vec{i}\frac{\partial\psi}{\partial x} + \vec{j}\frac{\partial\psi}{\partial y} + \vec{k}\frac{\partial\psi}{\partial z}\right)$$

$= \text{grad}\,\phi - \text{grad}\,\psi$

$\text{grad}\,(\phi \pm \psi) = \text{grad}\,\phi \pm \text{grad}\,\psi$

Hence proved,

Find the directional derivative of $\phi = x^2yz + 4xz^2 + xyz$ at $(1,2,3)$ in the direction of $2\vec{i}+\vec{j}-\vec{k}$.

Sol:-

Given $\phi = x^2yz + 4xz^2 + xyz$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= (2xyz + 4z^2 + yz)\vec{i} + (x^2z + xz)\vec{j} + (x^2y + 8xz + xy)\vec{k}$$

$$\nabla\phi\,(1,2,3) = 54\vec{i} + 6\vec{j} + 28\vec{k}$$

Given $\vec{a} = 2\vec{i} + \vec{j} - \vec{k}$

$$|\vec{a}| = \sqrt{4+1+1} = \sqrt{6}$$

Directional derivative $= \nabla\phi \cdot \frac{\vec{a}}{|\vec{a}|}$

$$= (54\vec{i} + 6\vec{j} + 28\vec{k}) \cdot \frac{2\vec{i}+\vec{j}-\vec{k}}{\sqrt{6}}$$

$$= \frac{1}{\sqrt{6}}(54\vec{i} + 6\vec{j} + 28\vec{k}) \cdot (2\vec{i}+\vec{j}-\vec{k})$$

$$= \frac{1}{\sqrt{6}}(108 + 6 - 28) = \frac{1}{\sqrt{6}}(86)$$

3. Find the angle between the surface $z = x^2+y^2-3$ and $x^2+y^2+z^2=9$ at $(2,-1,2)$

Sol:-

$$\phi_1 = x^2+y^2-z-3$$

$$\nabla\phi_1 = 2x\vec{i}+2y\vec{j}-\vec{k}$$

$$\nabla\phi_1 (2,-1,2) = 4\vec{i}-2\vec{j}-\vec{k}$$

$$|\nabla\phi_1| = \sqrt{16+4+1} = \sqrt{21}$$

$$\phi_2 = x^2+y^2+z^2-9$$

$$\nabla\phi_2 = 2x\vec{i}+2y\vec{j}+2z\vec{k}$$

$$\nabla\phi_2 (2,-1,2) = 4\vec{i}-2\vec{j}+4\vec{k}$$

$$|\nabla\phi_2| = \sqrt{16+4+16} = \sqrt{36} = 6$$

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$= \frac{(4\vec{i}-2\vec{j}-\vec{k}).(4\vec{i}-2\vec{j}+4\vec{k})}{\sqrt{21}.6}$$

$$= \frac{8}{3\sqrt{21}}$$

4. If $\vec{F} = x^2y\vec{i}+y^2z\vec{j}+z^2x\vec{k}$ find curl(curl$\vec{F}$).

Sol:- Given $\vec{F} = x^2y\vec{i}+y^2z\vec{j}+z^2x\vec{k}$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2y & y^2z & z^2x \end{vmatrix}$$

$= \vec{i}[0-y^2] - \vec{j}[z^2-0] + \vec{k}[0-x^2]$

$= -y^2\vec{i} - z^2\vec{j} - x^2\vec{k}$

curl (curl F) $= \nabla \times (\nabla \times \vec{F})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^2 & -z^2 & -x^2 \end{vmatrix}$$

$= \vec{i}[0+2z] - \vec{j}(-2x-0) + \vec{k}[0+2y]$

$= 2z\vec{i} + 2x\vec{j} + 2y\vec{k}$

$= 2[z\vec{i} + x\vec{j} + y\vec{k}]$

Hence solved.

5. prove that $\nabla \cdot \nabla\phi = \nabla^2\phi$

Sol:-

$$\nabla \cdot \nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)$$

$$\cdot \left(\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right)$$

$$= \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$= \cancel{\nabla\phi^2}\ \nabla^2\phi$

Hence proved,

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10 Marks:-

1. A particle Moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$ where t is the time find the Component of its velocity and acceleration at time $t = 1$ in the direction $\vec{i} - 3\vec{j} + 2\vec{k}$.

Sol:-

The component of velocity vector in the direction of $\vec{i} - 3\vec{j} + 2\vec{k}$

$$\frac{\vec{i} - 3\vec{j} + 2\vec{k}}{|\vec{i} - 3\vec{j} + 2\vec{k}|} \cdot V$$

Let $r = 2t^2\vec{i} + (t^2 - 4t)\vec{j} + (3t - 5)\vec{k}$

$$V = \frac{dr}{dt} = 4t\vec{i} + (2t - 4)\vec{j} + (3\vec{k})$$

$$V_{(t=1)} = 4\vec{i} - 2\vec{j} + 3\vec{k}$$

$$= \sqrt{16 + 4 + 9} = \sqrt{29}$$

$$\frac{A}{|A|} \cdot V = \frac{(\vec{i} - 3\vec{j} + 2\vec{k}) \cdot (4\vec{i} - 2\vec{j} + 3\vec{k})}{\sqrt{1 + 9 + 4}}$$

$$= \frac{4 + 6 + 6}{\sqrt{14}} = \frac{16}{\sqrt{14}}$$

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Acceleration $a = \frac{dv}{dt}$

$$a = 4\vec{i} - 12\vec{j} + 0$$

$$= \sqrt{16+4} = \sqrt{20}$$

$$\frac{A}{|\vec{A}|} \cdot a = \frac{(\vec{i} - 3\vec{j} + 2\vec{k})(4\vec{i} + 2\vec{j})}{\sqrt{1+9+4}}$$

$$= \frac{4-6}{\sqrt{14}} = \frac{-2}{\sqrt{14}} \text{ ''}$$

2\. P.T $\vec{F} = (2x+yz)\vec{i} + (4y+2x)\vec{j} - (6z-xy)\vec{k}$ is Solenoidal as well as irrotational and also find scalar potential of $\vec{F}$.

Sol:-

Solenoidal $= \nabla.\vec{F} = 0$

$$\nabla.\vec{F} = \frac{\partial}{\partial x} F_1 + \frac{\partial}{\partial y} F_2 + \frac{\partial}{\partial z} F_3$$

$$= \frac{\partial}{\partial x}(2x+yz) + \frac{\partial}{\partial y}(4y+zx) - \frac{\partial}{\partial z}(6z-xy)$$

$$= 2+4-6$$

$$= 0$$

Irrotational $= \text{curl}.\vec{F} = 0$

$$\text{curl}.\vec{F} = \nabla \times \vec{F}$$

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$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \partial/\partial y & \partial/\partial z \\ 2x+yz & 4y+zx & -6z+xy \end{vmatrix}$$

$$= \vec{i}(x-x) - \vec{j}(y-y) + \vec{k}(z-z)$$

$$= 0\vec{i} - 0\vec{j} + 0\vec{k}$$

Irrotational curl. $\vec{F} = \vec{0}$

Scalar potential :-

$$\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z} = (2x+yz)\vec{i} + (4y+zx)\vec{j} - (6z-xy)\vec{k}.$$

$$\frac{\partial\phi}{\partial x} = 2x+yz$$

$$\int\partial\phi = \int(2x+yz)\,dx$$

$$\phi_1 = \frac{2x^2}{2} + xyz$$

$$\phi_1 = x^2 + xyz$$

$$\frac{\partial\phi}{\partial y} = 4y+zx$$

$$\int\partial\phi = \int(4y+zx)\,\partial y$$

$$\phi_2 = \frac{4y^2}{2} + xyz$$

$$\phi_2 = 2y^2 + xyz$$

$$\frac{\partial\phi}{\partial z} = -6z+xy$$

$$\int\partial\phi = \int(-6z+xy)\,\partial z$$

$$\phi_3 = \frac{-6z^2}{2} + xyz$$

$$= -3z^2 + xyz$$

$$\phi = \phi_1 + \phi_2 + \phi_3$$

$$\phi = x^2 + 2y^2 - 3z^2 + xyz + xyz - xyz$$

$$= x^2 - 2y^2 - 3z^2 + xyz$$

3. Find a and b such that the surface $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at $(1,-1,2)$.

Sol:-

Let $\phi_1 = ax^2 - byz - (a+2)x$

$$\nabla\phi_1 = \vec{i}\,[2ax - (a+2)] + \vec{j}\,(-bz) + \vec{k}\,(-by)$$

$$\nabla\phi_1\,(1,-1,2) = \vec{i}\,[2a - a - 2] + \vec{j}\,(-2b) + b\vec{k}$$

$$= \vec{i}\,[a-2] - 2b\vec{j} + b\vec{k}$$

$$|\nabla\phi_1| = \sqrt{(a-2)^2 + (2b)^2 + b^2}$$

$$= \sqrt{(a-2)^2 + 4b^2 + b^2}$$

$$= \sqrt{(a-2)^2 + 5b^2}$$

$$\phi_2 = 4x^2y + z^3 - 4$$

$$\nabla\phi_2 = 8xy\vec{i} + 4x^2\vec{j} + 3z^2\vec{k}$$

$$(\nabla\phi_2)_{(1,-1,2)} = -8\vec{i} + 4\vec{j} + 12\vec{k}$$

$$|\nabla\phi_2| = \sqrt{64 + 16 + 144}$$

$$= \sqrt{224}$$

Given, orthogonally $(\theta = 90^\circ)$

$$\cos 90^\circ = 0$$

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$= \frac{-8(a-2)-8b+12b}{\sqrt{(a-2)^2+5b^2}\sqrt{224}}$$

$$0 = \frac{-8a+16-8b+12b}{\sqrt{(a-2)^2+5b^2}\sqrt{224}}$$

$$-8a+16-8b+12b = 0$$

$$-8a+16+4b = 0$$

$\div 4$

$$-2a+b+4 = 0$$

$$2a-b-4 = 0$$

Since the points $(1,-1,2)$ lies on the surface $\phi_1(x,y,z) = 0$ sub in

$$ax^2 - byz = (a+2)x$$

$$a+2b-(a+2) = 0$$

$$a+2b-a-2 = 0$$

$$2b-2 = 0$$

$$b = 2/2$$

$$b = 1$$

$$2a-(1)-4=0$$
$$2a-5=0$$
$$2a=5$$
$$a=5/2.$$

4. prove div (u grad v) = $u\nabla^2 v$ + (grad u). (grad v)

Sol:-

Given

$$\nabla.(u\nabla v) = u\nabla^2 v + \nabla u.\nabla v$$

$$u\nabla v = u\left[\vec{i}\frac{\partial v}{\partial x} + \vec{j}\frac{\partial v}{\partial y} + \vec{k}\frac{\partial v}{\partial z}\right]$$

$$= \vec{i}u\frac{\partial v}{\partial x} + \vec{j}u\frac{\partial v}{\partial y} + \vec{k}u\frac{\partial v}{\partial z}$$

$$\nabla.(u\nabla u) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right).\left(\vec{i}u\frac{\partial v}{\partial x} + \vec{j}u\frac{\partial v}{\partial y} + \vec{k}u\frac{\partial v}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}\left[u\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial y}\left[u\frac{\partial v}{\partial y}\right] + \frac{\partial}{\partial z}\left[u\frac{\partial v}{\partial z}\right]$$

$$= u\frac{\partial^2 v}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + u\frac{\partial^2 v}{\partial y^2} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + u\frac{\partial^2 v}{\partial z^2} + \frac{\partial u}{\partial z}\frac{\partial v}{\partial z}$$

$$= u\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial z}.$$

$$= u\nabla^2 v+\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}\rightarrow(1)$$

$$\nabla u=\vec{i}\frac{\partial u}{\partial x}+\vec{j}\frac{\partial u}{\partial y}+\vec{k}\frac{\partial u}{\partial z}$$

$$\nabla v=\vec{i}\frac{\partial v}{\partial x}+\vec{j}\frac{\partial v}{\partial y}+\vec{k}\frac{\partial v}{\partial z}$$

$$\nabla u.\nabla v=\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}+\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial z}\rightarrow(2)$$

From (1) and (2)

$$\nabla.(u\nabla v)=u\nabla^2 v+\nabla u.\nabla v$$

Hence proved.

5. A particle moves along the curve $x=e^{-t}$, $y=2\cos 3t$, $z=2\sin 3t$ determine the velocity and acceleration at any time t and the magnitude at $t=0$.

Sol:-

Let $r=e^{-t}\vec{i}+2\cos 3t\,\vec{j}+2\sin 3t\,\vec{k}$

$$V=\frac{dr}{dt}$$

$$\frac{dr}{dt}=-e^{-t}\vec{i}-2\sin 3t.3\vec{j}+2\cos 3t.3\vec{k}$$

$$V = -e^{-t}\vec{i} - 6\sin 3t\,\vec{j} + 6\cos 3t\,\vec{k}$$

$$V(t=0) = -\vec{i} - 0\vec{j} + 6\vec{k}$$

$$V(t=0) = -\vec{i} + 6\vec{k}$$

$$|V|_{t=0} = \sqrt{(-1)^2 + (6)^2} = \sqrt{1+36}$$

$$= \sqrt{37}$$

$$a = \frac{dv}{dt}$$

$$V = -e^{-t}\vec{i} - 6\sin 3t\,\vec{j} + 6\cos 3t\,\vec{k}$$

$$\frac{dv}{dt} = -(-e^{-t})\vec{i} - 6\cos 3t \cdot 3\vec{j} - 6\sin 3t \cdot 3\vec{k}$$

$$a = e^{-t}\vec{i} - 18\cos 3t\,\vec{j} - 18\sin 3t\,\vec{k}$$

$$a_{(t=0)} = \vec{i} - 18\vec{j} - 0\vec{k}$$

$$= \vec{i} - 18\vec{j}$$

$$|a|_{t=0} = \sqrt{(1)^2 + (-18)^2}$$

$$= \sqrt{1+324} = \sqrt{325}.$$