1

Unit-V

State and Prove cayley Hamilton Thm:-

Thm: 7.31 (Cayley Hamilton Thm)

Statement:

Any Square matrix A satisfies its characteristic equation. ie) $|A - xI| = 0$.

ie) If $a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is the characteristic polynomial of degree n of A Then

$$a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n = 0.$$

Proof:

T.P.T: $a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n = 0$

Let A be a square matrix of order n.

Let $|A - xI| = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n \rightarrow (1)$.

be the characteristic Polynomial of A.

Now, adj $(A - xI)$ is a Polynomial of degree $n-1$.

Since each entry of the matrix adj $(A - xI)$ is a cofactor of $A - xI$, and

Hence is a polynomial of degree $\leq n-1$.

$\therefore$ Let adj $(A - xI) = B_0 + B_1 x + B_2 x^2 + \cdots + B_{n-1} x^{n-1} \rightarrow (2)$

Now,

(4)

$(A - xI)\,\text{adj}(A - xI) = |A - xI|\,I$

$\boxed{[\because (\text{adj}A)A = A(\text{adj}A) = |A|I]}$

$(A - xI)(B_0 + B_1 x + \cdots + B_{n-1}x^{n-1}) = (a_0 + a_1 x + \cdots + a_n x^n)I$

(using (1) & (2))

$AB_0 + AB_1 x + \cdots + AB_{n-1}x^{n-1} - xB_0 - B_1 x^2 - \cdots - B_{n-1}x^{n-1+1}$

$= a_0 I + a_1 I x + a_2 x^2 I + \cdots + a_n x^n I$

$AB_0 + (AB_1 - B_0)x + \cdots - B_{n-1}x^n = a_0 I + a_1 I x + a_2 I x^2 + \cdots + a_n I x^n$

$\therefore$ Equating the coefficients of the corresponding powers of x we get

$\because AI = IA = A$

cons $AB_0 = a_0 I \rightarrow \times I \Rightarrow AB_0 I = a_0 I$

x coeffs $AB_1 - B_0 = a_1 I \longrightarrow \times^{ly} A \Rightarrow A^2 B_1 - AB_0 = a_1 A I$

$AB_2 - B_1 = a_2 I \longrightarrow \times^{ly} A^2 \Rightarrow A^3 B_2 - A^2 B_1 = a_2 A^2 I$

$\cdots$

$\cdots$

$AB_{n-1} - B_{n-2} = a_{n-1} I \rightarrow \times^{ly} A^{n-1} \Rightarrow A^n B_{n-1} - A^{n-1} B_{n-2} = a_{n-1} A^{n-1} I$

$- B_{n-1} = a_n I \rightarrow \times^{ly} A^n \Rightarrow - A^n B_{n-1} = a_n A^n I$

$\Rightarrow \boxed{a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n = 0}$

Hence Proved.

Note: If A is non-singular Matrix then

$A^{-1} = -\frac{1}{a_0}[a_1 I + a_2 A + \cdots + a_n A^{n-1}]$

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Solved problems

1. Find the characteristic eqn of the matrix

$$A = \begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$$

Soln:

To find: charac. eqn

The characteristic eqn of A is given by $|A - \lambda I| = 0$.

ie) $\begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$.

$(8-\lambda)[(7-\lambda)(3-\lambda) - 16] + 6[-6(3-\lambda) + 8] + 2(24 - 2(7-\lambda)) = 0$

$(8-\lambda)(\lambda^2 - 10\lambda + 5) + 6(6\lambda - 10) + 2(2\lambda + 10) = 0$

$\Rightarrow (8\lambda^2 - 80\lambda + \cancel{40} - \lambda^3 + 10\lambda^2 - 5\lambda) + 36\lambda - \cancel{60} + 4\lambda + \cancel{20} = 0$

$-\lambda^3 + 18\lambda^2 - 45\lambda = 0$

$\times - \Rightarrow \boxed{\lambda^3 - 18\lambda^2 + 45\lambda = 0}$

which represents the characteristic eqn of A.

(or)

Soln

$\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0. \rightarrow (1)$

where

S_1 = Sum of the main diagonal elements

$S_1 = 8 + 7 + 3 = 18 \Rightarrow \boxed{S_1 = 18}$

S_2 = Sum of the minors of the main diagonal elements

$= \begin{vmatrix} 7 & -4 \\ -4 & 3 \end{vmatrix} + \begin{vmatrix} 8 & 2 \\ 2 & 3 \end{vmatrix} + \begin{vmatrix} 8 & -6 \\ -6 & 7 \end{vmatrix}$

$= 21 - 16 + 24 - 4 + 56 - 36$

$\boxed{S_2 = 45}$

$S_3 = |A| = \begin{vmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{vmatrix}$

$= 8(21-16) + 6(-18+8) + 2(24-14)$

$= 8(5) + 6(-10) + 2(10)$

$|A| = 0 \Rightarrow \boxed{S_3 = 0}$

$S_1 = 18$; $S_2 = 45$; $S_3 = 0$ sub in (1)

$\Rightarrow \boxed{\lambda^3 - 18\lambda^2 + 45\lambda = 0}$

Problem - 7

Using Cayley Hamilton's Theorem for the matrix

$$A = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

find (i) A^{-1} (ii) A^4.
(iii) verify cayley H.T

Soln

Given $A = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}$

To find: (i) A^{-1} & (ii) A^4

(i) The characteristic eqn of A is $|A - \lambda I| = 0$.

$$\begin{vmatrix} 1-\lambda & 0 & -2 \\ 2 & 2-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

$(1-\lambda)[(2-\lambda)(2-\lambda)] - 0 - 2[0] = 0$

$(1-\lambda)[4 - 2\lambda - 2\lambda + \lambda^2] = 0$

$(1-\lambda)[\lambda^2 - 4\lambda + 4] = 0$

$\lambda^2 - 4\lambda + 4 - \lambda^3 + 4\lambda^2 - 4\lambda = 0$

$-\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$

$\times - \Rightarrow$ $\boxed{\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0}$

$\therefore$ by cayley Hamilton thm

$\lambda = A$ (Matrix)

$A^3 - 5A^2 + 8A - 4I = 0$ $\to$ (1)

$4I = A^3 - 5A^2 + 8A$.

$\times A^{-1} \Rightarrow$

$4A^{-1} = A^{-1}A^3 - 5A^2A^{-1} + 8AA^{-1}$

$[\because e^a \cdot e^b = e^{a+b}$
$AA^{-1} = I]$

$4A^{-1} = A^2 - 5A + 8I$

$\therefore A^{-1} = \frac{1}{4}[A^2 - 5A + 8I] \to$ (2)

$$A^2 = \begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1+0+0 & 0+0+0 & -2+0-4 \\ 2+4+0 & 0+4+0 & -4+8+8 \\ 0+0+0 & 0+0+0 & 0+0+4 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{pmatrix}$$

From (2) $\Rightarrow$

$$A^{-1} = \frac{1}{4}\left[\begin{pmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{pmatrix} - 5\begin{pmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{pmatrix} + 8\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]$$

$$A^{-1} = \frac{1}{4}\begin{pmatrix} 4 & 0 & 4 \\ 4 & 2 & -8 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\boxed{A^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & \frac{1}{2} & -2 \\ 0 & 0 & \frac{1}{2} \end{pmatrix}}$$

(ii) To find: A^4

$① \Rightarrow A^3 - 5A^2 + 8A - 4I = 0$

$\times A \Rightarrow A^4 - 5A^3 + 8A^2 - 4A = 0$

$A^4 = 5A^3 - 8A^2 + 4A \to ③$

To find: A^3, A^2 (student work).

Sub in ③

$$\boxed{A^4 = \begin{pmatrix} 1 & 0 & -30 \\ 30 & 16 & 60 \\ 0 & 0 & 16 \end{pmatrix}}$$

(iii) value of A^3, A^2, A sub in ①

$\Rightarrow \boxed{0 = 0}$

HW Pg.No: 7.27 – Problem – 2, 3, 4, 5, 6

7.8 Eigen values & Eigen Vectors

Defn:

Let A be an $n \times n$ Matrix. A number λ is called an eigen value of A if $\exists$ a non-zero vector $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ such that

$AX = \lambda X$ & X is called an eigen vector corresponding to the eigen value λ.

$$\underbrace{(A - \lambda I)}_{\text{Eigen value}} \overbrace{X}^{\text{eigen vector}} = 0$$

Characteristic Roots

Let $|A - \lambda I| = a_0\lambda^n + a_1\lambda^{n-1} + \dots + a_n$

The roots of this polynomial give the eigen values of A. Hence eigen values are also called characteristic roots.

Properties of Eigen values

Property - 1

Let X be an eigen vector corresponding to the eigen values λ_1 & λ_2. Then $\lambda_1 = \lambda_2$.

Proof:

To prove that: $\lambda_1 = \lambda_2$

We know that $X \neq 0$

$\Rightarrow AX = \lambda_1 X \to ①$ $\quad AX = \lambda_2 X \to ②$

eqn ① & ②.

$\Rightarrow \lambda_1 X = \lambda_2 X$

$\lambda_1 X - \lambda_2 X = 0$

$(\lambda_1 - \lambda_2) X = 0$

$\lambda_1 - \lambda_2 = 0$ or $X = 0$

$\because X \neq 0 \Rightarrow \lambda_1 - \lambda_2 = 0$

$\Rightarrow \boxed{\lambda_1 = \lambda_2}$

Hence proved.

Pro-2:

Let A be a square matrix. Then (i) The sum of the eigen values of A is equal to the sum of the diagonal elements (trace) of A.

(ii) Product of eigen values of A is $|A|$.

Proof:

T.P.T (i) $\lambda_1+\lambda_2+\dots+\lambda_n = a_{11}+a_{22}+\dots+a_{nn}$

(ii) $\lambda_1 \lambda_2 \dots \lambda_n = |A|$.

(i) Let $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$

W.K.T

The eigen values of A are the roots of the characteristic eqn

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix} \quad (1)$$

Let $|A-\lambda I| = a_0\lambda^n + a_1\lambda^{n-1} + \dots + a_n \quad (2)$

[Since $n=2 \Rightarrow \begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0$

$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda) - a_{12}a_{21} = 0$

$\Rightarrow a_{11}a_{22} - a_{11}\lambda - a_{22}\lambda + \lambda^2 - a_{12}a_{21} = 0$

$\Rightarrow \lambda^2 + a_{11}a_{22} - \lambda(a_{11}+a_{22}) - a_{12}a_{21} = 0 \quad$ (a)

$|A-\lambda I| = a_0\lambda^2 + a_1\lambda + a_2 \quad$ (b)

(a) & (b)

$\Rightarrow a_0 = 1;\ a_1 = -(a_{11}+a_{22})$

$\therefore a_0 = (-1)^2;\ a_1 = (-1)^{2-1}(a_{11}+a_{22})$]

(1) & (2) $\Rightarrow$ In General

$a_0 = (-1)^n$

$a_1 = (-1)^{n-1}(a_{11}+a_{22}+\dots+a_{nn}) \quad (3)$

Also by putting $\lambda = 0$ in (2)

$\Rightarrow a_n = |A|$.

Now let $\lambda_1, \lambda_2 \dots \lambda_n$ be the eigen values of A.

$\therefore \lambda_1, \lambda_2 \dots \lambda_n$ are the roots of (1)

$\therefore \lambda_1+\lambda_2+\dots+\lambda_n = \dfrac{-a_1}{a_0}$

$= \dfrac{-(-1)^{n-1}(a_{11}+a_{22}+\dots+a_{nn})}{(-1)^n}$

$= \dfrac{-(-1)^n \cdot (-1)^{-1}}{(-1)^n}(a_{11}+a_{22}+\dots+a_{nn})$

$= -(-(a_{11}+a_{22}+\dots+a_{nn}))$

$= a_{11}+a_{22}+\dots+a_{nn}$

$\therefore \boxed{\lambda_1+\lambda_2+\dots+\lambda_n = a_{11}+a_{22}+\dots+a_{nn}}$

$\therefore$ Sum of the eigen values $=$ Trace of A

Hence proved.

(ii) Product of the eigen values = Product of the roots.

$= \lambda_1 \lambda_2 \cdots \lambda_n$.

$= (-1)^n \frac{a_n}{a_0} = \frac{(-1)^n a_n}{(-1)^n}$

$= a_n$ $\quad \because a_n = |A|$

$$\boxed{\lambda_1 \lambda_2 \cdots \lambda_n = |A|}$$

Product of eigen value = $|A|$

Hence Proved.

— x —

Pro-3:

The eigen values of A & its Transpose A^T are the same.

Proof:

T.P.T: Eigen values of A = Eigen values of A^T

ie T.P.T: A & A^T have the same characteristic polynomial.

Since for any square matrix M. & $|M| = |M^T|$.

$|A - \lambda I| = |(A - \lambda I)^T|$

$= |A^T - (\lambda I)^T|$

$= |A^T - \lambda I|$

$\because (\lambda I)^T = \lambda I$

$\therefore$ Eigen values of A = Eigen value of A^T

Hence Proved.

— x —

Pro-4

If λ is an eigen values of a non singular matrix A Then $1/\lambda$ is an eigen value of A^{-1}.

Pf: T.P.T: $1/\lambda$ is an eigen value of A^{-1}

Let X be an eigen vector corresponding to λ.

Then $AX = \lambda X$

$\because A$ is non singular $|A| \neq 0$

$\therefore A^{-1}$ is exists.

$AX = \lambda X$.

$\times A^{-1} \Rightarrow A^{-1}AX = A^{-1}\lambda X$.

$(A^{-1}A)X = \lambda A^{-1} X$.

$IX = \lambda A^{-1} X$.

$A^{-1} X = \left(\frac{1}{\lambda}\right) X$.

$\therefore 1/\lambda$ is an eigen value of A^{-1}.

Hence proved.

— x —

Corollary: If $\lambda_1, \lambda_2 \ldots \lambda_n$ are the eigen values of a non singular matrix A then $1/\lambda_1, 1/\lambda_2, \ldots, 1/\lambda_n$ are the eigen values of A^{-1}.

Pro-5 : If λ is an eigen value of A Then $k\lambda$ is an eigen value of kA where k is a scalar.

Proof :

T.P.T : $k\lambda$ is an Eigen value of kA

Let x be an eigen vector corresponding to λ.

Then $AX = \lambda X \rightarrow (1)$.

$$\text{Now, } (kA)X = k(AX)$$
$$= k(\lambda X) \quad (\text{by } (1))$$
$$= (k\lambda)X.$$

Hence
$\therefore k\lambda$ is an eigen value of kA.

— X —

Pro-6

If λ is an eigen value of A then λ^k is an eigen value of A^k where k is any +ve integer.

Proof

T.P.T : λ^k is an eigen value of A^k.

Let X be an eigen vector corresponding to λ. Then

$$AX = \lambda X$$

$$\text{Now, } A^2X = (AA)X = A(AX)$$
$$= A(\lambda X) \quad (\text{by } (1))$$
$$= \lambda(AX)$$
$$= \lambda(\lambda X) \quad (\text{by } (1))$$
$$\boxed{A^2X = \lambda^2 X}$$

$\therefore \lambda^2$ is an eigen value of A^2

$$\text{Now, } A^kX = (A \cdot A \cdots k\text{ times})X$$
$$= (\lambda \cdot \lambda \cdots k\text{ time})X$$
$$\boxed{A^kX = \lambda^k X.}$$

$\therefore \lambda^k$ is an eigen vector of A^k. for any +ve integer

Hence Proved.

— X —

Pro-7

Eigen vectors corresponding to distinct eigen values of a matrix are linearly independent.

Proof :

T.P.T : Eigen vectors are linearly independent.

Let $\lambda_1, \lambda_2, \ldots \lambda_k$ be distinct eigen values of a matrix & let X_i be the eigen vector corresponding to λ_i.

Hence $AX_i = \lambda_i X_i$

$$(i = 1, 2, \ldots, k) \rightarrow (1)$$

Now, Suppose $x_1, x_2, \cdots x_k$ are linearly dependent.

Then There exist real numbers $\alpha_1, \alpha_2, \cdots \alpha_k$ not all zero, Such that $\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k = 0$

Among all such relations, we choose one of shortest length say j.

By rearranging the vectors $x_1, x_2, \cdots, x_k$ we may assume That

$$\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_j x_j = 0 \rightarrow (2)$$

$$\times A \Rightarrow A(\alpha_1 x_1) + A(\alpha_2 x_2) + \cdots + A(\alpha_j x_j) = 0$$

$$\Rightarrow \alpha_1 (A x_1) + \alpha_2 (A x_2) + \cdots + \alpha_j (A x_j) = 0$$

$$\Rightarrow \alpha_1 \lambda_1 x_1 + \alpha_2 \lambda_2 x_2 + \cdots + \alpha_j \lambda_j x_j = 0 \rightarrow (3)$$

Multiplying (2) by λ_1 & Subtracting from (3), We get

$$(2) \times \lambda_1 \Rightarrow \alpha_1 \lambda_1 x_1 + \alpha_2 \lambda_1 x_2 + \cdots + \alpha_j \lambda_1 x_j = 0 \rightarrow (4)$$

(4) - (3)

$$\Rightarrow \alpha_1 \cancel{\lambda_1} x_1 + \alpha_2 \lambda_1 x_2 + \cdots + \alpha_j \lambda_1 x_j - \alpha_1 \cancel{\lambda_1} x_1 - \alpha_2 \lambda_2 x_2 - \cdots - \alpha_j \lambda_j x_j = 0$$

$$\alpha_2 (\lambda_1 - \lambda_2) x_2 + \cdots + \alpha_j (\lambda_1 - \lambda_j) x_j = 0 \rightarrow (5)$$

Since $\lambda_1, \lambda_2, \cdots \lambda_j$ are distinct & $\alpha_2, \alpha_3 \cdots \alpha_j$ are non-zero,

$$\alpha_i (\lambda_1 - \lambda_i) \neq 0 \; ; i = 2, 3 \cdots j$$

(5) gives a relation whose length is $j-1$

$\Rightarrow \Leftarrow$ which is contra for our assumption ($\because j$ lay)

$\therefore x_1, x_2, \cdots x_k$ are linearly independent.

Hence proved.

— x —

Pro - 8 :

The characteristic roots of a Hermitian matrix are all real.

Pf:

T.P.T : Eigen values of Hermitian matrix are real.

Let A be a Hermitian matrix

w.k.t

$$A = \bar{A}^T \longrightarrow (1)$$

Let λ be a characteristic root of A &

Let x be a characteristic vector corresponding to λ.

$$\therefore Ax = \lambda x \longrightarrow (2)$$

$$(2) \times \bar{x}^T \Rightarrow \bar{x}^T A x = \lambda \bar{x}^T x$$

(10)

[Since $x^T A x$ is a 1×1 matrix
$\therefore (x^T A x) = (x^T A x)^T$]

$\Rightarrow (\bar{x}^T A x)^T = \lambda \bar{x}^T x$.

$\Rightarrow x^T A^T (\bar{x}^T)^T = \lambda \bar{x}^T x$.

$\Rightarrow x^T A^T \bar{x} = \lambda \bar{x}^T x$

Take conjugate on both sides

$\Rightarrow \overline{x^T A^T \bar{x}} = \overline{\lambda \bar{x}^T x}$

$\Rightarrow \bar{x}^T \bar{A}^T \bar{\bar{x}} = \bar{\lambda} \bar{\bar{x}}^T \bar{x}$ $\quad \because \bar{\bar{A}} = A$

$\Rightarrow \bar{x}^T \bar{A}^T x = \bar{\lambda} x^T \bar{x}$

$\Rightarrow \bar{x}^T A x = \bar{\lambda} x^T \bar{x}$ $\quad \because A = \bar{A}^T$

$\bar{x}^T \lambda x = \bar{\lambda} x^T \bar{x}$

$\because Ax = \lambda x$

$\lambda (\bar{x}^T x) = \bar{\lambda} (x^T \bar{x})$

Now, $\quad \because z\bar{z} = |z|^2$

$\bar{x}^T x = x^T \bar{x} = \bar{x}_1 x_1 + \cdots + \bar{x}_n x_n$

$= |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2$

$\therefore \bar{x}^T x = x^T \bar{x} \neq 0$.

$\Rightarrow \lambda (\cancel{\bar{x}^T x}) = \bar{\lambda} (\cancel{x^T \bar{x}})$

$\boxed{\lambda = \bar{\lambda}}$

$\therefore \lambda$ is real.

Hence proved.

— x —

Pro-9

The characteristic roots of a skew Hermitian matrix are either purely imaginary or zero.

Pf:

T.P.T: Eigen values of Hermitian matrix are either purely imaginary or zero.

Let A be a skew Hermitian matrix and λ be a characteristic root of A.

$\therefore |A - \lambda I| = 0$.

$\times i \Rightarrow |iA - i\lambda I| = 0$.

$\therefore i\lambda$ is a characteristic root of iA

Since A is skew Hermitian

$\Rightarrow iA$ is Hermitian.

by pro-9 $i\lambda$ is real.

$\therefore \lambda$ is purely imaginary (or) $i, 2i$

or zero

Hence proved.

— x —

Pro-10

Let λ be a characteristic root of an unitary matrix A. Then $|\lambda| = 1$

(or)

The characteristic roots of a unitary matrix are all the unit modulus.

Proof:

T.P.T: $|\lambda| = 1$

11

Let λ be a characteristic root of an unitary Matrix A and x be a characteristic vector corresponding to λ.

$\therefore \quad Ax = \lambda x \rightarrow (1)$

Taking conjugate & Transpose in (1)

$(\overline{Ax})^T = (\overline{\lambda x})^T$

$\therefore (\bar{A}\bar{x})^T = (\bar{\lambda}\bar{x})^T$

$\Rightarrow \bar{x}^T\bar{A}^T = \bar{\lambda}\bar{x}^T \rightarrow (2)$

Multiply (1) & (2)

$\Rightarrow (\bar{x}^T\bar{A}^T)Ax = \bar{\lambda}\bar{x}^T(\lambda x)$

$\bar{x}^T(\underbrace{\bar{A}^TA}_{I})x = \lambda\bar{\lambda}(\bar{x}^Tx)$

$[\because \bar{A}^TA = I]$

Hence $\bar{x}^Tx = \lambda\bar{\lambda}(\bar{x}^Tx)$

Since x is non-zero vector $\bar{x}^T$ is also non-zero vector.

$\therefore \bar{x}^Tx = \bar{x}_1x_1 + \dots + \bar{x}_nx_n$

$= |x_1|^2 + \dots + |x_n|^2$

$\bar{x}^Tx \neq 0$

$\therefore \quad \cancel{\bar{x}^Tx} = \lambda\bar{\lambda}(\cancel{\bar{x}^Tx})$

$\lambda\bar{\lambda} = 1 \qquad \because z\bar{z} = |z|^2$

$\Rightarrow |\lambda|^2 = 1$

$\Rightarrow |\lambda| = 1$

Hence proved.

Pro-11:

Zero is an eigen value of A iff A is a singular Matrix.

Proof:

The eigen values of A are the roots of the characteristic eqn $|A - \lambda I| = 0$.

Now, 0 is an eigen value of $A \iff |A - 0I| = 0$

$\iff |A| = 0$

$\iff A$ is a singular Matrix.

Hence proved.

Pro-12:

If A & B are two square matrices of the same order then AB & BA have the same eigen value.

Proof:

T.P.T AB & BA have the same eigen values.

Let λ be an eigen value of AB

i.e.) T.P.T: λ is also eigen value of BA.

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$\Rightarrow \lambda$ be an eigen value of AB & x be an eigen vector corresponding to λ.

$\therefore (AB)X = \lambda X$.

$\times B \Rightarrow \therefore B(AB)X = B(\lambda X). = \lambda(BX)$.

$(BA)(BX) = \lambda(BX)$.

$\therefore (BA)Y = \lambda Y$ where $Y = BX$.

Hence λ is an eigen value of BA.

Also BX is the corresponding eigen vector.

Hence proved.

— x —

Pro-13

If P & A are $n\times n$ matrices & P is a non singular matrix then A & $P^{-1}AP$ have the same eigen values.

Proof:

T.P.T: A & $P^{-1}AP$ have the same eigen values. Let $B = P^{-1}AP$

ie) **T.P.T**: A & B have the same eigen values.

ie) **T.P.T**: The characteristic polynomial of A and B are the same.

Now $|B - \lambda I| = |P^{-1}AP - \lambda I|$

$= |P^{-1}AP - P^{-1}(\lambda I)P|$

$= |P^{-1}(A - \lambda I)P|$

$= |P^{-1}|\,|A - \lambda I|\,|P|$. $\because P^{-1}P = I$

$= |P^{-1}|\,|P|\,|A - \lambda I|$

$= |P^{-1}P|\,|A - \lambda I|$

$= |I|\,|A - \lambda I|$

$\Rightarrow \lambda = |I|\,|A-\lambda I|$

$|B-\lambda I| = |A-\lambda I|.$

$\therefore$ The characteristic eqns. of A & $P^{-1}AP$ are the ~~ch~~ same.

$\therefore$ A & $P^{-1}AP$ have the same eigen values. Hence proved.

— x —

Pro-14

If λ is a characteristic root of A then $f(\lambda)$ is a characteristic root of the matrix $f(A)$. where $f(x)$ is any polynomial

proof:

T.P.T : $f(\lambda)$ is a characteristic root of $f(A)$.

Let $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$

where $a_0 \neq 0$ & $a_1, a_2, \ldots a_n$ are all real numbers.

$x = A$

$\therefore f(A) = a_0A^n + a_1A^{n-1} + \cdots + a_{n-1}A + a_nI$

Since λ is a characteristic root of A, λ^n is a characteristic root of A^n, 'n' is +ve integer (by pro-6).

$\therefore AX = \lambda X$ ~~$\Rightarrow$ x ly $a_0 \Rightarrow a_0$~~

$\therefore A^nX = \lambda^nX \Rightarrow$ x ly $a_0 \Rightarrow a_0A^nX = a_0\lambda^nX$

$+ \qquad +$

$A^{n-1}X = \lambda^{n-1}X \Rightarrow$ x ly $a_1 \Rightarrow a_1A^{n-1}X = a_1\lambda^{n-1}X$

$\vdots \qquad + \qquad +$

$AX = \lambda X \Rightarrow$ x ly $a_{n-1} \Rightarrow a_{n-1}AX = a_{n-1}\lambda X$

Adding the above eqns, we have

$a_0A^n X + a_1A^{n-1}X + \cdots + a_{n-1}AX = a_0\lambda^n X + a_1\lambda^{n-1}X + \cdots + a_{n-1}\lambda X.$

$(a_0A^n + a_1A^{n-1} + \cdots + a_{n-1}A)X = (a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda)X$

Adding a_nX on both sides.

$(\underbrace{a_0A^n + a_1A^{n-1} + \cdots + a_{n-1}A + a_nI}) X = (\underbrace{a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n}) X$

$$\boxed{f(A)\,X = f(\lambda)\,X}$$

Hence $f(\lambda)$ is a characteristic root of $f(A)$.

Hence proved.

Problems based on eigen values & eigen vectors

Problem-12

Find the eigen values & eigen vectors of the matrix.

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$$

Soln To find : (i) Eigen value
(ii) Eigen vector.

(i) The characteristic eqn of A is

$$|A - \lambda I| = 0.$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 1 & 3 \\ 1 & 5-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{vmatrix} = 0.$$

$$(1-\lambda)\left[(5-\lambda)(1-\lambda)-1\right] - 1\left[(1-\lambda)-3\right] + 1\left[1-3(5-\lambda)\right] = 0.$$

$$(1-\lambda)\left[5-5\lambda-\lambda+\lambda^2-1\right] - 1\left[-\lambda-2\right] + 3\left[1-15+3\lambda\right] = 0$$

$$(1-\lambda)\left[\lambda^2-6\lambda+4\right] + (\lambda+2) + 3(3\lambda-14) = 0$$

$$\lambda^2-6\lambda+4-\lambda^3+6\lambda^2-4\lambda+\lambda+2+9\lambda-42 = 0$$

$$-\lambda^3+7\lambda^2+0\lambda-36 = 0$$

$$x- \Rightarrow \boxed{\lambda^3-7\lambda^2+36=0}$$

which is characteristic eqn.

To find: characteristic root or eigen value

$$\begin{array}{r|rrrr} -2 & 1 & -7 & 0 & 36 \\ & 0 & -2 & 18 & -36 \\ \hline 3 & 1\lambda^2 & -9\lambda & 18_c & \boxed{0} \\ & 0 & 3 & -18 & \\ \hline 6 & 1 & -6 & \boxed{0} & \\ & 0 & 6 & & \\ \hline & 1 & \boxed{0} & & \end{array}$$

$18 = -6 \times -3$, $-9 = -6 + -3$

(or) $(\lambda+2)(\lambda^2-9\lambda+18)=0$

$\lambda+2=0$ | $\lambda^2-9\lambda+18=0$

$\boxed{\lambda=-2}$ | $(\lambda-6)(\lambda-3)=0$

$\lambda = 6, 3$

$$\boxed{\lambda = -2, 3, 6}$$

(ii) To find: Eigen vector.

case(i) If $\lambda=-2$ Then To find x.

let $x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ be an eigen vector corresponding to $\lambda=-2$.

Hence $AX = -2X$.

ie) $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = -2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -2x_1 \\ -2x_2 \\ -2x_3 \end{pmatrix}$

$\therefore x_1 + x_2 + 3x_3 = -2x_1 \Rightarrow 3x_1 + x_2 + 3x_3 = 0 \rightarrow (1)$

$x_1 + 5x_2 + x_3 = -2x_2 \Rightarrow x_1 + 7x_2 + x_3 = 0 \rightarrow (2)$

$3x_1 + x_2 + x_3 = -2x_3 \Rightarrow 3x_1 + x_2 + 3x_3 = 0 \rightarrow (3)$.

$(1) \neq (2) \neq (3)$

(1) & (2) using Cross Multiplication.

$3x_1 + x_2 + 3x_3 = 0$ &

$x_1 + 7x_2 + x_3 = 0$

$$\frac{x_1}{\begin{matrix} 1 & 3 \\ 7 & 1 \end{matrix}} \quad \frac{x_2}{\begin{matrix} 3 & 3 \\ 1 & 1 \end{matrix}} \quad \frac{x_3}{\begin{matrix} 3 & 1 \\ 1 & 7 \end{matrix}} \Rightarrow \frac{x_1}{1-3\times 7} = \frac{x_2}{3-3} = \frac{x_3}{21-1}$$

$$\Rightarrow \frac{x_1}{1-21} = \frac{x_2}{0} = \frac{x_3}{20}$$

$$\frac{x_1}{-20} = \frac{x_2}{0} = \frac{x_3}{20} \Rightarrow x_1 = -20;\ x_2 = 0$$

$x_3 = 20$.

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -20 \\ 0 \\ 20 \end{pmatrix}$$

$\div 20$ $\quad \div 10$

$$\Rightarrow \boxed{X = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \text{ (or) } \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}} \text{ (or) } X = (-2\ 0\ 2)^T$$

<u>Case (iii)</u> Eigen vector corresponding to $\lambda = 3$

Then $AX = \overset{\lambda X}{3}X$.

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3x_1 \\ 3x_2 \\ 3x_3 \end{pmatrix}$$

$x_1 + x_2 + 3x_3 = 3x_1 \Rightarrow -2x_1 + x_2 + 3x_3 = 0 \rightarrow (1)$

$x_1 + 5x_2 + x_3 = 3x_2 \Rightarrow x_1 + 2x_2 + x_3 = 0 \rightarrow (2)$

$3x_1 + x_2 + x_3 = 3x_3 \Rightarrow 3x_1 + x_2 - 2x_3 = 0 \rightarrow (3)$

$(1) \neq (2) \neq (3)$.

Take (1) & (2) using cross Multiplicati

$-2x_1 + x_2 + 3x_3 = 0$

$x_1 + 2x_2 + x_3 = 0$

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ -2 & 1 & 3 & -2 & 1 \\ 1 & 2 & 1 & 1 & 2 \end{array}$$

$$\frac{x_1}{1-6} = \frac{x_2}{3+2} = \frac{x_3}{-4-1} \Rightarrow \frac{x_1}{-5} = \frac{x_2}{5} = \frac{x_3}{-5}$$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -5 \\ 5 \\ -5 \end{pmatrix} \Rightarrow \boxed{X = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}} \text{ (or)}$$

$X = (-1 \quad 1 \quad -1)^T$

Case (iii)

Eigen vector corresponding to $\lambda = 6$.

$AX = 6X$

$$\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6x_1 \\ 6x_2 \\ 6x_3 \end{pmatrix}$$

$x_1 + x_2 + 3x_3 = 6x_1 \Rightarrow -5x_1 + x_2 + 3x_3 = 0 \rightarrow (1)$

$x_1 + 5x_2 + x_3 = 6x_2 \Rightarrow x_1 - x_2 + x_3 = 0 \rightarrow (2)$

$3x_1 + x_2 + x_3 = 6x_3 \Rightarrow 3x_1 + x_2 - 5x_3 = 0 \rightarrow (3)$

$(1) \neq (2) \neq (3)$

$\therefore$ Take (1) & (2)

$-5x_1 + x_2 + 3x_3 = 0$

$x_1 - x_2 + x_3 = 0$

$$\begin{array}{cccccc} & x_1 & & x_2 & & x_3 \\ -5 & 1 & 3 & -5 & 1 \\ 1 & -1 & 1 & 1 & -1 \end{array}$$

$$\frac{x_1}{1+3} = \frac{x_2}{3+5} = \frac{x_3}{5-1} \Rightarrow \frac{x_1}{4} = \frac{x_2}{8} = \frac{x_3}{4}$$

$\div 4$

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 8 \\ 4 \end{pmatrix} \Rightarrow \boxed{X = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}} \text{ (or)}$$

$$X = (1\ 2\ 1)^T$$

Result

Characteristic eqn	Eigen Value	Eigen vector
$\lambda^3 - 7\lambda^2 + 36 = 0$	$\lambda = -2$	$X = (-2\ 0\ 2)^T = \begin{pmatrix} -2 \\ 0 \\ 2 \end{pmatrix}$
	$\lambda = 3$	$X = (-1\ 1\ -1)^T = \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$
	$\lambda = 6$	$X = (1\ 2\ 1)^T = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$

Hw

Pro – 13, 14

Student work

Pro - 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11 (using Property)