Linear Algebra

Unit - I

Two Marks:

1 Vector space :

A non-empty set V is said to be a vector space over a field F if

* V is an abelian group under an operation called addition which we denoted by +.

* For every $\alpha \in F$ and $v \in V$. there is defined an element αv in V subject to the following conditions.

- $\alpha(u+v) = \alpha u + \alpha v \quad \forall\, u, v \in V$ and $\alpha \in F$
- $(\alpha+\beta)u = \alpha u + \beta u \quad \forall\, u \in V$ and $\alpha, \beta \in F$
- $\alpha(\beta u) = (\alpha\beta)u \quad \forall\, u \in V$ and $\alpha, \beta \in F$
- $1u = u \quad \forall\, u \in V$

2 Subspace :

Let V be a vector space over F. A non-empty subset W of V is a subspace of V iff W is closed with respect to vector addion and scalar multiplication over F under the operations of V.

3 Direct sum:

Let A and B be subspaces of a vector space V. Then V is called the direct sum of A and B if

(i) $A+B=V$ (ii) $A \cap B = \{0\}$

If V is the direct sum of A and B, we write

$$V = A \oplus B$$

4. Linear Transformation:

Let V and W be vector spaces over a field F. A mapping $T: V \to W$ is called a homomorphism if

* $T(u+v) = T(u) + T(v)$
* $T(\alpha u) = \alpha T(u)$ where $\alpha \in F$ and $u, v \in V$

A homomorphism T of vector spaces is also called linear transformation.

5. Kernel:

Let V and W be vector spaces over a field F and $T: V \to W$ be a linear transformation. Then the kernel of T is defined to be

$\{v / v \in V$ and $T(v) = 0\}$ and is denoted by $\ker T$

6. Linear combination:

Let V be a vector space over a field F. Let $v_1, v_2, \ldots, v_n \in V$. Then an element of the form $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n$ where $\alpha_i \in F$ is called a linear combination of the vectors $v_1, v_2, \ldots, v_n$.

7. Linear span:

Let S be a non-empty subset of a vector space V. Then the set of all linear combinations of finite sets of elements of S is called the linear span of S and is denoted by $L(S)$.

8. $\{0\}$ and V are subspaces of any vector space V.

$\{0\}$ which contains only the zero vector. Both vector addition and scalar multiplication are trivial. They are called trivial subspaces of V.

9. $W = \{(a,0,0) / a \in R\}$ is a subspace of R^3.

T.P.T : $\alpha u + \beta v \in W$

$\alpha, \beta \in R$; $u = (a,0,0)$, $v = (b,0,0) \in W$

$$\alpha u + \beta v = \alpha(a,0,0) + \beta(b,0,0)$$
$$= (\alpha a, 0, 0) + (\beta b, 0, 0)$$
$$= (\alpha a + \beta b, 0, 0) \in W$$

$\therefore$ W is subspace.

10. $W = \{(ka, kb, kc) / k \in R\}$ is a subspace of R^3.

T.P.T : $\alpha u + \beta v \in W$

$u = (k_1 a, k_1 b, k_1 c)$, $v = (k_2 a, k_2 b, k_2 c) \in W$

$$\alpha u + \beta v = \alpha(k_1 a, k_1 b, k_1 c) + \beta(k_2 a, k_2 b, k_2 c)$$
$$= (\alpha k_1 a, \alpha k_1 b, \alpha k_1 c) + (\beta k_2 a, \beta k_2 b, \beta k_2 c)$$
$$= (\alpha k_1 + \beta k_2)a, (\alpha k_1 + \beta k_2)b, (\alpha k_1 + \beta k_2)c \in W$$

$\therefore$ W is subspace.

11. $W = \{(a,b,0) / a,b \in R\}$ is a subspace of R^3.

T.P.T : $\alpha u + \beta v \in W$

$\alpha, \beta \in R$; $u = (a_1, b_1, 0)$, $v = (a_2, b_2, 0) \in W$

$$\alpha u + \beta v = \alpha(a_1, b_1, 0) + \beta(a_2, b_2, 0)$$
$$= (\alpha a_1, \alpha b_1, 0) + (\beta a_2, \beta b_2, 0)$$
$$= (\alpha a_1 + \beta a_2, \alpha b_1 + \beta b_2, 0) \in W$$

$\therefore$ W is a subspace.

12. Let W be set of all points in R^3 satisfying the equation $lx + my + nz = 0$. W is a subspace of R^3.

T.P.T : $\alpha u + \beta v \in W$

$\alpha, \beta \in R$; $u = (a_1, b_1, c_1)$, $v = (a_2, b_2, c_2) \in W$

$$la_1 + mb_1 + nc_1 = 0$$
$$la_2 + mb_2 + nc_2 = 0$$

$$\alpha u + \beta v \Rightarrow \alpha(la_1 + mb_1 + nc_1) + \beta(la_2 + mb_2 + nc_2) = 0$$

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$\Rightarrow (\alpha l a_1 + \beta l a_2) + (\alpha m b_1 + \beta m b_2) + (\alpha n c_1 + \beta n c_2) = 0$

$\Rightarrow l(\alpha a_1 + \beta a_2) + m(\alpha b_1 + \beta b_2) + n(\alpha c_1 + \beta c_2) = 0$

$\therefore$ W is a subspace

13 $W = \{ f / f \in F(x)$ and $f(a) = 0 \}$

W is a set of all polynomials in F(x).

F(x) having 'a' as a root , $a \in F$

W is a subspace over F.

$x - a \in W$, W is non-empty.

Let $f, g \in F(x)$ and $\alpha, \beta \in F$

T.P.T : a is root of $\alpha f + \beta g \in W$

$(\alpha f + \beta g)(a) = \alpha f(a) + \beta(g(a))$

$= \alpha 0 + \beta 0$ $\qquad \because \alpha 0 = 0$

$= 0$

$\therefore$ 'a' is a root of $\alpha f + \beta g$

$\alpha f + \beta g \in W$

$\therefore$ W is subspace.

14 $W = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} / a, b \in R \right\}$ is a subspace of $M_2(R)$

T.P.T : $\alpha u + \beta v \in W$

$\alpha, \beta \in R$; $u = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $v = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} \in W$

$\alpha u + \beta v = \alpha \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \beta \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$

$= \begin{bmatrix} \alpha a & 0 \\ 0 & \alpha b \end{bmatrix} + \begin{bmatrix} \beta c & 0 \\ 0 & \beta d \end{bmatrix}$

$= \begin{bmatrix} \alpha a + \beta c & 0 \\ 0 & \alpha b + \beta d \end{bmatrix} \in W$

$\therefore$ W is a subspace of $M_2(R)$.

15. Let V be a vector space over a field F and W a subspace of V. Then $T: V \to V/W$ defined by $T(v) = W + v$ is a linear transformation.

a) $T(u+v) = T(u) + T(v)$

$$T(v_1 + v_2) = W + (v_1 + v_2)$$
$$= (W + v_1) + (W + v_2)$$
$$= T(v_1) + T(v_2)$$

b) $T(\alpha u) = \alpha T(u)$

$$T(\alpha v_1) = W + \alpha v_1$$
$$= \alpha(W + v_1)$$
$$= \alpha T(v_1)$$

16. $T: V_3(R) \to V_3(R)$ defined by $T(a,b,c) = (a,0,0)$ is a linear transformation

a) $T(u+v) = T(u) + T(v)$

$$T((a_1,b_1,c_1) + (a_2,b_2,c_2)) = T(a_1+a_2, b_1+b_2, c_1+c_2)$$
$$= (a_1+a_2, 0, 0)$$
$$= (a_1,0,0) + (a_2,0,0)$$
$$= T(a_1,b_1,c_1) + T(a_2,b_2,c_2)$$
$$= T(u) + T(v)$$

b) $T(\alpha u) = \alpha T(u)$

$$T(\alpha(a,b,c)) = T(\alpha a, \alpha b, \alpha c)$$
$$= (\alpha a, 0, 0)$$
$$= \alpha(a,0,0)$$
$$= \alpha T(a,b,c)$$
$$= \alpha T(u)$$

17. Let V be the set of all polynomials of degree $\leq n$ in $R(x)$ including zero polynomial. $T: V \to V$ defined by $T(f) = \frac{df}{dx}$ is a linear transformation.

a) $T(u+v) = T(u) + T(v)$

$$T(f+g) = \frac{d(f+g)}{dx} = \frac{df}{dx} + \frac{dg}{dx}$$
$$= T(f) + T(g)$$

b) $T(\alpha f) = \alpha T(f)$

$$T(\alpha f) = \frac{d(\alpha f)}{dx} = \alpha \frac{df}{dx} = \alpha T(f)$$

6. Let $T: V \to V_{n+1}(R)$ is defined by $T(a_0 + a_1x + \dots + a_nx^n) = (a_0, a_1, \dots, a_n)$ is a linear transformation.

$$f = (a_0 + a_1x + \dots + a_nx^n), \quad g = (b_0 + b_1x + \dots + b_nx^n)$$

a) $T(f+g) = T(f) + T(g)$

$$f+g = [(a_0+b_0) + (a_1+b_1)x + \dots + (a_n+b_n)x^n]$$

$$T(f+g) = [(a_0+b_0), (a_1+b_1), \dots (a_n+b_n)]$$

$$= (a_0, a_1, \dots, a_n) + (b_0, b_1, \dots, b_n)$$

$$= T(a_0 + a_1x \dots a_nx^n) + T(b_0 + b_1x + \dots + b_nx^n)$$

$$= T(f) + T(g)$$

b) $T(\alpha f) = \alpha T(f)$

$$T[\alpha(a_0 + a_1x + \dots + a_nx^n)] = T[\alpha a_0 + \alpha a_1x + \dots + \alpha a_nx^n]$$

$$= \alpha a_0, \alpha a_1, \dots \alpha a_n$$

$$= \alpha T(a_0 + a_1x + \dots + a_nx^n)$$

$$= \alpha T(f)$$

7. Let V denote the set of all sequences in R. $T: V \to V$ defined by $T(a_1, a_2, \dots a_n, \dots) = (0, a_1, a_2, \dots, a_n, \dots)$ is a linear transformation.

a) $T(u+v) = T(u) + T(v)$

$$T(u+v) = T[(a_1, a_2, \dots, a_n \dots) + (b_1, b_2, \dots b_n, \dots)]$$

$$= T[(a_1+b_1), (a_2+b_2) \dots (a_n+b_n) \dots]$$

$$= [0, (a_1+b_1), (a_2+b_2) \dots (a_n+b_n) \dots]$$

$$= (0, a_1, a_2 \dots a_n \dots) + (0, b_1, b_2 \dots b_n)$$

$$= T(u) + T(v)$$

b) $T(\alpha u) = \alpha T(u)$

$$T(\alpha u) = T(\alpha(a_1, a_2 \dots a_n \dots))$$

$= T[\alpha a_1, \alpha a_2, \cdots, \alpha a_n \cdots]$

$= [0, \alpha a_1, \alpha a_2 \cdots \alpha a_n \cdots]$

$= \alpha(0, a_1, a_2 \cdots a_n \cdots)$

$= \alpha T(u)$

Five Marks:

1. $R \times R$ is a vector space over R under addition and scalar multiplication defined by $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and $\alpha(x_1, x_2) = (\alpha x_1, \alpha x_2)$

a) Closure:

If $(x_1, x_2), (y_1, y_2) \in R \times R$

$(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2) \in R \times R$

b) Associative:

If $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in R \times R$

$[(x_1, x_2) + (y_1, y_2)] + (z_1, z_2) = (x_1, x_2) + [(y_1, y_2) + (z_1, z_2)]$

$(x_1 + y_1, x_2 + y_2) + (z_1, z_2) = (x_1, x_2) + (y_1 + z_1, y_2 + z_2)$

$(x_1 + y_1 + z_1, x_2 + y_2 + z_2) = (x_1 + y_1 + z_1, x_2 + y_2 + z_2)$

c) Identity:

If $(x_1, x_2) \in R \times R \ \exists\ (0,0) \in R \times R$

$(x_1, x_2) + (0,0) = (0,0) + (x_1, x_2)$

$(x_1 + 0, x_2 + 0) = (0 + x_1, 0 + x_2)$

$(x_1, x_2) = (x_1, x_2)$

d) Inverse:

If $(x_1, x_2) \in R \times R \ \exists\ (-x_1, -x_2) \in R \times R$

$(x_1, x_2) + (-x_1, -x_2) = (-x_1, -x_2) + (x_1, x_2)$

$(x_1 - x_2, x_2 - x_2) = (-x_1 + x_1, -x_2 + x_2)$

$(0,0) = (0,0)$

e) Commutative:

If $(x_1, x_2), (y_1, y_2) \in R \times R$

$$(x_1, x_2) + (y_1, y_2) = (y_1, y_2) + (x_1, x_2)$$

$$(x_1 + y_1, x_2 + y_2) = (x_1 + y_1, x_2 + y_2)$$

$\therefore (R \times R, +)$ is an abelian.

(i) $\alpha(u+v) = \alpha u + \alpha v$

$u = (x_1, x_2), v = (y_1, y_2)$; $\alpha \in R$

$$\alpha[(x_1, x_2) + (y_1, y_2)] = \alpha[x_1 + y_1, x_2 + y_2]$$
$$= \alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2$$
$$= (\alpha x_1, \alpha x_2) + (\alpha y_1, \alpha y_2)$$
$$= \alpha(x_1, x_2) + \alpha(y_1, y_2)$$

(ii) $u(\alpha + \beta) = u\alpha + u\beta$

$u = (x_1, x_2)$; $\alpha, \beta \in R$

$$(x_1, x_2)(\alpha + \beta) = [(\alpha + \beta)x_1, (\alpha + \beta)x_2]$$
$$= [\alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2]$$
$$= (\alpha x_1, \alpha x_2) + (\beta x_1, \beta x_2)$$
$$= \alpha(x_1, x_2) + \beta(x_1, x_2)$$

(iii) $\alpha(\beta u) = \alpha\beta(u)$

$u = (x_1, x_2)$

$$\alpha(\beta(x_1, x_2)) = (\alpha\beta x_1, \alpha\beta x_2)$$
$$= \alpha\beta(x_1, x_2)$$

(iv) $1.u = u$

$u = (x_1, x_2)$

$$1.(x_1, x_2) = (x_1, x_2)$$

$\therefore R \times R$ is a Vector space.

Hence Proved.

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2. $R^n = \{(x_1, x_2, \ldots, x_n) / x_i \in R, 1 \leq i \leq n\}$. Then R^n is a vector space over R under addition and scalar multiplication defined by

$$(x_1, x_2, \ldots x_n) + (y_1, y_2, \ldots, y_n) = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n) \text{ and}$$

$$\alpha(x_1, x_2, \ldots, x_n) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$$

a) Closure:

If $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in R^n$

$$(x_1, x_2, \ldots x_n) + (y_1, y_2, \ldots, y_n) = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$$

b) Associative:

If $(x_1, x_2 \ldots x_n), (y_1, y_2, \ldots, y_n), (z_1, z_2, \ldots, z_n) \in R^n$

$$[(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n)] + (z_1, z_2, \ldots, z_n)$$
$$= (x_1+y_1, x_2+y_2, \ldots x_n+y_n) + (z_1, z_2, \ldots, z_n)$$
$$= (x_1+y_1+z_1, x_2+y_2+z_2, \ldots, x_n+y_n+z_n) \longrightarrow ①$$

$$(x_1, x_2, \ldots, x_n) + [(y_1, y_2, \ldots, y_n) + (z_1, z_2, \ldots, z_n)]$$
$$= (x_1, x_2, \ldots x_n) + (y_1+z_1, y_2+z_2, \ldots, y_n+z_n)$$
$$= (x_1+y_1+z_1, x_2+y_2+z_2, \ldots x_n+y_n+z_n) \longrightarrow ②$$

① = ②

c) Identity:

If $(x_1, x_2, \ldots, x_n) \in R^n \ \exists\ (0, 0, \ldots 0) \in R^n$

$$(x_1, x_2, \ldots, x_n) + (0, 0, \ldots, 0) = (0, 0, \ldots, 0) + (x_1, x_2, \ldots, x_n)$$
$$(x_1+0, x_2+0, \ldots, x_n+0) = (0+x_1, 0+x_2 \ldots 0+x_n)$$
$$(x_1, x_2, \ldots x_n) = (x_1, x_2, \ldots, x_n)$$

d) Inverse:

If $(x_1, x_2, \ldots, x_n) \in R^n \ \exists\ (-x_1, -x_2, \ldots, -x_n) \in R^n$

$$(x_1, x_2, \ldots, x_n) + (-x_1, -x_2 \ldots, -x_n) = (-x_1, -x_2, \ldots, -x_n) + (x_1, x_2 \ldots x_n)$$
$$(x_1-x_1, x_2-x_2, \ldots, x_n-x_n) = (-x_1+x_1, -x_2+x_2, \ldots, -x_n+x_n)$$
$$(0, 0, \ldots, 0) = (0, 0, \ldots, 0)$$

e) Commutative:

If $(x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in R^n$

$$(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) = (y_1, y_2, \cdots y_n) + (x_1, x_2 \cdots, x_n)$$

$$(x_1+y_1, x_2+y_2, \cdots, x_n+y_n) = (x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$$

$\therefore (R^n, +)$ is an abelian.

(i) $\alpha(u+v) = \alpha u + \alpha v$

$u = (x_1, x_2, \cdots, x_n)$, $v = (y_1, y_2, \cdots, y_n)$; $\alpha \in R^n$

$$\alpha[(x_1, x_2, \cdots, x_n) + (y_1, y_2, \cdots, y_n)] = \alpha[x_1+y_1, x_2+y_2, \ldots, x_n+y_n]$$

$$= \alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2 \cdots \alpha x_n + \alpha y_n$$

$$= (\alpha x_1, \alpha x_2, \cdots \alpha x_n) + (\alpha y_1, \alpha y_2, \cdots \alpha y_n)$$

$$= \alpha(x_1, x_2, \cdots x_n) + \alpha(y_1, y_2, \cdots y_n)$$

(ii) $u(\alpha+\beta) = u\alpha + u\beta$

$u = (x_1, x_2, \cdots x_n)$; $\alpha, \beta \in R^n$

$$(x_1, x_2, \cdots, x_n)(\alpha+\beta) = (\alpha+\beta)x_1, (\alpha+\beta)x_2, \cdots, (\alpha+\beta)x_n$$

$$= \alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2, \cdots, \alpha x_n + \beta x_n$$

$$= \alpha(x_1, x_2, \cdots, x_n) + \beta(x_1, x_2, \cdots, x_n)$$

(iii) $\alpha(\beta u) = (\alpha\beta)u$

$u = (x_1, x_2, \cdots, x_n)$; $\alpha, \beta \in R^n$

$$\alpha(\beta(x_1, x_2, \cdots, x_n)) = \alpha\beta x_1, \alpha\beta x_2, \cdots, \alpha\beta x_n$$

$$= \alpha\beta(x_1, x_2, \cdots, x_n)$$

(iv) $1 \cdot u = u$

$u = (x_1, x_2, \cdots, x_n)$

$$1 \cdot (x_1, x_2, \cdots, x_n) = (x_1, x_2, \cdots, x_n)$$

$\therefore R^n$ is a Vector Space.

Hence Proved.

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3 Let R^+ be the set of all +ve real numbers. Define addition and scalar multiplication as follows $u+v = uv$ for all $u,v \in R^+$; $\alpha u = u^\alpha$ for all $u \in R^+$ and $\alpha \in R$. Then R^+ is a real vector space.

a) Closure:

If $u, v \in R^+$

$$u+v = uv \in R^+$$

b) Associative:

If $u, v, w \in R^+$

$$u+(v+w) = (u+v)+w$$

$$u+vw = uv+w$$

$$uvw = uvw$$

c) Identity:

If $u \in R^+$ $\exists$ $e \in R^+$

$$u+e = u$$

$$ue = u$$

$$e = 1$$

d) Inverse:

If $u \in R^+$ $\exists$ $u' \in R^+$

$$u+u' = e$$

$$uu' = 1$$

$$u' = 1/u$$

e) Commutative:

If $u, v \in R^+$

$$u+v = v+u$$

$$uv = uv$$

$\therefore$ $(R^+, +)$ is an abelian.

i) $\alpha(u+v) = \alpha u + \alpha v$

$$\alpha(u+v) = \alpha(uv)$$

$$= (uv)^\alpha$$

$$\alpha(u+v) = u^\alpha v^\alpha \rightarrow ①$$

$$\alpha u + \alpha v = \alpha u . \alpha v$$

$$= u^\alpha v^\alpha \rightarrow ②$$

$$① = ②$$

(ii) $(\alpha+\beta)u = \alpha u + \beta u$

$$(\alpha+\beta)u = u^{\alpha+\beta}$$
$$= u^{\alpha}u^{\beta} \rightarrow ①$$

$$\alpha u + \beta u = \alpha u \, \beta u$$
$$= u^{\alpha} u^{\beta} \rightarrow ②$$

$$① = ②$$

(iii) $\alpha(\beta u) = \alpha\beta(u)$

$$\alpha(\beta u) = \alpha(u^{\beta})$$
$$= u^{\alpha\beta} \rightarrow ①$$
$$\alpha\beta(u) = u^{\alpha\beta} \rightarrow ②$$
$$① = ②$$

(iv) $1.u = u$

$$1.u = u^{1}$$
$$= u$$

$\therefore R^{+}$ is a Vector space

Hence Proved.

4. Let V be a vector space over a field F. Then

(i) $\alpha 0 = 0$ for all $\alpha \in F$

$$\alpha 0 = \alpha(0+0)$$
$$= \alpha 0 + \alpha 0$$
$$\alpha 0 - \alpha 0 = \alpha 0$$
$$\boxed{\alpha 0 = 0}$$

(ii) $0v = 0$ for all $v \in V$

$$0v = (0+0)v$$
$$= 0v + 0v$$
$$0v - 0v = 0v$$
$$\boxed{0v = 0}$$

(iii) $(-\alpha)v = \alpha(-v) = -(\alpha v)$ for all $\alpha \in F$ and $v \in V$

T.P.T : $(-\alpha)v = -(\alpha v)$

Wkt, $0v = 0$

$[\alpha + (-\alpha)]v = 0$

$\alpha v + (-\alpha)v = 0$

$(-\alpha)v = -(\alpha v) \rightarrow (1)$

T.P.T : $\alpha(-v) = -(\alpha v)$

Wkt, $\alpha 0 = 0$

$[v + (-v)]\alpha = 0$

$\alpha v + \alpha(-v) = 0$

$\alpha(-v) = -(\alpha v) \rightarrow (2)$

from (1) and (2)

$(-\alpha)v = \alpha(-v) = -(\alpha v)$

(iv) $\alpha v = 0 \Rightarrow \alpha = 0$ or $v = 0$

Let $\alpha v = 0$. If $\alpha = 0$ there is nothing to prove

Let $\alpha \neq 0$, $v = 0$, then $\alpha^{-1} \in F$

$v = 1.v = (\alpha^{-1}\alpha).v = \alpha^{-1}(\alpha v) = \alpha^{-1}(0) = 0$

5. Let V be a vector space over F. A non-empty subset W of V is a subspace of V iff W is closed with respect to addition and scalar multiplication in V.

W is a subspace of V.

T.P.T : W is closed w.r. to vector addition & scalar multiplication

W is vector space.

If $u, v \in W$

$u + v \in W$

If $u \in W$; $\alpha \in F$

$\alpha u \in W$

$\therefore$ W is closed w.r. to vector addition & scalar multiplication

Conversely, W is closed w.r.to vector addition and scalar Multiplication

T.P.T : W is subspace

(ie) W is a vector space.

Closure:

$$\text{If } u, v \in W \Rightarrow u+v \in W$$

Associative:

$$\text{If } u, v, w \in W \Rightarrow (u+v)+w = u+(v+w)$$

$$u+v+w = u+v+w$$

Identity:

$$\text{If } u \in W \Rightarrow 0+u = u+0 = u$$

$$u \cdot 0 = 0 \cdot u = 0 \in W$$

Inverse:

$$\text{If } v \in W \Rightarrow v+(-v) = (-v)+v = 0$$

$$v = (-1)v = -v \in W$$

Commutative:

$$\text{If } u, v \in W \Rightarrow u+v = v+u$$

Scalar Multiplication:

$$\text{If } u \in W \; ; \; \alpha \in F \Rightarrow \alpha u \in W$$

$\therefore$ W is a Vector Space.

6 Let V be a vector space over a field F. A non-empty subset W of V is a subspace of V iff $u, v \in W$ and $\alpha, \beta \in F \Rightarrow \alpha u + \beta v \in W$.

W is a subspace of V.

T.P.T : $\alpha u + \beta v \in W$

$$\text{If } \alpha, \beta \in F \; ; \; u, v \in W$$

$$\alpha u, \beta v \in W$$

$$\alpha u + \beta v \in W$$

Conversely, If $\alpha, \beta \in F \; : \; u, v \in W$

$$\alpha u + \beta v \in W \rightarrow (1)$$

T.P.T : W is a subspace.

If $\alpha=1$, $\beta=1$ Sub in ①

$\alpha u+\beta v = 1.u+1.v$

$= u+v \in W$

If $\beta=0$, $\alpha u+0 \in W$

$\alpha u \in W$

$\therefore$ W is closed

$\therefore$ W is Subspace.

7. P.T the intersection of two subspaces of a vector space is a subspace.

Let A and B be two subspaces of a vector space.

WCT, $A \cap B$ is a subspace of V.

Clearly, $0 \in A \cap B$ and hence $A \cap B$ is non-empty.

Let $u, v \in A \cap B$; $\alpha, \beta \in F$

$u, v \in A$ and $u, v \in B$

$\because$ A is subspace.

$\therefore \alpha u+\beta v \in A$.

$\because$ B is subspace

$\therefore \alpha u+\beta v \in B$ [$\because$ A and B are subspaces]

$\therefore \alpha u+\beta v \in A \cap B$

Hence $A \cap B$ is a subspace of V.

8. If A and B are subspaces of V prove that $A+B = \{v \in V / v = a+b, a \in A, b \in B\}$ is a subspace of V. Further show that A+B is the smallest subspace containing A and B.

T.P.T : A+B is a subspace.

Let $v_1, v_2 \in A+B$ and $\alpha \in F$

$v_1 = a_1+b_1$, $v_2 = a_2+b_2$ where $a_1, a_2 \in A$ and $b_1, b_2 \in B$

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$v_1+v_2 = (a_1+b_1)+(a_2+b_2)$

$= (a_1+a_2)+(b_1+b_2) \in A+B$

$\alpha(a_1+b_1) = \alpha a_1+\alpha b_1 \in A+B$

$\therefore A+B$ is a subspace of V. Clearly $A \subseteq A+B$ & $B \subseteq A+B$

Let W be any subspace of V containing A and B.

We P.T $A+B \subseteq W$

Let $v \in A+B$. Then $v = a+b$ where $a \in A$ and $b \in B$

$\because A \subseteq W$, $a \in W$, Similarly $B \subseteq W$, $b \in W$.

$\therefore a+b = v \in W$

$\therefore A+B \subseteq W$ so that $A+B$ is the smallest subspace of V containing A and B.

9. Let A and B be subspaces of a vector space V. Then $A \cap B = \{0\}$ iff every vector $v \in A+B$ can be uniquely expressed in the form $v = a+b$ where $a \in A$ and $b \in B$.

Let $A \cap B = \{0\}$. Let $v \in A+B$

$v = a_1+b_1 = a_2+b_2$ where $a_1, a_2 \in A$ and $b_1, b_2 \in B$

Then $a_1-a_2 = b_2-b_1$

$a_1-a_2 \in A$, $b_2-b_1 \in B$

$a_1-a_2, b_2-b_1 \in A \cap B$

$\because A \cap B = \{0\}$, $a_1-a_2 = 0$; $b_2-b_1 = 0$

$a_1 = a_2$; $b_2 = b_1$

Hence v in the form of $a+b$, $a \in A$, $b \in B$ is unique.

Conversely, $A+B$ in the form of $a+b$; $a \in A$, $b \in B$

WCT, $A \cap B = \{0\}$

If $A \cap B \neq \{0\}$. Let $v \in A \cap B$ and $v \neq 0$

$$0 = v - v = 0 + 0$$

0 is in the form of $a+b$

10. Let $T: V \to W$ be a linear transformation. Then $T(V) = \{T(v) / v \in V\}$ is a subspace of W.

Let w_1 and $w_2 \in T(V)$ and $\alpha \in F$. Then there exist $v_1, v_2 \in V$ such that $T(v_1) = w_1$ and $T(v_2) = w_2$

$$w_1 + w_2 = T(v_1) + T(v_2)$$

$$w_1 + w_2 = T(v_1 + v_2) \in T(V)$$

$$\alpha w_1 = \alpha T(v_1)$$

$$= T\alpha(v_1) \in T(V)$$

$\therefore T(V)$ is a subspace of W.

Ten Marks:

1. Let V be a vector space over F and W be a subspace of V. Let $V/W = \{W + v / v \in V\}$. Then V/W is a vector space over F under the following operations.

(i) $(W+v_1)+(W+v_2) = W + v_1 + v_2$

(ii) $\alpha(W+v_1) = W + \alpha v_1$

a) Closure:

If $(W+v_1), (W+v_2) \in V/W$; $v_1, v_2 \in V$

$$(W+v_1)+(W+v_2) = W + v_1 + v_2$$

b) Associative:

If $(W+v_1), (W+v_2), (W+v_3) \in V/W$; $v_1, v_2, v_3 \in V$

$$[(W+v_1)+(W+v_2)] + (W+v_3) = (W+v_1) + [(W+v_2)+(W+v_3)]$$

$$(W+v_1+v_2) + W + v_3 = W + v_1 + (W + v_2 + v_3)$$

$$W + v_1 + v_2 + v_3 = W + v_1 + v_2 + v_3$$

c) Identity:

If $(W+v_1) \in V/W$ $\exists$ $(W+0) \in V/W$

$$(W+v_1)+(W+0) = (W+0)+(W+v_1)$$

$$W+v_1 = W+v_1$$

d) Inverse:

If $(W+v_1) \in V/W$ $\exists$ $(W-v_1) \in V/W$

$$(W+v_1)+(W-v_1) = (W-v_1)+(W+v_1)$$

$$W+v_1-v_1 = W-v_1+v_1$$

$$W+0 = W+0$$

e) Commutative:

If $(W+v_1), (W+v_2) \in V/W$

$$(W+v_1)+(W+v_2) = (W+v_2)+(W+v_1)$$

$$W+v_1+v_2 = W+v_1+v_2$$

$\therefore$ $(V/W, +)$ is an abelian.

(i) $\alpha(u+v) = \alpha u+\alpha v$

$$\alpha[(W+v_1)+(W+v_2)] = \alpha(W+v_1+v_2)$$

$$= W+\alpha v_1+\alpha v_2$$

$$= W+\alpha v_1+W+\alpha v_2$$

$$= \alpha(W+v_1)+\alpha(W+v_2)$$

(ii) $u(\alpha+\beta) = u\alpha+u\beta$

$$(W+v_1)(\alpha+\beta) = W+(\alpha+\beta)v_1$$

$$= W+\alpha v_1+\beta v_1$$

$$= W+\alpha v_1+W+\beta v_1$$

$$= \alpha(W+v_1)+\beta(W+v_1)$$

(iii) $\alpha(\beta u) = (\alpha\beta)u$

$$\alpha(\beta(W+v_1)) = \alpha(W+\beta v_1)$$

$$= (W+\alpha\beta v_1) = \alpha\beta(W+v_1)$$

(iv) $1.u = u$

$$1.(W+V_1) = W + V_1$$

$\therefore V/W$ is a vector space.

V/W is called Quotient space of V.

8. Let V and W be vector spaces over a field F and $T: V \to W$ be an epimorphism. (i) $\ker T = V_1$ is a subspace of V and (ii) $V/V_1 \cong W$.

(i) $V_1 = \ker T$

$= \{v \mid v \in V \text{ and } T(v) = 0\}$

Clearly, $T(0) = 0$. Hence $0 \in \ker T = V_1$

$\therefore V_1$ is non-empty subset of V.

Let $u, v \in \ker T$; $\alpha, \beta \in F$

$\therefore T(u) = 0$, $T(v) = 0$

$$\begin{aligned} T(\alpha u + \beta v) &= T(\alpha u) + T(\beta v) \\ &= \alpha T(u) + \beta T(v) \\ &= \alpha 0 + \beta 0 \\ &= 0 \end{aligned}$$

$\therefore \alpha u + \beta v \in \ker T$

$\therefore \ker T$ is a subspace of V.

(ii) $\phi : \frac{V}{V_1} \to W$ by

$$\phi(V_1 + v) = T(v)$$

ϕ is well defined. Let $V_1 + v = V_1 + w$

$\therefore v \in V_1 + w$

$v = V_1 + w$ where $v_1 \in V_1$

$\therefore T(v) = T(v_1 + w) = T(v_1) + T(w)$

$= 0 + T(w)$

$= T(w)$

$\therefore \phi(V_1 + v) = \phi(V_1 + w)$

ϕ is 1-1:

$\phi(V_1 + v) = \phi(V_1 + w)$

$T(v) = T(w)$

$T(v) - T(w) = 0$

$T(v) + T(-w) = 0$

$T(v - w) = 0$

$v - w \in \ker T = V_1$

$v \in V_1 + w$

$V_1 + v \in V_1 + w$

$V_1 + v = V_1 + w$

ϕ is onto:

Let $w \in W$. Since T is onto there exists $v \in V$ such that $T(v) = w$

$\therefore \phi(V_1 + v) = w$

$$\phi[(V_1 + v) + (V_1 + w)] = \phi[V_1 + (v + w)]$$
$$= T(v + w)$$
$$= T(v) + T(w)$$
$$= \phi(V_1 + v) + \phi(V_1 + w)$$

$$\phi[\alpha(V_1 + v)] = \phi(V_1 + \alpha v)$$
$$= T(\alpha v)$$
$$= \alpha T(v)$$
$$= \alpha T(V_1 + v)$$

$\therefore V/V_1 \cong W.$

3 Let V be a vector space over a field F. Let A and B be subspaces of V. Then $\frac{A+B}{A} \cong \frac{B}{A \cap B}$

Wkt, $A+B$ is a subspace of V containing A. Hence $\frac{A+B}{A}$ is also vector space over F.

An element of $\frac{A+B}{A}$ is of the form $A+(a+b)$ where $a \in A$ and $b \in B$, but $A+a = A$.

Hence an element of $\frac{A+B}{A}$ is of the form $A+b$

Now, consider $f : B \to \frac{A+B}{A}$ defined by $f(b) = A+b$

Clearly, f is onto.

$$f(b) = f(a)$$
$$A+b = A+a$$
$$b = a$$
$$\therefore f \text{ is } 1\text{-}1.$$

Linear Transformation:

(i) $f(b_1+b_2) = A+(b_1+b_2) = (A+b_1)+(A+b_2)$
$= f(b_1) + f(b_2)$

(ii) $f(\alpha b_1) = A+\alpha b_1 = \alpha(A+b_1) = \alpha f(b_1)$

Hence f is a linear transformation.

Let K be the kernel of f.

Then $K = \{ b / b \in B, A+b = A \}$

Now, $A+b = A$ iff $b \in A$, then $K = A \cap B$

Hence Proved.

4. Let V and W be vector spaces over a field F. Let $L(V,W)$. represent the set of all linear transformations from V to W. Then $L(V,W)$ itself is a vector space over F under addition and scalar multiplication defined by $(f+g)(v) = f(v) + g(v)$ and $(\alpha f)(v) = \alpha f(v)$

a) Closure:

Let $f, g \in L(V,W)$ and $v_1, v_2 \in V$

$$(f+g)(v_1+v_2) = f(v_1+v_2) + g(v_1+v_2)$$
$$= f(v_1) + f(v_2) + g(v_1) + g(v_2)$$
$$= (f+g)(v_1) + (f+g)v_2$$

$$f+g \in L(V,W)$$

b) Associative:

Let $f, g, h \in L(V,W)$ and $v_1, v_2, v_3 \in V$

$$((f+g)+h)((v_1+v_2)+v_3) = (f+g)(v_1+v_2) + h(v_1+v_2) + (f+g)v_3 + h(v_3)$$
$$= f(v_1) + g(v_2) + h(v_1) + h(v_2) + f(v_3) + g(v_3) + h(v_3) \rightarrow ①$$

$$(f+(g+h))(v_1+(v_2+v_3)) = f(v_1) + f(v_2+v_3) + (g+h)(v_1) + (g+h)(v_2+v_3)$$
$$= f(v_1) + f(v_2) + f(v_3) + g(v_1) + h(v_1) + g(v_2) + h(v_3) \rightarrow ②$$

$$① = ②$$

c) Identity:

The function $f: V \to W$ defined by $f(v) = \vec{0}$ for all $v \in V$ is clearly a linear transformation and is the additive identity $L(V,W$.

d) Inverse:

Further $(-f): V \to W$ defined by $(-f)(v) = -f(v)$ is the additive inverse of f.

e) Commutative:

Let $f, g \in L(V,W)$ & $v_1, v_2 \in V$

$$(f+g)(v_1+v_2) = (v_1+v_2)(f+g)$$

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$$f(v_1)+g(v_1)+f(v_2)+g(v_2) = f(v_1)+g(v_1)+f(v_2)+g(v_2)$$

Also, $(f+g)(\alpha v) = f(\alpha v)+g(\alpha v)$

$= \alpha f(v) + \alpha g(v)$

$= \alpha[f(v)+g(v)]$

$= \alpha(f+g)v$

Hence $f+g \in L(V,W)$

Now, $(\alpha f)(v_1+v_2) = \alpha f(v_1) + \alpha f(v_2)$

$= \alpha[f(v_1)+f(v_2)]$

$= \alpha f(v_1+v_2)$

Also, $(\alpha f)(\beta v) = \alpha[f\beta(v)] = \alpha[\beta f(v)]$

$= \beta[\alpha f(v)]$

Hence $L(V,W)$ is a vector space over F.