Complex Analysis.

1. P.T under a continuous mapping the image of any connected set is connected

A function f is a continuous function

A mapping $f: S \to S'$ is said to be one-to-one if $f(x) = f(y)$ only for $x = y$

It is said to be onto if $f(S) = S'$. A mapping with both these properties has an inverse f^{-1}, defined on S; satisfies $f^{-1}(f(x)) = x$ and $f(f^{-1}(x')) = x'$. Therefore both f and f' are continuous. Hence proved

2. Length and Area:

Let consider the length of a differentiable arc γ with the equation $z = z(t) = x(t) + iy(t)$, $a \le t \le b$ is given by

$$L(\gamma) = \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt = \int_a^b |z'(t)|\, dt.$$

The image curve γ' is defined by $w = w(t) = f(z(t))$ with the derivative $w'(t) = f'(z(t))\, z'(t)$.

Its length is

$$L(\gamma') = \int_a^b |f'(z(t))||z'(t)|\, dt.$$

$$L(\gamma) = \int_\gamma |dz|, \quad L(\gamma') = \int_\gamma |f'(z)||dz|.$$

Now let E be a point set in the plane whose area

$$A(E) = \iint_E dx\,dy$$

If $f(z) = u(x, y) + iv(x+y)$

The image of the area is $E' = f(E)$ is

$$A(E') = \iint_E |u_x v_y - u_y v_x| \, dx\, dy.$$

But if $f(z)$ is conformal mapping of a open set containing E, then $u_x v_y - u_y v_x = |f'(z)|^2$

$$A(E') = \iint_E |f'(z)|^2 dx dy$$

3. $\int_\gamma x \, dz$ where γ is the directed line segment from 0 to $1+i$

Soln:

$$z(t) = (1+i)t \implies z(0) = 0, \; z(1) = 1+i.$$

observe $Re(z(t)) = t$ and $z'(t) = 1+i$.

$$\int_\gamma x \, dz = \int_0^1 t(1+i)\, dt.$$

$$= \frac{1+i}{2}\left[t^2\right]_0^1$$

$$= \frac{1+i}{2}.$$

$$\int_\gamma x dz = \frac{1+i}{2}.$$

4 **Singularities:**

A point a is called a singular point or a singularity of a function $f(z)$ if $f(z)$ is not analytic at a and f is analytic at some point of every disc $|z-a| < r$.

Eg: $f(z) = 1/z$.

$z = 0$ is the singularity

5. **Simply connected and Multiply connected**

A region is simply connected if its complement with respect to the extended plane is connected. A region which is not simply connected is called multiply connected region

6. **Argument principle:**

If $f(z)$ is meromorphic in Ω with the zeros a_j and the poles b_k, Then

$$\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}\,dz = \sum_{j} n(\gamma, a_j) - \sum_{k} n(\gamma, b_k)$$

for every cycle γ which is homologous to zero in Ω and does not pass through any of the zeros or poles

7. **Harmonic function:**

A real-valued function $u(z)$ or $u(x,y)$ defined and single valued in a region Ω is said to be harmonic in Ω, or a potential function, if it is continuous together with its partial derivatives of the first two orders and satisfies Laplace's equation

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

8. **Laurent series:**

$f(z)$ be a analytic function whose region $R_1 < |z-a| < R_2$, $f(z) = \sum_{n=-\infty}^{+\infty} A_n (z-a)^n$

where $A_n = \frac{1}{2\pi i}\int_{|s-a|=r}\frac{f(s)}{(s-a)^{n+1}}$.

5- mark.

1. Suppose that the real line R is represented as the union $R = A \cup B$ of two disjoint closed sets, we may assume that $a_1 < b_1$, we bisect the interval (a_1, b_1) and note that two halves has its left end point in A and its right end point in B. we denote this interval by (a_2, b_2) and continue the process indefinitely. In this way we obtain a sequence of nested interval (a_n, b_n) with $a_n \in A$, $b_n \in B$ are closed c would have to be a common point of A and B. This contradiction shows that either A or B is empty and hence R is connected.

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2. Suppose that f is continuous on a compact set X

$$f : X \rightarrow X.$$

For every $y \in X$, there is a ball $B(y, \rho)$ such that $d'(f(x), f(y)) < \epsilon/2$.

for $x \in B(y, \rho)$

here ρ may depend on y.

consider the covering of X by a smaller balls $B(y, \rho/2)$. There exists a finite subcovering :

$X \subset B(y_1, \rho_1/2) \cup \cdots \cup B(y_m, \rho_m/2)$.

Let δ be the smallest of the numbers $\rho_1/2, \rho_2/2, \ldots, \rho_m/2$ and suppose that $d(x_1, x_2) < \delta$. There is a y_k with $d(x_1, y_k) < \rho_k/2$ and we obtain

$$d(x_2, y_k) < \rho_k/2 + \delta \leq \delta_k$$

Hence $d'(f(x_1), f(y_k)) < \epsilon/2$ and

$$d'(f(x_2), f(y_k)) < \epsilon/2.$$

$$\therefore d'(f(x_1), f(x_2)) < \epsilon \text{ as desired.}$$

3. Morera's Thm :

If $f(z)$ is defined and continuous in a region Ω, and if $\int_\gamma f\,dz = 0$ for all closed curves γ in Ω, then $f(z)$ is analytic in Ω

Proof :

If $f(z)$ is defined on Ω.

Let $f(z)\,dz = f(z)\,dx + if(z)$.

Then $\frac{\partial F(z)}{\partial x} = f(z)$, $\frac{\partial F(z)}{\partial y} = if(z)$.

Then it satisfy Cauchy Riemann equation

$$\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y}$$

$\therefore$ $f(z)$ is continuous. otherwise $\int_\gamma f\,dz$ is not defined

$\therefore F(z)$ is analytic with the derivative $f(z)$.

The integral $\int_\gamma f\,dz$, with continuous f, depends only on the end points & if and only if is the derivative of an analytic function in Ω

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4. Taylor's Thm:

If $f(z)$ is analytic in a region Ω, conta -ining it is possible.

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \cdots + \frac{f^{n-1}(a)}{(n-1)!}(z-a)^{n-1}$$

$$+ f_n(z)(z-a)^n$$

where $f_n(z)$ is analytic in Ω.

Proof:

Let us define the function

$$F(z) = \frac{f(z)-f(a)}{z-a}.$$

It is not defined for $z=a$, then

$\lim_{z \to a} (z-a) F(z) = 0$. The limit of $F(z)$ as z tends to a is $f'(a)$.

Hence there exists an analytic function which is equal to $F(z)$ for $z \neq a$ and equal to $f'(a)$ for $z=a$.

Repeating this process for $f_2(z)$

$(f_1(z) - f_1(a))/(z-a)$ for $z \neq a$

and $f_1'(a)$ for $z=a$ and so on.

The recursive scheme by which $f_n(z)$ is defined

$$f(z) = f(a) + (z-a)f_1(z)$$
$$f_1(z) = f_1(a) + (z-a)f_2(z)$$
$$\cdots$$
$$f_{n-1}(z) = f_{n-1}(a) + (z-a)f_n(z)$$

Then

$$f(z) = f(a) + (z-a)f_1(a) + \cdots + (z-a)^{n-1} f_{n-1}(a) + (z-a)^n f_n(z)$$

Diff n times and setting $z=a$

$$f^n(a) = n!\, f_n(a).$$

5 Schwarz lemma:

If $f(z)$ is analytic for $|z|<1$ and satisfies the conditions $|f(z)|<1$, $f(0)=0$, then $|f(z)| \le |z|$ and $|f'(0)| \le 1$. If $|f(z)| = |z|$ for some $z \ne 0$ or if $|f'(0)| = 1$, then $f(z) = cz$ with a constant c of absolute value 1.

Proof:

Let us consider a function $f_1(z)$ is equal to $f(z)/z$ for $z \ne 0$ and to $f'(0)$ for $z=0$. On the circle $|z| = r < 1$ it is of absolute value $\le 1/r$ and hence

$$|f_1(z)| \le 1/r \text{ for } |z| \le r.$$

Letting r tend to 1 then $|f_1(z)| \le 1$.

we obtain $|f(Rz)| \le |z|$

$$|f(z)| \le |z|/R$$

We can replace the condition $f(0)=0$

by an arbitrary condition $f(z_0) = w_0$.

where $|z_0| < R$ and $|w_0| < M$. Let

$\zeta = Tz$ be a linear transformation which maps $|z| < R$ onto $|\zeta| < 1$ with z_0 going into the origin.

Let Sw be a linear transformation with $Sw_0 = 0$ which maps $|w| < M$ onto $|Sw| < 1$.

A function $Sf(T^{-1}\zeta)$ satisfies

$|Sf(T^{-1}\zeta)| \leq |\zeta|$ or $|Sf(z)| \leq |Tz|$

$$\left| \frac{M(f(z) - w_0)}{M^2 - \bar{w}_0 f(z)} \right| \leq \left| \frac{R(z - z_0)}{R^2 - \bar{z}_0 z} \right|$$

Residue Theorem:

Let $f(z)$ be analytic except for isolated singularities a_j in a region Ω. Then

$$\frac{1}{2\pi i} \int_{\gamma} f(z)\,dz = \sum_j n(\gamma, a_j) \operatorname*{Res}_{z \to a_j} f(z)$$

for any cycle γ which is homologous to zero in Ω and does not pass through any of the points a_j.

That $n(\gamma, a_j)$ is either 0 or 1. Then

$$\frac{1}{2\pi i} \int_{\gamma} f(z)\,dz = \sum_j \operatorname*{Res}_{z = a_j} f(z).$$

where the sum is extended over all singularities enclosed by γ.

Proof :

A function $f(z)$ which is analytic in a region Ω except for isolated singularities. Assume that there are only a finite number of singular points denoted by $a_1, a_2, \dots, a_n$.

To each a_j there exists a $\delta_j > 0$ such that the doubly connected region

$0 < |z - a_j| < \delta_j$ is contained in Ω'.

Draw a circle C_j about a_j of radius $< \delta_j$ and let

$$R_j = \frac{P_j}{2\pi i}$$

$$P_j = \int_{C_j} f(z)\, dz$$

be the corresponding period of $f(z)$.

The function $\frac{1}{(z-a_j)}$ has the period $2\pi i$

$$f(z) - \frac{R_j}{z-a_j}$$

Let γ be a cycle in Ω' which is homologous to zero with respect to Ω. Then γ satisfies the homology.

$$\gamma \sim \sum_j n(\gamma, a_j) C_j$$

Then

$$\int_\gamma f\, dz = \sum_j n(\gamma, a_j) P_j$$

and since $P_j = 2\pi i$. Then

$$\frac{1}{2\pi i} \int_\gamma f\, dz = \sum_j n(\gamma, a_j) R_j$$

7. $\int_0^{\pi} \log \sin\theta \, d\theta$.

consider $1 - e^{2iz} = -2ie^{iz} \sin z$.

$1 - e^{2iz} = 1 - e^{-2y} (\cos 2x + i\sin 2x)$,

There exist $\delta \to 0$. Then

$|1 - e^{2iz}| / |z| \to 2$ for $z \to 0$

$\int_0^{\pi} \log(-2ie^{ix} \sin x) dx = 0$.

$\log e^{ix} = ix$ lies between 0 and π

$\log(-i) = -\pi i/2$.

$-\int_0^{\pi} [\log 2 + \log i + \log e^{ix} + \log \sin x] dx = 0$

$\left[x \log 2 - \pi/2 \, ix + ix^2/2\right]_0^{\pi}$

$+ \int_0^{\pi} \log \sin x \, dx = 0$.

$+ \left[(\pi \log 2 - \frac{\pi^2 i}{2} + \frac{i\pi^2}{2}\right] + \int_0^{\pi} \log \sin x \, dx = 0$.

$+ \pi \log 2 + \int_0^{\pi} \log \sin x \, dx = 0$

$\therefore \int_0^{\pi} \log \sin x \, dx = -\pi \log 2$.

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8. Suppose that $f(z)$ is not identically zero. The zeros of $f(z)$ are in any case isolated.

For any point $z_0 \in \Omega$ there is

therefore a number $r>0$ such that

$f(z)$ is defined and $\neq 0$ for $0<|z-z_0|\leq r$.

In particular, $|f(z)|$ has a positive minimum on the circle $|z-z_0|=r$ which we denote by C. It follows that $1/f_n(z)$ converges uniformly to $1/f(z)$ on C.

Then

$$f_n'(z) \to f'(z), \text{ uniformly on } C$$

$$\lim_{n\to\infty} \frac{1}{2\pi i}\int_C \frac{f_n'(z)}{f_n(z)}\,dz = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz.$$

The integrals on the left are zero. The number of roots of the eqn $f_n(z)=0$ inside of C. The integral on the right side are zero and $f(z_0)\neq 0$. since z_0 was arbitrary

9. Schwartz reflecting principle:

Let Ω^+ be the part in the upper half plane of a symmetric region Ω and let σ be the part of the real axis in Ω. Suppose that $v(x)$ is continuous in $\Omega^+\cup\sigma$, harmonic in Ω^+, and zero on σ. Then v has a harmonic extension to Ω which satisfies the symmetry relation $v(\bar{z})=-v(z)$. if v is the imaginary part of an analytic function $f(z)$ in Ω^+, then $f(z)$ has an analytic extension which satisfies $f(z)=\overline{f(\bar{z})}$

Proof:

Let $V(z)$ equal to $v(z)$ in Ω^+, 0 on σ, and equal to $-v(\bar{z})$

in the mirror image of Ω^+.

To show: v is harmonic on σ.

Let $x_0 \in \sigma$. consider a disk containing x_0 in Ω.

Let P_v denote the poisson integral with the boundary values V.

Then $V - P_v$ is harmonic in the upper half of the disk. It vanishes on the half circle. and also on the diameter, v tends to 0. P_v vanishes. By maximum principle.

$$V = P_v$$ in the upper half disk.

v has a conjugate harmonic function $-u_0$ in the same disk

$$u_0 = \operatorname{Re} f(z)$$

$$U_0(z) = u_0(z) - u_0(\bar{z}).$$

Then $\frac{\partial U_0}{\partial x} = 0 \Rightarrow \frac{\partial U_0}{\partial y} = 2\frac{\partial u_0}{\partial y} = -2\frac{\partial v}{\partial x} = 0.$

Then $\partial U_0/\partial x - i\partial U_0/\partial y$ vanishes on the real axis. Then U_0 is constant and zero.

$u_0(z) = u_0(\bar{z})$.

$\therefore$ Hence proved.

Part - C

16. Prove that A Set is compact if and only if it is complete and totally bounded?

Proof:-

To prove the other part, assume that the metric space S is complete and totally bounded. Suppose that there exists an open covering which does not contain any finite Subcovering. Write $\varepsilon_n = 2^{-n}$. We know that S can be covered by finitely many $B(x, \varepsilon_1)$. If each had a finite subcovering, the same would be true of S; hence there exists a $B(x_1, \varepsilon_1)$ which does not admit a finite subcovering. Because $B(x_2, \varepsilon_2)$ has no finite subcovering. It is clear how to continue the construction: we obtain a sequence x_n with the property that $B(x_n, \varepsilon_n)$ has no finite subcovering and $x_{n+1} \in B(x_n, \varepsilon_n)$. The second property implies $d(x_n, x_{n+1}) < \varepsilon_n$ and hence $d(x_n, x_{n+p}) < \varepsilon_n + \varepsilon_{n+1} + \cdots + \varepsilon_{n+p-1} < 2^{-n+1}$. It follows than x_n is a cauchy sequence. It converges to a limit y, and this y belongs to one of the open sets U in the given covering. Because U is open, it contains a ball $B(y, \delta)$ choose n so large that $d(x_n, y) < \delta/2$ and $\varepsilon_n < \delta/2$. Then $B(x_n, \varepsilon_n) \subset B(y, \delta)$, for $d(x, x_n) < \varepsilon_n$ implies $d(x, y) \leq d(x, x_n) + d(x_n, y) < \delta$. Therefore $B(x_n, \varepsilon_n)$ admits a finite subcovering, namely by the single set U. This is a contradiction, and we conclude that S has the Heine-Borel property.

17. State and prove the Cauchy's theorem for a rectangle:-

Statement:-

If the function $f(z)$ is analytic on R, then

$$\int_{\partial R} f(z)\,dz = 0 \rightarrow (1)$$

<u>proof</u> :-

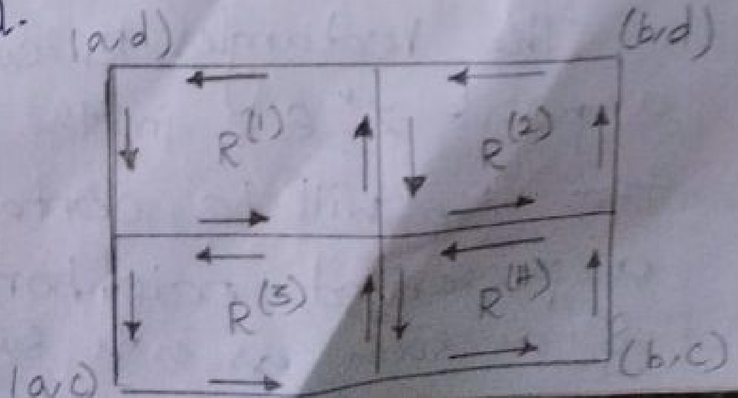
The proof is based on the method of bisection. Let us introduce the notation.

$$\eta(R) = \int_{\partial R} f(z)\,dz$$

Which we will also use for any rectangle contained in the given one. If R is divided into four congruent rectangles $R^{(1)}, R^{(2)}, R^{(3)}, R^{(4)}$, we find that

$$\eta(R) = \eta(R^{(1)}) + \eta(R^{(2)}) + \eta(R^{(3)}) + \eta(R^{(4)}), \longrightarrow (2)$$

for the integrals over the common sides cancel each other. It is important to note that this fact can be verified explicitly and does not make illicit use of geometric intuition. Nevertheless, a reference is helpful.



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It follows from ② that at least one of the rectangles $R^{(k)}$, $k = 1, 2, 3, 4$, must satisfy the condition

$$|\eta(R^{(k)})| \geq \frac{1}{4}|\eta(R)|.$$

We denote this rectangle by R_1; if several $R(k)$ have this property, the choice shall be made according to some definite rule.

This process can be repeated indefinitely, and we obtain a sequence of nested rectangles $R \supset R_1 \supset R_2 \supset \cdots \supset R_n \supset \cdots$ with the property

$$|\eta(R_n)| \geq 1/4\,|\eta(R_{n-1})|$$

and hence

$$|\eta(R_n)| \geq 4^{-n}\,|\eta(R)| \rightarrow ③$$

The rectangles R_n converge to a point $z^* \in R$ in the sense that R_n will be contained in a prescribed neighborhood $|z - z^*| < \delta$ as soon as n is sufficiently

By virtue of these equations 6 we are able to write.

$$\eta(R_n) = \int_{\partial R_n} [f(z) - f(z^*) - (z - z^*) f'(z^*)] dz,$$

and it follows by ④ that

$$|\eta(R_n)| \leq \varepsilon \int_{\partial R_n} |z - z^*| \cdot |dz|. \quad ⑤$$

In the last integral $|z - z^*|$ is at most equal to the length d_n of the diagonal of R_n. If L_n denotes the length of the perimeter of R_n, the integral is hence $\leq d_n L_n$. But if d and L be the corresponding quantities for the original rectangle R, it is clear that $d_n = 2^{-n} d$ and $L_n = 2^{-n} L$. By ⑤ we have hence

$$|\eta(R_n)| \leq 4^{-n} d L \varepsilon,$$

and comparison with ③ yields.

$$|\eta(R)| \leq d L \varepsilon.$$

Since ε is arbitrary, we can only have $\eta(R) = 0$, and the large. First of all, we choose δ so small that $f(z)$ is defined and analytic in $|z-z^*|<\delta$. Secondly if $\varepsilon>0$ is given, we can choose δ so that

$$\left|\frac{f(z)-f(z^*)}{z-z^*}-f'(z^*)\right|<\varepsilon$$

(or)

$$|f(z)-f(z^*)-(z-z^*)f'(z^*)|<\varepsilon|z-z^*| \rightarrow 2$$

for $|z-z^*|<\delta$. We assume that δ Satisfies both conditions and that R_n is contained in $|z-z^*|<\delta$.

We make now the Observation that.

$$\int_{\partial R_n} dz=0.$$

$$\int_{\partial R_n} z\,dz=0.$$

These trivial Special cases of our theorem have already been proved in sec. We recall that the proof depended on the fact that 1 and z are the derivative of z and $z^2/2$, resp.

theorem is proved

13. State and prove the local correspondence theorem:-

Statement:-

Suppose that $f(z)$ is analytic at z_0, $f(z_0) = w_0$, and that $f(z) - w_0$ has a zero of order n at z_0. If $\epsilon > 0$ is sufficiently small, there exists a corresponding $\delta > 0$ such that for all a with $|a - w_0| < \delta$ the equation $f(z) = a$ has exactly n roots in the disk $|z - z_0| < \epsilon$.

Proof:

We can choose ϵ so that $f(z)$ is defined and analytic for $|z - z_0| \leq \epsilon$ and so that z_0 is the only zero of $f(z) - w_0$ in this disk. Let γ be the circle $|z - z_0| = \epsilon$ and Γ its image under the mapping $w = f(z)$. Since w_0 belongs to the complement of the closed set Γ, there exists a neighborhood $|w - w_0| < \delta$ which does not intersect it. It follows immediately that all values a of this neighborhood are taken the same number of times inside of γ. The equation $f(z)=w_0$ has exactly n coinciding roots inside of γ, and hence every value a is taken n times.

It is understood that multiple roots are counted according to their multiplicity, but if ε is sufficiently small we can assert that all roots of the equation $f(z)=a$ are simple for $a \neq w_0$. Indeed, it is sufficient to choose ε so that $f'(z)$ does not vanish for $0<|z-z_0|<\varepsilon$.

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19. State and prove Rouche's Theorem:-

Statement:-

Let γ be homologous to zero in Ω and such that $n(\gamma, z)$ is either 0 or 1 for any point

z not on γ. Suppose that $f(z)$ and $g(z)$ are analytic in Ω and satisfy the inequality $|f(z)-g(z)| < |f(z)|$ on γ. Then $f(z)$ and $g(z)$ have the same number of zeros enclosed by γ.

Proof:-

The assumption implies that $f(z)$ and $g(z)$ are zero-free on γ. Moreover, they satisfy the inequality

$$\left|\frac{g(z)}{f(z)} - 1\right| < 1$$

on γ. The values of $F(z) = g(z)/f(z)$ on γ are thus contained in the open disk of center 1 and radius 1. When theorem 18 applied to $F(z)$, we have thus $n(\Gamma, 0) = 0$, and the assertion follows.

A typical application of Rouche's theorem would be the following. Suppose that we wish to find the number of zeros of a function $f(z)$ in the disk $|z| \leq R$. Using Taylor's theorem we can write

$$f(z) = P_{n-1}(z) + z^n f_n(z)$$ 10

Where P_{n-1} is a polynomial of degree $n-1$. For a suitably chosen n it may happen that we can prove the inequality $R^n |f_n(z)| < |P_{n-1}(z)|$ on $|z| = R$. Then $f(z)$ has the same number of zeros in $|z| \leq R$ as $P_{n-1}(z)$, and this number can be determined by approximate solution of the polynomial equation $P_{n-1}(z) = 0$.

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20. State and prove the poissons Formula :-

<u>Statement</u> :-

If $u(z)$ is continuous on a closed bounded set E and harmonic on the interior of E, then it is uniquely determined by its values on the boundary of E. Indeed if u_1 and u_2 are two such functions with the same boundary values, then $u_1 - u_2$ is harmonic with the boundary

Value 0. By the maximum and 11 minimum principle the difference $u_1 - u_2$ must then be identically zero on E.

Proof:-

Suppose that $u(z)$ is harmonic in the closed disk $|z| \leq R$. The linear transformation

$$z = S(\zeta) = \frac{R(R\zeta + a)}{R + \bar{a}\zeta}$$

maps $|\zeta| \leq 1$ onto $|z| \leq R$ with $\zeta = 0$ corresponding to $z = a$. The function $u(S(\zeta))$ is harmonic in $|\zeta| \leq 1$ and by we obtain

$$u(a) = \frac{1}{2\pi} \int_{|\zeta|=1} u(S(\zeta)) \, d\arg\zeta.$$

From

$$\zeta = \frac{R(z-a)}{R^2 - \bar{a}z}.$$

We compute.

$$d\arg\zeta = -i \frac{d\zeta}{\zeta} = -i \left(\frac{1}{z-a} + \frac{\bar{a}}{R^2 - \bar{a}z} \right) dz$$

$$=\left(\frac{z}{z-a}+\frac{\bar{a}z}{R^2-\bar{a}z}\right)d\theta$$ 12

on substituting $R^2 = z\bar{z}$ the coefficient of $d\theta$ in the last expression can be rewritten as

$$\frac{z}{z-a}+\frac{\bar{a}}{\bar{z}-\bar{a}}=\frac{R^2-|a|^2}{|z-a|^2}$$

Or, equivalently, as

$$\frac{1}{2}\left(\frac{z+a}{z-a}+\frac{\bar{z}+\bar{a}}{\bar{z}-\bar{a}}\right)=\operatorname{Re}\frac{z+a}{z-a}$$

We obtain the two forms.

$$u(a)=\frac{1}{2\pi}\int_{|z|=R}\frac{R^2-|a|^2}{|z-a|^2}u(z)d\theta$$

$$=\frac{1}{2\pi}\int_{|z|=R}\operatorname{Re}\frac{(z+a)}{z-a}u(z)d\theta$$

of poisson's formula. In polar coordinates,

$$u(re^{i\varphi})=\frac{1}{2\pi}\int_0^{2\pi}\frac{R^2-r^2}{R^2-2rR\cos(\theta-\varphi)+r^2}u$$