Unit - IV

General Preliminaries on Banach algebra

Algebra

A linear space with another internal operations called Vector Multiplication and under certain conditions is called an algebra.

Banach Algebra

A Banach Algebra is a Complex Banach Space which is also an algebra with identity one and in which the multiplicative structure is related to the norm by the following conditions

1) $\|xy\| \leq \|x\|\|y\|$

2, $\|1\| = 1$

Note:

Vector Multiplication is jointly Continuous

Proof:

If $x_n \to x$ and $y_n \to y$, we've to prove $x_n y_n \to xy$

Now, $\|x_n y_n - xy\| = \|x_n y_n - x_n y + x_n y - xy\|$

$= \|x_n(y_n - y) + (x_n - x)y\|$

$\leq \|x_n(y_n - y)\| + \|(x_n - x)y\|$

$\leq \|x_n\|\|y_n - y\| + \|x_n - x\|\|y\|$ By Cond (1) of vector Multiplication

$\to 0$

$\therefore$ Since $y_n \to y$ and $x_n \to x$

$\therefore x_n y_n \to xy$

* Eg of Banach Algebra

1, $\mathcal{B}(B)$, the set of all continuous linear operators on a Banach space B is a Banach Algebra

2, $\mathcal{B}(H)$. the set of all continuous linear Operators on a Hilbert space H is a Banach Algebra

Definition.

The Banach Subalgebra :

The Banach Subalgebras of A, (a Banach algebra) are those subsets of A, which are themselves Banach Algebras. w.r.to the same Algebraic operations, the same Identity and the same norm.

Eg. of Banach Subalgebra

The Set of all set adjoint operators. on H (the Hilbert Space) is a Banach Subalgebra of the Banach algebra $\mathcal{B}(H)$

Definition

An elt $x \in A$, a Banach algebra is said to be Regular if $\exists$ inverse of a x (vector) w.r. to multiplication.

ie, if $\exists$ an elt $y \in A$ such that $xy = yx =$

Otherwise it is called a Singular element.

Note :

1, Inverse of x is denoted by x^{-1}.

2. The Set of all regular elements of a Banach algebra A is denoted by G. which will be a group under multiplication. It's complement consisting of all singular elements. is denoted by S.

Thrm:

Every element x for which $\|x-1\|<1$ is regular and the inverse of such an element is given by the formula $x^{-1} = 1+\sum_{n=1}^{\infty}(1-x)^n$.

Proof

If we put $r = \|x-1\|$ So that $r<1$, then

$\|(1-x)^n\| = \|(1-x)(1-x)\cdots n \text{ times}\|$

$\leq \|1-x\|\,\|1-x\| \cdots n \text{ times}$.

Since $\|xy\| \leq \|x\|\|y\|$ is $\|(1-x)^n\| \leq \|1-x\|^n$

$= r^n$.

The above shows the partial sums of the series $\left\{\sum_{n=1}^{\infty}(1-x)^n\right\}$ form a Cauchy Sequ in A

Since, A is complete, these partial sums converges to an element of A which we denoted by $\sum_{n=1}^{\infty}(1-x)^n$.

we define y by $y = 1+\sum_{n=1}^{\infty}(1-x)^n$

Now, $y - xy = (1-x)y = (1-x)\left[1+\sum_{n=1}^{\infty}(1-x)^n\right]$

$$= (1-x) + \sum_{n=1}^{\infty} (1-x)^{n+1}$$

$$= (1-x) + \sum_{n=2}^{\infty} (1-x)^{n}$$

$$\sum_{n=1}^{\infty} (1-x)^{n} = y-1 \text{ , by defn of } y\text{'s}$$

ie $y - xy = y - 1 \Rightarrow -xy = -1 \Rightarrow xy = 1$

III^{ly} we can prove $yx = 1$

$\therefore y$ is the inverse of x.

ie x has inverse and hence regular.

Inverse of x ie, y is given by

$$y = 1 + \sum_{n=1}^{\infty} (1-x)^{n}$$

2. Thm

G is an open set and $\therefore S$ is a closed set.

Proof

Let x_0 be any elem. in G. Let x be any element such that:

$$\| x - x_0 \| \le \frac{1}{\| x_0^{-1} \|}$$

Now.

$$\| x_0^{-1} x - 1 \| = \| x_0^{-1} (x - x_0) \|$$

$$\le \| x_0^{-1} \| \| x - x_0 \|$$

$$\le \| x_0^{-1} \| \cdot \frac{1}{\| x_0^{-1} \|}$$

$\therefore \|x_0^{-1}x - 1\| < 1.$

$\therefore$ by prel- thrm. $x_0^{-1}x$ is regular and hence $x_0^{-1}x \in G$,

Already, $x_0 \in G$.

Since G is a group, by closure property $x_0 \in G$ and $x_0^{-1}x \in G \Rightarrow x_0(x_0^{-1}x) \in G$

$\Rightarrow x \in G$.

$\therefore$ Any element x with the condition $\|x - x_0\| < \frac{1}{\|x_0^{-1}\|}$ is an element of G.

$\therefore$ we can construct an open sphere around every point of G which is contained in G. $\therefore G$ is open.

Thrm 3 :

The mapping : $x \to x^{-1}$ of G into G is continuous and hence a homomorphism of G onto itself.

Proof

Let $f : G \to G$ given by $x \to x^{-1}$

is $f(x) = x^{-1}$. $\forall x \in G$

We've to prove f is continuous.

ie, " .. $x \to x_0 \Rightarrow f(x) \to f(x_0)$

Let $x \to x_0$. Then $\|x - x_0\| < \dfrac{1}{2\|x_0^{-1}\|}$ $\to$ ①

Now,

$$\|x_0^{-1}x - 1\| = \|x_0^{-1}(x - x_0)\|$$

$$\leq \|x_0^{-1}\|\|x - x_0\|$$

$$< \|x_0^{-1}\| \dfrac{1}{2\|x_0^{-1}\|} = \dfrac{1}{2}$$

$\therefore \|x_0^{-1}x - 1\| < \dfrac{1}{2} \to$ ②

$\Rightarrow \|x_0^{-1}x - 1\| < 1$

$\Rightarrow x_0^{-1}x$ is regular and its inverse is given by

$$(x_0^{-1}x)^{-1} = 1 + \sum_{n=1}^{\infty}(1 - x_0^{-1}x)^n \text{, by thrm ①}$$

ie $x^{-1}x_0 = 1 + \sum_{n=1}^{\infty}(1 - x_0^{-1}x)^n \to$ ③

Now,

$$\|f(x) - f(x_0)\| = \|x^{-1} - x_0^{-1}\|$$

$$= \|(x^{-1}x_0 - 1)x_0^{-1}\|$$

$$\leq \|x^{-1}x_0 - 1\|\|x_0^{-1}\|$$

$$= \left\|\sum_{n=1}^{\infty}(1 - x_0^{-1}x)^n\right\|\|x_0^{-1}\|, \text{ using ③}$$

$$\leq \sum_{n=1}^{\infty}\|(1 - x_0^{-1}x)\|^n \cdot \|x_0^{-1}\|$$

$\because \|x^n\| \leq \|x\|^n$.

$$= (\|1 - x_0^{-1}x\| + \|1 - x_0^{-1}x\|^2 + \|1 - x_0^{-1}x\|^3 + \dots)\|x_0^{-1}\|.$$

$$= \| 1 - x_0^{-1}x \| \left(1 + \|1 - x_0^{-1}x\| + \|1 - x_0^{-1}x\|^2 + \cdots \right) \|x_0^{-1}\|$$

$$= \|1 - x_0^{-1}x\| \left(\frac{a}{1-r}\right) \|x_0^{-1}\|$$

where $a = 1$,
$r = \|1 - x_0^{-1}x\|$

$$= \|1 - x_0^{-1}x\| \frac{1}{1 - \|1 - x_0^{-1}x\|} \cdot \|x_0^{-1}\|$$

$$\therefore \|f(x) - f(x_0)\| \leq \frac{\|x_0^{-1}\| \|1 - x_0^{-1}x\|}{1 - \|1 - x_0^{-1}x\|} \longrightarrow (4)$$

(2) is $\|x_0^{-1}x - 1\| < 1/2$.

$$\rightarrow -\|x_0^{-1}x - 1\| > -1/2$$

$$\rightarrow 1 - \|x_0^{-1}x - 1\| > 1 - 1/2$$

$$\Rightarrow 1 - \|x_0^{-1}x - 1\| > 1/2$$

$$\Rightarrow \frac{1}{1 - \|x_0^{-1}x - 1\|} > 2$$

$\therefore$ (4) becomes,

$$\|f(x) - f(x_0)\| \leq 2 \|x_0^{-1}\| \cdot \|1 - x_0^{-1}x\|$$

$$= 2\|x_0^{-1}\| \|x_0^{-1}(x_0 - x)\|$$

$$\leq 2\|x_0^{-1}\| \|x_0^{-1}\| \cdot \|x_0 - x\|$$

$$= 2\|x_0^{-1}\|^2 \|x_0 - x\| \rightarrow 0,$$

since $x \rightarrow x_0$

$$\therefore f(x) \rightarrow f(x_0)$$

$$\therefore x \rightarrow x_0 \Rightarrow f(x) \rightarrow f(x_0)$$

$\therefore$ The mapping : $x \rightarrow x_0$ of G into G is continuous.

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By defn 1- it is one - to - one and onto

∴ It is a homeomorphism

Defn 1-

An element z in a Banach algebra A is called a topological divisor of zero. if there exists a sequence $\{z_n\}$ in A such that $\|z_n\| = 1$ for every 'n' and either $zz_n \to 0$ or $z_n z \to 0$

The set of all topological divisor of zero is denoted by Z.

Thrm: Z is a subset of S.

Proof

Let z be an element of Z or z is a topological divisor of zero.

Then by defn there exists a sequence $\{z_n\}$ such that $\|z_n\| = 1$ and $zz_n \to 0$ (say)

we've to prove $z \in S$

Let us assume that $z \in G$ or z is regular

Then z^{-1} exists. we've $zz_n \to 0$.

$\therefore z^{-1}(zz_n) \to 0$

ie, $(z^{-1}z)z_n \to 0$

ie, $z_n \to 0$ which contradiction the fact that $\|z_n\| = 1 \; \forall n$.

$\therefore z \notin G \Rightarrow z \in$ complement of G

$\Rightarrow z \in S$.

$\therefore z \in Z \Rightarrow z \in S$

$\therefore Z$ is a subset of S.

(8) Thrm 5

The boundary of S is a subset of Z.

Proof.

Let $z \in$ boundary of S

We've to show $z \in Z$

Since, S is closed, every point in its boundary will be a limit point of some Cgt. Seqn in G. by defn/- of closed set.

$\therefore$ There exists a Seq $\{x_n\}$ in G. Such that $x_n \to z$ and hence $\|z - x_n\| < \frac{1}{k} \to (1)$

We prove that $\{x_n^{-1}\}$ is unbdd.

Let us assume that $\{x_n^{-1}\}$ be bdd

Then, $\|x_n^{-1}\| < K \to (2)$

Now, $\|x_n^{-1} z - 1\| = \|x_n^{-1}(z - x_n)\|$

$\leq \|x_n^{-1}\| \|z - x_n\|$

$< K \cdot \frac{1}{k}$, using (1) & (2)

$\therefore \|x_n^{-1} z - 1\| < 1$

$\therefore x_n^{-1} z$ is regular, by thrm (1)

$\Rightarrow x_n^{-1} z \in G.$

Already, $x_n \in G$

$\therefore x_n (x_n^{-1} z) \in G$, G being a group.

$\Rightarrow z \in G.$ which is a contradiction.

$\therefore \{x_n^{-1}\}$ is unbdd.

$\therefore$ we may assume that $\|r_n^{-1}\| \to \infty$

we define z_n as follows: $z_n = \dfrac{r_n^{-1}}{\|r_n^{-1}\|}$

So that $\|z_n\| = 1$

Also, $$z z_n = \frac{z r_n^{-1}}{\|r_n^{-1}\|} = \frac{1 + z r_n^{-1} - 1}{\|r_n^{-1}\|}$$

$$= \frac{1 + (z - r_n) r_n^{-1}}{\|r_n^{-1}\|}$$

$$= \frac{1}{\|r_n^{-1}\|} + \frac{1}{\|r_n^{-1}\|} r^{-1}(z - r_n)$$

$$= \frac{1}{\|r_n^{-1}\|} + z_n (z - r_n)$$

$$\because \frac{r_n^{-1}}{\|r_n^{-1}\|} = z_n$$

$\to 0$ as $\|r_n^{-1}\| \to \infty$ and $r_n \to z$

for given $z \in$ boundary of S, $\exists$ a Sequence $\{z_n\}$ in A such that $\|z_n\| = 1$ and $z z_n \to 0$

$\therefore z$ is a topological divisor of Zero

$\Rightarrow z \in Z$

$\therefore z \in$ boundary of S

$\Rightarrow z \in Z$

Boundary of S is a subset of Z.

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Spectrum of an element:

The Spectrum of an element x in a Banach algebra A is defined as the following Subset of complex plane.

$$f(x) = \{\lambda : x - \lambda_1 \text{ is singular}\}$$

Note

To express the dependence of $\sigma(x)$ on A, we use the notation $\sigma_A(x)$

Since the set of singular elements in A ie, S is closed. $\sigma(x)$ is also closed.

Resolvent ~~or~~ Resolvent set of x:

The resolvent ~~or~~ resolvent set of x is denoted by $\rho(x)$ is the complement of $\sigma(x)$ and clearly it is open.

The resolvent of x is the function with values in A. defined on $\rho(x)$ by $X(\lambda) = (x-\lambda_1)^{-1}$ is $x : \rho(x) \to A$, $\lambda \to (x - \lambda_1)^{-1}$ is ~~(x)~~

$$x(\lambda) = (x - \lambda_1)^{-1}$$

Note:

1. If λ and μ are both in $\rho(x)$, then $x(\lambda) = (x - \lambda_1)^{-1}$ and $x(\mu) = (x - \mu_1)^{-1}$

(or) $x(\lambda)(x - \lambda_1) = 1$ and $x(\mu)(x - \mu_1)^{*} = 1$

Consider $x(\lambda) = x(\lambda) \cdot 1$

$= x(\lambda) \cdot x(\mu)(x-\mu_1)$

$= x(\lambda) x(\mu)[x-\lambda_1+\lambda_1-\mu_1]$

$= x(\lambda) x(\mu)[(x-\lambda I)+(\lambda-\mu)I]$

$= [x(\lambda)(x-\lambda_1)+(\lambda-\mu)x(\lambda)]x(\mu)$

$= [1+(\lambda-\mu)(x(\lambda)]\, x(\mu)$.

ie, $x(\lambda) = x(\mu)+(\lambda-\mu)x(\lambda)x(\mu)$

ie, $x(\lambda) - x(\mu) = (\lambda-\mu)x(\lambda)x(\mu)$

~~ie~~. This relation is called the resolvent equ.

2.

WKT. the mapping: $x \to x^{-1}$ is constant.

$\therefore x = \lambda \to (x-\lambda_1)^{-1}$ is also constant.

Also,

$$x(\lambda) = (x-\lambda)^{-1}$$

$$= \lambda^{-1}\left(\frac{x}{\cancel{\lambda}}-1\right)^{-1}$$

$$\therefore x(\lambda) \to 0 \text{ as } \lambda \to \infty$$

6. Thrm: 6

$\sigma(x)$ is non empty

Proof

Let f be a functional on A is $f \in A^*$ and define $f(\lambda) = f(x(\lambda))$

It is clear that $f(\lambda)$ is a complex function which is defined and constant on resolvent set $f(x)$.

The ~~set~~ resolvent eqn is $x(\lambda) = (\lambda-\mu)\, x(\lambda)(x/G)$

ie.,
$$\frac{x(\lambda) - x(\mu)}{\lambda-\mu} = x(\lambda)\, x(\mu)$$

$$\therefore \frac{f(x(\lambda)) - F(x(\mu))}{\lambda-\mu} = f(x(\lambda)\, x(\mu))$$

by continuity and elements of f.

$$\therefore \lim_{\lambda \to \mu} \frac{f(x(\lambda)) - f(x(\mu))}{\lambda-\mu} = f\left(x/(\mu)^2\right).$$

ie,
$$\lim_{\lambda \to \mu} \frac{f(\lambda) - f(\mu)}{\lambda-\mu} = f\left(x(\mu)^2\right),$$

by def of f.

$\therefore f(\lambda)$ has derivative at each point of $f(x)$.

Let us assume that $\sigma(x)$ be empty. ~~point of $f(x)$~~. So that $f(x)$ is the entire Complex plane.

$\therefore$ We can say $f(\lambda)$ has derivative at each point of C and hence an entire function.

Already it is bounded

$f(\lambda)$ is constant for all λ $\rightarrow$ (1)

Now,

$$|f(\lambda)| = |f(x(\lambda))| \le \|f\| \, \|x(\lambda)\| \to$$

$\to 0$ as $\lambda \to \infty$ (by note 2)

$\therefore f(\lambda) \to 0$ as $\lambda \to \infty \longrightarrow$ (2)

from (1) & (2). $f(\lambda) = 0$ for all λ

ie, $f(x(\lambda)) = 0$

ie, $x(\lambda) = 0$. Since $f \in A^*$, is arbitrary

ie $(x - \lambda_1)^{-1} = 0$

This is impossible.

Since no inverse can equal to 0.

$\sigma(x)$ is non-empty.

<u>Division Algebra:</u>

A division algebra is an algebra with identity in which each non-zero element is regular.

<u>Thrm 7</u>

If A is a division algebra then it equal the set of all scalar multiples of the identity.

<u>Proof:</u>

We have to show if x is an element of A, then $x = \lambda_1$ for some scalar λ

Suppose $x \ne \lambda_1$ for every λ

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Then

$x - \lambda 1 \neq 0$ for $\forall \lambda$.

$\therefore x - \lambda 1$ is a non-zero element in the division algebra A and hence regular, by definition of division algebra.

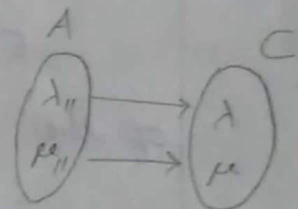
$\therefore x - \lambda 1$ is not singular for every λ.

and hence $\sigma(x)$ is empty, by definition of $\sigma(x)$

This is a contradiction.

$\therefore$ Our assumption $x \neq \lambda 1, \forall \lambda$ is wrong

$\therefore x = \lambda 1$ for some λ.



Note:

The Mapping : $\lambda_1 \rightarrow \lambda$ from the set of all Scalars multiples of the identity onto C is an isometric isomorphism.

$\therefore$ The Set of all Scalar multiples of the identity can be identified with C.

Thrm 8:

If 0 is the only topological divisor of Zero, in A. then A = C.

Proof

Let x be any element in A. Then $\sigma(x)$ is non-empty and also closed.

$\therefore$ It has a boundary point λ.

(by defn. of closed set)

$\therefore x - \lambda I$ is boundary point which is also Singular (by defn of $\sigma(x)$)

ie, $x - \lambda_1$ is boundary point of S. the set of Singular elements of A.

But, boundary of $S \subseteq Z$

$\therefore x - \lambda_1$ is a topological divisor of zero in A is 0.

$\therefore x - \lambda 1 = 0$ is $x = \lambda_1$

$\therefore$ Every element of A is a scalar Multiple of the identity.

$\therefore A$ can be identified with C or $A = C$.

Thrm 9

If the norm in A satisfies the inequality $\|xy\| \geq k\|x\|\|y\|$ for some +ve Constant C, then $A = C$.

Proof

Let z be a topological divisor of zero in A

Then, $\exists$ a sequence $\{z_n\}$ in A such that

$\|z_n\| = 1$ and $zz_n \longrightarrow 0$ (say)

$$zz_n \longrightarrow 0 \rightarrow \|zz_n\| \longrightarrow 0$$

By data,

$$\|zz_n\| \geq k\|z\|\|z_n\|, \quad (k, +ve)$$

ie, $\|zz_n\| \geq k\|z\|$

$\therefore \|zz_n\| \to 0$ and $\|zz_n\| \geq k\|z\|$.

where k is +ve

The above $\Rightarrow z = 0$

the only topology divisor of zero is 0

$\therefore$ By previous thm $A = C$

<u>Thrm 10</u>.

If A is a Banach Subalgebra of a Banach algebra A'. then the spectra of an x in A and A' are valated as follows. ① $\sigma_{A'}(x) \subseteq \sigma_A(x)$

② each boundary point of $\sigma_A(x)$ is also a boundary point of $\sigma_{A'}(x)$

<u>Proof</u>

1) Let $\lambda \in \sigma_{A'}(x)$ then by defn- of $\sigma_{A'}(x)$, $x - \lambda I$ is singular in A'. Since, $A \subseteq A'$, $x - \lambda I$ is singular in A also.

$\therefore \lambda \in \sigma_A(x)$, by defn. of Spectrum

Hence $\sigma_{A'}(x) = \sigma_A(x)$

2) Let λ be a boundary point of $\sigma_A(x)$. Then Correspondingly. there is a singular element $x - \lambda I$, Which is also a boundary point of S.

$\therefore x - \lambda_1$ is a topological divisor of zero in A (by thrm 5)

$\therefore x - \lambda_1$ is a topological divisor of zero in A' also since $A \subseteq A'$

$\Rightarrow x - \lambda_2$ is singular in A' (by thrm ④)

$\Rightarrow \lambda \in \sigma_{A'}(x)$. But $\sigma_{A'}(x) \subseteq \sigma_A(x)$, by Pre. proof

$\therefore \lambda$ is a boundary point of $\sigma_{A'}(x)$ also

Definition

Spectral radius

If x is an element of a Banach algebra A. then the spectral radius of x is defined as

$$r(x) = \sup\{|\lambda| : \lambda \in \sigma_A(x)\}$$

Note:

$$0 \le r(x) \le \|x\|$$

Proof

If λ is a complex no. such that $|\lambda| > \|x\|$ then $\|x/\lambda\| < 1$ or $\|1-(1-x/\lambda)\| < 1$

$\therefore$ By thrm (1), $1 - \frac{x}{\lambda}$ is regular and hence $x - \lambda 1$ is regular.

$\therefore$ if $|\lambda| > \|x\|$, then $x - \lambda 1$ is regular

$\Rightarrow$ if $|\lambda| \le \|x\|$, then $x - \lambda 1$ is regular and hence $\lambda \in \sigma_A(x)$ or it conversely if $\lambda \in \sigma_A(x)$ then $|\lambda| \le \|x\|$

~~$\therefore$ Sup H~~

$\therefore \sup\{|\lambda| : \lambda \in \sigma_A(x)\} \le \|x\|$ is

$r(x) \le \|x\|$.

The formula for Spectral radius

Let x be an element in banach algebra A is a Spectral radius is defined by

$$r(x) = \sup \{|\lambda| : \lambda \in \sigma_A(x)\}$$

Now, let A' be the banach subalgebra of A is generated by x.

The closure of the set of all polynomial of x. $r(x)$ has the same value if it is complete with respect to A.

$$r(x) = \sup \{|\lambda| : \lambda \in \sigma_{A'}(x)\}$$

lemma

$$\sigma(x^n) = (\sigma(x))^n$$

Proof:

Let λ be the non-zero complex number and $\lambda_1, \lambda_2, \ldots \lambda_n$ its distinct n^{th} roots then,

Consider,

$$x^n - \lambda I = (x - \lambda_1 I)(x - \lambda_2 I)\ldots(x - \lambda_n I)$$

$\therefore$ we have, $(x^n - \lambda I)$ is singular iff atleast one $(x - \lambda_i I)$ is singular

$$\text{ie, } \lambda \in \sigma(x^n) \text{ iff } \lambda_i \in \sigma(x) \text{ iff}$$

$$\lambda = \lambda_i^n \text{ then } \lambda_i \in \sigma(x)$$

$$\sigma(x^n) = (\sigma(x))^n$$

Hence proved.

Spectral radius theorem

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$$r(x) = \lim \|x^n\|^{1/n}$$

P.T the formula for the spectral radius.

Proof:

From the lemma, we have

$$\sigma(x^n) = (\sigma(x))^n$$

$$r(x^n) = (r(x))^n \rightarrow (*)$$

By the defn of spectral radius of x.

W.K.T $0 \le r(x) \le \|x\|$

Replace x by x^n

$$r(x^n) \le \|x^n\|$$

$$(r(x))^n \le \|x^n\|$$

$$(r(x))^{n \cdot \frac{1}{n}} = \|x^n\|^{1/n}$$

$$r(x) \le \|x^n\|^{1/n}, \forall n \rightarrow (A)$$

To conclude the Proof

It sufficies to prove that,

If a is any real number $\ni r(x) < a$

Then, $\|x^n\|^{1/n} \le a \; \forall n \ge$

But n is finite

Consider,

$$x(\lambda) = (x - \lambda I)^{-1}, \; \lambda$$

$$= \lambda^{-1}\left(\frac{x}{\lambda} - I\right)^{-1}$$

$$= \lambda^{-1}\left(\frac{x}{\lambda} - 1\right)^{-1}$$

$$= -\lambda^{-1}(1 - x/\lambda)^{-1}$$

$$= -\lambda^{-1}\left(1 + \frac{x}{\lambda} + \frac{x^2}{\lambda^2} + \cdots\right)$$

$$x(\lambda) = -\lambda^{-1}\left(1+\sum_{n=1}^{\infty}\frac{x^n}{\lambda^n}\right) \to ①$$

Let f be any function on A, Then

$$F(x(\lambda)) = -\lambda^{-1}\left(f(1)+\sum_{n=1}^{\infty} F(x^n)\lambda^{-n}\right) \to ②$$

This is true for all λ

using laurent Series,

for $|\lambda| \geq \|x\|$

$f(x(\lambda))$ is analytic function in the region $|\lambda| \geq r(x)$

Eqn (2) $\Rightarrow$ is the laurent expansion for $|\lambda| \geq \|x\|$

We know from complex analysis. to expansion is valid for $|\lambda| > r(x)$

let α be any real number $\ni: r(x) < \alpha < a$

Then the Series,

$\Sigma f\left(\frac{x^n}{\alpha^n}\right)$ converges for every $f \in A^*$ and hence its term form a bounded Sequence.

ie, $\left\{\frac{x^n}{\alpha^n}\right\}$ is a bounded set

$$\left\|\frac{x^n}{\alpha^n}\right\| \leq k, \forall n.$$

$$\|x^n\| < k\|\alpha^n\|$$

$$\Rightarrow \|x^n\|^{1/n} \leq k^{1/n}\|\alpha^n\|^{1/n}$$

$\|x^n\|^{1/n} < K^{1/n} \|\alpha^n\| < G$

$\|x^n\|^{1/n} \leq a$ for sufficiently large 'n'

$\|x^n\|^{1/n} \leq a, \forall n$ but n is finite

$\Rightarrow \|x^n\|^{1/n} \leq a \wedge 1$

$\rightarrow r(x) \geq \|x^n\|^{1/n} \longrightarrow (B)$

from (A) & (B)

$r(x) = \lim \|x^n\|^{1/n}$

Hence proved

It is Spectral radius thrm

Radial and semi Simplicity ideals.

Ideals:

An ideal in A was defined to be a subset I with the following 3 properties.

1) I is a linear subspace

2) $i \in I \rightarrow x_i \in I$, $\forall$ element $x \in A$

3) $i \in I \Rightarrow ix \in I, x \in A$.

If I statisfy the (1) & (2) condition then it is called "left ideal". If I statisfies (1) & (3)

If I satisfies all the 3 conditions then it is called "two sided ideal".

Regular:

An element $x \in A$ is regular then $\exists$ an element $y \ni: xy = yx = 1$

x is called left regular if $\exists$ an element $y \ni: yx = 1$.

x is called right regular if $\exists$ an element $y \ni : xy = 1$

$R = \cap MLI$ left Ideal, $\cap MRI \rightarrow$ right Ideal

lemma: 1

If r is a left ideal on R then $(1-r)$ is a left regular.

Proof:

Let $r \in R$

Suppose $(1-r)$ is left singular

let

$L = x(1-r)$

$L = x - xr,\ x \in A$

ie, L is proper left ideal which contains $1-r$, we must imbed L is a maximal left ideal M which contains $(1-r)$.

Since, $r \in R$

$1-r \in M$. and Since

$1 = (1-r) + r \in M$

$\Rightarrow 1 \in M$

which is contradiction

$1-r$ is left regular

lemma: 2

If r is an element of R then $(1-r)$ is a regular.

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Proof

By lemma 1,

If r is a left ideal of R. then $1-r$ is a left-regular

Since $r \in R$, $1-r$ is left regular

$\exists$: an element $s \in A \ni: s(1-r) = 1$

$\Rightarrow s$ is right regula

$$s(1-r) = 1$$

$$s - sr = 1$$

$$s = 1 + sr$$

$$s = 1 - (-sr)$$

$\therefore$ Since R is left ideal,

$x \in R \Rightarrow (-s) r \in R$

$1-(-sr) = s$ is left regular

$\therefore$ s is left regular

$\therefore$ s is both rights left regular with inverse $(1-r)$. So $(1-r)$ is also regular.

$\therefore$ Hence the Proof.

Lemma 3:

If r is an element of R, then $(1-xr)$ is regular for every x.

Proof:-

Since R is left ideal, $\forall x \in R$

So $xr \in R$.

But, W.K.T

If $x \in R$, then $(1-r)$ is regular now, $xr \in R$

$\therefore (1-xr)$ is regular. $\forall x$.

Lemma : 4

If r is an element of A with the proper that $(1-xr)$ is regular, $\forall x$ then r is in R.

Proof:

Suppose r does not belong to R then r is not in some maximal left ideal M.

Consider the set,

$M + Ar = \{M + xr, m \in M, x \in A\}$

To Prove:

$M + Ar$ is a left ideal containing both M & r

for let,

$M + xr \in M + Ar$ and $m + y_r \in M + Ar$

where

$m + x_r + m + y_r = m + (x+y)r \in M + Ar$

$\{\therefore x, y \in A, m \in M\}$.

Also,

If $M + x_r \in M + A_r$ & $y \in A$ - then,

$y(m + x_r) = ym + (xy)r = m + Ar$

$\Rightarrow y(m + x_r) \in M + Ar$

$x_r \in M + Ar$

$M + Ar = A$

Hence, $M + Ar = 1, \forall m$ & x

$m + xr = 1$ for some M & x

$(1 - xr) = m$ is a regular element in M

This is impossible for no proper ideal can contain any regular element.

Hence our assumption that $x \notin R$ is wrong

$\therefore x \in R$

Hence proved.

Note:

The effect of these lemma is to establish the equality of two sets

Intersection of maximal left ideal $R = \cap M$ and

$I = \{x : (1 - xr)\}$ is regular $\forall x \rightarrow ①$

Similarly,

we get by applying to the maximal right ideal we get

$\cap = \{r : 1 - rx$ is regular $\forall x\} \rightarrow ②$

This is maximal ideal.

We prove that,

All of these set are the same of showing that the two sets on the right of eqn (1) & (2) all equal to the one another.

This is called radial.

Lemma: 5.

If $1 - x_r$ is regular then $1 - rx$ is regular.

Proof:

let $1 - x_r$ be regular with inverse

$S = (1 - xr)^{-1}$

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$$(1-xr)S = S(1-xr) = 1$$

Consider,

$$(1-rx)^{-1}$$

$$(1-rx)^{-1} = 1 + rx + r^2x^2 + \cdots$$

$$= 1 + r[x + (xr)x + \cdots]$$

$$= 1 + rx[1 + xr + \cdots]$$

$$(1-rx)^{-1} = 1 + rx[1-xr]^{-1}$$

$$(1-rx)^{-1} = 1 + rSx$$

Now,

$$(1-rx)(1+rSx) = 1$$

III^{ly}

we can prove

$$(1+rSx)(1-rx) = 1$$

$\therefore (1-rx)$ is equal with inverse given $(-1+rSx)$

Theorem A:

The radical R of A equals each of the four sets is $\cap MLI = \{r ; 1-xr \text{ is regular}, \forall x\}$, $\cap MRI = \{r ; 1-rx \text{ is regular}, \forall x\}$ is therefore a proper two sided ideals.

Proof:

$$R = \cap MLI = \{r ; 1-xr \text{ is regular}, \forall x\}$$

$$R = \cap MRI = \{r ; 1-rx \text{ is regular}, \forall x\}$$

ie, R is multiplication of all left ideals and right ideal and hence, R is a proper two sided ideal.

Semi Simple

A is said to be semi simple if its radical equals the zero ideal $\{0\}$

ie, if each non-zero element of A is Outside Some maximal left ideal.

Thm - B

Every maximal left ideal in A is closed

Proof

WKT,

If I is an ideal in A then by the joint. Continunity of the algebraic equation is closure

To Prove A is closed

ie, I is an ideal of some kind

ie, I is an maximal left ideal.

Let M be the maximal left ideal.

Our Claim:

Every element of M is singular for it $x \in M$ is not singular it must be regular

x^{-1} exist & $x x^{-1} \in M$

$\Rightarrow 1 \in M$

$\rightarrow M = A$

which is $\rightarrow\leftarrow$

$\therefore x$ is a singular element

ie, M is a subset of S

ie. $M \subset S$

ie, The element of M are Singular where S is the set of singular element $M \subset S$

$\overline{M} \subset \overline{S}$ $\{\overline{S} = S, 1 \notin S\}$

$\bar{M} \subset S$

$1 \notin \bar{M}$

$\therefore \bar{M}$ is a proper ideal

Hence $M \neq \bar{M}$, $M = \bar{M}$

{ $\because$ because M is maximal $\subset \bar{M}$ }

$\therefore M$ is closed

A is closed

Hence proved.

Thrm: C

The radical R of A is a proper closed two sided ideal

Proof

Thrm(A) & Thrm(B).

Thrm D:

If I is a proper closed two sided ideal in A then the quotient algebra A/I is a banach algebra.

Proof

first we have to prove that : $A/_I$ is a banach space

let us define

$\|x+I\| = \inf\{\|x+i\|, i \in I\}$

i) $\|x+I\| \geq 0$

ii) Since I is closed

$\|x+I\| = 0$ iff $\exists$ a sequence $\{x_i\}$ in I

$\ni$: $\|x_i + I\| \to 0$

ie, iff $x \in I$

ie, $x + I = I$

$= \text{zero element of } A/I$

$x + I = 0$

Consider,

iii) $\| (x+I) + (y+I) \| = \| (x+y) + I \|$

$= \inf \{ \| x+y+i \| ; i \in I \}$

$= \inf \{ \| x+y+i+i' \| , i, i' \in I \}$

$= \inf \{ \| (x+i) + (y+i') \| , i, i' \in I \}$

$\leq \inf \{ \| x+i \| ; i \in I \} + \inf \| y+i' \| , i' \in I \}$

$= \| x + I \| + \| y + I \|$

$\| x+I \| + \| y+I \| = \| x+I \| + \| y+I \|$

iv) $\| \alpha (x+I) \| = \inf \{ \| \alpha (x+i) \| ; i \in I \}$

$= \inf \{ |\alpha| \| x+i \| , i \in I \}$

$= |\alpha| \{ \inf \| x+i \| ; i \in I \}$

$= \| x+I \| \, |\alpha|$

Hence A/I is normal linear space

Now we have to prove that A/I is Complete Space assuming A is complete.

We shall prove that

Let $\{ x_n + I \}$ be a Cauchy sequence in $\frac{A}{I}$.

Then we can find a sequence $\{ x_n + I \} \ni$:

$\| (x_1 + I) - (x_2 + I) \| \leq 1/2$

$$\|(x_2+I)-(x_3+I)\| \le 1/2^2$$

.....

.....

$$\|(x_n+I)-(x_{n+1}+I)\| \le 1/2^n$$

We shall prove that this eqn is Convergent in $\frac{A}{I}$.

Let $y_1 \in x_1+I$. Then we select it $y_2 \in x_2+I \ni \|y_1-y_2\| < 1/2$

Similarly, we have $y_3 \in x_3+I \ni \|y_2-y_3\| < \frac{1}{2^2}$

Continuing this way we get

$$\|y_n - y_{n+1}\| \le \frac{1}{2^n}$$

Hence, if $m<n$, we have

$$\|y_m-y_n\| = \{\|y_m-y_{m+1}+y_{m+1}-y_{m-2}+\cdots + y_{n-1}-y_n\|\}$$

$$\le \|y_m-y_{m+1}\| + \|y_{m+1}-y_{m+2}\| + \cdots + \|y_{n-1}-y_n\|$$

$$\le \frac{1}{2^m} + \frac{1}{2^{m+1}} + \cdots + \frac{1}{2^{n-1}}$$

$$= \frac{1}{2^m}\left[1-\frac{1}{2}\right] = \frac{1}{2^m}\left(1/2^{-1}\right)$$

$$\le 1/2^{m-1}$$

So the sequence, $\{y_n\}$ is convergent

$\therefore$ The sequence $\{y_n\}$ is a Cauchy sequence

by completeness of A $\exists$ a vector y in A

$\ni$ y_n. Now, we have to prove that,

$$x_n + I \to y + I$$

$$\|(x_n+I)-(y+I)\| = \|x_n - y + I\|$$

$$= \inf\{\|(x_n - y) + i\| ; i \in I\}$$

$$\leq \{\|(x_n + i) - y\| ; i \in I\}$$

$$\leq \|x_n + I - y\|$$

$$\|(x_n+I)-(y+I)\| \leq \|y_n - y\|$$

where,

$$y_n \to y$$

$$\|(x_n+I)-(y+I)\| \to 0$$

$$x_n + I \to y + I \in \frac{A}{I}$$

Thus every Cauchy sequence has a convergent subsequence in A/I and

$\frac{A}{I}$ is complete

$\to$ A/I is banach space

Hence we have to prove that,

$\frac{A}{I}$ is a banach algebra.

i) We shall prove that

$$\|1 + I\| = 1$$

$$\|1+I\| = \inf\{\|1+i\|;\ i \in I\}$$
$$\leq \|1\|$$
$$\|1+I\| \leq 1 \longrightarrow (1)$$

Consider,

$$\|(x+I)(y+I)\| = \|xy+I\|$$
$$= \inf\{\|xy+i\|;\ i \in I\}.$$

$$\|(x+R)(y+I)\| = \inf\{\|(x+i_1)(y+i_2)\|,\ i_1, i_2 \in I\}$$
$$\leq \inf\{\|x+i\| \cdot i_1 \in I\}.$$
$$\inf\{\|y+i_2\|;\ i_2 \in I\}$$

$$\|(x+I)(y+I)\| \leq \|x+I\|\,\|y+I\|$$
$$\|1+I\| \leq \|(1+I)^2\|$$
$$= \|(1+I)(1+I)\|$$
$$\|1+I\| \leq \|1+I\|^2$$
$$1 \leq \|1+I\| \longrightarrow (2)$$

from (1) & (2)

$$\|1+I\| = 1$$

$\frac{A}{I}$ is a banach Algebra.

Thrm. E

$\frac{A}{R}$ is a Semi-Simple banach algebra

Proof:

In thrm / D replace I by R and i by r then.

$\frac{A}{R}$ is a banach algebra

In Order to Prove that

$\frac{A}{R}$ is a semi simple.

To prove that: The radical of $\frac{A}{R}$ is $\{0\}$

Consider the homeomorphism $x \to x+R$ of $\frac{A}{R}$

Then the homeomorphism contain 1-1 & onto y.

If M is a maximal left ideal of A then

$R \subset M \subset A$

$\{\therefore R = \cap MLI \cdot \subset M \subset A\} \longrightarrow (*)$

$\therefore \frac{M}{R}$ is left ideal in $\frac{A}{R}$

Also, $\frac{M}{R}$ is a maximal left ideal

Thus there is 1-1 correspondence between the maximal left ideal in A this $\frac{A}{R}$ and radical of $A = \cap MLI$ & by $(*)$

$\exists$: a 1-1 correspondence b/w the radical of A and radicals of A/R

$\cap$ MLI in A $\longrightarrow$ $\cap$ MLI in $\frac{A}{R}$

If A is semi simple then the radical of $A = \{0\}$

Radical of A $\rightarrow$ Radical of $\frac{A}{R}$ is between A and radical of $\frac{A}{R}$.