Fuzzy Sets and their Applications.

UNIT - IV

4.1 Fuzzy Numbers:

Consider the fuzzy sets defined on the set R of real numbers. The membership function of such type of fuzzy sets are defined by

$$A : R \to [0,1]$$

clearly, such type of fuzzy sets have a quantitative meaning and under certain conditions may be viewed as fuzzy numbers or fuzzy intervals.

A fuzzy set A on R, which satisfy atleast the following three properties is called a fuzzy number.

(i) A must be a normal fuzzy set.

(ii) ${}^{\alpha}A$ must be a closed interval for every $\alpha \in [0,1]$

(iii) The support of A, ${}^{0+}A$ must be bounded.

The significance of the bounded support of a fuzzy number and its α-cuts for $\alpha \neq 0$ must be closed interval is that it allow us to define meaningful arithmetic operations on fuzzy numbers in terms of standard arithmetic operations on closed intervals. since all the α-cuts of any fuzzy number are required to be closed intervals $\forall\ \alpha \in (0,1]$.

Thus, every fuzzy number is a convex fuzzy set. However, the converse is not necessarily true, because α-cuts of some convex fuzzy sets may be open or half-open intervals.

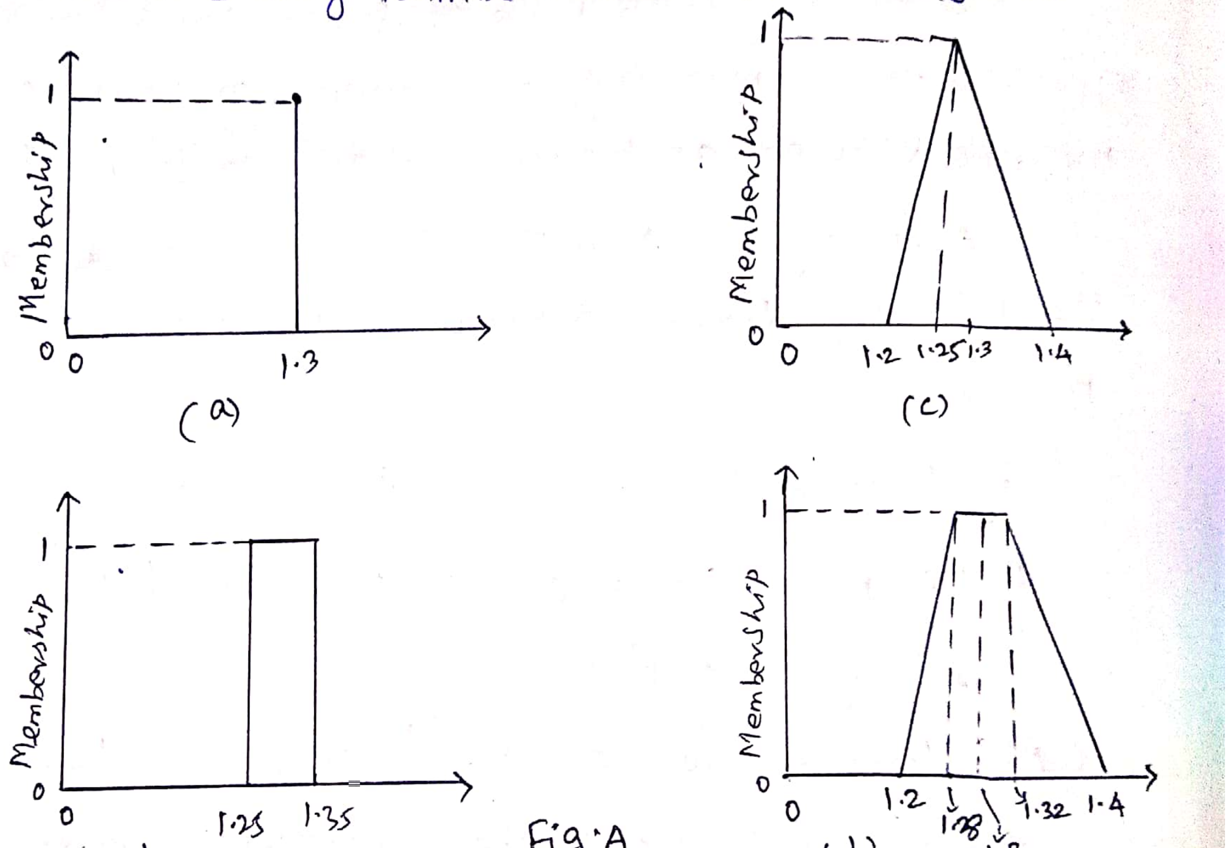
Special case of fuzzy numbers include ordinary real numbers and intervals of real numbers, as illustrated in Fig A :.

(a) an ordinary real no 1.3

(b) an ordinary (crisp) closed interval [1.25, 1.35]

(c) a fuzzy number expressing the proposition "close to 1.3"

(d) a fuzzy number with a flat region.



(b) Fig:A (d)

Although the triangular and trapezoidal shapes of membership functions shown in Fig (A) are used most often for representing fuzzy numbers, other shapes may be preferable in some applications.

Furthermore, membership functions of fuzzy numbers need not be symmetric as are those in Fig(A).

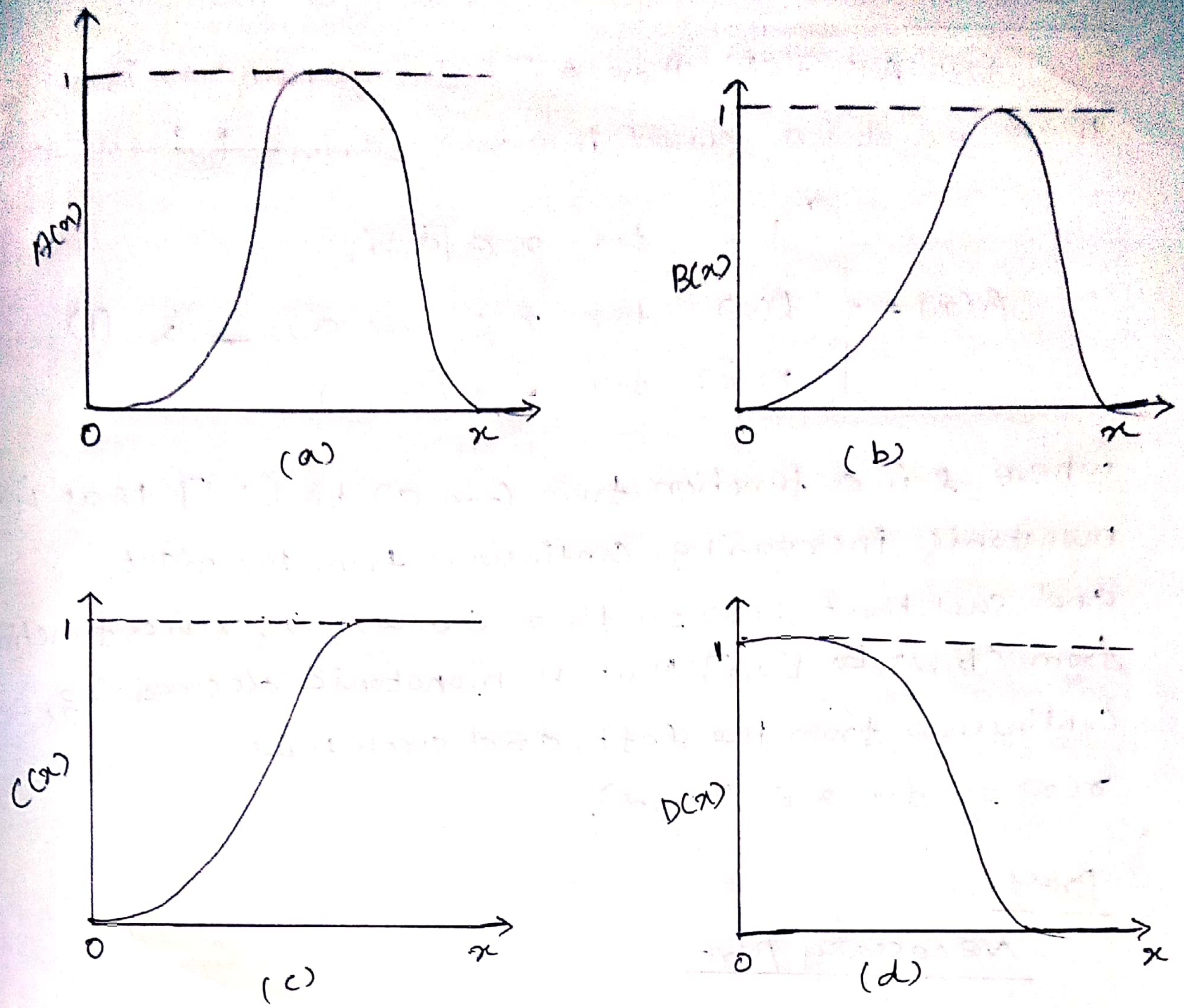


Fig: B

Family typical are so-called "bell-shaped" membership functions, as exemplified by the functions in fig (B) (a) symmetric. (b) asymmetric membership functions which only increase (fig (B) (c)) or only decrease (fig B, d) also quality as fuzzy numbers.

They capture our conception of a large number or a small number in the content of each particular application.

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Theorem 1

Let $A \in \mathcal{F}(\mathbb{R})$. Then, A is a fuzzy number iff there exists a closed interval $[a, b] \neq \phi$ such that

$$A(x) = \begin{cases} 1 & \text{for } x \in [a, b] \\ l(x) & \text{for } x \in (-\infty, a) \\ r(x) & \text{for } x \in (b, \infty) \end{cases} \longrightarrow (1)$$

where l is a function from $(-\infty, a)$ to $[0, 1]$ that is monotonic increasing, continuous from the right, and such that $l(x) = 0$ for $x \in (-\infty, \omega_1)$; r is a function from (b, ∞) to $[0, 1]$ that is monotonic decreasing, continuous from the left, and such that $r(x) = 0$ for $x \in (\omega_2, \infty)$.

Proof

Necessary part:

Since A is a fuzzy number, ${}^{\alpha}A$ is a closed interval for every $\alpha \in (0, 1]$. For $\alpha = 1$, ${}^{1}A$ is a non empty closed interval because A is normal. Hence, there exist a pair $a, b \in \mathbb{R}$ such that ${}^{1}A = [a, b]$, where $a \leq b$.

i.e, $A(x) = 1$ for $x \in [a, b]$ and $A(x) < 1$ for $x \notin [a, b]$.

Now, let $l(x) = A(x)$ for any $x \in (-\infty, a)$.

Then, $0 \le l(x) < 1$ since $0 \le A(x) < 1$ for every $x \in (-\infty, a)$.

Let $x \le y < a$; then $A(y) \ge \min[A(x), A(a)] = A(x)$.

By known theorem

"A fuzzy set A on $\mathbb{R}$ is convex iff

$A(\lambda x_1 + (1-\lambda)x_2) \ge \min[A(x_1), A(x_2)]$

$\forall x_1, x_2 \in \mathbb{R}$ and all $\lambda \in [0,1]$, where min

denotes the minimum operator."

Since A is convex and $A(a) = 1$. Hence, $l(y) \ge l(x)$

i.e, l is monotonic increasing.

Assume that $l(x)$ is not continuous from the right. This means that for some $x_0 \in (-\infty, a)$ there exists a sequence of numbers $\{x_n\}$ such that $x_n \ge x_0$ for any n and $\lim_{n \to \infty} x_n = x_0$

but $\lim_{n\to\infty} l(x_n) = \lim_{n\to\infty} A(x_n) = \alpha > l(x_0) = A(x_0)$

Now, $x_n \in {}^{\alpha}A$ for any n since ${}^{\alpha}A$ is a closed interval and hence, also $x_0 \in {}^{\alpha}A$.

$\therefore l(x_0) = A(x_0) \ge \alpha$, which is a contradiction.

i.e, $l(x)$ is continuous from the right.

The proof that function r in Thm 1 is monotonic decreasing and continuous from the left is similar.

$\because$ A is a fuzzy number, ${}^{0+}A$ is bounded.

Hence, $\exists$ a pair $\omega_1, \omega_2 \in \mathbb{R}$ of finite numbers such that $A(x) = 0$ for $x \in (-\infty, \omega_1) \cup (\omega_2, \infty)$.

Sufficient part:

Every fuzzy set A defined by eqn(1) is clearly normal, and its support, ${}^{0+}A$, is bounded, since ${}^{0+}A \subseteq [\omega_1, \omega_2]$. It remains to prove that ${}^{\alpha}A$ is a closed interval for any $\alpha \in (0,1]$.

Let $\quad x_\alpha = \inf\{x / l(x) \geq \alpha ; x < a\}$

$\quad\quad y_\alpha = \sup\{x / r(x) \geq \alpha , x > b\}$

for each $\alpha \in (0,1]$. We need to P.T ${}^{\alpha}A = [x_\alpha, y_\alpha]$ $\forall \alpha \in (0,1]$.

For any $x_0 \in {}^{\alpha}A$, if $x_0 < a$, then $l(x_0) = A(x_0) \geq \alpha$. i.e $x_0 \in \{x / l(x) \geq \alpha, x < a\}$, and consequently, $x_0 \geq \inf\{x / l(x) \geq \alpha, x < a\} = x_\alpha$.

If $x_0 > b$, then $r(x_0) = A(x_0) \geq \alpha$. i.e $x_0 \in \{x / r(x) \geq \alpha, x > b\}$ and, consequently, $x_0 \leq \sup\{x / r(x) \geq \alpha, x > b\} = y_\alpha$.

Obviously, $x_\alpha \leq a$ and $y_\alpha \geq b$; i.e $[a, b] \subseteq [x_\alpha, y_\alpha]$.

$\therefore x_0 \in [x_\alpha, y_\alpha]$ and hence, ${}^{\alpha}A \subseteq [x_\alpha, y_\alpha]$.

It remains to prove that $x_\alpha, y_\alpha \in {}^{\alpha}A$.

By the definition of x_α, there must exist a sequence $\{x_n\}$ in $\{x / l(x) \geq \alpha, x < a\}$ such that $\lim_{n \to \alpha} x_n = x_\alpha$, where $x_n \geq x_\alpha$ for any n. $\because l$ is continuous from the right, we've

$$l(x_\alpha) = l(\lim_{n \to \alpha} x_n) = \lim_{n \to \alpha} l(x_n) \geq \alpha.$$

Hence, $x_\alpha \in {}^{\alpha}A$. We can prove that $y_\alpha \in {}^{\alpha}A$ in a similar way.

Hence the theorem.

Fuzzy cardinality

Given a fuzzy set A defined on a finite universal set X, its fuzzy cardinality, $|\tilde{A}|$, is a fuzzy number defined on N by the formula $|\tilde{A}|(|{}^{\alpha}A|) = \alpha$ $\forall \alpha \in \Lambda(A)$.

4.2, Linguistic Variables.

The variables whose states are fuzzy numbers representing linguistics concepts, such as very large, large, medium, small and so on, as interpreted in a particular content are called linguistic variables.

Base variable

A base variable is a variable in the classical sense, examplified by any physical variable (ex., temperature, pressure, speed, humidity etc) as well as any other numerical variables. (ex. age, interest rate, performance, salary, blood count, probability, reliability, etc)

Note :

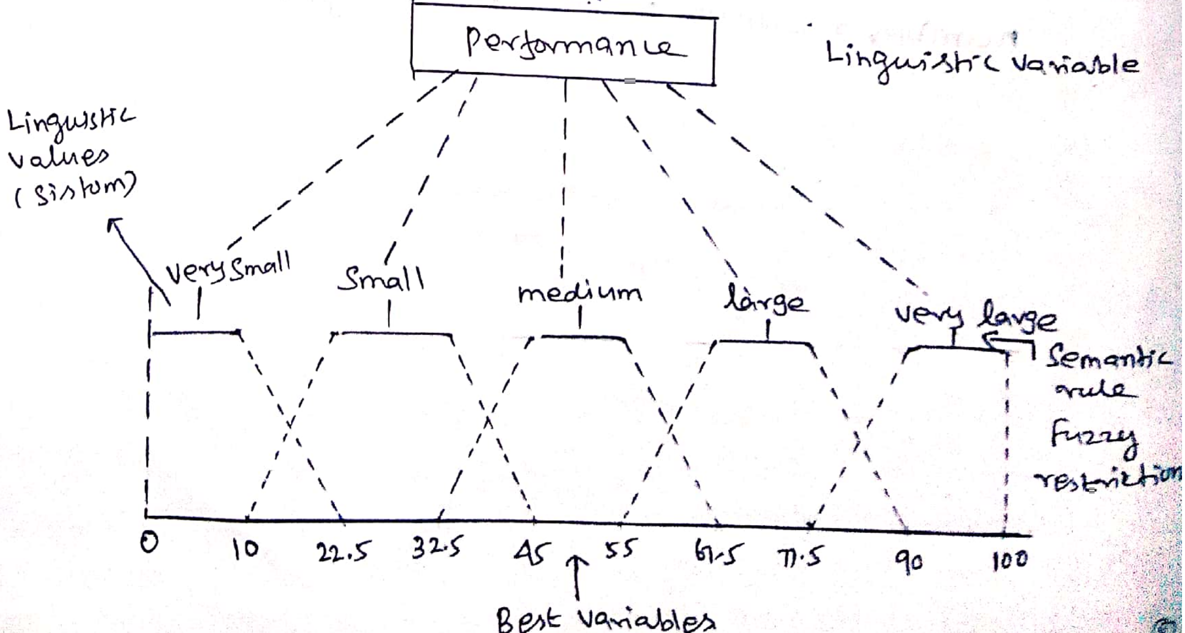
1. The states of a linguistic variables are expressed by linguistic terms interpreted as specify fuzzy number is defined in terms of a base variables the values of which are real numbers within a specific range.

2. In a linguistic variable, linguistic terms representing approximate values of a base variable, germane to a particular application, are captured by appropriate fuzzy numbers.

3. Each linguistic variable is fully characterized by a quintuple (v, T, X, g, m) in which v is the name of the variable, T is the set of linguistic terms of v that refer to a base variable whose values range over a universal set X, g is a syntactic rule for generating linguistic terms, and m is a semantic rule that assign to each linguistic term $t \in T$, its meaning, $m(t)$, which is a fuzzy set on x (i.e $m: T \to F(x)$).

For Example

A linguistic variable expresses the performance of a goal-oriented entity in a given concept by 5 basic linguistic terms - very small, small, medium, large, very large - as well as linguistic terms generated by a syntactic rule, such as not very small, large or very large, very very small and so on. Each of the basic linguistic terms is assigned one of five fuzzy numbers by a semantic rule. The fuzzy numbers, whose membership functions have the usual trapezoidal shapes, are defined on the interval [0, 100]



A.3. Fuzzy Arithmetic operations on Intervals

Fuzzy arithmetic is based on the following two properties of fuzzy numbers.

(i) Each fuzzy set and thus also each fuzzy number, can fully and uniquely be represented by its α-cuts.

(ii) α-cuts of each fuzzy number are closed intervals of real numbers $\forall \alpha \in [0,1]$.

The above two properties states that the Arithmetic operations on fuzzy numbers can be define in terms of arithmetic operations on their α-cuts, i.e arithmetic operations on closed intervals.

Let $*$ denote any of the four arithmetic operations. i.e addition $(+)$, substraction $(-)$, multiplication $(\cdot)$ and division $(/)$, on closed intervals, then

$$[a,b] * [d,e] = \{ f * g : a \leq f \leq b,\ d \leq g \leq e \}$$

is a general property of all arithmetic operations on closed intervals, except that $[a,b] / [d,e]$ is not defined when $0 \in [d,e]$.

Note:

The result of an arithmetic operation on closed intervals is again a closed interval.

The four arithmetic operations on closed intervals are defined as follows.

(i) Addition : $[a,b] + [d,e] = [a+d, b+e]$

(ii) Subtraction: $[a,b] - [d,e] = [a-e, b-d]$.

(iii) Multiplication:

$[a,b] \cdot [d,e] = [\min(ad, ae, bd, be), \max(ad, ae, bd, be)]$

(iv) Division:

$[a,b] / [d,e] = [a,b] \cdot [\frac{1}{e}, \frac{1}{d}]$

$= [\min(\frac{a}{d}, \frac{a}{e}, \frac{b}{d}, \frac{b}{e}), \max(\frac{a}{d}, \frac{a}{e}, \frac{b}{e}, \frac{b}{d})]$

Provided $0 \notin [d,e]$.

The above four operations are also called interval valued Arithmetic operations.

Note

1. A real number $a \in R$, may also be regarded as a special interval $[a,a]$.

2. $\tilde{0} = [0,0]$, $\tilde{1} = [1,1]$.

For ex:

$[3,5] + [2,4] = [5,9]$

$[3,5] - [2,4] = [-1,3]$

$[3,4] \cdot [2,2] = [6,8]$

$[-1,1] / [-2,-0.5] = [-2,2]$.

Theorem 2

Let $A = [a_1, a_2]$, $B = [b_1, b_2]$, $C = [c_1, c_2]$, $\tilde{0} = [0, 0]$, $\tilde{1} = [1, 1]$. Then show that

(i) $A+B = B+A$, $A \cdot B = B \cdot A$ – Commutativity

(ii) $(A+B)+C = A+(B+C)$, $(A \cdot B) \cdot C = A \cdot (B \cdot C)$ – Associativity

(iii) $A = \tilde{0} + A = A + \tilde{0}$, $A = \tilde{1} \cdot A = A \cdot \tilde{1}$ – Identity

(iv) $A \cdot (B+C) \subseteq A \cdot B + A \cdot C$ – subdistributivity

(v) If $b, c \geq 0$ for every $b \in B$ and $c \in C$, then

$A \cdot (B+C) = A \cdot B + A \cdot C$ – Distributivity

(vi) $0 \in A - A$ and $1 \in A/A$

(vii) If $A \subseteq E$ and $B \subseteq F$, then

$A+B \subseteq E+F$, $A-B \subseteq E-F$, $A \cdot B \subseteq E \cdot F$,

$A/B \subseteq E/F$ – Inclusion monotonicity.

Proof:

Given $A = [a_1, a_2]$, $B = [b_1, b_2]$

(i) $A+B = [a_1, a_2] + [b_1, b_2] = [a_1 + b_1, a_2 + b_2]$

$= [b_1 + a_1, b_2 + a_2]$

Thus $\boxed{A+B = B+A}$

$A \cdot B = [a_1, a_2] \cdot [b_1, b_2]$

$= [\min(a_1b_1, a_1b_2, a_2b_1, a_2b_2),$
$\max(a_1b_1, a_1b_2, a_2b_1, a_2b_2)]$

$= [\min(b_1a_1, b_2a_1, b_1a_2, b_2a_2),$
$\max(b_1a_1, b_2a_1, b_1a_2, b_2a_2)]$

Thus $\boxed{A \cdot B = B \cdot A}$

(ii) $(A+B)+C = ([a_1, a_2] + [b_1, b_2]) + [c_1, c_2]$

$= [a_1 + b_1, a_2 + b_2] + [c_1, c_2]$

$= [a_1, a_2] + [b_1 + c_1, b_2 + c_2]$

$= [a_1, a_2] + ([b_1 + b_2] + [c_1 + c_2])$

$\therefore (A+B)+C = A+(B+C)$

Illy $(A \cdot B) \cdot C = A \cdot (B \cdot C)$

(iii) $\bar{0} = [0,0]$ and $\bar{1} = [1,1]$

$A = [a_1, a_2] = [a_1 + 0, a_2 + 0]$

$= [a_1, a_2] + [0,0] = A + \bar{0}$

and $A = [a_1, a_2] = [0 + a_1, 0 + a_2]$

$= [0,0] + [a_1, a_2] = \bar{0} + A$

Thus $A = \bar{0} + A = A + \bar{0}$

Similarly $1 \cdot A = [1,1] \cdot [a_1, a_2]$

$= [\min(1 \cdot a_1, 1 \cdot a_2, 1 \cdot a_1, 1 \cdot a_2),$

$\max(1 \cdot a_1, 1 \cdot a_2, 1 \cdot a_1, 1 \cdot a_2)]$

$= [\min(a_1, a_2, a_1, a_2), \max(a_1, a_2, a_1, a_2)]$

$= [\min(a_1, a_2), \max(a_1, a_2)]$

$= [a_1, a_2] = A.$

Thus. $A = 1 \cdot A = A \cdot 1$

(iv) we have $A \cdot (B + C) = \{a \cdot (b + c) / a \in A, b \in B, c \in C\}$

$= \{a \cdot b + a \cdot c / a \in A, b \in B, c \in C\}$

Thus $A \cdot (B + C) \subseteq \{a \cdot b + a' \cdot c / a, a' \in A, b \in B, c \in C\}$

$= A \cdot B + A \cdot C$

Hence $A \cdot (B + C) \subseteq A \cdot B + A \cdot C$

(v) Let us assume that $b_1 \geq 0$ and $c_1 \geq 0$. Then, we've to consider the following 3 cases.

(a) If $a_1 \geq 0$ then

$A \cdot (B + C) = [a_1 \cdot (b_1 + c_1), a_2 \cdot (b_2 + c_2)]$

$= [a_1 \cdot b_1, a_2 \cdot b_2] + [a_1 \cdot c_1, a_2 \cdot c_2]$

$A \cdot (B + C) = A \cdot B + A \cdot C$

(b) If $a_1 < 0$ and $a_2 \leq 0$ then $-a_2 \geq 0$, $(-A) = [-a_2, -a_1]$, and

$(-A) \cdot (B+C) = (-A) \cdot B + (-A) \cdot C$

Hence $A \cdot (B+C) = A \cdot B + A \cdot C$

(c) If $a_1 < 0$ and $a_2 > 0$, then

$$A \cdot (B+C) = [a_1 \cdot (b_2 + c_2), a_2 \cdot (b_2 + c_2)]$$

$$= [a_1 \cdot b_2, a_2 \cdot b_2] + [a_1 \cdot c_2, a_2 \cdot c_2]$$

$$= A \cdot B + A \cdot C$$

(vi) $A - A = [a_1, a_2] - [a_1, a_2]$

$$= [a_1 - a_2, a_2 - a_1]$$

$$= [-(a_2 - a_1), a_2 - a_1]$$

Let $a = (a_2 - a_1)$

$\Rightarrow A - A = [-a, a]$

Thus, $0 \in [-a, a] = A - A$

Hence $0 \in A - A$

Similarly $1 \in A/A$

$$\frac{A}{A} = \left[\min\left(\frac{a_1}{a_1}, \frac{a_1}{a_2}, \frac{a_2}{a_1}, \frac{a_2}{a_2}\right), \max\left(\frac{a_1}{a_1}, \frac{a_1}{a_2}, \frac{a_2}{a_1}, \frac{a_1}{a_2}\right)\right]$$

Case (i) If $a_1 < a_2$

$\Rightarrow \frac{a_1}{a_2} < 1$ and $\frac{a_2}{a_1} > 1$

$\Rightarrow \frac{A}{A} = \left[\frac{a_1}{a_2}, \frac{a_2}{a_1}\right] \Rightarrow 1 \in A/A$

Case (ii) If $a_1 = a_2 \Rightarrow \frac{A}{A} = [1, 1] \Rightarrow 1 \in \frac{A}{A}$

Case (iii) If $a_1 > a_2 \Rightarrow \frac{a_1}{a_2} > 1$ and $\frac{a_2}{a_1} < 1$

$\Rightarrow \frac{A}{A} = \left[\frac{a_2}{a_1}, \frac{a_1}{a_2}\right] \Rightarrow 1 \in \frac{A}{A}$.

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(vii) Let $A = [a_1, a_2]$; $B = [b_1, b_2]$

$E = [e_1, e_2]$ and $F = [f_1, f_2]$

Since $A \subseteq E$

$\Rightarrow [a_1, a_2] \subseteq [e_1, e_2]$

$\Rightarrow e_1 \leq a_1$ and $a_2 \leq e_2$ — (1)

also $B \subseteq F$

$\Rightarrow [b_1, b_2] \subseteq [f_1, f_2]$

$f_1 \leq b_1$ and $b_2 \leq f_2$ — (3)

from eqn (1) & (2), we've

$e_1 + f_1 \leq a_1 + b_1$ and $a_2 + b_2 \leq e_2 + f_2$

$\Rightarrow [a_1 + b_1, a_2 + b_2] \subseteq [e_1 + f_1; e_2 + f_2]$

$\Rightarrow [a_1, a_2] + [b_1, b_2] \subseteq [e_1, e_2] + [f_1, f_2]$

$A + B \subseteq E + F$

Now, from eqn (1), we've

$e_1 \leq a_1$ and $a_1 \leq e_2$ — (3)

from eqn (2), we've

$f_1 \leq b_1$ and $b_2 \leq f_2$

$\Rightarrow -f_1 \geq -b_1$ and $-b_2 \geq -f_2$

$\Rightarrow -f_2 \leq -b_2$ and $-b_1 \leq -f_1$ — (4)

on adding eqn (3) and (4), we've

$e_1 - f_2 \leq a_1 - b_2$ and $a_2 - b_1 \leq e_2 - f_1$

$\Rightarrow [a_1 - b_2, a_2 - b_1] \subseteq [e_1 - f_2, e_2 - f_1]$

$\Rightarrow [a_1, a_2] - [b_1, b_2] \subseteq [e_1, e_2] - [f_1, f_2]$

$\Rightarrow A - B \subseteq E - F$

we've $A = [a_1, a_2]$, $B = [b_1, b_2]$

$E = [e_1, e_2]$, $F = [f_1, f_2]$

$A \subseteq E$ and $B \subseteq F$

$\Rightarrow [a_1, a_2] \subseteq [e_1, e_2]$ and $[b_1, b_2] \subseteq [f_1, f_2]$

$\Rightarrow e_1 \leq a_1$ and $a_2 \leq e_2$ and $f_1 \leq b_1$ and $b_2 \leq f_2$ — (5)

$\Rightarrow \min\{e_1f_1, e_1f_2, e_2f_1, e_2f_2\} \leq \min\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}$

& $\max\{a_1b_1, a_1f_2, a_2b_1, a_2b_2\} \leq \max\{e_1f_1, e_1f_2, e_2f_1, e_2f_2\}$

Thus.

$[\min\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}, \max\{a_1b_1, a_1b_2, a_2b_1, a_2b_2\}]$

$\subseteq [\min\{e_1f_1, e_1f_2, e_2f_1, e_2f_2\}, \max\{e_1f_1, e_1f_2, e_2f_1, e_2f_2\}]$

$\Rightarrow A \cdot B \subseteq E \cdot F$

Now from eqn (5), we've

$e_1 \leq a_1$, $a_2 \leq e_2$, $f_1 \leq b_1$, $b_2 \leq f_2$

$\Rightarrow e_1 \leq a_1$, $a_2 \leq a_2$, $\frac{1}{b_1} \leq \frac{1}{f_1}$, $\frac{1}{f_1} \leq \frac{1}{b_2}$

Thus, $\min\left\{\frac{e_1}{f_1}, \frac{e_1}{f_2}, \frac{e_2}{f_1}, \frac{e_2}{f_2}\right\} \leq \min\left\{\frac{a_1}{b_1}, \frac{a_1}{b_2}, \frac{a_2}{b_1}, \frac{a_2}{b_2}\right\}$

$\max\left\{\frac{a_1}{b_1}, \frac{a_1}{b_2}, \frac{a_2}{b_1}, \frac{a_2}{b_2}\right\} \leq \max\left\{\frac{e_1}{f_1}, \frac{e_1}{f_2}, \frac{e_2}{f_1}, \frac{e_2}{f_2}\right\}$

Thus $\left[\min\left\{\frac{a_1}{b_1}, \frac{a_1}{b_2}, \frac{a_2}{b_1}, \frac{a_2}{b_2}\right\}, \max\left\{\frac{a_1}{b_1}, \frac{a_1}{b_2}, \frac{a_2}{b_1}, \frac{a_2}{b_2}\right\}\right]$

$\subseteq \left[\min\left\{\frac{e_1}{f_1}, \frac{e_1}{f_2}, \frac{e_2}{f_1}, \frac{e_2}{f_2}\right\}, \max\left\{\frac{e_1}{f_1}, \frac{e_1}{f_2}, \frac{e_2}{f_1}, \frac{e_2}{f_2}\right\}\right]$

$\Rightarrow \frac{A}{B} \subseteq E/F$

Hence proved.

4.4 Arithmetic operations on Fuzzy Numbers

In general, there are two methods for developing fuzzy arithmetic:

(i) First method is based on interval arithmetic

(ii) Second method is based on extension Principle by which operations on real numbers are extended to operations on fuzzy numbers.

Method 1: Interval Fuzzy Arithmetic

Let A and B are fuzzy numbers and let $*$ denote any of four basic arithmetic-operations.

Then a fuzzy set $A * B$ on R and its α-cut, ${}^{\alpha}(A * B)$ is denfined as

$${}^{\alpha}(A * B) = {}^{\alpha}A * {}^{\alpha}B \quad \text{for any } \alpha \in (0,1]$$

and $$A * B = \bigcup_{\alpha \in [0,1]} {}^{\alpha}(A * B)$$

$\because$ ${}^{\alpha}(A * B)$ is closed interval for each $\alpha \in [0,1]$ and A, B are fuzzy numbers, thus $A * B$ is also a fuzzy number.

Example

Consider two triangular-shape fuzzy numbers A and B defined as follows.

$$\begin{array}{lll} 0 & \text{for} & x \leq -2,\ x > 4 \\ \frac{x+2}{3} & \text{for} & -2 < x \leq 1 \\ 4-x/3 & \text{for} & 1 < x \leq 4 \end{array}$$

$$\begin{array}{lll} 0 & \text{for} & x \leq 2 \text{ and } x \geq 8 \\ x-2/3 & \text{for} & 2 < x \leq 5 \\ 8-x/3 & \text{for} & 5 < x \leq 8 \end{array}$$

Find $A+B$, $A-B$, $A \cdot B$ and A/B and represent them by Graphs.

Solution:

The α-cuts of given, fuzzy sets A and B are

$$\alpha_A = [3\alpha-2, 4-3\alpha], \alpha_B = [3\alpha+2, 8-3\alpha]$$

we have to find the sum, difference, product and division of A and B.

For this, we've

$$\alpha(A+B) = \alpha_A + \alpha_B$$

$$= [3\alpha-2, 4-3\alpha] + [3\alpha+2, 8-3\alpha]$$

$$= [6\alpha, 12-6\alpha]$$

$$\alpha(A-B) = \alpha_A - \alpha_B$$

$$= [3\alpha-2, 4-3\alpha] - [3\alpha+2, 8-3\alpha]$$

$$= [6\alpha-10, 2-6\alpha]$$

$$\alpha(A \cdot B) = \alpha_A \cdot \alpha_B$$

$$= [3\alpha-2, 4-3\alpha] \cdot [3\alpha+2; 8-3\alpha]$$

$$= [\min[9\alpha^2-4, -9\alpha^2+6\alpha+8, -9\alpha^2+30\alpha-16, 9\alpha^2-36\alpha+32],$$

$$\max[9\alpha^2-4, -9a^2+6\alpha+8, -9\alpha^2+30\alpha-16, 9\alpha^2-36\alpha+32]]$$

(or)
$$\alpha(A \cdot B) = \begin{cases} [-9\alpha^2+30\alpha-16, 9\alpha^2-36\alpha+32] \text{ for } \alpha \in [0, .5] \\ [-9\alpha^2+30\alpha-16, 9\alpha^2-36\alpha+32] \text{ for } \alpha \in [.5, 1] \end{cases}$$

$$\alpha(A/B) = \begin{cases} \left[\frac{3\alpha-2}{3\alpha+2}, \frac{4-3\alpha}{3\alpha+2}\right] \text{ for } \alpha \in [0, .5] \\ \left[\frac{3\alpha-2}{8-3\alpha}, \frac{4-3\alpha}{3\alpha+2}\right] \text{ for } \alpha \in [0.5, 1] \end{cases}$$

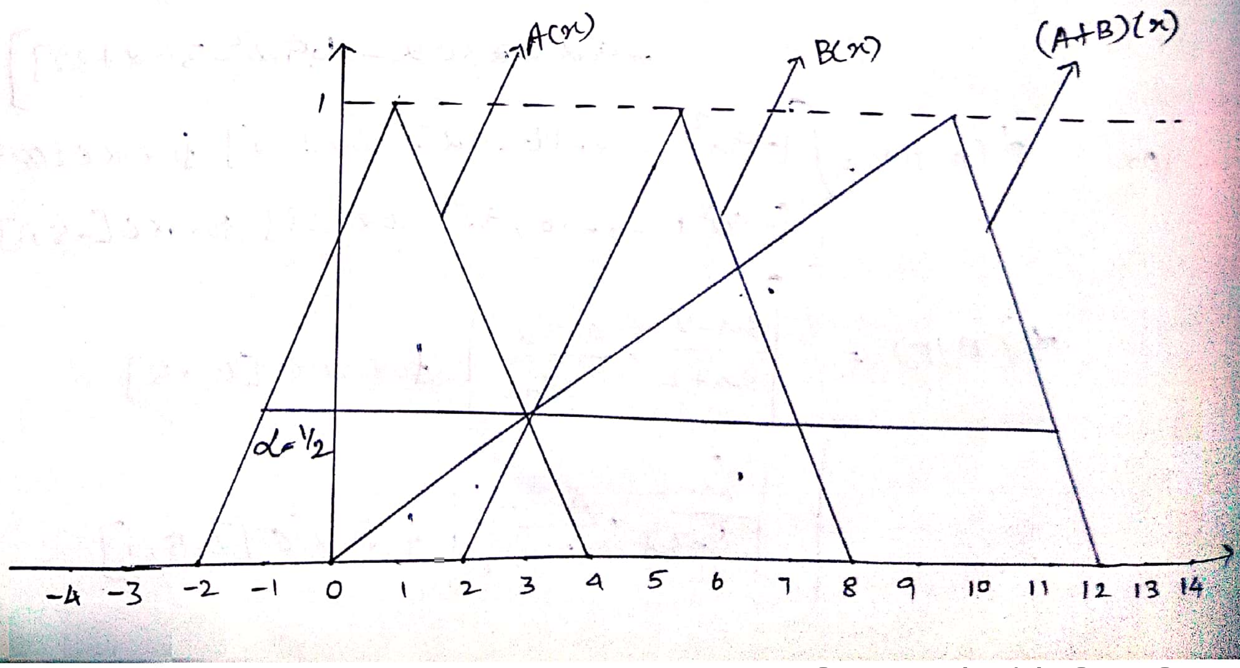
and the corresponding fuzzy numbers are given by

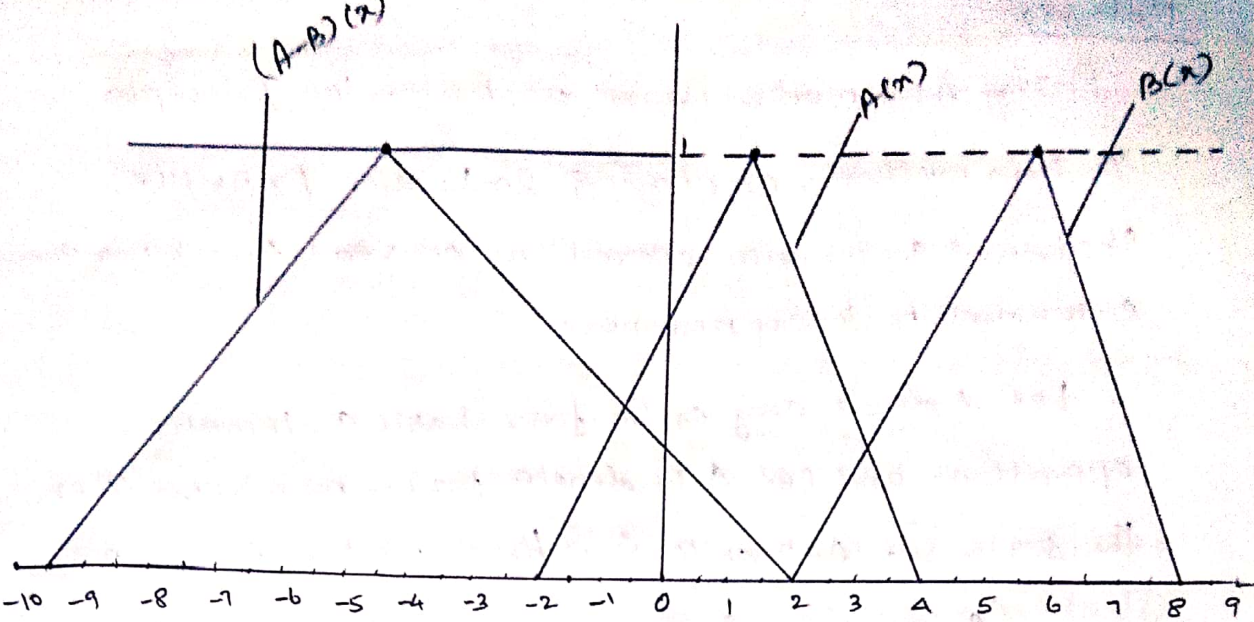
$$(A+B)(x) = \begin{cases} 0 & \text{for } x \le 0, x \ge 12 \\ x/6 & 0 < x \le 6 \\ \dfrac{12-x}{6} & 6 < x \le 12 \end{cases}$$

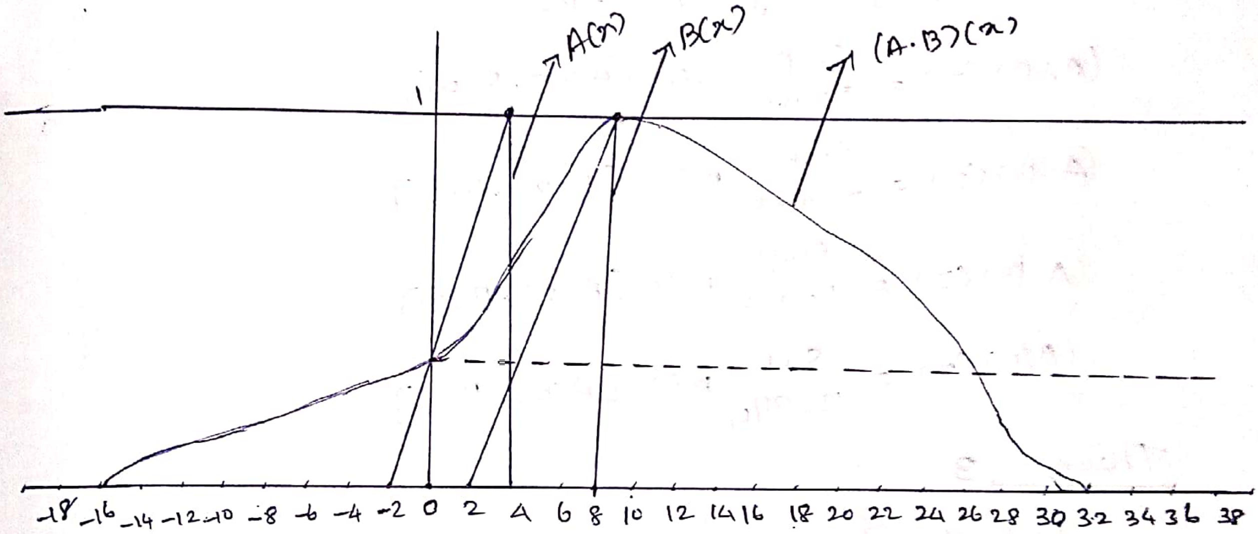
$$(A-B)(x) = \begin{cases} 0 & \text{for } x \le -10 \\ x+10/6 & -10 < x \le -4 \\ 2-x/6 & -4 < x \le 2 \end{cases}$$

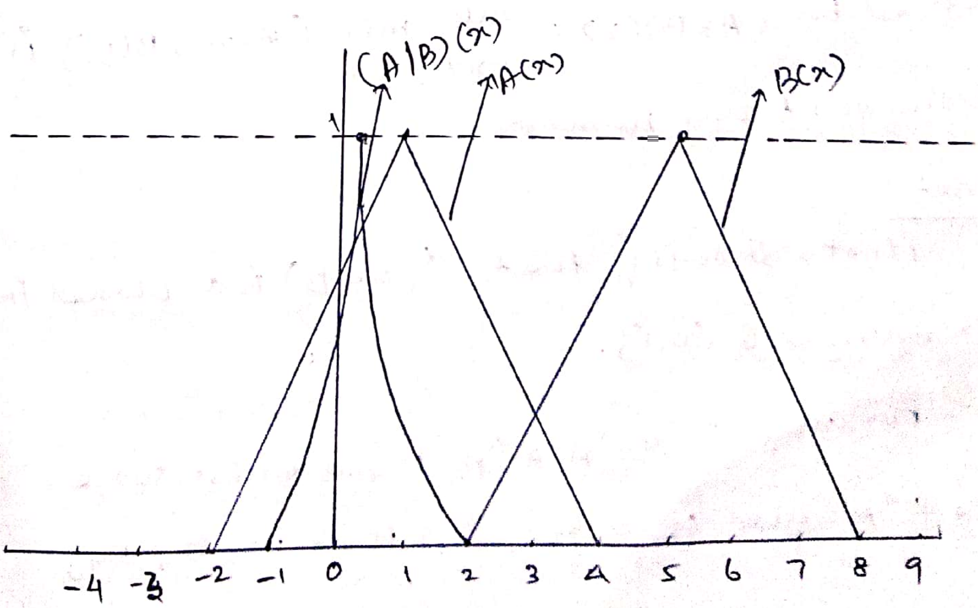
$$(A \cdot B)(x) = \begin{cases} 5-\sqrt{9-x}/3 & \text{for } -16 < x \le 0 \\ \sqrt{4+x}/3 & 0 < x \le 5 \\ 6-\sqrt{4+x}/3 & 5 < x \le 32 \\ 0 & x \le -16, x \ge 32 \end{cases}$$

$$(A/B)(x) = \begin{cases} 0 & \text{for } x \le -1, x \ge 2 \\ 2(x+1)/3(1-x) & -1 < x \le 0 \\ 8x+2/3(1+x) & 0 < x \le 1/5 \\ 4-2x/3(1+x) & \frac{1}{5} < x \le 2 \end{cases}$$




(A-B)(x)
A(x)
B(x)
1
-10 -9 -8 -7 -6 -5 -4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9


A(x)
B(x)
(A.B)(x)
1
-18 -16 -14 -12 -10 -8 -6 -4 -2 0 2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38


(A/B)(x)
A(x)
B(x)
1
-4 -3 -2 -1 0 1 2 3 4 5 6 7 8 9

Method 2:

Fuzzy Arithmetic Based on Extension Principle

In this method, employing extension principle, Standard arithmetic operations on real numbers are extended to fuzzy numbers.

Let $*$ denote any of the four basic arithmetic operations and let A, B denote fuzzy numbers. Then the fuzzy set $A*B$ on $\mathbb{R}$ is defined by the equation.

$$(A*B)(z) = \sup_{z=x*y} \min[A(x), B(y)] \quad \forall z \in \mathbb{R}.$$

More specifically, $\forall z \in \mathbb{R}$

$$(A+B)(z) = \sup_{z=x+y} \min[A(x), B(y)]$$

$$(A-B)(z) = \sup_{z=x-y} \min[A(x), B(y)]$$

$$(A \cdot B)(z) = \sup_{z=x \cdot y} \min[A(x), B(y)]$$

$$(A/B)(z) = \sup_{z=x/y} \min[A(x), B(y)]$$

Theorem: 3

Let $* \in \{+, -, \cdot, /\}$, and let A, B denote Continuous fuzzy numbers. Then, the fuzzy set $A*B$ defined by $(A*B)(z) = \sup_{z=x*y} \min[A(x), B(y)]$ is Continuous fuzzy number.

Proof

First, showing that ${}^{\alpha}(A*B)$ is a closed interval for every $\alpha \in (0,1]$.

For any $z \in {}^{\alpha}A * {}^{\alpha}B$, there exist some $x_0 \in {}^{\alpha}A$ and $y_0 \in {}^{\alpha}B$ such that $z = x_0 * y_0$

Thus, $(A*B)(z) = \sup_{z=x*y} \min[A(x), B(y)]$

$\geq \min[A(x_0), B(y_0)]$

$\geq \alpha$

Hence, $z \in {}^{\alpha}(A*B)$ and consequently,

${}^{\alpha}A * {}^{\alpha}B \subseteq {}^{\alpha}(A*B)$ for any $z \in {}^{\alpha}(A*B)$, we've

$$(A*B)(z) = \sup_{z=x*y} \min[A(x), B(y)] \geq \alpha$$

Moreover, for any $n > [1/\alpha]+1$, where $[1/\alpha]$ denotes the largest integer that is less than or equal to $1/\alpha$, there exist x_n and y_n such that $z = x_n * y_n$ and

$\min[A(x_n), B(y_n)] > \alpha - 1/n$

i.e, $x_n \in {}^{\alpha-1/n}A$, $y_n \in {}^{\alpha-1/n}B$ and we may consider two sequences $\{x_n\}$ and $\{y_n\}$.

$\because \alpha - \frac{1}{n} \leq \alpha - \frac{1}{n+1}$

we've

$${}^{\alpha-1/(n+1)}A \subseteq {}^{\alpha-1/n}A, \quad {}^{\alpha-1/(n+1)}B \subseteq {}^{\alpha-1/n}B$$

Hence, $\{x_n\}$ and $\{y_n\}$ fall into some ${}^{\alpha-1/n}A$ and ${}^{\alpha-1/n}B$, respectively. Since the latter are closed intervals, $\{x_n\}$ and $\{y_n\}$ are bounded sequences. Thus, there exists a convergent subsequence $\{x_n\}$ such that $x_{n,i} \to x_0$. To the corresponding subsequence $\{y_{n,i}\}$ there also exists a convergent subsequence $\{x_{n,i,j}\}$, such that $y_{n,i,j} \to y_0$.

If we take the corresponding subsequence, $\{x_{n,i,j}\}$ from $\{x_{n,i}\}$, then $x_{n,i,j} \to x_0$.

Thus, we have two sequences $\{x_{n,i,j}\}$ and $\{y_{n,i,j}\}$

Such that $x_{n,i,j} \to x_0$, $y_{n,i,j} \to y_0$ and

$x_{n,i,j} * y_{n,i,j} = z$.

Now, $\because$ $*$ is continuous,

$$z = \lim_{j \to \infty} x_{n,i,j} * y_{n,i,j} = (\lim_{j \to \infty} x_{n,i,j}) * (\lim_{j \to \infty} y_{n,i,j})$$

$$= x_0 * y_0$$

Also, $\because$ $A(x_{n,i,j}) > \alpha - \frac{1}{n_{i,j}}$ and $B(y_{n,i,j}) > \alpha - \frac{1}{n_{i,j}}$

$$A(x_0) = A(\lim_{j \to \infty} x_{n,i,j}) = \lim_{j \to \infty} A(x_{n,i,j}) \geq \lim_{j \to \infty} (\alpha - \frac{1}{n_{j,j}}) = \alpha$$

$$B(y_0) = B(\lim_{j \to \infty} y_{n,i,j}) = \lim_{j \to \infty} B(y_{n,i,j}) \geq \lim_{j \to \infty} (\alpha - \frac{1}{n_{i,j}}) = \alpha$$

$\therefore$ $\exists$ $x_0 \in {}^{\alpha}A$, $y_0 \in {}^{\alpha}B$ such that $z = x_0 * y_0$.

i.e $z \in {}^{\alpha}A * {}^{\alpha}B$. Thus ${}^{\alpha}(A*B) \subseteq {}^{\alpha}A * {}^{\alpha}B$

and consequently ${}^{\alpha}(A*B) = {}^{\alpha}A * {}^{\alpha}B$

Now we prove that $A*B$ must be continuous.

Assume $A*B$ is not continuous at z_0,

i.e

$$\lim_{z \to z_0} (A*B)(z) < (A*B)z_0 = \sup_{z_0 = x*y} \min[A(x), B(y)]$$

Then, there must exist x_0 and y_0 such that

$z_0 = x_0 * y_0$ and

$$\lim_{z \to z_0^-} (A * B)(z) < \min [A(x_0), B(y)] \quad \text{--- (1)}$$

Since the operation $* \in \{+, -, \cdot, /\}$ is monotonic with respect to the first and second arguments respectively, we can always find two sequences $\{x_n\}$ and $\{y_n\}$ such that $x_n \to x_0$, $y_n \to y_0$ as $n \to \infty$.

and $x_n * y_n < z_0$ for any n.

Let $z_n = x_n * y_n$, then $z_n \to z_0$ as $n \to \infty$.

Thus,

$$\lim_{z \to z_0^-} (A * B)(z) = \lim_{n \to \infty} (A * B)(z_n) = \lim_{n \to \infty} \sup_{z_n = x * y} \min [A(x), B(y)]$$

$$\geq \lim_{n \to \infty} \min [A(x_n), B(y_n)]$$

$$= \min \left[A(\lim_{n \to \infty} x_n), B(\lim_{z \to z_0} y_n) \right]$$

$$= \min [A(x_0), B(y_0)]$$

This contradicts equation (1). and

$\therefore$ $A * B$ must be a continuous fuzzy number.

This completes the proof.