Unit - 1

Partial differential equation of 1^{st} order.

Definition: PDE:

Any differential equation contains two or more than independent variables is called PDE.

Eg: i) $\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial \theta}{\partial t}$

is second order equation in two variables

ii) $x \frac{\partial \theta}{\partial x} + y \frac{\partial \theta}{\partial y} + \frac{\partial \theta}{\partial t} = 0$

is a first order equation in three variables.

iii) $\left(\frac{\partial \theta}{\partial x}\right)^3 + \left(\frac{\partial \theta}{\partial t}\right) = 0$

is a first order in two variables

Here,

$\frac{\partial z}{\partial x} = p$. $\frac{\partial z}{\partial y} = q$, $\frac{\partial^2 z}{\partial x^2} = r$, $\frac{\partial^2 z}{\partial x \partial y} = s$

$\frac{\partial^2 z}{\partial y^2} = t$

x, y are independent va
a dependent variables.

origin of 1st order partial
equation. (problem) 2m & 5m

1) $x^2 + y^2 + (z-c)^2 = a^2$

soln:

Given $x^2 + y^2 + (z-c)^2 = a^2$

where x, y are independent v
and z is a dependent variable.
a arbitrary constant.

Let $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$, $\frac{\partial^2 z}{\partial x^2} = r$

Diff ...(1) p. w. r. x

$$2x + 2(z-c)\frac{\partial z}{\partial x} = 0$$

$$2x + 2(z-c)p = 0 \rightarrow \cancel{(2)}$$

$$x + (z-c)p = 0$$

$$x = -(z-c)p \rightarrow (2)$$

P.W.D.W.r to (1) y

$$2y + 2(z-c)\frac{\partial z}{\partial y} = 0$$

$$2y + 2(z-c)q = 0$$

$$y + (z-c)q = 0 \rightarrow (3)$$

$$y = -(z-c)q$$

$\frac{(2)}{(3)} \Rightarrow \frac{x}{y} = \frac{-(z-c)p}{-(z-c)q}$

$x/y = p/q \Rightarrow xq = yp$

$yp - xq = 0$ (or) $\cancel{xq - yp = 0}$

Eliminate the arbitrary constant.

2) $x^2 + y^2 = (z-c)^2 \tan^2\alpha$.

Soln:

Given $x^2+y^2 = (z-c)^2 \tan^2\alpha \rightarrow (1)$

where x and y are independent variable z is a dependent variable and c is a arbitrary constant.

Diff. (1) p.w.r to (1) in x

$$2x + 0 = 2(z-c)\tan^2\alpha \cdot \frac{\partial z}{\partial x}$$

$$2x = 2(z-c)\, p \tan^2\alpha$$

$$x = (z-c)\, p \tan^2\alpha \rightarrow (2)$$

Diff. (1) p.w.r to y

$$2y = 2(z-c)\frac{\partial z}{\partial y}\tan^2\alpha.$$

$$2y = 2(z-c)\, q \tan^2\alpha$$

$$y = (z-c)\, q \tan^2\alpha \longrightarrow (3)$$

$$\frac{(2)}{(3)} = \frac{x}{y} = \frac{(z-c)\, p\tan^2\alpha}{(z-c)\, q\tan^2\alpha}$$

$$\frac{x}{y} = \frac{p}{q}$$

$$yp - xq = 0$$

which is required partial differential equation.

Eliminating arbitrary constant:

i) Let a relation

$$x^2 + y^2 + (z-c)^2 = a^2,$$

$$x^2 + y^2 = (z-c)^2 \tan^2\alpha.$$

are both of the type $F(x, y, z, a, b) = 0 \rightarrow (1)$

ii) where a and b denote arbitrary constants and differentiate (1) and partial with respect to x and y.

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial x} = 0$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} \cdot p \rightarrow (2)$$

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \cdot \frac{\partial z}{\partial y} = 0$$

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} \cdot q = 0 \rightarrow (3)$$

In this (3) eqns involving two arbitrary constant a and b. It will be possible to eliminate a and b from these equations to obtain a relation of the form

$$F(x, y, z, p, q) = 0.$$

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3) $z = (x+a)(y+b)$

Soln:

Given $z = (x+a)(y+b) \rightarrow (1)$

where x and y are independent variable and z are dependent variable and a and b are arbitrary constant.

Diff (1) p.w.r to x.

$$\frac{\partial z}{\partial x} = (1+0)(y+b)$$

$$p = \frac{\partial z}{\partial x} = y+b \rightarrow (2)$$

Diff (1) p.w.r to y

$$\frac{\partial z}{\partial y} = (x+\cancel{b}a)(1+0)$$

$$q = (x+a) \rightarrow (3)$$

(2) and (3) are multiply

$$p \cdot q = (y+b)(x+a)$$

$$pq = z.$$

which is required P.D.E

4) $z^2 = (x+a)^2 + (y+b)^2$

Given $z^2 = (x+a)^2 + (y+b)^2 \rightarrow (1)$

where x and y are independent variable and z are dependent variable, a and b are arbitrary constant.

Diff. (1) w.r to x

$$2z\frac{\partial z}{\partial x} = 2(x+a)$$

$$2zp = 2(x+a)$$

$$zp = (x+a)$$

squaring on both sides

$$z^2p^2 = (x+a)^2 \rightarrow (2)$$

diff. (2) w.r. to y

$$2z\frac{\partial z}{\partial y} = 2(y+b)$$

$$z2q = 2(y+b)$$

$$zq = (y+b)$$

squaring on both sides

$$z^2q^2 = (y+b)^2 \rightarrow (3)$$

Adding on both sides

$$z^2p^2 + q^2z^2 = (x+a)^2 + (y+b)^2$$

$$z^2p^2 + q^2z^2 = z^2$$

(or)

$$p^2 + q^2 = 1$$

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5) $ax^2+by^2+z^2=1$

Soln.

Given $ax^2+by^2+z^2=1 \rightarrow (1)$

$$z^2 = 1 - ax^2 - by^2 \rightarrow (2)$$

where

Diff. (2) w.r.to x

$$2z \cdot \frac{\partial z}{\partial x} = -2ax$$

$$2z \cdot p = -2ax$$

$$zp = -ax$$

$$a = -\frac{zp}{x} \rightarrow (3)$$

Diff. (1) w.r.to y

$$2z \cdot \frac{\partial z}{\partial y} = -2by$$

$$2zq = -2by$$

$$-zq = by$$

$$b = -\frac{zq}{y} \rightarrow (4)$$

Put (3) and (4) in (1)

$$\left(-\frac{zp}{x}\right)x^2 + \left(-\frac{zq}{y}\right)y^2 + z^2 = 1$$

$$z\left[z - px - qy\right] = 1$$

b) $2z = (ax+y)^2 + b$

Soln.

Given $2z = (ax+y)^2 + b \longrightarrow (1)$

D. iff (1). w.r. to x

$$2\frac{\partial z}{\partial x} = 2(ax+y)(a) + 0$$

$$2p = 2(ax+y)(a)$$

$$p = (ax+y)\,a \longrightarrow (2)$$

Diff. (1) w.r. to y

$$2\frac{\partial z}{\partial y} = 2(ax+y) + 0$$

$$2q = 2(ax+y)$$

$$q = (ax+y)$$

Diff. (2) in w.r. to x

$$\frac{\partial z}{\partial x} = (ax+y)a$$

$$\frac{\partial z}{\partial x}\left(\frac{\partial}{\partial y}\right) = (0+1)a$$

$$\frac{\partial^2 z}{\partial x \partial y} = a$$

$$s = a$$

$$p = qs.$$

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From the partial differential equations eliminatry the arbitrary function ϕ from $\phi(u,v)=0$ where u and v are the function of (x,y,z).

Soln:

$u \to x, y, z$
$v \to x, y, z$

$$\phi(u,v)=0 \quad \to (1)$$

Diff-(1) p. w. r to x

$$\frac{\partial\phi}{\partial u}\left\{\frac{\partial u}{\partial x}(1)+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right\}+\frac{\partial\phi}{\partial v}\left\{\frac{\partial v}{\partial x}(1)+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right\}=0$$

$$\frac{\partial\phi}{\partial u}\left\{\frac{\partial u}{\partial x}+p\cdot\frac{\partial u}{\partial z}\right\}+\frac{\partial\phi}{\partial v}\left\{\frac{\partial v}{\partial x}+\cancel{q}\,p\cdot\frac{\partial v}{\partial z}\right\}=0$$
$\to (2)$

Diff (1) w. r. to y

$$\frac{\partial\phi}{\partial u}\left\{\frac{\partial u}{\partial y}(1)+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial y}\right\}+\frac{\partial\phi}{\partial v}\left\{\frac{\partial v}{\partial y}(1)+\frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial y}\right\}=0$$

$$\frac{\partial\phi}{\partial u}\left\{\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right\}+\frac{\partial\phi}{\partial v}\left\{\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right\}=0 \to (3)$$

From (2)

$$\frac{\partial\phi}{\partial u}\left\{\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right\}=-\frac{\partial\phi}{\partial v}\left\{\frac{\partial v}{\partial x}+p\cdot\frac{\partial v}{\partial z}\right\} \to (4)$$

From (3)

$$\frac{\partial\phi}{\partial u}\left\{\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right\}=-\frac{\partial\phi}{\partial v}\left\{\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right\} \to (5)$$

$$\frac{(4)}{(5)} \Rightarrow \frac{\frac{\partial\phi}{\partial u}\left\{\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right\}}{\frac{\partial\phi}{\partial u}\left\{\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right\}} = \frac{-\frac{\partial\phi}{\partial v}\left\{\frac{\partial v}{\partial x}+p\cdot\frac{\partial v}{\partial z}\right\}}{-\frac{\partial\phi}{\partial v}\left\{\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right\}}$$

$$\frac{\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}}{\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}} = \frac{\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}}{\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}}$$

$$\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right) = \left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right)$$

$$\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}+q\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}+p\cdot\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}+pq\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial z} =$$

$$\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}+p\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}+q\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}+pq\frac{\partial u}{\partial z}\frac{\partial v}{\partial z}$$

$$\Rightarrow p\left(\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}\right)+q\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}\right)$$

$$= \left(\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}\right)$$

$$= Pp + Qq = R.$$

where

$$P = \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z} \text{ (or) } \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial y}$$

$$Q = \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x} \text{ (or) } \frac{\partial u}{\partial z}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}$$

$$R = \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} \text{ (or) } \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$p\frac{\partial(u,v)}{\partial(y,z)}+q\frac{\partial(u,v)}{\partial(z,x)} = \frac{\partial(u,v)}{\partial(x,y)}$$

$$\begin{vmatrix} P & Q & R \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \partial v & \partial v & \partial v \end{vmatrix}$$

Eliminating of arbitrary function f from the equations.

$$z = xy + f(x^2+y^2).$$ $py - qx$

soln:

(2) $z = xy + f(x+y) \Rightarrow p - q = y - x$

Given $z = xy + f(x^2+y^2). \rightarrow (1)$

Diff. (1) w.r.to x

$$\frac{\partial z}{\partial x} = y + f'(x^2+y^2)\,2x$$

$$\frac{\partial z}{\partial x} = y + f'(x^2+y^2)\,2x$$

$$p = 2x \cdot f'(x^2+y^2) \rightarrow (2)$$

Diff. (1) w.r.to y.

$$\frac{\partial z}{\partial y} = f'(x^2+y^2)\,2y.$$

$$q = f'(x^2+y^2)\,2y \rightarrow (3)$$

$$\frac{(2)}{(3)} \Rightarrow \frac{p}{q} = \frac{2x\,f'(x^2+y^2)}{2y f'(x^2+y^2)}$$

$$\frac{p}{q} = x/y$$

$$py = xq$$

$$py - xq = 0.$$

which is required P.D.E.

Eliminate the arbitrary function from $z = xy + f(x+y)$.

Soln:

Given $z = xy + f(x+y) \longrightarrow (1)$

Diff. (1) p.w.r. to x.

$$\frac{\partial z}{\partial x} = y + f'(x+y).(1)$$

$$p - y = f'(x+y) \longrightarrow (2).$$

Diff. (1) p.w.r. to y

$$\frac{\partial z}{\partial y} = x + f'(x+y)(1).$$

$$q - x = f'(x+y) \longrightarrow (3).$$

From (2) and (3).

$$p - y = q - x.$$

which is required P.D.E.

Eliminate the arbitrary function from $f(xy - z, x+y) = 0$.

solution: $f(xy - z, x+y) = 0$

$u = xy - z$ $\quad$ $v = x + y$

$u_x = y$ $\quad$ $v_x = 1$

$u_y = x$ $\quad$ $v_y = 1$

$u_z = -1$ $\quad$ $v_z = 0$

$$\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$$

$$\begin{vmatrix} p & q & -1 \\ y & x & -1 \\ 1 & 1 & 0 \end{vmatrix} = 0$$

$$p(0+1) - q(0+1) - 1(y-x) = 0$$

$$p - q - y + x = 0$$

$$p - y = q - x.$$

which is required P.D.E.

Eliminate the arbitrary function from $f(x^2+y^2+z^2,\ x+y+z)=0$.

Soln:

Given $f(x^2+y^2+z^2,\ x+y+z)=0$

This eqn is of the form $\phi(u,v)=0$

Here $u = x^2+y^2+z^2$ $\qquad v = x+y+z$.

$u_x = 2x$ $\qquad v_x = 1$

$u_y = 2y$ $\qquad v_y = 1$

$u_z = 2z$ $\qquad v_z = 1$.

$$\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$$

$$\begin{vmatrix} p & q & -1 \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$p(y-z) - q(x-z) - 1(x-y) = 0$$

$$py - pz - qx + qz - x + y = 0$$

$$p(y-z) - q(x-z) = x - y.$$

which is required P.D.E.

Eliminate the arbitrary function from the following:

(i) $f(x^2+y^2+z^2, z^2-xy) = 0.$

Soln:

Given $f(x^2+y^2+z^2, z^2-xy) = 0.$

W.K.T $\phi(u, v) = 0$

$u = x^2+y^2+z^2$ $\qquad v = z^2 - xy$

$u_x = 2x$ $\qquad v_x = -y$

$u_y = 2y$ $\qquad v_y = -x$

$u_z = 2z$ $\qquad v_z = 2z.$

$$\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$$

$$\begin{vmatrix} p & q & -1 \\ 2x & 2y & 2z \\ -y & -x & 2z \end{vmatrix} = 0$$

$$\begin{vmatrix} p & q & -1 \\ x & y & z \\ -y & -x & 2z \end{vmatrix} = 0.$$

$p(2yz + z \cdot x) - q(2xz + zy) - 1(-x^2 + y^2)$

$p(2yz + xz) - q(2xz + zy) = y^2 - x^2$

which is required P.D.E.

ii) $z = xy + f(x-y)$.

Soln:

Given $z = xy + f(x-y) \rightarrow (1)$

Diff. (1) p.w.r to x.

$$\frac{\partial z}{\partial x} = y + f'(x-y)(1)$$

$$p - y = f'(x-y) \rightarrow (2)$$

Diff. (1). p.w.r. to y

$$\frac{\partial z}{\partial y} = x + f'(x-y)(1)$$

$$q = x - f'(x-y).$$

$$f'(x-y) = x - q \rightarrow (3)$$

From (2) and (3).

$$p - y = x - q.$$

which is required.

iii) $z = f(xy/z)$.

Soln: $z = f\left(\frac{xy}{z}\right)$

Let $F\left(\frac{xy}{z}, -z\right) = 0$

This eqn is of the form.

$$\phi(u, v) = 0$$

Here $u = xy/z$ $\quad v = -z$

$u_x = y/z$ $\quad v_x = 0$

$u_y = x/z$ $\quad v_y = 0$

$u_z = xy(-z^{-2}) = \frac{-xy}{z^2}$ $\quad v_z = -1.$

$$\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$$

$$\begin{vmatrix} p & q & -1 \\ y/z & x/z & -xy/z^2 \\ 0 & 0 & -1 \end{vmatrix} = 0$$

$$\begin{vmatrix} p & q & -1 \\ y & z & -xy/z \\ 0 & 0 & -1 \end{vmatrix} = 0.$$

$$p(-x+0) - q(-y+0) - 1(0+0) = 0.$$

$$-px + qy = 0.$$

i.e., $px - qy = 0.$

which is required P.D.E.

Cauchy problem for first order Equation:

Complete integral:

The relation of the type $F(x, y, z, a, b) = 0$ is a partial differential equation of the 1st any such relation which contains two arbitrary constant a and b is a solution of a P.D.E of the 1st order. is said to be complete solution or a complete integral of that equation.

General Integral:

The relation of the type $F(u, v) = 0$ involving an arbitrary function F connecting two known functions u and v of x, y, z and providing a solution of a 1st order D.E is called a general solution or general integral of that equation.

For example 1:

Find the complete integral of $z = xp + yq + pq$

The given eqn is the form of

$$z = xp + yq + pq.$$

put $p = a$, $q = b$

$$z = xa + yb + ab$$

2) $z = px + qy + \log pq.$

The complete integral is

$$z = ax + by + \log ab.$$

Find the complete integral of $z(xp - yq) = y^2 - x^2$.

soln: $z(xp - yq) = y^2 - x^2$.

$$xp - yq = \frac{y^2 - x^2}{z}$$

The given equation is a forms!

$$Pp + Qq = R.$$

The Lagrange's Equation.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{\frac{y^2 - x^2}{z}}$$

consider the (1) and (2) ratio.

$$\frac{dx}{x} = \frac{dy}{-y}$$

Jing on both sides

$$\int \frac{dx}{x} = \int -\frac{dy}{y}.$$

$\log x = -\log y + \log c_1$

$\log x + \log y = \log c_1$

$xy = c_1 \rightarrow (1)$

(1) + (2) = (3)

consider,

$$\frac{dx+dy}{x-y} = \frac{dz}{\frac{y^2-x^2}{z}}$$

$$\frac{dx+dy}{x-y} = \frac{z\,dz}{(y+x)(y-x)}$$

$$\frac{dx+dy}{x-y} = \frac{-z\,dz}{(y+x)(x-y)}$$

$$(x+y)\,(dx+dy) = -z\,dz$$

$$(x+y)\,d(x+y) = -z\,dz.$$

Jing on both sides,

$$\int (x+y)\,d(x+y) = -\int z\,dz$$

$$\frac{(x+y)^2}{2} = -\frac{z^2}{2} + c_2$$

$$\frac{(x+y)^2}{2} + \frac{z^2}{2} = c_2$$

$$(x+y)^2 + z^2 = 2c_2.$$

$\therefore f(xy, (x+y)^2+z^2)$. general

which is the required integral P.D.E.