Vivekananda College of Arts and Science For Women,
Thenpathi, Sirkali.

Department of Mathematics

Complex Analysis

Class : III B.Sc. (Mathematics) **Notes of Lessons** **Subject Code : 16SCCMM13**

Objectives: To enable the students to

1. Understand the functions of complex variables, continuity and differentiation of complex variable functions, $C - R$ equations of analytic functions.
2. Learn about elementary transformation concepts in complex variable.
3. Know about complex Integral functions with Cauchy's Theorem, power series expansions of Taylor's and Laurant's series.
4. Understand the singularity concepts and residues, solving definite integrals using the residue concepts.

Unit 1: Functions of a Complex variable – Limits – Theorems on Limits – Continuous functions – Differentiability – Cauchy−Riemann equations – Analytic functions – Harmonic functions.

Unit 2: Elementary transformations – Bilinear transformations – Cross ratio – fixed points of Bilinear Transformation – Some special bilinear transformations.

Unit 3: Complex integration – definite integral – Cauchy's Theorem – Cauchy's integral formula – Higher derivatives.

Unit 4: Series expansions – Taylor's series – Laurant's Series – Zeroes of analytic functions – singularities.

Unit 5: Residues – Cauchy's Residue Theorem – Evaluation of definite integrals.

Text Book(s)

(1). S. Arumugam, A. Thangapandi Isaac, and A. Somasundaram, *Complex Analysis*, New Scitech Publications (India) Pvt Ltd, 2002.

Unit–1: Analytic Functions

Introduction

In this Unit we study in detail the concepts of limit and continuity for functions of a complex variable. We also introduce the notion of differentiability for functions of a complex variable and see how the derivative of a complex function of one complex variable sometimes behaves like the derivative of a real function of one real variable and other times is comparable to the partial derivatives of a real function of two variables.

Functions of a Complex Variable

We use the letters z and w to denote complex variables. Thus to denote a complex valued function of a complex variable we use the notation $w = f(z)$. Throughout this unit we shall consider functions whose domain of definition is a region of the complex plane.

The function $w = iz + 3$ is defined in the entire complex plane. The function $w = \frac{1}{z^2+1}$ is defined at all points of the complex plane except at $z = \pm i$.

The function $w = |z|$ is defined in the entire complex plane and this is a real valued function of the complex variable z.

If $a_0, a_1, \ldots, a_n$ are complex constants the function $P(z) = a_0 + a_1 z + \cdots + a_n z^n$ is defined in the entire complex plane and is called a polynomial in z.

If $P(z)$ and $Q(z)$ are polynomials the quotient $\frac{P(z)}{Q(z)}$ is called a rational function and it is defined for all z with $Q(z) \neq 0$.

The function $f(z) = x^4 + y^4 + i\left(x^2 + y^2\right)$ is defined over the entire complex plane. In general if $u(x, y)$ and $v(x, y)$ are real valued functions of two variables both defined on a region S of the complex plane then $f(z) = u(x, y) + iv(x, y)$ is a complex valued function defined on S.

Conversely each complex function $w = f(z)$ can be put in the form

$$w = f(z) = u(x, y) + iv(x, y)$$

where u and v are real valued functions of the real variables x and y.

$u(x, y)$ is called the real part and $v(x, y)$ is called the imaginary part of the function $f(z)$.

For example, $f(z) = z^2 = (x + iy)^2 = \left(x^2 - y^2\right) + i(2xy)$ so that $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$.

Thus a complex function $w = f(z)$ can be viewed as a function of the complex variable z or as a function of two real variables x and y.

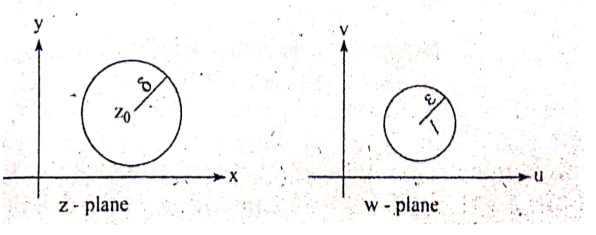
To have a geometric representation of the function $w = f(z)$ it is convenient to draw separate complex planes for the variables z and w so that corresponding to each point $z = x + iy$ of the z -plane there is a point $w = u + iv$ in the w-plane.

Limits

Let $w = f(z)$ be a function defined in some region containing a point z_0 except perhaps at the point z_0. It may happen that as z approaches z_0 the value $f(z)$ of the function is arbitrarily close to a complex number l. Then we say that the limit of the function $f(z)$ as z approaches z_0 is l. This idea is expressed in a precise form in the following definition.

Definition

A function $w = f(z)$ is said to have the limit l as z tends to z_0 if given $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |z - z_0| < \delta \Rightarrow |f(z) - l| < \varepsilon$. In this case we write $\lim_{z \to z_0} f(z) = l$.



Geometrically the definition states that given any open disc with centre l and radius ε there exists an open disc with centre z_0 and radius δ such that for every point $z\,(\neq z_0)$ in the disc $|z - z_0| < \delta$ the image $w = f(z)$ lies in the disc $|w - l| < \varepsilon$.

Lemma 1. *When the limit of a function $f(z)$ exists as z tends to z_0 then the limit has a unique value.*

Proof. Suppose that $\lim_{z \to z_0} f(z)$ has two values l_1 and l_2. Then given $\varepsilon > 0$ there exists δ_1 and $\delta_2 > 0$ such that

$$0 < |z - z_0| < \delta_1 \Rightarrow |f(z) - l_1| < \frac{\varepsilon}{2} \quad \textit{and}$$
$$0 < |z - z_0| < \delta_2 \Rightarrow |f(z) - l_2| < \frac{\varepsilon}{2}$$

Now let $\delta = \min\{\delta_1, \delta_2\}$. Then if $0 < |z - z_0| < \delta$ we have

$$\begin{aligned} |l_1 - l_2| &= |l_1 - f(z) + f(z) - l_2| \\ &\leq |f(z) - l_1| + |f(z) - l_2| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary $|l_1 - l_2| = 0$ so that $l_1 = l_2$. □

Example 2. *Let* $f(z) = \begin{cases} z^2 & \text{if } z \neq i \\ 0 & \text{if } z = i \end{cases}$. *As z approaches i, $f(z)$ approaches $i^2 = -1$. Hence we expect that* $\lim_{z \to i} f(z) = -1$.

Solution. To prove that we must show that given $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |z - i| < \delta \Rightarrow \left|z^2 + 1\right| < \varepsilon$. Now

$$\left|z^2 + 1\right| = |(z + i)(z - i)|$$

$$= |z+i||z-i| \tag{1}$$

Note that if we can find a $\delta > 0$ satisfying the requirements of the definition then we can choose another $\delta \leq 1$ satisfying the requirements of the definition. Now

$$\begin{aligned} 0 < |z-i| < 1 \Rightarrow |z+i| &= |z-i+2i| \\ &\leq |z-i| + |2i| \\ &< 1+2 = 3 \end{aligned}$$

Therefore $|z+i| < 3$. Using this in (1), we obtain $0 < |z-i| < 1 \Rightarrow \left|z^2+1\right| < 3|z-i|$. Hence if we choose $\delta = \min\left\{1, \frac{\varepsilon}{3}\right\}$, we get

$$0 < |z-i| < \delta \Rightarrow \left|z^2+1\right| < \varepsilon$$

Therefore $\lim\limits_{z \to i} f(z) = -1$. ■

Example 3. $\lim\limits_{z \to 2} \frac{z^2-4}{z-2} = 4$.

Solution. Let $f(z) = \frac{z^2-4}{z-2}$. Hence $f(z)$ is not defined at $z = 2$ and when $z \neq 2$ we have

$$f(z) = \frac{(z+2)(z-2)}{z-2} = z+2$$

Therefore $|f(z) - 4| = |z+2-4| = |z-2|$ when $z \neq 2$. Now given $\varepsilon > 0$, we choose $\delta = \varepsilon$. Then $0 < |z-2| < \delta \Rightarrow |f(z) - 4| < \varepsilon$. Therefore

$$\lim_{z \to 2} f(z) = 4$$

■

Example 4. *The function* $f(z) = \frac{\overline{z}}{z}$ *does not have a limit as* $z \to 0$.

Solution. Given

$$f(z) = \frac{\overline{z}}{z} = \frac{x-iy}{x+iy}$$

Suppose $z \to 0$ along the path $y = mx$. Along this path

$$f(z) = \frac{x-imx}{x+imx} = \frac{1-im}{1+im} \; as \; x \neq 0$$

Hence if $z \to 0$ along the path $y = mx$, $f(z)$ tends to $\frac{1-im}{1+im}$ which is different for different values of m. Hence $f(z)$ does not have a limit as $z \to 0$. ■

Example 5. *Let* $f(z) = \frac{x^2y^2}{(x+y^2)^3}, z \neq 0$. *Then* $f(z)$ *does not have a limit as* $z \to 0$.

Solution. Along the parabola $y^2 = mx$ we have

$$f(z) = \frac{mx^2}{(x+mx)^3} = \frac{m}{(1+m)^3}$$

Hence if $z \to 0$ along the parabola $y^2 = mx$, $f(z)$ tends to $\frac{m}{(1+m)^3}$ which depends on m. Hence $f(z)$ does not have a limit as $z \to 0$. ■

Definition

We say $\lim_{z\to\infty} f(z) = l$ if given $\varepsilon > 0$ there exists a number $m > 0$ such that

$$|z| > m \Rightarrow |f(z) - l| < \varepsilon.$$

We say that $\lim_{z\to z_0} f(z) = \infty$ if for given $n > 0$ there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \Rightarrow |f(z)| > n.$$

We say that $\lim_{z\to\infty} f(z) = \infty$ if for given $n > 0$ there exists $m > 0$ such that

$$|z| > m \Rightarrow |f(z)| > n.$$

Theorems on Limit

We state without proof the following theorem on the limits of sum, product and quotient of two functions. The proof is analogues to that of real functions.

Theorem 1

Let f and g be two functions whose limits at z_0 exist. Let $\lim_{z\to z_0} f(z) = l$ and $\lim_{z\to z_0} g(z) = m$. Then

(1). $\lim_{z\to z_0} [f(z) + g(z)] = l + m$.

(2). $\lim_{z\to z_0} f(z)g(z) = lm$.

(3). $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{1}{m}$ provided $m \neq 0$.

Theorem 2

(1). If $\lim_{z\to z_0} f(z) = l$, then $\lim_{z\to z_0} \overline{f(z)} = \bar{l}$.

(2). If $\lim_{z\to z_0} f(z) = l$, then $\lim_{z\to z_0} |f(z)| = |l|$.

(3). $\lim_{z\to z_0} f(z) = l$ iff $\lim_{z\to z_0} \operatorname{Re} f(z) = \operatorname{Re} l$ and $\lim_{z\to z_0} \operatorname{Im} f(z) = \operatorname{Im}(l)$.

Proof.

(1). Let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that

$$0 < |z - z_0| < \delta \Rightarrow |f(z) - l| < \varepsilon$$

Now $|\overline{f(z)} - \bar{l}| = |\overline{f(z) - l}| = |f(z) - l|$. Hence $0 < |z - z_0| < \delta \Rightarrow |\overline{f(z)} - \bar{l}| < \varepsilon$ so that $\lim_{z \to z_0} \overline{f(z)} = \bar{l}$.

(2). $||f(z)| - |l|| \leq |f(z) - l|$ and hence

$$0 < |z - z_0| < \delta \Rightarrow ||f(z)| - |l|| < \varepsilon$$

Therefore $\lim_{t \to z_0} |f(z)| = |l|$.

(3). Let $\lim_{z \to z_0} f(z)| = l$. Since Re $f(z) = \frac{1}{2}[f(z) + f\overline{(z)}]$, we have

$$\begin{aligned}\lim_{z \to z_0} \operatorname{Re} f(z) &= \frac{1}{2}\left[\lim_{z \to z_0} f(z) + \lim_{z \to z_0} f\overline{(z)}\right] \\ &= \frac{1}{2}(l + \bar{l}) \\ &= \operatorname{Re} l\end{aligned}$$

Similarly $\lim_{z \to z_0} \operatorname{Im} f(z) = \operatorname{Im} l$.

Conversely, let $\lim_{z \to z_0} \operatorname{Re} f(z) = \operatorname{Re} l$ and let $\lim_{z \to z_0} \operatorname{Im} f(z) = \operatorname{Im} l$. Since $f(z) = \operatorname{Re} f(z) + i \operatorname{lm} f(z)$ it follows that $\lim_{z \to z_0} f(z) = \operatorname{Re} l + i \operatorname{Im} l = l$.

□

Continuous Functions

Definition

Let f be a complex valued function defined on a region D of the complex plane. Let $z_0 \in D$. Then f is said to be continuous at z_0 if $\lim_{z \to z_0} f(z) = f(z_0)$. Thus f is continuous at z_0 if given $\varepsilon > 0$ there exists a $\delta > 0$ such that $|z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$. f is said to be continuous in D if it is continuous at each point of D.

Theorem 3

(1). If f and g are continuous at z_0 then $f + g, fg$ and $\overline{f}$ are continuous at z_0 and f/g is continuous at z_0 if $g(z_0) \neq 0$.

(2). If f is continuous at z_0 then $|f|$ is also continuous at z_0.

(3). If f is continuous at z_0 iff Re f and Im f are continuous at z_0.

(4). Any polynomial $P(z)$ is continuous at each point of the complex plane and any rational function $\frac{P(z)}{Q(z)}$ is continuous at all points where $Q(z) \neq 0$.

Differentiability

Definition

Let f be a complex function defined in a region D and let $z \in D$. Then f is said to be differentiable at z if $\lim\limits_{h\to 0} \frac{f(z+h)-f(z)}{h}$ exists and is finite. This limit is denoted by $f'(z)$ or $\frac{df}{dz}$ and is called the derivative of $f(z)$ at z. The function is said to be differentiable in D if it is differentiable at all points of D.

Example 6. *The function $f(z) = z^2$ is differentiable at every point and $f'(z) = 2z$.*

Solution.

$$\begin{aligned}\frac{f(z+h)-f(z)}{h} &= \frac{(z+h)^2 - z^2}{h} \\ &= 2z + h \\ Hence\ \lim_{h\to 0}\frac{f(z+h)-f(z)}{h} &= \lim_{h\to 0}(2z+h) \\ &= 2z \\ \therefore\ f'(z) &= 2z\end{aligned}$$

■

Example 7. *The function $f(z) = \overline{z}$ is nowhere differentiable.*

Solution.

$$\begin{aligned}\frac{f(z+h)-f(z)}{h} &= \frac{\overline{(z+h)} - \overline{z}}{h} \\ &= \frac{\overline{z} + \overline{h} - \overline{z}}{h} \\ &= \frac{\overline{h}}{h}\end{aligned}$$

$\lim\limits_{h\to 0} \frac{\overline{h}}{h}$ does not exist. Therefore $f(z) = \overline{z}$ is nowhere differentiable. ■

Remark 1

If $f(z)$ is differentiative at a point z then it is continuous at that point.

Proof.

$$\begin{aligned}\lim_{h\to 0}[f(z+h) - f(z)] &= \lim_{h\to 0}\left[\frac{f(z+h)-f(h)}{h}\right] \times \lim_{h\to 0} h \\ &= f'(z) \times 0 \\ &= 0\end{aligned}$$

Therefore $\lim\limits_{h\to 0} f(z+h) = f(z)$ so that f is continuous at z. □

The converse of the above result is not true.

For example, $f(z) = \overline{z}$ is continuous everywhere but it is nowhere differentiable.

The definition of derivative for complex functions is identical to the definition for real functions and the following formal rules of differentiation are true for complex functions also and the proof is left as an exercise.

Theorem 4

Let $f(z)$ and $g(z)$ be differentiable at a point z. Then

(1). $(f+g)'(z) = f'(z) + g'(z)$.

(2). $(fg)'(z) = f(z)g'(z) + f'(z)g(z)$.

(3). $\left(\frac{f}{g}\right)'(z) = \frac{f'(z)g(z) - f(z)g'(z)}{[g(z)]^2}$ provided $g(z) \neq 0$.

(4). Suppose g is differentiable at z and f is differentiable at $g(z)$. Let $F(z) = f(g(z))$. Then $F'(z) = f'(g(z))g'(z)$. (This is the usual chain rule for the derivative of composite functions).

(5). Let n be any positive integer. The function $f(z) = z^n$ is differentiable at every point and $f'(z) = nz^{n-1}$.

(6). The polynomial $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$ is differentiable at every point and $P'(z) = a_1 + 2a_2 z + \cdots + na_n z^{n-1}$.

(7). If n is a negative integer $f(z) = z^n$ is differentiable at every point $z \neq 0$ and $f'(z) = nz^{n-1}$.

The Cauchy-Riemann Equations

The existence of the derivative of a complex function of a complex variable $f(z)$ requires $\frac{f(z+h)-f(z)}{h}$ to approach to the same limit as $h \to 0$ along any path. This has some far reaching consequences. In this section we derive some important properties of the real and imaginary parts of the differentiable function $f(z) = u(x,y) + iv(x,y)$.

Theorem 5

Let $f(z) = u(x,y) + iv(x,y)$ be differentiable at a point $z_0 = x_0 + iy_0$. Then $u(x,y)$ and $v(x,y)$ have first order partial derivatives $u_x(x_0,y_0)$, $u_y(x_0,y_0)$, $v_x(x_0,y_0)$ and $v_y(x_0,y_0)$ at (x_0,y_0) and these partial derivatives satisfy the Cauchy-Riemann equations (C.R equations) given by

$$\begin{aligned} u_x(x_0,y_0) &= v_y(x_0,y_0) \text{ and } u_y(x_0,y_0) = -v_x(x_0,y_0) \\ \text{Also, } f'(z_0) &= u_x(x_0,y_0) + iv_x(x_0,y_0) \\ &= v_y(x_0,y_0) - iu_y(x_0,y_0) \end{aligned}$$

Proof. Since $f(z) = u(x,y) + iv(x,y)$ is differentiable at $z_0 = x_0 + iy_0$ $\lim\limits_{h \to 0} \frac{f(z_0+h)-f(z_0)}{h}$ exists and hence the limit is independent of the path in which h approaches zero. Let $h = h_1 + ih_2$. Now

$$\begin{aligned}\frac{f(z_0+h)-f(z_0)}{h} &= \frac{u(x_0+h_1\cdot y_0+h_2)+iv(x_0+h_1,y_0+h_2)-u(x_0,y_0)-iv(x_0,y_0)}{h_1+ih_2}\\ &= \left[\frac{u(x_0+h_1,y_0+h_2)-u(x_0,y_0)}{h_i+ih_2}\right]+i\left[\frac{v(x_0+h_1,y_0+h_2)-v(x_0,y_0)}{h_1+ih_2}\right]\end{aligned}$$

Suppose $h \to 0$ along the real axis so that $h = h_1$. Then

$$\begin{aligned}f'(z_0) &= \lim_{h_1\to 0}\left[\frac{f(z_0+h_1)-f(z_0)}{h_1}\right]\\ &= \lim_{h_1\to 0}\left[\frac{u(x_0+h_1,y_0)-u(x_0,y_0)}{h_1}\right]+i\lim_{h_1\to 0}\left[\frac{v(x_0+h_1,y_0)-v(x_0,y_0)}{h_1}\right]\\ &= u_x(x_0,y_0)+iv_x(x_0,y_0) \qquad (1)\end{aligned}$$

Now, suppose $h \to 0$ along the imaginary axis so that $h = ih_2$. Therefore

$$\begin{aligned}f'(z_0) &= \lim_{ih_2\to 0}\left[\frac{f(z_0+h_2)-f(z_0)}{ih_2}\right]\\ &= \lim_{h_2\to 0}\left[\frac{u(x_0,y_0+h_2)-u(x_0,y_0)}{ih_2}\right]+i\lim_{h_2\to 0}\left[\frac{v(x_0,y_0+h_2)-v(x_0,y_0)}{ih_2}\right]\\ &= \left[\frac{u_y(x_0,y_0)}{i}\right]+i\left[\frac{v_y(x_0,y_0)}{i}\right]\\ &= \frac{1}{i}u_y(x_0,y_0)+v_y(x_0,y_0)\\ &= -iu_y(x_0,y_0)+v_y(x_0,y_0) \qquad (2)\end{aligned}$$

From (1) and (2) we get

$$f'(z_0) = u_x(x_0,y_0)+iv_x(x_0,y_0) = v_y(x_0,y_0)-iu_y(x_0,y_0)$$

Equating real and imaginary parts we get

$$\begin{aligned}u_x(x_0,y_0) &= v_y(x_0,y_0)\\ u_y(x_0,y_0) &= -v_x(x_0,y_0)\end{aligned}$$

□

Example 8. *Let* $f(z) = \begin{cases}\frac{xy}{x^2+y^2} & \text{if } z \neq 0\\ 0 & \text{if } z = 0\end{cases}$. *Here* $u(x,y) = \begin{cases}\frac{xy-}{x^2+y^2} & \text{if } (x,y) \neq (0,0)\\ 0 & \text{if } (x,y) = (0,0)\end{cases}$ *and* $v(x,y) = 0$.

Solution. Now

$$\begin{aligned}u_x(0,0) &= \lim_{h\to 0}\left[\frac{u(h,0)-u(0,0)}{h}\right]\\ &= \lim_{h\to 0}\left[\frac{0-0}{h}\right] = 0\end{aligned}$$

Similarly $u_y(0,0)=0$. Also $v_x(0,0)=0$ and $v_y(0,0)=0$. Hence the C.R equations are satisfied at $z=0$. Now, along the path $y=mx$.

$$f(z)=\frac{xmx}{x^2+m^2x^2}=\frac{m}{1+m^2}\quad if \quad x\neq 0$$

Hence if $z\to 0$ along the path $y=mx$, $f(z)\to\frac{m}{1+m^2}$ which is different for different values of m. Hence $f(z)$ does not have a limit as $z\to 0$ so that $f(z)$ is not even continuous at $z=0$. Thus $f(z)$ is not differentiable at $z=0$. ■

Theorem 6

Let $f(z)=u(x,y)+iv(x,y)$ be a function defined in a region D such that u,v and their first order partial derivatives are continuous in D. If the first order partial derivatives of u,v satisfy the Cauchy-Riemann equations at a point $(x,y)\in D$ then f is differentiable at $z=x+iy$.

Proof. Since $u(x,y)$ and its first order partial derivatives are continuous at (x,y) we have by the mean value theorem for functions of two variables

$$u\left(x+h_1,y+h_2\right)-u(x,y)=h_1u_x(x,y)+h_2u_y(x,y)+h_1\varepsilon_1+h_2\varepsilon_2 \tag{1}$$

where ε_1 and $\varepsilon_2\to 0$ as h_1 and $h_2\to 0$. Similarly

$$v\left(x+h_1,y+h_2\right)-v(x,y)=h_1v_x(x,y)+h_2v_y(x,y)+h_1\varepsilon_3+h_2\varepsilon_4 \tag{2}$$

where $\varepsilon_3,\varepsilon_4\to 0$ as h_1 and $h_2\to 0$. Let $h=h_1+ih_2$. Then

$$\begin{aligned}
\frac{f(z+h)-f(z)}{h}&=\frac{1}{h}\left[u\left(x+h_1,y+h_2\right)-u(x,y)+iv\left(x+h_1,y+h_2\right)-v(x,y)\right]\\
&=\frac{1}{h}\left[\left\{h_1u_x(x,y)+h_2u_y(x,y)+h_1\varepsilon_1+h_2\varepsilon_2\right\}+i\left\{h_1v_x(x,y)+h_2v_y(x,y)+h_1\varepsilon_3+h_2\varepsilon_4\right\}\right]\\
&=\frac{1}{h}\left[h_1\left\{u_x(x,y)+iv_x(x,y)\right\}+h_2\left\{u_y(x,y)+iv_y(x,y)\right\}+h_1\left(\varepsilon_1+i\varepsilon_3\right)+h_2\left(\varepsilon_2+i\varepsilon_4\right)\right]\\
&=\frac{1}{h}\left[\left(h_1+ih_2\right)u_x(x,y)-i\left(h_1+ih_2\right)u_y(x,y)+h_1\left(\varepsilon_1+i\varepsilon_3\right)+h_2\left(\varepsilon_2+i\varepsilon_4\right)\right]\\
&=\frac{1}{h}\left[hu_x(x,y)-ihu_y(x,y)+h_1\left(\varepsilon_1+i\varepsilon_3\right)+h_2\left(\varepsilon_2+i\varepsilon_4\right)\right]\\
&=u_x(x,y)-iu_y(x,y)+\frac{h_1}{h}\left(\varepsilon_1+i\varepsilon_3\right)+\frac{h_2}{h}\left(\varepsilon_2+i\varepsilon_4\right)
\end{aligned}$$

Now, since $\left|\frac{h_1}{h}\right|\leq 1$, $\frac{h_1}{h}\left(\varepsilon_1+i\varepsilon_3\right)\to 0$ as $h\to 0$. Similarly $\frac{h_2}{h}\left(\varepsilon_2+i\varepsilon_4\right)\to 0$ as $h\to 0$. Therefore

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=u_x(x,y)-iu_y(x,y).$$

Hence f is differentiable. □

Example 9. *Let* $f(z)=e^x(\cos y+i\sin y)$. *Therefore* $u(x,y)=e^x\cos y$ *and* $v(x,y)=e^x\sin y$. *Then* $u_x(x,y)=e^x\cos y=v_y(x,y)$ *and* $u_y(x,y)=-e^x\sin y=-v_x(x,y)$.

Solution. Thus the first order partial derivatives of u and v satisfy the Cauchy-Riemann equations at every point. Further $u(x, y)$ and $v(x, y)$ and their first order partial derivatives are continuous at every point. Hence f is differentiable at every point of the complex plane. ■

Alternate forms of Cauchy - Riemann equations

In the following theorem we express the Cauchy-Riemann equations in complex form.

Theorem 7: Complex form of C-R equations

Let $f(z) = u(x, y) + iv(x, y)$ be differentiable. Then the C.R equations can be put in the complex form as $f_x = -if_y$.

Proof. Let $f(z) = u(x, y) + iv(x, y)$. Then $f_x = u_x + iv_x$ and $f_y = u_y + iv_y$. Hence

$$\begin{aligned} f_x &= -if_y \\ \Leftrightarrow u_x + iv_x &= -i\left(u_y + iv_y\right) \\ \Leftrightarrow u_x + iv_x &= v_y - iu_y \\ \Leftrightarrow u_x &= v_y \text{ and } v_x = -u_y \end{aligned}$$

Thus the two C.R equations are equivalent to the equation $f_x = -if_y$. □

In the following theorem we express the Cauchy-Riemann equations and the derivative of a complex function in terms of it polar coordinates.

Theorem 8: C.R equations in polar coordinates

Let $f(z) = u(r, \theta)+iv(r, \theta)$ be differentiable at $z = re^{i\theta} \neq 0$. Then $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$. Further $f'(z) = \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right)$.

Proof. We have $x = r\cos\theta$ and $y = r\sin\theta$. Hence

$$\begin{aligned} \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \\ &= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta \qquad (1) \end{aligned}$$

Also

$$\begin{aligned} \frac{\partial v}{\partial \theta} &= \frac{\partial v}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta} \\ &= \frac{\partial v}{\partial x}(-r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta) \\ \frac{1}{r}\frac{\partial v}{\partial \theta} &= -\frac{\partial v}{\partial x}\sin\theta + \frac{\partial v}{\partial y}\cos\theta \\ &= \frac{\partial u}{\partial y}\sin\theta + \frac{\partial u}{\partial x}\cos\theta \text{ (using C.R equations)} \end{aligned}$$

$$= \frac{\partial u}{\partial r} \text{ (using (1))}$$

$$Thus\ \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

Similarly we can prove that $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$. Now

$$\begin{aligned} r\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) &= r\left[\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}\right) + i\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}\right)\right] \\ &= r\left[\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right) + i\left(\frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta\right)\right] \\ &= r\cos\theta\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + r\sin\theta\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) \\ &= x\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) + iy\left(\frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}\right) \\ &= xf'(z) + iyf'(z) \\ &= (x+iy)f'(z) \\ &= zf'(z) \\ f'(z) &= \frac{r}{z}\left(\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right) \end{aligned}$$

We now proceed to express C.R equations in yet another form. Let $f(z) = u(x,y) + iv(x,y)$. Since $x = \frac{z+\overline{z}}{2}$ and $y = \frac{z-\overline{z}}{2i}$, we have

$$f(z) = u\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) + iv\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right)$$

□

Example 10. *Verify Cauchy-Riemann equations for the function* $f(z) = z^3$.

Solution.

$$\begin{aligned} f(z) &= z^3 = (x+iy)^3 \\ &= \left(x^3 - 3xy^2\right) + i\left(3x^2y - y^3\right) \\ u(x,y) &= x^3 - 3xy^2 \text{ and } v(x,y) = 3x^2y - y^3 \end{aligned}$$

Therefore

$$\begin{aligned} u_x &= 3x^2 - 3y^2 \ \ and \ \ v_x = 6xy \\ u_y &= -6xy \qquad and \ \ v_y = 3x^2 - 3y^2 \end{aligned}$$

Here $u_x = v_y$ and $u_y = -v_x$. Hence the Cauchy-Riemann equations are satisfied. ■

Example 11. *Find constants a and b so that the function $f(z) = a\left(x^2 - y^2\right) + ibxy + c$ is differentiable at every point.*

Solution. Here $u(x,y) = a\left(x^2 - y^2\right) + c$ and $v(x,y) = bxy$. $u_x = 2ax$; $v_x = b_y$; $u_y = -2ay$ and $v_y = bx$. Clearly $u_x = v_y$ and $u_y = -v_x$ iff $2a = b$. Therefore C-R equations are satisfied at all points iff $2a = b$. Therefore the function $f(z)$ is differentiable for all values of a, b with $2a = b$. ■

Analytic Functions

Definition

A function f defined in a region D of the complex plane is said to be **analytic at a point** $a \in D$ if f is differentiable at every point of some neighbourhood of a. Thus f is analytic at a if there exists $\varepsilon > 0$ such that f is differentiable at every point of the disc $S(a, \varepsilon) = \{z/|z - a| < \varepsilon\}$. If f is analytic at every point of a region D then f is said to be **analytic** in D . A function which is analytic at every point of the complex plane is called an **entire function** or **integral function**.

Theorem 9

An analytic function in a region D with its derivative zero at every point of the domain is a constant.

Proof. Let $f(z) = u(x, y) + iv(x, y)$ be analytic in D and $f'(z) = 0$ for all $z \in D$. Since $f'(z) = u_x + iv_x = v_y - iu_y$ we have $u_x = u_y = v_y = v_y = 0$. Therefore $u(x, y)$ and $v(x, y)$ are constant functions and hence $f(z)$ is constant. □

Example 12. *Any analytic function $f(z) = u + iv$ with arg $f(z)$ constant is itself a constant function.*

Solution. arg $f(z) = \tan^{-1}(v/u) = c$, where c is a constant. Therefore $\frac{v}{u} = k$ where k is a constant. Therefore $v = ku$. Hence $v_x = ku_x$ and $v_y = ku_y$. Eliminating k from the above equations we get $u_x v_y = v_x u_y$. Therefore $u_x v_y - u_y v_x = 0$. Therefore $u_x^2 + u_y^2 = 0$ (by C.R. equations). Therefore $u_x = 0$ and $u_y = 0$ and hence u is constant. Similarly we can prove that v is constant. Therefore $f = u + iv$ is constant. ■

Harmonic Functions

Definition

Let $u(x, y)$ be a function of two real variables x and y defined in a region D. $u(x, y)$ is said to be a **harmonic function** if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ and this equation is called **Laplace's equation**.

Theorem 10

The real and imaginary parts of an analytic function are harmonic functions.

Proof. Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function. Then u and v have continuous partial derivatives of first order which satisfy the C.R equations given by $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. Further

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} \text{ and } \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x}$$

Now

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial v}{\partial x}\right)$$
$$= \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial y \partial x}$$
$$= 0$$

Thus u is a harmonic function. Similarly we can prove that v is a harmonic function. □

Definition

Let $f = u + iv$ be an analytic function in a region D. Then v is said to be a **conjugate harmonic** function of u .

Theorem 11

Let $f = u + iv$ be an analytic function in a region D. Then v is a harmonic conjugate of u if and only if u is a harmonic conjugate of $-v$.

Proof. Let v be a harmonic conjugate of u. Then $f = u + iv$ is analytic. Therefore $if = iu - v$ is also analytic. Hence u is a harmonic conjugate of $-v$ The proof for the converse is similar. □

Theorem 12

Any two harmonic conjugates of a given harmonic function u in a region D differ by a real constant.

Proof. Let u be a harmonic function. Let v and v^* be two harmonic conjugates of u. $u + iv$ and $u + iv^*$ are analytic in D. Hence by the Cauchy-Riemann equation we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = \frac{\partial v^*}{\partial y}$$
$$\text{and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = -\frac{\partial v^*}{\partial x}$$
$$\therefore \quad \frac{\partial v}{\partial y} = \frac{\partial v^*}{\partial y} \text{ and } \frac{\partial v}{\partial x} = \frac{\partial v^*}{\partial x}$$

Hence $\frac{\partial}{\partial y}(v - v^*) = 0$ and $\frac{\partial}{\partial x}(v - v^*) = 0$. Therefore $v = v^* + c$ where c is a real constant. □

Milne-Thompson method

Let $u(x, y)$ be a given harmonic function. Let $f(z) = u(x, y) + iv(x, y)$ be an analytic function. Then

$$f'(z) = u_x(x, y) + iv_x(x, y)$$
$$= u_x(x, y) - iu_y(x, y)$$

Let $\varphi_1(x,y) = u_x(x,y)$ and $\varphi_2(x,y) = u_y(x,y)$. We have $x = \frac{z+\overline{z}}{2}$ and $y = \frac{z-\overline{z}}{2i}$. Hence

$$f'(z) = \varphi_1\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) - i\varphi_2\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right)$$

Putting $z = \overline{z}$ we obtain $f'(z) = \varphi_1(z,0) - i\varphi_2(z,0)$. Hence

$$f(z) = \int [\varphi_1(z,0) - i\varphi_2(z,0)]\, dz + c$$

Example 13. *Prove that $u = 2x - x^3 + 3xy^2$ is harmonic and find its harmonic conjugate. Also find the corresponding analytic function.*

Solution. Given $u = 2x - x^3 + 3xy^2$. Therefore

$$\begin{aligned} u_x &= 2 - 3x^2 + 3y^2 \\ u_{xx} &= -6x \\ u_y &= 6xy \\ u_{yy} &= 6x \end{aligned}$$

Therefore $u_{xx} + u_{yy} = 0$. Hence u is harmonic. Let v be a harmonic conjugate of u. Therefore $f(z) = u + iv$ is analytic. By Cauchy-Riemann equations we have

$$v_y = u_x = 2 - 3x^2 + 3y^2$$

Therefore integrating with respect to y we get

$$v = 2y - 3x^2y + y^3 + \lambda(x) \tag{1}$$

where $\lambda(x)$ is an arbitrary function of x. Therefore $v_x = -6xy + \lambda'(x)$. Now $v_x = -u_y$ gives $-6xy + \lambda'(x) = -6xy$. Hence $\lambda'(x) = 0$ so that $\lambda(x) = c$ where c is a constant. Thus $v = 2y - 3x^2y + y^3 + c$ [from (1)]. Now

$$\begin{aligned} f(z) &= \left(2x - x^3 + 3xy^2\right) + i\left(2y - 3x^2y + y^3\right) + ic \\ &= 2(x+iy) - \left[\left(x^3 - 3xy^2\right) + i\left(3x^2y - y^3\right)\right] + ic \\ &= 2z - z^3 + ic \end{aligned}$$

Therefore $f(z) = 2z - z^3 + ic$ is the required analytic function. ■

Unit–2: Bilinear Transformations

Introduction

A function $f : C \to C$ can be thought of as a transformation from one complex plane to another complex plane. Hence the nature of a complex function can be described by the manner in which it maps regions and curves from one complex plane to another. In this chapter we shall discuss bilinear transformations and see how various regions are transformed by these transformations.

Elementary Transformations

(1). **Translation:** $w = z + b$
Consider the transformation $w = z + b$. If $z = x + iy, w = u + iv$ and $b = b_1 + ib_2$ then the image of the point (x, y) in the z -plane is the point $(x + b_1, y + b_2)$ in the w -plane.
Under this transformation the image of any region is simply a translation of that region. Hence the two regions have the same shape, size and orientation. In particular the image of a straight line is a straight line and the image of a circle with centre a and radius r is a circle with centre $a + b$ and radius r.
We note that ∞ is the only fixed point of this transformation when $b \neq 0$.

(2). **Rotation:** $w = az$ **where** $|a| = 1$
Consider the transformation $w = az$ where $|a| = 1$. Let $z = re^{i\theta}$ and $a = e^{i\alpha}$ so that $|a| = 1$. Therefore $w = az = e^{i\alpha}\left(re^{i\theta}\right) = re^{i(\theta+\alpha)}$. Therefore a point with polar coordinates (r, θ) in the z -plane is mapped to the point $(r, \theta + \alpha)$ in the w -plane. Hence this transformation represents a rotation through an angle $\alpha = \arg a$ about the origin.
Under this transformation also straight lines are mapped into straight lines and circles are mapped into circles. We note that 0 and ∞ are the two fixed points of this transformation.

(3). **Magnification or Contraction:** $w = bz(b > 0, real)$
Consider the transformation $w = bz$ where b is real and $b > 0$. Then a point with polar coordinates (r, θ) in the z-plane is mapped into the point (br, θ) in the $w-$ plane. Hence this transformation represents a magnification or contraction by the factor according as $b > 1$ or $b < 1$.
Under this transformation also straight lines are mapped into straight lines and circles are mapped into circles. We note that 0 and ∞ are the fixed points of this transformation.

(4). **Inversion:** $w = \frac{1}{z}$
Consider the transformation $w = \frac{1}{z}$. Put $z = re^{i\theta}$. Therefore $w = (1/r)e^{-i\theta}$. This transformation can be expressed as a product of two transformations $T_1(z) = (1/r)e^{i\theta}$ and $T_2(z) = re^{-i\theta} = \overline{z}$. For,

$$
\begin{aligned}
(T_1 \circ T_2)(z) &= T_1(T_2(z)) \\
&= T_1\left(re^{-i\theta}\right)
\end{aligned}
$$

$$= \left(\frac{1}{r}\right) e^{-i\theta}$$
$$= \frac{1}{z}$$

The transformation $T_1(z) = (1/r)e^{i\theta}$ represents the inversion with respect to the unit circle $|z| = 1$ and $T_2(z) = \overline{z}$ represents reflection about the real axis.

Hence the transformation $w = \frac{1}{z}$ is the inversion w.r.t the unit circle followed by the reflection about the real axis. Here points outside the unit circle are mapped into points inside the unit circle and vice versa. Points on the circle are reflected about the real axis. In terms of cartesian coordinates the above transformation can be expressed in the form

$$w = u + iv = \frac{1}{x + iy} = \frac{x - iy}{x^2 + y^2}$$
$$u = \frac{x}{x^2 + y^2} \;\; and \;\; v = \frac{-y}{x^2 + y^2}$$

Similarly from $z = \frac{1}{w}$ we get

$$x = \frac{u}{u^2 + v^2} \;\; and \;\; y = \frac{-v}{u^2 + v^2} \tag{1}$$

Now, consider the equation

$$a\left(x^2 + y^2\right) + bx + cy + d = 0 \tag{2}$$

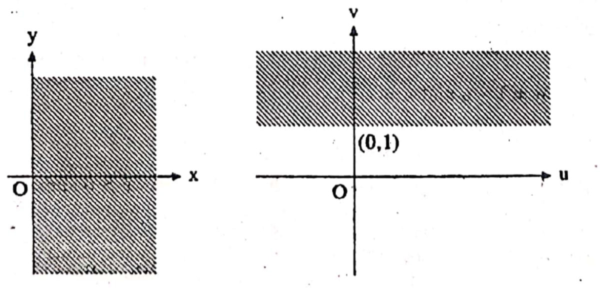
where a, b, c, d are real. This equation represents a circle or a straight line according as $a \neq 0$ or $a = 0$. Using (1) in (2) we get

$$d\left(u^2 + v^2\right) + bu - cv + a = 0 \tag{3}$$

Now, suppose $a \neq 0$; $d \neq 0$.
In this case both (2) and (3) represent circles not passing through the origin. Hence circles not passing through the origin are mapped into circles not passing through the origin.

Similarly, a circle passing through the origin is mapped into a straight line not passing through the origin. A straight line not passing through the origin is mapped into a circle passing through the origin. A straight line passing through the origin is again mapped into a line passing through the origin.
Thus we see that under the transformation $w = \frac{1}{z}$ the image of a circle need not be a circle and the image of a straight line need not be a straight line. However the family of circles and lines are again mapped into the family of circles and lines.
We note that the fixed points of the transformation $u = \frac{1}{z}$ are 1 and -1.

Example 14. *Under the transformation $w = iz + i$ show that the half plane $x > 0$ maps onto the half plane $v > 1$.*

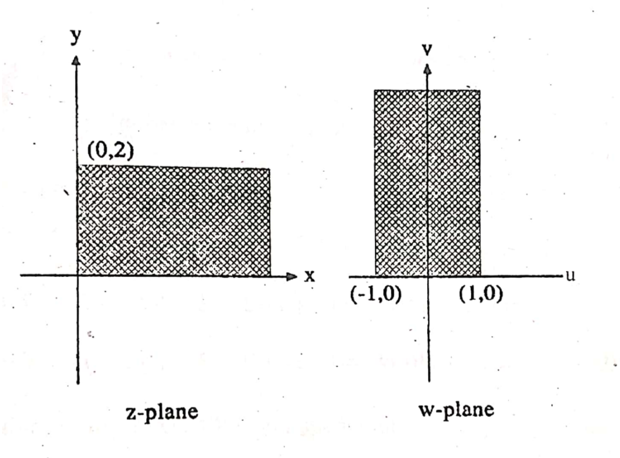


Solution. Let $z = x + iy$ and $w = u + iv$

$$w = iz + i \Rightarrow w = i(x + iy) + i = -y + i(x + 1)$$

Therefore $u + iv = -y + i(x + 1)$. Therefore $u = -y$ and $v = x + 1$. Therefore $x > 0 \iff v > 1$. Therefore the half plane $x > 0$ is mapped into the half plane $v > 1$. ■

Example 15. *Show that the region in the z-plane given by $x > 0$ and $0 < y < 2$ is mapped into the region in the w -plane given by $-1 < u < 1$ and $v > 0$ under the transformation $w = iz + 1$.*



Solution. Let $z = x + iy$ and $w = u + iv$.

$$\begin{aligned} w = iz + 1 \quad &\Rightarrow w = i(x + iy) + 1 \\ &\Rightarrow u + iv = (-y + 1) + ix \end{aligned}$$

Therefore $u = 1 - y$ and $v = x$. Therefore $x > 0$ and $0 < y < 2 \iff v > 0$ and $-1 < u < 1$. Hence the given region is mapped into the region $v > 0$ and $-1 < u < 1$ as shown in the figure. ■

Bilinear Transformations

A transformation of the form

$$w = T(z) = \frac{az + b}{cz + d} \tag{1}$$

where a, b, c, d are complex constants and $ad - bc \neq 0$, is called a bilinear transformation or Mobius transformation.
We define $T(\infty) = \frac{a}{c}$ and $T\left(\frac{-d}{c}\right) = \infty$. Hence T becomes a $1-1$ onto map of the extended complex plane onto itself. The inverse of (1) is given by

$$z = T^{-1}(w) = \frac{-dw + b}{cw - a}$$

which is also a bilinear transformation.

Theorem 13

Any bilinear transformation can be expressed as a product of translation, rotation, magnification or contraction and inversion.

Proof. Let

$$w = T(z) = \frac{az+b}{cz+d} \text{ where } ad - bc \neq 0 \tag{1}$$

be the given bilinear transformation.
Case 1: $c = 0$. Hence $d \neq 0$ (since $ad - bc \neq 0$). Therefore

$$\begin{aligned}(1) \Rightarrow w &= \frac{az+b}{d} \\ &= \left(\frac{a}{d}\right) z + \left(\frac{b}{d}\right)\end{aligned}$$

Now, let $T_1(z) = \left(\frac{a}{d}\right) z$ and $T_2(z) = z + \left(\frac{b}{d}\right)$. T_1 and T_2 are elementary transformations and

$$\begin{aligned}(T_2 \circ T_1)(z) &= T_2\left[\left(\frac{a}{d}\right) z\right] \\ &= \left(\frac{a}{d}\right) z + \left(\frac{b}{d}\right) \\ &= T(z)\end{aligned}$$

Case 2: $c \neq 0$.

$$\begin{aligned}w = \frac{az+b}{cz+d} &= \frac{a\left[z + \left(\frac{d}{c}\right)\right] + b - \left(\frac{ad}{c}\right)}{c\left[z + \left(\frac{d}{c}\right)\right]} \\ &= \frac{a}{c} + \frac{b - \left(\frac{ad}{c}\right)}{cz+d}\end{aligned}$$

Now, let

$$\begin{aligned}T_1(z) &= cz + d \\ T_2(z) &= \frac{1}{z} \\ T_3(z) &= \left(b - \frac{ad}{c}\right) z \\ T_4(z) &= z + \left(\frac{a}{c}\right)\end{aligned}$$

Then $T(z) = (T_4 \circ T_3 \circ T_2 \circ T_1)(z)$. Hence the theorem. □

Example 16. *Show that the transformation* $w = \frac{5-4z}{4z-2}$ *maps the unit circle* $|z| = 1$ *into a circle of radius unity and centre* $-\frac{1}{2}$.

Solution.

$$\begin{aligned} w &= \frac{5-4z}{4z-2} \\ \therefore \quad 4wz - 2w &= 5 - 4z \\ \therefore \quad (4w+4)z &= 5 + 2w \\ \therefore \quad z &= \frac{5+2w}{4w+4} \end{aligned}$$

Now, $|z| = 1 \Rightarrow z\overline{z} = 1$.

$$\begin{aligned} &\Rightarrow \left(\frac{5+2w}{4w+4}\right)\left(\frac{5+2\overline{w}}{4\overline{w}+4}\right) = 1 \\ &\Rightarrow 25 + 4w\overline{w} + 10w + 10\overline{w} = 16w\overline{w} + 16 + 16(w+\overline{w}) \\ &\Rightarrow 12w\overline{w} + 6\overline{w} + 6w - 9 = 0 \\ &\Rightarrow w\overline{w} + \frac{1}{2}\overline{w} + \frac{1}{2}w - \frac{3}{4} = 0 \end{aligned}$$

This represents the equation of the circle with centre $-\frac{1}{2}$ and radius $\sqrt{\frac{1}{4}+\frac{3}{4}} = 1$. Hence the result.
■

Cross Ratio

Definition

Let z_1, z_2, z_3, z_4 be four distinct points in the extended complex plane. The cross ratio of these four points denoted by (z_1, z_2, z_3, z_4) is defined by

$$(z_1, z_2, z_3, z_4) = \begin{cases} \frac{(z_1-z_3)(z_2-z_4)}{(z_1-z_4)(z_2-z_3)} & \text{if none of } z_1, z_2, z_3, z_4 \text{ is } \infty \\ \frac{z_1-z_3}{z_1-z_4} & \text{if } z_2 \text{ is } \infty \\ \frac{z_2-z_4}{z_1-z_4} & \text{if } z_3 \text{ is } \infty \\ \frac{z_2-z_3}{z_2-z_3} & \text{if } z_4 \text{ is } \infty \\ \frac{z_2-z_4}{z_2-z_3} & \text{if } z_1 \text{ is } \infty \end{cases}$$

Theorem 14

Any bilinear transformation preserves cross ratio.

Proof. Let $w = \frac{az+b}{cz+d}, ad - bc \neq 0$ be the given bilinear transformation. Let z_1, z_2, z_3, z_4 be four distinct points. Let their images under this transformation be w_1, w_2, w_3, w_4 respectively. We assume that all the z_i and w_i are different from ∞. We claim that $(z_1, z_2, z_3, z_4) = (w_1, w_2, w_3, w_4)$. We have

$$w_i = \frac{az_i + b}{cz_i + d} \quad (i = 1, 2, 3, 4)$$

Now,

$$\begin{aligned} w_1 - w_3 &= \frac{az_1 + b}{cz_1 + d} - \frac{az_3 + b}{cz_3 + d} \\ &= \frac{(ad - bc)(z_1 - z_3)}{(cz_1 + d)(cz_3 + d)} \\ &= k_1 (z_1 - z_3) \quad (say) \end{aligned}$$

Similarly $w_2 - w_4 = k_2 (z_2 - z_4)$. Therefore

$$\begin{aligned} (w_1 - w_3)(w_2 - w_4) &= k_1 k_2 (z_1 - z_3)(z_2 - z_4) \\ &= k (z_1 - z_3)(z_2 - z_4) \end{aligned}$$

Similarly we can prove that

$$\begin{aligned} (w_1 - w_4)(w_2 - w_3) &= k (z_1 - z_4)(z_2 - z_3) \\ \frac{(w_1 - w_3)(w_2 - w_4)}{(w_1 - w_4)(w_2 - w_3)} &= \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)} \end{aligned}$$

The proof is similar if one of the z_i or w_i is ∞. □

Example 17. *Find the bilinear transformation which maps the points* $z_1 = 2, z_2 = i\ z_3 = -2$, *onto* $w_1 = 1, w_2 = i, w_3 = -1$ *respectively.*

Solution. Let the image of any point z under the required transformation be w. The required bilinear transformation is given by the equation

$$\begin{aligned} (w, 1, i, -1) &= (z, 2, i, -2) \\ \frac{(w - i)(1 + 1)}{(w + 1)(1 - i)} &= \frac{(z - i)(2 + 2)}{(z + 2)(2 - i)} \\ \frac{2(w - i)}{(w + 1)(1 - i)} &= \frac{4(z - i)}{(z + 2)(2 - i)} \\ \frac{(w - i)}{w - iw + 1 - i} &= \frac{2(z - i)}{2z - iz + 4 - 2i} \\ iwz + 6w - 3z - 2i &= 0 \\ w(iz + 6) &= 3z + 2i \\ w &= \frac{3z + 2i}{iz + 6} \end{aligned}$$

This is the required bilinear transformation. ■

Example 18. *Find the bilinear transformation which maps the points* $z = -1, 1, \infty$ *respectively on* $w = -i, -1, i$.

Solution. Let the image of any point z under the required bilinear transformation be w. Since bilinear transformation preserves cross ratio we have

$$\begin{aligned}(z,-1,1,\infty) &= (w,-i,-1,i)\\ \frac{z-1}{-1-1} &= \frac{(w+1)(-i-i)}{(w-i)(-i+1)}\\ (z-1)(w-iw-i-1) &= 4iw+4i\\ w[z-1-i(z-1)-4i] &= 4i+(i+1)(z-1)\\ w &= \frac{(i+1)z+3i-1}{(1-i)z-3i-1}\end{aligned}$$

■

Fixed Points of Bilinear Transformations

If $w=f(z)$ is any transformation from the z -plane to w -plane, the fixed points of the transformation are the solutions of the equation $z=f(z)$.
Consider a bilinear transformation given by

$$w=\frac{az+b}{cz+d} \text{ where } ad-bc\neq 0$$

The fixed points or invariant points of the bilinear transformation are given by the roots of the equation $z=\frac{az+b}{cz+d}$ (i.e.) $cz^2+(d-a)z-b=0$.
Case 1: $c\neq 0$. In this case the fixed points are given by

$$z=\frac{(a-d)\pm\sqrt{[(d-a)^2+4bc]}}{2c}$$

When $(d-a)^2+4bc\neq 0$, the given bilinear transformation has two finite fixed points and when $(d-a)^2+4bc=0$ it has only one finite fixed point.
Case 2: $c=0$. In this case the bilinear transformation becomes $w=\left(\frac{a}{d}\right)z+\frac{b}{d}$. Clearly ∞ is one fixed point. Other fixed point is determined by the equation $z=\left(\frac{a}{d}\right)z+\frac{b}{d}$ (i.e.) $(d-a)z-b=0$. Therefore if $d-a\neq 0$ we get a finite fixed point $\frac{b}{d-a}$. If $d-a=0$ then ∞ is the only fixed point.
Thus we have
Case (i): $c\neq 0$; $(d-a)^2+4bc\neq 0\Rightarrow$ 2 finite fixed points.
Case (ii): $c\neq 0$; $(d-a)^2+4bc=0\Rightarrow$ one finite fixed point.
Case (iii): $c=0$; $a\neq d\Rightarrow\infty$ and one finite fixed point.
Case (iv): $c=0$; $a=d\Rightarrow\infty$ is the only fixed point.

Theorem 15

Any bilinear transformation having two finite fixed points α and β can be written in the form $\frac{w-\alpha}{w-\beta}=k\left(\frac{z-\alpha}{z-\beta}\right)$.

Proof. Let T be the given bilinear transformation having α and β as fixed points. Let the image of any point γ under T be δ. Then the bilinear transformation T is given by $(w,\delta,\alpha,\beta)=(z,\gamma,\alpha,\beta)$. Therefore

$$\frac{(w-\alpha)(\delta-\beta)}{(w-\beta)(\delta-\alpha)}=\frac{(z-\alpha)(\gamma-\beta)}{(z-\beta)(\gamma-\alpha)}$$

$$\frac{w-\alpha}{w-\beta} = k\left(\frac{z-\alpha}{z-\beta}\right) \quad where \quad k = \frac{(\gamma-\beta)(\delta-\alpha)}{(\gamma-\alpha)(\delta-\beta)}$$

□

Definition

Let T be a bilinear transformation with two finite fixed points α, β. If k given by (1) is real T is called hyperbolic and if $|k| = 1, T$ is called elliptic.

Theorem 16

Any bilinear transformation having ∞ and $\alpha \neq \infty$ as fixed points can be written in the form $w - \alpha = k(z - \alpha)$.

Proof. Let T be the given bilinear transformation having ∞ and α as fixed points. Let the image of any point γ under T be δ. Then the bilinear transformation is given by $(w, \delta; \alpha, \infty) = (z, \gamma; \alpha, \infty)$. Therefore

$$\frac{w-\alpha}{\delta-\alpha} = \frac{z-\alpha}{\gamma-\alpha}$$

$$w - \alpha = k(z-\alpha) \quad where \quad k = \frac{\delta-\alpha}{\gamma-\alpha}$$

□

Definition

A bilinear transformation with only one finite fixed point is called parabolic.

Theorem 17

Any bilinear transformation having ∞ as the only fixed point is a translation.

Proof. Let $w = \frac{az+b}{cz+d}$ be the bilinear transformation having ∞ as the only fixed point. Then $c = 0$ and $a = d$. Therefore the bilinear transformation reduces to the form $w = \frac{az+b}{a}$. Therefore $w = z + \left(\frac{b}{a}\right)$ which is a translation. □

Example 19. *Find the invariant points of the transformations (i)* $w = \frac{1+z}{1-z}$ *(ii)* $w = \frac{1}{z-2i}$.

Solution.

(i). The invariant points of $w = f(z)$ are got from $f(z) = z$. Therefore

$$\begin{aligned} f(z) = z \Rightarrow z &= \frac{1+z}{1-z} \\ \Rightarrow z - z^2 &= 1 + z \\ \Rightarrow 1 + z^2 &= 0 \\ \Rightarrow z &= \pm i \end{aligned}$$

Therefore i and $-i$ are the two fixed points of the transformation.

(ii). $f(z) = z \Rightarrow z = \dfrac{1}{z-2i}$

$$\Rightarrow z^2 - 2iz - 1 = 0$$

$$\Rightarrow \quad (z-i)^2 = 0$$

Hence i is the (only) fixed point.

■

Some Special Bilinear Transformations

In this section we shall determine the general form of the transformations which map

(i). the real axis onto itself.

(ii). the unit circle onto itself.

(iii). the real axis onto the unit circle.

Theorem 18

A bilinear transformation $w = \frac{az+b}{cz+d}$, where $ad - bc \neq 0$ maps the real axis into itself if and only if a, b, c, d are real.
Further this transformation maps the upper half plane $Im\ z \geq 0$ into the upper half plane $Im\ w \geq 0$ if and only if $ad - bc > 0$.

Proof. Suppose a, b, c, d are real. Then obviously, z is real $\Rightarrow w$ is also real. Therefore the real axis is mapped into itself.
Conversely consider any bilinear transformation T that maps the real axis into itself. Therefore there exist real numbers x_1, x_2, x_3 such that $T(x_1) = 1$, $T(x_2) = 0$ and $T(x_3) = \infty$. Therefore the bilinear transformation T is given by

$$(z, x_1, x_2, x_3) = (w, 1, 0, \infty)$$
$$\frac{(z-x_2)(x_1-x_3)}{(z-x_3)(x_1-x_2)} = \frac{w-0}{1-0} = w$$

Therefore $w = \frac{az+b}{cz+d}$ where $a = x_1 - x_3$; $b = -x_2(x_1 - x_3)$; $c = x_1 - x_2$ and $d = -x_3(x_1 - x_2)$. Since x_1, x_2, x_3 are real a, b, c, d are also real. Now

$$2i\,\mathrm{Im}\,w = w - \overline{w} = \frac{az+b}{cz+d} - \frac{a\overline{z}+b}{c\overline{z}+d}$$
$$= \frac{(ad-bc)(z-\overline{z})}{|cz+d|^2}$$
$$= 2i\left(\frac{ad-bc}{|cz+d|^2}\right) Imz$$

Therefore $lm\ w = \frac{(ad-bc)}{|cz+d|^2}\ Im\ z$. Therefore the upper half plane $Im\ z \geq 0$ is mapped onto the upper half plane $lm\ w \geq 0 \Leftrightarrow ad - bc > 0$. □

Theorem 19

Any bilinear transformation which maps the real axis onto unit circle $|w| = 1$ can be written in the form $w = e^{i\lambda}\left(\frac{z-\alpha}{z-\overline{\alpha}}\right)$ where λ is real.
Further this transformation maps the upper half plane Im $z \geq 0$ onto the unit circular disc $|w| \leq 1$ iff Im $\alpha > 0$.

Proof. Let $w = \frac{az+b}{cz+d}$ where $ad-bc \neq 0$ be any bilinear transformation which maps the real axis onto the unit circle $|w| = 1$. 0 and ∞ are inverse points with respect to the unit circle $|w| = 1$. Hence their pre-images $-(b/a)$ and $-(d/c)$ are reflection points with respect to the real axis. Therefore if $\alpha = -\left(\frac{b}{a}\right)$ then $\overline{\alpha} = -\left(\frac{d}{c}\right)$. Now

$$\begin{aligned} w &= \frac{az+b}{cz+d} \\ &= \left(\frac{a}{c}\right)\left[\frac{z+(b/a)}{z+(d/c)}\right] \\ &= \left(\frac{a}{c}\right)\left(\frac{z-\alpha}{z-\overline{\alpha}}\right) \end{aligned}$$

Now, suppose z is real. Hence $|w| = 1$. Therefore

$$\left|\frac{a}{c}\right| \frac{|z-\alpha|}{|z-\overline{\alpha}|} = 1$$

Now, since z is real $z = \overline{z}$ and hence

$$|z-\alpha| = |\overline{z-\alpha}| = |\overline{z}-\overline{\alpha}| = |z-\overline{\alpha}|$$

Therefore $\left|\frac{a}{c}\right| = 1$. Hence $\frac{a}{c} = e^{i\lambda}$ where λ is real. Therefore

$$w = e^{i\lambda}\left(\frac{z-\alpha}{z-\overline{\alpha}}\right),$$

where λ is real, is the required transformation. Now

$$\begin{aligned} w\overline{w} - 1 &= e^{i\lambda}\left(\frac{z-\alpha}{z-\overline{\alpha}}\right)e^{-i\lambda}\left(\frac{\overline{z}-\overline{\alpha}}{\overline{z}-\alpha}\right) - 1 \\ &= \left(\frac{z-\alpha}{z-\overline{\alpha}}\right)\left(\frac{\overline{z}-\overline{\alpha}}{\overline{z}-\alpha}\right) - 1 \\ &= \frac{-4\,\mathrm{Im}\,z\,\mathrm{Im}\,\alpha}{|z-\alpha|^2} \text{ (on simplification)} \end{aligned}$$

Therefore the bilinear transformation maps the upper half plane $\mathrm{Im}\,z \geq 0$ onto the disc $|w| \leq 1$ iff $\mathrm{Im}\,\alpha > 0$. □

Example 20. *Prove that the transformation given by $\overline{a}wz - bw - \overline{b}z + a = 0$ maps the unit circle $|z| = 1$ onto the unit circle $|w| = 1$ if $|b| \neq |a|$.*

Proof. $\overline{a}wz - bw - \overline{b}z + a = 0$. Therefore

$$w = \frac{\overline{b}_z - a}{\overline{a}z - b}$$

Now

$$\begin{aligned} w\overline{w} - 1 &= \left(\frac{\overline{b}z - a}{\overline{a}z - b}\right)\left(\frac{b\overline{z} - \overline{a}}{a\overline{z} - \overline{b}}\right) - 1 \\ &= \frac{(z\overline{z} - 1)\left(|b|^2 - |a|^2\right)}{|\overline{a}z - b|^2} \text{ (on simplification)} \end{aligned}$$

If $|b| \neq |a|$ then $w\overline{w} - 1 = 0 \Leftrightarrow z\overline{z} - 1 = 0$. Therefore the unit circle $|z| = 1$ is mapped onto the unit circle $|w| = 1$ if $|b| \neq |a|$. □

Unit–3: Complex Integration

Introduction

In this chapter we develop the theory of integration for complex functions. We assume that the reader is familiar with the Riemann integral of a function defined on $[a, b]$. Using this we define the integral of a complex valued function defined on $[a, b]$ and the integral of a function $f : D \to C$ where D is a region in C, along a curve C lying in D. We prove Cauchy's fundamental theorem and study the various consequences of this theorem.

Definite Integral

Definition

Let $f(t) = u(t) + iv(t)$ be a continuous complex valued function defined on $[a, b]$. We define

$$\int_a^b f(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt.$$

Remark 21. *The following properties of the definite integral can be easily verified*

(1). $\text{Re}\int\limits_a^b f(t)dt = \int\limits_a^b \text{Re}[f(t)]dt.$

(2). $\text{Im}\int\limits_a^b f(t)dt = \int\limits_a^b \text{Im}[f(t)]dt.$

(3). $\int\limits_a^b [f(t) + g(t)]dt = \int\limits_a^b f(t)dt + \int\limits_a^b g(t)dt.$

(4). $\int\limits_a^b cf(t)dt = c\int\limits_a^b f(t)dt$ *where* c *is any complex constant.*

Lemma 22. $\left|\int\limits_a^b f(t)dt\right| \le \int\limits_a^b |f(t)|dt.$

Proof. Let $\int_a^b f(t)dt = re^{i\theta}$. Therefore

$$\begin{aligned}\left|\int_a^b f(t)dt\right| &= r = e^{-i\theta}\int_a^b f(t)dt \\ &= \text{Re}\left(e^{-i\theta}\int_a^b f(t)dt\right) \quad \text{(since } r \text{ is real)}\end{aligned}$$

$$= \text{Re}\left(\int_a^b e^{-i\theta} f(t)dt\right) \quad \text{(using 4)}$$
$$= \int_a^b \text{Re}\left(e^{-i\theta} f(t)dt\right) \quad \text{(using 1)}$$
$$\leq \int_a^b \left|e^{-i\theta} f(t)\right| dt$$
$$= \int_a^b \left|e^{-i\theta}\right| |f(t)|dt$$
$$= \int_a^b |f(t)|dt$$

Thus $\left|\int\limits_a^b f(t)dt\right| \leq \int\limits_a^b |f(t)|dt$. □

Definition

Let C be a piecewise differentiable curve given by the equation $z = z(t)$ where $a \leq t \leq b$. Let $f(z)$ be a continuous complex valued function defined in a region containing the curve C. We define

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt.$$

Example 23. *Consider $\int\limits_C f(z)dz$ where $f(z) = \frac{1}{z}$ and C is the circle $|z| = r$ described in the positive sense. The parametric equation of the circle $|z| = r$ is given by $z = re^{it}$ where $0 \leq t \leq 2\pi$ and $z'(t) = ire^{it}$. Therefore*

$$\int_C f(z)dz = \int_C \frac{dz}{z} = \int_0^{2\pi} \frac{ire^{it}}{re^{it}}dt$$
$$= i\int_0^{2\pi} dt$$
$$= 2\pi i$$

Theorem 20

$\int\limits_{-C} f(z)dz = -\int\limits_C f(z)dz.$

Proof. Suppose the equation of C is given by $z = z(t)$ where $a \leq t \leq b$. We know that the equation of $-C$ is given by

$$z(t) = z(b + a - t) \;\; where \;\; a \leq t \leq b$$

Now,

$$\int\limits_{-C} f(z)dz = \int_a^b f(z(b + a - t))z'(b + a - t)(-dt)$$

Put $b+a-t=u$. Then $-dt=du$. Also $t=a \Rightarrow u=b$ and $t=b \Rightarrow u=a$. Therefore

$$\begin{aligned}\int\limits_{-C} f(z)dz &= \int_b^a f(z(u))z'(u)du \\ &= -\int_a^b f(z(u))z'(u)du \\ &= -\int\limits_C f(z)dz\end{aligned}$$

□

Definition

Let C_1 be a differentiable curve with origin z_1 and terminus z_2. Let C_2 be another differentiable curve with origin z_2 and terminus z_3. Then the curve C which consists of C_1 followed by C_2 is a piecewise differentiable curve with origin z_1 and terminus z_3. This curve is denoted by $C_1 + C_2$.

Definition

Let C be a piecewise differentiable curve given by the equation $z = z(t)$ where $a \leq t \leq b$. Then the length l of C is defined by

$$l = \int_a^b \left|z'(t)\right| dt$$

Theorem 21

$\left|\int\limits_C f(z)dz\right| \leq Ml$ where $M = \max\{|f(z)|/z \in C\}$ and l is the length of C.

Proof. Suppose C is given by the equation $z = z(t)$ where $a \leq t \leq b$. By definition of M we have

$$|f(z(t))| \leq M \quad \forall \quad t; a \leq t \leq b \tag{1}$$

Now

$$\begin{aligned}\left|\int\limits_C f(z)dz\right| &= \left|\int_a^b f(z(t))z'(t)dt\right| \\ &\leq \int_a^b \left|f(z(t))z'(t)\right| dt \\ &= \int_a^b \left|f\left(z(t)\|z'(t)\right)\right| dt \\ &\leq \int_a^b M\left|z'(t)\right| dt \quad \text{(using (1))}\end{aligned}$$

$$= M \int_a^b |z'(t)|\,dt$$
$$= Ml$$
$$\left| \int_C f(z)dz \right| \leq Ml$$

□

Example 24. *Evaluate* $\int_C f(z)dz$ *where* $f(z) = y - x - i3x^2$ *and* C *is the line segment from* $z = 0$ *to* $z = 1 + i$.

Solution. The equation of the line segment C joining $z = 0$ and $z = 1 + i$ is given by $y = x$. Therefore the parametric equation of C can be taken as $x = t$ and $y = t$ where $0 \leq t \leq 1$. Hence $z(t) = x(t) + iy(t) = t + it$ so that $z'(t) = (1 + i)$. Now

$$f(z(t)) = t - t - i3t^2 = -i3t^2$$

Therefore

$$\int_C f(z)dz = \int_0^1 f(z(t))z'(t)dt$$
$$= \int_0^1 -i3t^2(1+i)dt$$
$$= -3i(1+i)\left[\frac{t^3}{3}\right]_0^1$$
$$= 1 - i$$

■

Cauchy's Theorem

In this section we prove the fundamental theorem of integration known as Cauchy's theorem which forms the basis for the theory of complex integration

Definition

Let $p(x, y)$ and $q(x, y)$ be two real valued functions. Then the differential equation $p(x, y)dx + q(x, y)dy = 0$ is said to be exact if there exists a function $u(x, y)$ such that $\frac{\partial u}{\partial x} = p$ and $\frac{\partial u}{\partial y} = q$.

Theorem 22: Cauchy's Theorem

Let f be a function which is analytic at all points inside and on a simple closed curve C. Then $\int_C f(z)dz = 0$.

Proof. Let D be the closed region consisting of all points interior to C together with the points on C. Let $\varepsilon > 0$ be given. Let. C_j $(j = 1, 2, \ldots, n)$ denote the boundaries of the squares and partial squares covering D such that there exists a point z_j lying inside or on C_j satisfying

$$\left|\frac{f(z) - f(z_j)}{z - z_j} - f'(z_j)\right| < \varepsilon \tag{1}$$

for all z distinct from z_j and lying within or on C_j. Let

$$\delta_j(z) = \begin{cases} \frac{f(z)-f(z_j)}{z-z_j} - f'(z_j) & \text{if } z \neq z_j \\ 0 & \text{if } z = z_j \end{cases}$$

Clearly $\delta_j(z)$ is a continuous function and

$$\begin{aligned} f(z) &= f(z_j) - z_j f'(z_j) + z f'(z_j) + (z - z_j)\,\delta_j(z) \\ \int_{C_j} f(z)dz &= \int_{C_j} f(z_j)\,dz - \int_{C_j} z_j f'(z_j)\,dz + \int_{C_j} z f'(z_j)\,dz + \int_{C_j} (z - z_j)\,\delta_j(z)dz \\ &= f(z_j)\int_{C_j} dz - z_j f'(z_j)\int_{C_j} dz + f'(z_j)\int_{C_j} zdz + \int_{C_j} (z - z_j)\,\delta_j(z)dz \\ &= \int_{C_j} (z - z_j)\,\delta_j(z)dz \left(\text{since} \int_{C_j} dz = 0 \; and \; \int_{C_j} zdz = 0\right) \end{aligned}$$

Therefore

$$\sum_{j=1}^{n}\int_{C_j} f(z)dz = \sum_{j=1}^{n}\int_{C_j} (z - z_j)\,\delta_j(z)dz \tag{2}$$

Now, in the sum $\sum\limits_{j=1}^{n}\int\limits_{C_j} f(z)dz$ the integrals along the common boundary of every pair of adjacent subregions cancel each other.

Hence only the integrals along the arcs which are the parts of C remain. Therefore

$$\sum_{j=1}^{n}\int_{C_j} f(z)dz = \int_{C} f(z)dz$$

Therefore from (2),

$$\begin{aligned} \int_C f(z)dz &= \sum_{j=1}^{n}\int_{C_j} (z - z_j)\,\delta_j(z)dz \\ \left|\int_C f(z)dz\right| &= \left|\sum_{j=1}^{n}\int_{C_j} (z - z_j)\,\delta_j(z)dz\right| \\ &\leq \sum_{j=1}^{n}\int_{C_j} \left|(z - z_j)\,\delta_j(z)\right| dz \\ &= \sum_{j=1}^{n}\int_{C_j} |z - z_j|\,|\delta_j(z)|\,dz \end{aligned}$$

$$\left|\int_C f(z)dz\right| \leq \sum_{j=1}^{n} \int_{C_j} |z - z_j||\delta_j(z)|\, dz \tag{3}$$

Now if C_j is a square and s_j is the length of its side then $|z - z_j| < \sqrt{2}s_j$ for all z on C_j . Also from (1) we have $|\delta_j(z)| < \varepsilon$ and hence

$$\begin{aligned}\int_{C_j} |z - \dot{z}_j|\, |\delta_j(z)|\, dz &< \left(\sqrt{2}s_j\varepsilon\right)(4s_j) \\ &= 4\sqrt{2}A_j\varepsilon \end{aligned} \tag{4}$$

where A_j is the area of the square C_j. Similarly for a partial square with boundary C_j if l_j is the length of the arc of C which forms a part of C_j . We have

$$\begin{aligned}\int_{C_j} |z - z_j|\, |\delta_j(z)|\, dz &< \sqrt{2}s_j\varepsilon\,(4s_j + l_j) \\ &< 4\sqrt{2}A_j\varepsilon + \sqrt{2}Sl_j \end{aligned} \tag{5}$$

where S is the length of a side of some square containing the entire region D as well as all the squares originally used in covering D. We observe that the sum of all A_j 's that occur in the right hand side of (4) and (5) do not exceed S^2 and the sum of all the l_j^s is equal to L (the length of C). Using (4) and (5) in (3) we obtain

$$\begin{aligned}\left|\int_C f(z)dz\right| &< \left(4\sqrt{2}S^2 + \sqrt{2}SL\right)\varepsilon \\ &= k\varepsilon\end{aligned}$$

where $k = 4\sqrt{2}S^2 + \sqrt{2}SL$ is a constant. Thus

$$\left|\int_C f(z)dz\right| < k\varepsilon$$

Since ε is arbitrary we have $\int\limits_C f(z)dz = 0$. □

Definition

A region D is said to be simply connected if every simple closed curve lying in D encloses only points of D.

Cauchy's Integral Formula

In this section we establish another fundamental result known as Cauchy's integral formula using Cauchy's theorem.

Theorem 23

Let $f(z)$ be a function which is analytic inside and on a simple closed curve C. Let z_0 be any point in the interior of C. Then

$$f\left(z_0\right)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz$$

Proof. Choose a circle C_0 with centre z_0 and radius r_0 such that C_0 lies in the interior of C. Now, z_0 is the only point inside C at which the function $\frac{f(z)}{z-z_0}$ is not analytic and hence is analytic. in the region D consisting of all points inside and on C except the points interior to C_0. Hence

$$\begin{aligned}
\int_C \frac{f(z)dz}{z-z_0} &= \int_{C_0}\frac{f(z)dz}{z-z_0}\\
&= \int_{C_0}\left(\frac{f(z)-f\left(z_0\right)+f\left(z_0\right)}{z-z_0}\right)dz\\
&= \int_{C_0}\left(\frac{f(z)-f\left(z_0\right)}{z-z_0}\right)dz+\int_{c_0}\frac{f\left(z_0\right)}{z-z_0}dz\\
&= \int_{C_0}\left(\frac{f(z)-f\left(z_0\right)}{z-z_0}\right)dz+f\left(z_0\right)\int_{C_0}\frac{dz}{z-z_0}\\
&= \int_{C_0}\left(\frac{f(z)-f\left(z_0\right)}{z-z_0}\right)dz+f\left(z_0\right)(2\pi i)
\end{aligned}$$

Thus

$$\int_C \frac{f(z)dz}{z-z_0}=\int_{C_0}\left(\frac{f(z)-f\left(z_0\right)}{z-z_0}\right)dz+2\pi i f z_0 \tag{1}$$

We now claim that

$$\int_{C_0}\left(\frac{f(z)-\dot{f}\left(z_0\right)}{z-z_0}\right)dz=0$$

Since $f(z)$ is analytic inside and on C it is continuous at z_0. Therefore given $\varepsilon>0$ there exists $\delta>0$ such that

$$\left|z-z_0\right|<\delta \Rightarrow \left|f(z)-f\left(z_0\right)\right|<\varepsilon$$

If we choose $r_0<\delta$, then $\left|z-z_0\right|<r_0 \Rightarrow \left|f(z)-f\left(z_0\right)\right|<\varepsilon$. Hence

$$\begin{aligned}
\left|\int_{C_0}\left(\frac{f(z)-f\left(z_0\right)}{z-z_0}\right)dz\right| &< \left(\frac{\varepsilon}{r_0}\right)\left(2\pi r_0\right)\\
&= 2\pi\varepsilon
\end{aligned}$$

Thus

$$\left|\int_{C_0}\left(\frac{f(z)-f\left(z_0\right)}{z-z_0}\right)dz\right|<2\pi\varepsilon$$

since ε is arbitrary we have

$$\int_{C_0}\left(\frac{f(z)-f\left(z_0\right)}{z-z_0}\right)dz=0$$

$\therefore$ From (1) we get

$$\int_C \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$$

Therefore

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}dz$$

□

Example 25. *Consider* $\int_C \frac{dz}{z-3}$ *where* C *is the circle* $|z-2| = 5$.

Solution. Let $f(z) = 1$. The point $z = 3$ lies inside C. Hence by Cauchy's integral formula $\int\limits_C \frac{dz}{z-3} = 2\pi i f(3) = 2\pi i$. ■

Theorem 24: Maximum Modulus Theorem

Let $f(z)$ be continuous in a closed and bounded region D and analytic and nonconstant in the interior of D. Then $|f(z)|$ attains its maximum value on the boundary of D and never in the interior of D.

Proof. Since f is continuous in a closed and bounded region $D, |f(z)|$ is bounded and attains its bound. Therefore there exists a positive real number M such that

$$|f(z)| \leq M \ \ \forall \ \ z \in D \tag{1}$$

and equality holds for at least one point z in D. Suppose that there exists an interior point $z_0 \in D$ such that

$$|f(z_0)| = M \tag{2}$$

Choose a circle with centre z_0 and radius r such that the circular disc $|z - z_0| \leq r$ is contained in D. Then we have

$$f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f\left(z_0 + re^{i\theta}\right) e^{i\theta} d\theta$$

Therefore

$$|f(z_0)| \leq \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(z_0 + re^{i\theta}\right)\right| d\theta \tag{3}$$

Also from (1) and (2) we have $\left|f\left(z_0 + re^{i\theta}\right)\right| \leq |f(z_0)|$. Therefore

$$\int_0^{2\pi} \left|f\left(z_0 + re^{i\theta}\right)\right| d\theta \leq 2\pi |f(z_0)|$$

$$|f(z_0)| \geq \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(z_0 + re^{i\theta}\right)\right| d\theta \tag{4}$$

From (3) and (4) we get

$$|f(z_0)| = \frac{1}{2\pi}\int_0^{2\pi} \left|f\left(z_0 + re^{i\theta}\right)\right| d\theta$$

$$2\pi |f(z_0)| = \int_0^{2\pi} \left|f\left(z_0 + re^{i\theta}\right)\right| d\theta$$

$$\int_0^{2\pi} |f(z_0)|\, d\theta = \int_0^{2\pi} \left| f\left(z_0 + re^{i\theta}\right)\right| d\theta$$

$$\int_0^{2\pi} \left[|f(z_0)| - \left| f\left(z_0 + re^{i\theta}\right)\right| \right] d\theta = 0$$

Since the integrand in the above expression is continuous and non-negative we have $|f(z_0)| - \left|f\left(z_0 + re^{i\theta}\right)\right| = 0$ (ie) $|f(z_0)| = \left|f\left(z_0 + re^{i\theta}\right)\right|$ for all z in the circular disc $|z - z_0| < r$ (ie) $|f(z_0)| = |f(z)|$ for all z in the circular disc. Therefore $f(z)$ is constant in a neighbourhood of z_0. Since $f(z)$ is continuous it follows that $f(z)$ is constant throughout D which is a contradiction. Therefore the maximum of $|f(z)|$ is not attained at any of the interior points of D. Hence the theorem. □

Example 26. *Evaluate using Cauchy's integral formula*

$$\frac{1}{2\pi i}\int_C \frac{z^2+5}{z-3} dz \ \ where \ \ C \ \ is \ \ |z| = 4$$

Solution. $f(z) = z^2 + 5$ is analytic inside and on $|z| = 4$ and $z = 3$ lies inside it. Therefore by Cauchy's integral formula $\frac{1}{2\pi i}\int\limits_C \frac{z^2+5}{z-3}dz = f(3) = 3^2 + 5 = 14.$ ■

Higher Derivatives

In this section we shall prove that an analytic function has derivatives of all orders. It follows, in particular, that the derivative of an analytic function is again an analytic function. Consider a function $f(z)$ which is analytic in a region D. Let $z \in D$. Let C be any circle with centre z such that the circle and its interior is contained in D. By Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

We now proceed to prove that

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta$$

and in general

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^{n+1}} d\zeta.$$

Theorem 25

Let f be analytic inside and on a simple closed curve C. Let z be any point inside C. Then

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{(\zeta - z)^2} d\zeta.$$

Proof. By Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z} d\zeta$$

Therefore

$$\begin{aligned}\frac{f(z+h)-f(z)}{h} &= \frac{1}{h(2\pi i)}\int_C \left(\frac{f(\zeta)}{\zeta-z-h}-\frac{f(\zeta)}{\zeta-z}\right)d\zeta \\ &= \frac{1}{h2\pi i}\int_C \left[\frac{hf(\zeta)}{(\zeta-z-h)(\zeta-z)}\right]d\zeta \\ &= \frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z-h)(\zeta-z)} \qquad (1)\end{aligned}$$

Now

$$\begin{aligned}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z-h)(\zeta-z)}-\int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^2} &= \int_C \left[\frac{f(\zeta)}{(\zeta-z-h)(\zeta-z)}-\frac{f(\zeta)}{(\zeta-z)^2}\right]d\zeta \\ &= \int_C \frac{f(\zeta)}{(\zeta-z)}\left(\frac{1}{\zeta-z-h}-\frac{1}{\zeta-z}\right)d\zeta \\ &= \int_C \frac{f(\zeta)}{(\zeta-z)}\left[\frac{h}{(\zeta-z-h)(\zeta-z)}\right]d\zeta \\ &= h\int_C \frac{f(\zeta)d\zeta}{(\zeta-z-h)(\zeta-z)^2}\end{aligned}$$

Therefore

$$\frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z-h)(\zeta-h)}-\frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^2}=\frac{h}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z-h)(\zeta-z)^2}$$

Therefore

$$\frac{f(z+h)-f(z)}{h}-\frac{1}{2\pi i}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z)^2}=\frac{h}{2\pi|i|}\int_C \frac{f(\zeta)d\zeta}{(\zeta-z-h)(\zeta-z)^2} \qquad (2)$$

Now, let M denote the maximum value of $|f(\zeta)|$ on C. Let L be the length of C and d be the shortest distance from z to any point on the curve C. Therefore for any point ζ on C we have

$$|\zeta-z|\geq d \;\; and \;\; |\zeta-z-h|\geq|\zeta-z|-|h|\geq d-|h|$$

□

Theorem 26: Liouville's Theorem

A bounded entire function in the complex plane is constant.

Proof. Let $f(z)$ be a bounded entire function. Since $f(z)$ is bounded there exists a real number M such that $|f(z)|\leq M$ for all z. Let z_0 be any complex number and $r>0$ be any real number. By Cauchy's inequality we have $|f'(z_0)|\leq\frac{M}{r}$. Taking the limit as $r\to\infty$ we get $f'(z_0)=0$. Since z_0 is arbitrary $f'(z)=0$ for all z in the complex plane. Therefore $f(z)$ is a constant function. □

Theorem 27: Fundamental theorem of algebra

Every polynomial of degree ≥ 1 has atleast one zero (root) in C.

Proof. Let $f(z)$ be a polynomial of degree ≥ 1. Suppose $f(z)$ has no zero in C. Then $f(z) \neq 0$ for all z Further $f(z)$ is an entire function in the complex plane. Therefore $\frac{1}{f(z)}$ is also an entire function. Also as $z \to \infty, f(z) \to \infty$. Therefore $\frac{1}{f(z)} \to 0$ as $z \to \infty$. Therefore $\frac{1}{f(z)}$ is a bounded function. Hence by Liouville's theorem $\frac{1}{f(z)}$ is a constant function. Therefore $f(z)$ is a constant function and hence it is a polynomial of degree zero which is a contradiction. Hence $f(z)$ has at least one root in C. Hence the theorem. □

Theorem 28: Morera's theorem

If $f(z)$ is continuous in a simply connected domain D and if $\int_C f(z)dz = 0$ for every simple closed curve C lying in D then $f(z)$ is analytic in D.

Proof. By known results there exists an analytic function $F(z)$ such that $F'(z) = f(z)$ in D. Also we know the derivative of an analytic function is an analytic function. Hence $F'(z)$ is analytic in D. Therefore $f(z)$ is analytic in D. □

Example 27. *Evaluate* $\int_C \frac{\sin z}{(z-\pi/2)^2} dz$ *where* C *is the circle* $|z| = 2$.

Solution. Let $f(z) = \sin z$. Hence $f'(z) = \cos z\cdot$ Also $\pi/2$ lies inside $|z| = 2$. Hence

$$\begin{aligned}\int_C \frac{\sin z dz}{(z-\pi/2)^2} &= 2\pi i f'(\pi/2) \\ &= 2\pi i(\cos \pi/2) \\ &= 0\end{aligned}$$

■

Unit–4: Series Expansions

Introduction

In this chapter we consider the problem of representing a given function as a power series. We prove that if a function is analytic at a point z_0 then it can be expanded as a power series called Taylor's series consisting of non-negative powers of $z - z_0$ and the expansion is valid in some neighbourhood of z_0. We also prove that a function $f(z)$ which is analytic in an annular region $a < |z - z_0| < b$ can be expanded as a series called Laurent's series consisting of positive and negative powers of $z - z_0$. We also introduce the concept of singular points of a function and classify the singular points and discuss the behaviour of the function in the neighbourhood of a singularity.

Taylor's Series

Theorem 29: (Taylor's Theorem)

Let $f(z)$ be analytic in a region D containing z_0. Then $f(z)$ can be represented as a power series in $z - z_0$ given by

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n + \cdots$$

The expansion is valid in the largest open disc with centre z_0 contained in D.

Proof. Let $r > 0$ be such that the disc $|z - z_0| < r$ is contained in D. Let $0 < r_1 < r$. Let C_1 be the circle $|z - z_0| = r_1$. By Cauchy's integral formula we have

$$f(z) = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)}{(\zeta - z)} d\zeta \tag{1}$$

Also by theorem on higher derivatives we have

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C_1} \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}} \tag{1}$$

Now

$$\begin{aligned} \frac{1}{\zeta - z} &= \frac{1}{(\zeta - z_0) - (z - z_0)} \\ &= \frac{1}{(\zeta - z_0)\left[1 - \frac{z - z_0}{\zeta - z_0}\right]} \end{aligned}$$

$$= \frac{1}{\zeta - z_0}\left[1 + \left(\frac{z-z_0}{\zeta-z_0}\right) + \left(\frac{z-z_0}{\zeta-z_0}\right)^2 + \cdots + \left(\frac{z-z_0}{\zeta-z_0}\right)^{n-1} + \frac{\left(\frac{z-z_0}{\zeta-z_0}\right)^n}{1-\left(\frac{z-z_0}{\zeta-z_0}\right)}\right]$$

$$= \frac{1}{\zeta - z_0} + \frac{z-z_0}{(\zeta-z_0)^2} + \frac{(z-z_0)^2}{(\zeta-z_0)^3} + \ldots + \frac{(z-z_0)^{n-1}}{(\zeta-z_0)^n} + \frac{(z-z_0)^n}{(\zeta-z_0)^n(\zeta-z)}$$

Now, multiplying throughout by $\frac{f(\zeta)}{2\pi i}$, integrating over C_1 and using (1) and (2) we get

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots + \frac{f^{(n-1)}(z_0)}{(n-1)!}(z-z_0)^{n-1} + R_n \qquad (3)$$

where

$$R_n = \frac{(z-z_0)^n}{2\pi i}\int_{c_1}\frac{f(\zeta)d\zeta}{(\zeta-z)(\zeta-z_0)^n}$$

Here ζ lies on C_1 and z lies in the interior of C_1 so that $|\zeta - z_0| = r_1$ and $|z - z_0| < r_1$. Therefore

$$|\zeta - z| = |(\zeta - z_0) - (z - z_0)| \geq |\zeta - z_0| - |z - z_0| = r_1 - |z - z_0|$$

Therefore

$$\frac{1}{|\zeta - z|} \leq \frac{1}{r_1 - |z - z_0|}$$

Let M denote the maximum value of $|f(z)|$ on C_1. Then

$$|R_n| \leq \frac{|z-z_0|^n}{2\pi}\frac{M(2\pi r_1)}{(r_1 - |z-z_0|)r_1^n}$$
$$= \frac{M|z-z_0|}{(r_1 - |z-z_0|)}\left(\frac{|z-z_0|}{\dot{r}_1}\right)^{n-1}$$

Also $\left|\frac{z-z_0}{r_1}\right| < 1$. Hence $\lim_{n\to\infty} R_n = 0$. Therefore taking limit as $n \to \infty$ in (3) we get

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \cdots + \frac{f^{(n)}(z_0)}{n!}(z-z_0)^n + \cdots$$

□

Example 28. *The Taylor's series for* $f(z) = \frac{1}{z}$ *about* $z = 1$ *is given by*

$$\frac{1}{z} = f(1) + \frac{f'(1)}{1!}(z-1) + \frac{f''(1)}{2!}(z-1)^2 + \frac{f'''(1)}{3!}(z-1)^3 + \cdots$$

Solution. Now,

$$f(z) = \frac{1}{z} \Rightarrow f(1) = 1$$
$$f'(z) = -\frac{1}{z^2} \Rightarrow f'(1) = -1$$
$$f''(z) = \frac{2}{z^3} \Rightarrow f''(1) = 2$$

$$f'''(z) = -\frac{6}{z^4} \Rightarrow f'''(1) = -6$$

. . . .

Hence the Taylor's series expansion for $\frac{1}{z}$ about 1 is

$$\frac{1}{z} = 1 - (z-1) + (z-1)^2 - (z-1)^3 + \cdots$$

This expansion is valid in the disc $|z-1| < 1$. Similarly the Taylor's series for $f(z) = \frac{1}{z}$ about $z = i$ is given by

$$\frac{1}{z} = \frac{1}{i} - \frac{z-i}{i^2} + \frac{(z-i)^2}{i^3} - \frac{(z-i)^3}{i^4} + \dots$$

and the expansion is valid in the disc $|z-i| < 1$. ■

Maclaurin's series expansion of some of the standard functions are given below.

(1). $e^{-z} = 1 - \frac{z}{1!} + \frac{z^2}{2!} - \cdots + (-1)^n \frac{z^n}{n!} + \cdots (|z| < \infty)$.

(2). $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots + (-1)^{n-1} \frac{z^{2n-1}}{(2n-1)!} + \cdots (|z| < \infty)$.

(3). $\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots + (-1)^{n-1} \frac{z^{2n-2}}{(2n-2)!} + \cdots (|z| < \infty)$.

(4). $\sinh z = \frac{1}{1!} + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots + \frac{z^{2n-1}}{(2n-1)!} + \cdots (|z| < \infty)$.

(5). $\cosh z = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots + \frac{z^{2n}}{(2n)!} + \cdots (|z| < \infty)$.

(6). $\frac{1}{1+z} = 1 - z + z^2 - z^3 + \cdots + (-1)^n z^n + \cdots (|z| < 1)$.

(7). $\frac{1}{1-z} = 1 + z + z^2 + z^3 + \cdots + z^n + \cdots (|z| < 1)$.

(8). $\log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots (-1)^{n-1} \frac{z^n}{n} + \cdots (|z| < 1)$.

(9). $\log(1-z) = -z - \frac{z^2}{2} - \frac{z^3}{3} - \cdots - \frac{z^n}{n} - \cdots (|z| < 1)$.

Example 29. *Expand $f(z) = \sin z$ in a Taylor's series about $z = \frac{\pi}{4}$ and determine the region of convergence of this series.*

Solution. The Taylor's series for $f(z)$ about $z = \frac{\pi}{4}$ is

$$f(z) = f(\pi/4) + \frac{(z-\pi/4)}{1!} f'(\pi/4) + \frac{(z-\pi/4)^2}{2!} f''(\pi/4) + \cdots$$

Here $f(z) = \sin z$. Hence $f(\pi/4) = \frac{1}{\sqrt{2}}$.

$$f'(z) = \cos z. \ \ Hence \ \ f'(\pi/4) = \frac{1}{\sqrt{2}}$$

$$f''(z) = -\sin z. \ \ Hence \ \ f''(\pi/4) = -\frac{1}{\sqrt{2}}$$

$$f'''(z) = -\cos z. \ \ Hence \ \ f'''(\pi/4) = -\frac{1}{\sqrt{2}}$$

The Taylor's series for $\sin z$ about $z = \pi/4$ is

$$\begin{aligned}\sin z &= \frac{1}{\sqrt{2}} + \frac{(z-\pi/4)}{1!}\left(\frac{1}{\sqrt{2}}\right) - \frac{(z-\pi/4)^2}{2!}\left(\frac{1}{\sqrt{2}}\right) + \cdots \\ &= \frac{1}{\sqrt{2}}\left[1 + \frac{(z-\pi/4)}{1!} - \frac{(z-\pi/4)^2}{2!} - \frac{(z-\pi/4)^3}{3!} + \cdots\right]\end{aligned}$$

The expansion is valid in the entire complex plane. ■

Laurent's Series

A series of the form

$$\sum_{n=1}^{\infty} \frac{b_n}{z_n} \tag{1}$$

can be considered as an ordinary power series in the variable $\frac{1}{z}$. Hence if the radius of convergence of the power series $\sum_{n=1}^{\infty} b_n z^n$ is r and $r < \infty$ the series $\sum_{n=1}^{\infty} \frac{b_n}{z^n}$ converges in the region $|z| > r$. The convergence is uniform in every region $|z| \geq \rho > r$ and the series represents an analytic function in $|z| > r$. If the series (1) is combined with the usual power series we get a more general series of the form

$$\sum_{-\infty}^{\infty} a_n z^n \tag{2}$$

This series is said to converge at a point if the part of the series consisting of the negative powers of z and the part of the series consisting of non-negative powers of z are separately convergent. We know that the series consisting of non-negative powers of z converges in a disc $|z| < r_2$ and the series consisting of negative powers of z converges in a region $|z| > r_1$. Therefore if $r_1 < r_2$ the series represented by (2) converges in the region $r_1 < |z| < r_2$ and in this annulus region it represents an analytic function.

We shall now prove that the converse situation is also true. i.e., any function which is analytic in a region containing the annulus $r_1 < |z - z_0| < r_2$ can be represented in a series of the form $\sum_{-\infty}^{\infty} a_n (z - z_0)^n$.

Theorem 30: Laurent's Theorem

Let C_1 and C_2 denote respectively the concentric circles $|z-z_0|=r_1$ and $|z-z_0|=r_2$ with $r_1<r_2$. Let $f(z)$ be analytic in a region containing the circular annulus $r_1<|z-z_0|<r_2$. Then $f(z)$ can be represented as a convergent series of positive and negative powers of $z-z_0$ given by

$$f(z)=\sum_{n=1}^{\infty}\frac{b_n}{(z-z_0)^n}+\sum_{n=0}^{\infty}a_n(z-z_0)^n$$

where

$$b_n=\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)d\zeta}{(\zeta-z_0)^{-n+1}}\quad and$$
$$a_n=\frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)d\zeta}{(\zeta-z_0)^{n+1}}$$

Proof. Let z be any point in the circular annulus $r_1<|z-z_0|<r_2$. Then by known theorem we have,

$$f(z)=\frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)d\zeta}{\zeta-z}-\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)d\zeta}{\zeta-z}$$

Therefore

$$f(z)=\frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)d\zeta}{\zeta-z}+\frac{1}{2\pi i}\int_{C_1}\frac{f(\zeta)d\zeta}{z-\zeta}\tag{1}$$

As in the proof of Taylor's theorem, we have

$$\frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)}{\zeta-z}d\zeta=a_0+a_1(z-z_0)+a_2(z-z_0)^2+\dot{+}a_{n-1}(z-z_0)^{n-1}+R_n(z)\tag{2}$$

where

$$a_n=\frac{1}{2\pi i}\int_{C_2}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}d\zeta\quad and$$
$$R_n(z)=\frac{(z-z_0)^n}{2\pi i}\int_{C_2}\frac{f(\zeta)d\zeta}{(\zeta-z_0)^n(\zeta-z)}$$

Now,

$$\begin{aligned}\frac{1}{z-\zeta}&=\frac{1}{z-z_0+z_0-\zeta}\\&=\frac{1}{(z-z_0)-(\zeta-z_0)}\\&=\frac{1}{(z-z_0)\left[1-\frac{\zeta-z_0}{z-z_0}\right]}\\&=\frac{1}{z-z_0}\left[1+\left(\frac{\zeta-z_0}{z-z_0}\right)+\left(\frac{\zeta-z_0}{z-z_0}\right)^2+\cdots+\left(\frac{\zeta-z_0}{z-z_0}\right)^{n-1}+\frac{\left(\frac{\zeta-z_0}{z-z_0}\right)^n}{1-\left(\frac{\zeta-z_0}{z-z_0}\right)}\right]\end{aligned}$$

Multiplying by $\frac{f(\zeta)}{2\pi i}$ and integrating over C_1 we get

$$\int_{C_1} \frac{f(\zeta)d\zeta}{z-\zeta} = \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_{n-1}}{(z-z_0)^{n-1}} + S_n(z) \tag{3}$$

where

$$b_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(\zeta)d\zeta}{(\zeta-z_0)^{-n+1}}; \quad S_n = \frac{1}{2\pi i\,(z-z_0)^n} \int_{C_1} \frac{f(\zeta)\,(\zeta-z_0)^n\,d\zeta}{z-\zeta}$$

From (1), (2) and (3) we get

$$\begin{aligned} f(z) &= a_0 + a_1(z-z_0) + \cdots\cdots + a_{n-1}(z-z_0)^{n-1} \\ &\quad + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_{n-1}}{(z-z_0)^{n-1}} + R_n(z) + S_n(z) \end{aligned} \tag{4}$$

The required result follows if we can prove that $R_n \to 0$ and $S_n \to 0$ as $n \to \infty$. Now, if $\zeta \in C_1$ then $|\zeta - z_0| = r_1$ and

$$|z-\zeta| = |(z-z_0) - (\zeta - z_0)| \geq |z-z_0| - r_1$$

If $\zeta \in C_2$ then $|\zeta - z_0| = r_2$ and

$$|\zeta - z| = |(\zeta - z_0) - (z-z_0)| \geq r_2 - |z-z_0|$$

Now let M denote the maximum value of $|f(z)|$ in $C_1 \cup C_2$. Then

$$\begin{aligned} |R_n| &\leq \frac{|z-z_0|^n}{2\pi} \frac{M\,(2\pi r_2)}{r_2^n\,(r_2 - |z-z_0|)} \\ &\leq \frac{M\,|z-z_0|}{(r_2 - |z-z_0|)} \left(\frac{|z-z_0|}{r_2}\right)^{n-1} \end{aligned}$$

Since $\frac{|z-z_0|}{r_2} < 1$, $R_n \to 0$ as $n \to \infty$. Also

$$\begin{aligned} |S_n| &\leq \frac{1}{|z-z_0|^n\,2\pi} \frac{M r_1^n\,(2\pi r_1)}{(|z-z_0| - r_1)} \\ &\leq \frac{M r_1}{(|z-z_0| - r_1)} \left(\frac{r_1}{|z-z_0|}\right)^n \end{aligned}$$

Since $\frac{r_1}{|z-z_0|} < 1, S_n \to 0$ as $n \to \infty$. Hence, by taking limit $n \to \infty$ in (4) we get

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n\,(z-z_0)^n$$

Hence the theorem. □

Example 30. *Find the Laurent's series expansion of $f(z) = z^2 e^{1/z}$ about $z = 0$.*

Solution. $f(z) = z^2 e^{1/z}$. Clearly $f(z)$ is analytic at all points $z \neq 0$. Now

$$\begin{aligned} f(z) &= z^2 \left[1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots\right] \\ &= z^2 + z + \frac{1}{2} + \frac{1}{3!z} + \frac{1}{4!z^2} + \cdots \end{aligned}$$

This is the required Laurent's series expansion for $f(z)$ at $z = 0$. ■

Zeros of an Analytic Function

Definition

Let $f(z)$ be a function which is analytic in a region D. Let $a \in D$. Then a is said to be a zero of order r (where r is a positive integer) for $f(z)$ if $f(z) = (z-a)^r \varphi(z)$ where $\varphi(z)$ is analytic at a and $\varphi(a) \neq 0$.

Example 31. *Consider $f(z) = \sin z$. We know that*

$$\begin{aligned}\sin z &= z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots \\ &= z\left(1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots\right) \\ &= z\varphi(z)\end{aligned}$$

where $\varphi(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots$. Obviously $\varphi(z)$ is analytic and $\varphi(0) = 1 \neq 0$. $z = 0$ is a zero of order 1 for $\sin z$.

Theorem 31

Suppose $f(z)$ is analytic in a region D and is not identically zero in D . Then the set of all zeros of $f(z)$ is isolated.

Proof. Let $a \in D$ be a zero for $f(z)$. We shall prove that there exists a neighbourhood $|z-a| < \delta$ such that this neighbourhood does not contain any other zero for $f(z)$. Suppose a is a zero of order r for $f(z)$. Then

$$f(z) = (z-a)^r \varphi(z) \tag{1}$$

where $\varphi(z)$ is analytic at a and $\varphi(a) \neq 0$. Now, since φ is analytic at a, φ is continuous at a. Therefore we can find a $\delta > 0$ such that

$$|z-a| < \delta \Rightarrow |\varphi(z) - \varphi(a)| < \frac{|\varphi(a)|}{2}$$

We claim that the neighbourhood $|z-a| < \delta$ does not contain any other zero of $f(z)$. Suppose $b \neq a$ is another zero for $f(z)$ in this neighbourhood. Then $|b-a| < \delta$ and $f(b) = 0$. Therefore $(b-a)^r \varphi(b) = 0$. Now, since $b \neq a, (b-a)^r \neq 0$. Therefore $\varphi(b) = 0$. Further

$$|b-a| < \delta \Rightarrow |\varphi(b) - \dot{\varphi}(a)| < \frac{|\varphi(a)|}{2} \Rightarrow |\varphi(a)| < \frac{|\varphi(a)|}{2}$$

which is a contradiction. Thus the neighbourhood $|z-a| < \delta$ contains no other zero of $f(z)$ and hence the set of all zeros of $f(z)$ is isolated. □

Singularities

Definition

A point a is called a singular point or a singularity of a function $f(z)$ if $f(z)$ is not analytic at a and f is analytic at some point of every disc $|z-a| < r$.

Example 32. *Consider the function* $f(z) = \frac{1}{z}$. *Then* $f'(z) = -\frac{1}{z^2}$ *for all* $z \neq 0$. *Thus* $f(z)$ *is analytic except at* $z = 0$. *Therefore* $z = 0$ *is a singular point of* $f(z)$.

Definition

A point a is called an isolated singularity for $f(z)$ if

(1). $f(z)$ is not analytic at $z = a$ and

(2). there exists $r > 0$ such that $f(z)$ is analytic in $0 < |z-a| < r$.

(i.e) the neighbourhood $|z-a| < r$ contains no singularity of $f(z)$ except a.

Example 33. $f(z) = \frac{z+1}{z^2(z^2+1)}$ *has three isolated singularities* $z = 0, i, -i$.

Definition

Let a be an isolated singularity for $f(z)$. Then a, is called a removable singularity if the principal part of $f(z)$ at $z = a$ has no terms.

Example 34. *Let* $f(z) = \frac{\sin z}{z}$. *Clearly 0 is an isolated singular point for* $f(z)$. *Now*

$$\begin{aligned} \frac{\sin z}{z} &= \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right) \\ &= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \cdots \end{aligned}$$

Here the principal part of $f(z)$ *at* $z = 0$ *has no terms. Hence* $z = 0$ *is a removable singularity. Also* $\lim_{z\to 0} \frac{\sin z}{z} = 1$. *Hence the singularity can be removed by defining* $f(0) = 1$ *so that the extended function becomes analytic at* $z = 0$.

Unit–5: Series Expansions

Introduction

In this chapter we introduce the concept of the residue of a function $f(z)$ at an isolated singular point and prove Cauchy's residue theorem. Using this theorem we evaluate certain types of real definite integrals.

Residues

Definition

Let a be an isolated singularity for $f(z)$. Then the residue of $f(z)$ at a is defined to be the coefficient of $\frac{1}{z-a}$ in the Laurent's series expansion of $f(z)$ about a and is denoted by Res $\{f(z);a\}$.
Thus Res $\{f(z);a\} = \frac{1}{2\pi i}\int\limits_C f(z)dz = b_1$ where C is a circle $|z-a| = r$ such that f is analytic in $0 < |z-a| < r$.

Example 35. *Consider*

$$\begin{aligned} f(z) &= \frac{e^z}{z^2} \\ \frac{e^z}{z^2} &= \frac{1}{z^2}\left(1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots\cdots\right) \\ &= \frac{1}{z^2} + \frac{1}{z} + \frac{1}{2!} + \frac{z}{3!} + \frac{z^2}{4!} + \cdots \end{aligned}$$

Therefore $f(z)$ has a double pole at $z = 0$. Therefore $\text{Res}\{f(z);0\}$ = *coefficient of* $\frac{1}{z} = 1$.

Lemma 36. *If $z = a$ is a simple pole for $f(z)$ then*

$$\text{Res}\{f(z);a\} = \lim_{z\to a}(z-a)f(z)$$

Proof. Since $z = a$ is a simple pole for $f(z)$ the Laurent's series expansion for $f(z)$ about $z = a$ is given by $f(z) = \frac{b_1}{z-a} + a_0 + a_1(z-a) + \cdots\cdots$. Now, $(z-a)f(z) = b_1 + a_0(z-a) + a_1(z-a)^2 + \cdots$. Therefore

$$\begin{aligned} \lim_{z\to a}(z-a)f(z) &= b_1 \\ &= \text{Res}\{f(z);a\} \end{aligned}$$

□

Lemma 37. *If a is a simple pole for $f(z)$ and $f(z) = \frac{g(z)}{z-a}$ where $g(z)$ is analytic at a and $g(a) \neq 0$ then* $\text{Res}\{f(z);a\} = g(a)$.

Proof. By above Lemma, $\text{Res}\{f(z); a\} = \lim_{z\to a}(z-a)f(z) = \lim_{z\to a} g(z) = g(a)$. □

Lemma 38. *If a is a simple pole for $f(z)$ and if $f(z)$ is of the form $\frac{h(z)}{k(z)}$ where $h(z)$ and $k(z)$ are analytic at a and $h(a) \neq 0$ and $k(a) = 0$ then*

$$\text{Res}\{f(z); a\} = \frac{h(a)}{k'(a)}$$

Proof.

$$\begin{aligned}
\text{Res}\{f(z); a\} &= \lim_{z\to a}(z-a)f(z) \\
&= \lim_{z\to a}(z-a)\frac{h(z)}{k(z)} \\
&= \lim_{z\to a} h(z) \lim_{z\to a} \frac{(z-a)}{k(z)} \\
&= \lim_{z\to a} h(z) \lim_{z\to a} \left[\frac{z-a}{k(z)-k(a)}\right] \quad (since \;\; k(a) = 0) \\
&= h(a)\left[\frac{1}{k'(a)}\right] \\
&= \frac{h(a)}{k'(a)}
\end{aligned}$$

□

Lemma 39. *Let a be a pole or order $m > 1$ for $f(z)$ and let $f(z) = \frac{g(z)}{(z-a)^m}$ where $g(z)$ is analytic at a and $g(a) \neq 0$. Then*

$$\text{Res}\{f(z); a\} = \frac{g^{(m-1)}(a)}{(m-1)!}$$

Proof. $g^{(m-1)}(a) = \frac{(m-1)!}{2\pi i} \int\limits_C \frac{g(z)dz}{(z-a)^m}$ (by theorem on higher derivatives) where C is a circle $|z-a| = r$ such that $f(z)$ is analytic in $0 < |z-a| < r$. Therefore

$$\frac{g^{(m-1)}(a)}{(m-1)!} = \frac{1}{2\pi i}\int_C f(z)dz = \text{Res}\{f(z); a\}$$

□

Example 40. *Calculate the residue of $\frac{z+1}{z^2-2z}$ at its poles.*

Solution. Let $f(z) = \frac{z+1}{z^2-2z} = \frac{z+1}{z(z-2)}$. $z = 0$ and $z = 2$ are simple poles for $f(z)$

$$\begin{aligned}
\text{Res}\{f(z); 0\} &= \lim_{z\to 0}(z-0)\left[\frac{z+1}{z(z-2)}\right] \\
&= \lim_{z\to 0}\frac{z+1}{z-2} = -\frac{1}{2} \\
\text{Res}\{f(z); 2\} &= \lim_{z\to 2}(z-2)\left[\frac{z+1}{z(z-2)}\right] \\
&= \lim_{z\to 2}\frac{z+1}{z} = \frac{3}{2}
\end{aligned}$$

■

Cauchy's Residue Theorem

Theorem 32: Cauchy's Residue Theorem

Let $f(z)$ be a function which is analytic inside and on a simple closed curve C except for a finite number of singular points $z_1, z_2, \ldots\ldots, z_n$ inside C. Then

$$\int_C f(z)dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}\{f(z); z_j\}.$$

Proof. Let $C_1, C_2, \cdots, C_n$ be circles with centres $z_1, z_2, \ldots\ldots, z_n$ respectively such that all circles are interior to C and are disjoint with each other (refer figure). By Cauchy's theorem for multiply connected regions we have

$$\begin{aligned}\int_C f(z)dz &= \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \cdots + \int_{C_n} f(z)dz\\ &= 2\pi i \operatorname{Res}\{f(z); z_1\} + 2\pi i \operatorname{Res}\{f(z); z_2\} + \cdots + 2\pi i \operatorname{Res}\{f(z); z_n\}\\ &= 2\pi i \sum_{j=1}^{n} \operatorname{Res}\{f(z); z_j\}.\end{aligned}$$

Hence the theorem. □

Example 41. *Evaluate* $\int\limits_C \frac{z^2 dz}{(z-2)(z+3)}$ *where* C *is the circle* $|z| = 4$.

Solution. Let $f(z) = \frac{z^2}{(z-2)(z+3)}$. $z = 2$ and $z = -3$ are simple poles for $f(z)$ and both of them lie inside $|z| = 4$. Now,

$$\operatorname{Res}\{f(z); 2\} = \lim_{z-2}(z-2)\left[\frac{z^2}{(z-2)(z+3)}\right] = \frac{4}{5}$$

$$\operatorname{Res}\{f(z); -3\} = \lim_{z\to -3}(z+3)\left[\frac{z^2}{(z-2)(z+3)}\right] = -\frac{9}{5}$$

Therefore by Residue theorem

$$\begin{aligned}\int_C f(z)dz &= 2\pi i\left[\frac{4}{5} + \left(-\frac{9}{5}\right)\right]\\ &= -2\pi i\end{aligned}$$

$$\therefore \int_C \frac{z^2 dz}{(z-2)(z+3)} = -2\pi i$$

■

Theorem 33: Argument Theorem

Let f be a function which is analytic inside and on a simple closed curve C except for a finite number of poles inside C. Also let $f(z)$ have no zeros on C. Then $\frac{1}{2\pi i}\int\limits_C \frac{f'(z)}{f(z)}dz = N - P$ where N is the number of zeros of $f(z)$ inside C and P is the number of poles of $f(z)$ inside C (A pole or zero of order m is counted n times).

Proof. We observe that the singularities of the function $\frac{f'(z)}{f(z)}$ inside C are the poles and zeros of $f(z)$ lying inside C. Let z_0 be a zero of order n for $f(z)$. Let C_1 be a circle with centre z_0 such that is the only zero of $f(z)$ inside C_1. Then $f(z) = (z - z_0)^n\, g(z)$ where $g(z)$ is analytic and nonzero inside C_1. Hence $f'(z) = n\,(z - z_0)^{n-1}\, g(z) + (z - z_0)^n\, g'(z)$. Therefore

$$\frac{f'(z)}{f(z)} = \frac{n}{z - z_0} + \frac{g'(z)}{g(z)} \tag{1}$$

Since $g(z)$ is analytic and non zero inside C_1, $\frac{g'(z)}{g(z)}$ is also analytic and hence can be expanded as a Taylor's series about z_0. Therefore

$$\begin{aligned}\text{Res}\left\{\frac{f'(z)}{f(z)}; z_0\right\} &= \text{ coefficient of } \frac{1}{z - z_0} \text{ in } (1)\\ &= n\end{aligned}$$

Similarly if z_1 is a pole of order p for $f(z)$, then $\text{Res}\left\{\frac{f'(z)}{f(z)}; z_1\right\} = -p$. Hence by Cauchy's residue theorem, $\frac{1}{2\pi i}\int\limits_C \frac{f'(z)}{f(z)}dz = N - P$ where N is the number of zeros and P is the number of poles of $f'(z)$ within C. □

Corrolary 42. *If $f(z)$ is analytic inside and on C and not zero on C, then $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N$ where N is the number of zeros lying inside C.*

Theorem 34: Rouche's Theorem

If $f(z)$ and $g(z)$ are analytic inside and on a simple closed curve C and if $|g(z)| < |f(z)|$ on C then $f(z) + g(z)$ and $f(z)$ have the same number of zeros inside C.

Proof. $f(z) + g(z) = f(z)\left[1 + \frac{g(z)}{f(z)}\right] = f(z)\varphi(z)$, where $\varphi(z) = 1 + \frac{g(z)}{f(z)}$. Hence $[f(z) + g(z)]' = f'(z) + g'(z) = f'(z)\varphi(z) + f(z)\varphi'(z)$. Therefore

$$\begin{aligned}\frac{f'(z) + g'(z)}{f(z) + g(z)} &= \frac{f'(z)\varphi(z) + f(z)\varphi'(z)}{f(z)\varphi(z)}\\ &= \frac{f'(z)}{f(z)} + \frac{\varphi'(z)}{\varphi(z)}\\ \frac{1}{2\pi i}\int\limits_C \left[\frac{f'(z) + g'(z)}{f(z) + g(z)}\right]\cdot dz &= \frac{1}{2\pi i}\int\limits_C \frac{f'(z)}{f(z)}dz + \frac{1}{2\pi i}\int\limits_C \frac{\varphi'(z)}{\varphi(z)}dz \qquad (1)\end{aligned}$$

Now, by hypothesis $|g(z)| < |f(z)|$ and hence $\left|\frac{g(z)}{f(z)}\right| < 1$ on C. Therefore $|\varphi(z) - 1| < 1$ on C. Hence by maximum modulus theorem, $|\varphi(z) - 1| < 1$ for every point z inside C. Therefore $\varphi(z) \neq 0$ for every point inside C. Hence

$$\begin{aligned}\int\limits_C \frac{\varphi'(z)}{\varphi(z)}dz &= \text{ Number of zeros of } \varphi(z) \text{ within } C\\ &= 0\end{aligned}$$

Hence from (1), we have

$$\frac{1}{2\pi i}\int\limits_C \left[\frac{f'(z)+g'(z)}{f(z)+g(z)}\right] dz = \frac{1}{2\pi i}\int\limits_C \frac{f'(z)}{f(z)} dz$$

Therefore $N_1 = N_2$, where N_1 and N_2 denote respectively the number of zeros of $f(z)+g(z)$ and $f(z)$ inside C. Hence the theorem. □

Theorem 35: Fundamental Theorem of Algebra

A polynomial of degree n with complex coefficients has n zeros in C.

Proof. Let $a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$, where $a_n \neq 0$, be a polynomial of degree n. Let $f(z) = a_n z^n$ and $g(z) = a_0 + a_1 + \cdots + a_{n-1} z^{n-1}$. Clearly $\lim\limits_{z\to\infty} \frac{g(z)}{f(z)} = 0$. Hence there exists a positive real number r such that $\left|\frac{g(z)}{f(z)}\right| < 1$ for all z with $|z| > r$. Hence by Rouche's theorem $f(z)$ and $f(z)+g(z)$ have the same number of zeros inside the circle $|z| = r+1$. But 0 is a zero of multiplicity n for $f(z)$. Hence the given polynomial $f(z)+g(z)$ also has n zeros. □

Example 43. *Evaluate* $\int\limits_C \frac{dz}{2z+3}$ *where* C *is* $|z| = 2$.

Solution. $z = -\frac{3}{2}$ is the simple pole of $f(z)$ which lies inside the circle $|z| = 2$.

$$\operatorname{Res}\left\{f(z); -\frac{3}{2}\right\} = \lim_{z\to -3/2} \frac{h(z)}{k'(z)}$$

where $h(z) = 1$ and $k(z) = 2z+3$. Therefore

$$\operatorname{Res}\left\{f(z); -\frac{3}{2}\right\} = \frac{1}{2}$$

Therefore by residue theorem $\int\limits_C f(z)dz = 2\pi i\left(\frac{1}{2}\right) = \pi i$. ■

Evaluation of Definite Integrals

We use Cauchy's residue theorem for evaluating certain types of real definite integrals.

TYPE 1: $\int\limits_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$, where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$.

To evaluate this type of integral we substitute $z = e^{i\theta}$. As θ varies from 0 to 2π, z describes the unit circle $|z| = 1$. Also,

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{z+z^{-1}}{2} \text{ and}$$
$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} = \frac{z-z^{-1}}{2i}$$

Substituting these values in the given integrand the integral is transformed into $\int\limits_C \theta(z)dz$ where $\theta(z) = f\left[\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right]$ and C is the positively oriented unit circle $|z| = 1$. The integral $\int\limits_C \theta(z)dz$ can be evaluated using the residue theorem.

Example 44. *Evaluate* $\int\limits_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$.

Solution. Let $I = \int\limits_0^{2\pi} \frac{d\theta}{5+4\sin\theta}$. Put $z = e^{i\theta}$. Then $dz = izd\theta$ and $\sin\theta = \frac{z-z^{-1}}{2i}$. The given integral is transformed to

$$\begin{aligned} I &= \int\limits_C \frac{dz}{iz\left[5+4\left(\frac{z-z^{-1}}{2i}\right)\right]} \\ &= \int\limits_C \frac{dz}{2z^2+5iz-2} \end{aligned}$$

where C is the unit circle $|z| = 1$. Let

$$f(z) = \frac{1}{2z^2+5iz-2} = \frac{1}{2(z+2i)(z+i/2)}$$

Therefore $-2i$ and $-i/2$ are simple poles of $f(z)$ and the pole $-i/2$ lies inside C. Also

$$\text{Res}\{f(z); -i/2\} = \lim_{z\to i/2} \frac{1}{2(z+2i)} = \frac{1}{3i}$$

Hence by Cauchy's Residue Theorem $I = 2\pi i\left(\frac{1}{3i}\right) = \frac{2\pi}{3}$. ■

TYPE 2: $\int\limits_{-\infty}^{\infty} f(x)dx$ where $f(x) = \frac{g(x)}{h(x)}$ and $g(x), h(x)$ are polynomials in x and the degree of $h(x)$ exceeds that of $g(x)$ by at least two.

To evaluate this type of integral we take $f(z) = \frac{g(z)}{h(z)}$. The poles of $f(z)$ are determined by the zeros of the equation $h(z) = 0$.

Case (i): No pole of $f(z)$ lies on the real axis.

We choose the curve C consisting of the interval $[-r, r]$ on the real axis and the semi circle $|z| = r$ lying in the upper half of the plane. Here r is chosen sufficiently large so that all the poles lying in the upper half of the plane are in the interior of C. Then we have

$$\int\limits_C f(z)dz = \int\limits_{-r}^{r} f(x)dx + \int\limits_{C_1} f(z)dz$$

where C_1 is the semi circle. Since $\deg h(x) - \deg f(x) \geq 2$ it follows that $\int\limits_{C_1} f(z)dz \to 0$ as $r \to \infty$ and hence $\int\limits_C f(z)dz = \int\limits_{-\infty}^{\infty} f(x)dx$. Therefore $\int\limits_{-\infty}^{\infty} f(x)dx$ can be evaluated by evaluating $\int\limits_C f(z)dz$ which in turn can be evaluated by using Cauchy's residue theorem.

Case (ii): $f(z)$ has poles lying on the real axis.

Suppose a is a pole lying on the real axis. In this case we indent the real axis by a semi-circle C_2 of radius ε with centre a lying in the upper half plane where ε is chosen to be sufficiently small (refer figure).

Such an indenting must be done for every pole of $f(z)$ lying on the real axis. It can be proved that $\int\limits_{C_2} f(z)dz = -\pi i \operatorname{Res}\{f(z); a\}$. By taking limit as $r \to \infty$ and $\varepsilon \to 0$ we obtain the value of $\int\limits_{-\infty}^{\infty} f(x)dx$.

Example 45. *Use Contour integration method to evaluate* $\int\limits_0^{\infty} \frac{dx}{1+x^4}$.

Solution. Let $f(z) = \frac{1}{1+z^4}$. The poles of $f(z)$ are given by the roots of the equation $z^4 + 1 = 0$, which are the four fourth roots of -1. By De Moivre's theorem they are given by $e^{i\pi/4}; e^{i3\pi/4}; e^{i5\pi/4}; e^{i7\pi/4}$ and all are simple poles. We choose the contour C consisting of the interval $[-r, r]$ on the real axis and the upper semi-circle $|z| = r$ which we denote by C_1. Therefore

$$\int_C f(z)dz = \int_{-r}^{r} f(x)dx + \int_{C_1} f(z)dz \qquad (1)$$

The poles of $f(z)$ lying inside the contour C are obviously $e^{i\pi/4}$ and $e^{i3\pi/4}$ only. We find the residues of $f(z)$ at these points.

$$\operatorname{Res}\left\{f(z); e^{i\pi/4}\right\} = \frac{h\left(e^{i\pi/4}\right)}{k'\left(e^{i\pi/4}\right)}$$

where $h(z) = 1$ and $k(z) = z^4 + 1$ so that $k'(z) = 4z^3$. Therefore

$$\operatorname{Res}\left\{f(z); e^{i\pi/4}\right\} = \frac{1}{4e^{i3\pi/4}} = \frac{e^{-i3\pi/4}}{4}$$

Similarly

$$\operatorname{Res}\left\{f(z); e^{i3\pi/4}\right\} = \frac{e^{-i9\pi/4}}{4}$$

By residue theorem

$$\begin{aligned}
\int_C f(z)dz &= 2\pi i(\text{ sum of the residues at the poles}) \\
&= 2\pi i\left[\frac{e^{-i3\pi/4}}{4} + \frac{e^{-i9\pi/4}}{4}\right] \\
&= \frac{\pi i}{2}[(\cos(3\pi/4) - i\sin(3\pi/4)) + (\cos(9\pi/4) - i\sin(9\pi/4))] \\
&= \frac{\pi i}{2}\left[\left(-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right) + \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)\right] \\
&= \frac{\pi i}{2}\left(\frac{-2i}{\sqrt{2}}\right) = \frac{\pi}{\sqrt{2}}
\end{aligned}$$

From (1),

$$\int_{-r}^{r} \frac{dx}{1+x^4} + \int_{C_1} f(z)dz = \frac{\pi}{2}$$

As $r \to \infty$, $\int\limits_{C_1} f(z)dz \to 0$.

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$$
$$2\int_{0}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$$
$$\int_{0}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$$

■

TYPE 3: $\int\limits_{-\infty}^{\infty} \frac{g(x)}{h(x)} \cos axdx$ or $\int\limits_{-\infty}^{\infty} \frac{g(x)}{h(x)} \sin axdx$, where $g(x)$ and $h(x)$ are real polynomials such that degree of $h(x)$ exceeds that of $g(x)$ by at least one and $a > 0$.

Case (i): $h(x)$ has no zeros on the real axis.

In this case take $f(z) = \frac{g(z)}{h(z)} e^{iaz}$. Therefore $f(z)$ has no poles on the real axis. Choose the contour as in Type 2 and proceeding as in Type 2 we get the value of $\int\limits_{-\infty}^{\infty} \frac{g(x)}{h(x)} e^{iax} dx$.

Taking the real and imaginary parts of $\frac{g(x)}{h(x)} e^{iax} dx$ we obtain the values of $\int\limits_{-\infty}^{\infty} \frac{g(x)}{h(x)} \cos axdx$ and $\int\limits_{-\infty}^{\infty} \frac{g(x)}{h(x)} \sin axdx$.

Case (ii): $h(x)$ has zeros of order one on the real axis.

Take $f(z) = \frac{g(z)}{h(z)} e^{iaz}$. We notice that $f(z)$ has real poles. Suppose a is a real zero of $h(x)$ on the real axis. In this case we indent the real axis as Case (ii) of Type 2 and evaluate the integral.

To prove that the integral over the upper semicircle tends to zero as $r \to \infty$, we use the following lemma.

Lemma 46 (Jordan's Lemma). *Let $f(z)$ be a function of the complex variable z satisfying the following conditions.*

(1). $f(z)$ is analytic in upper half plane except at a finite number of poles.

(2). $f(z) \to 0$ uniformly as $|z| \to \infty$ with $0 \leq \arg z \leq \pi$.

(3). a is a positive integer.

Then $\lim\limits_{r\to\infty} \int\limits_{C} f(z)e^{i\alpha z} dz = 0$ where C is the semi circle with centre at the origin and radius r.

Example 47. *Prove that $\int\limits_{0}^{\infty} \frac{\cos x}{1+x^2} dx = \frac{\pi}{2e}$.*

Solution. Let $f(z) = \frac{e^{iz}}{1+z^2}$. The poles of $f(z)$ are given by i and $-i$. Choose the contour C as shown in the figure. The pole of $f(z)$ that lies within C is i. Hence by residue theorem

$$\int_C f(z)dz = 2\pi i \operatorname{Res}\{f(z); i\}$$
$$= 2\pi i \frac{h(i)}{k'(i)} \text{ where } h(z) = e^{iz} \text{ and } k(z) = 1 + z^2$$

$$= \frac{2\pi i e^{-1}}{2i}$$

$$= \frac{\pi}{e}$$

Therefore

$$\int_{-r}^{r} \frac{e^{iax}}{x^2+1} dx + \int_{C_1} \frac{e^{iaz}}{z^2+1} dz = \frac{\pi}{e}$$

When $r \to \infty$ the integral over C_1 tends to zero. Therefore

$$\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1} dx = \frac{\pi}{e}$$

Equating real parts we get

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2} dx = \frac{\pi}{e}$$

$$2\int_{0}^{\infty} \frac{\cos x}{1+x^2} dx = \frac{\pi}{e} \left(\text{since } \frac{\cos x}{1+x^2} \text{ is an even function} \right)$$

$$\int_{0}^{\infty} \frac{\cos x}{1+x^2} dx = \frac{\pi}{2e}$$

■