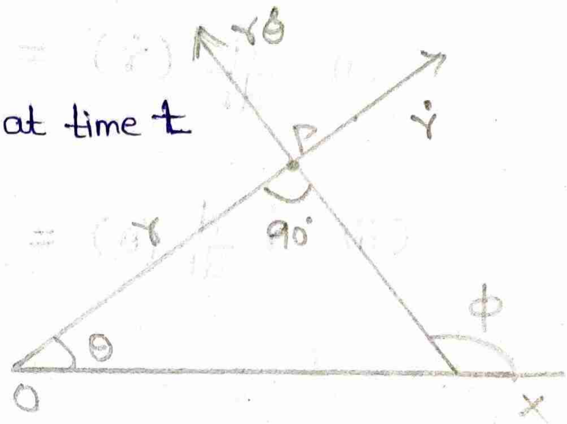
DYNAMICS

Unit - 5

Motion under the action of Central forces

§. 11.2 Velocity and acceleration in polar Coordinates

Let



P - Position of a moving particle at time t

O - Pole

OX - initial line

$\vec{OP} (= r)$ - position Vector of P

Let the polar Coordinates of P be (r, θ).

Hence the Velocity of $P = \frac{d}{dt}(r)$.

Since r has modulus r and amplitude θ, $\frac{d}{dt}(r)$ will have Components

(i) $\dot{r}$ along OP

(ii) $r\dot{\theta}$ $\perp$ to OP

Hence the velocity vector V at P has Components

(i) $\dot{r}$ along OP in the direction in which r increases

(ii) $r\dot{\theta}$ $\perp$ to OP in the direction in which θ increases.

These are respectively called the radial and transverse Components of V.

The acceleration vector at P is the derivative of the Velocity vector V.

The radial component of v is a vector with modulus $\dot{r}$ and amplitude θ.

Hence the derivative of $\dot{r}$ will have components

(i) $\frac{d}{dt}(\dot{r}) = \ddot{r}$ along OP in the direction in which r increases

(ii) $\dot{r}\frac{d}{dt}(\theta) = \dot{r}\dot{\theta}$ $\perp$ to OP in the direction in which θ increases.

The transverse component of v is a vector with modulus $r\dot{\theta}$ and amplitude $\phi = \frac{\pi}{2} + \theta$.

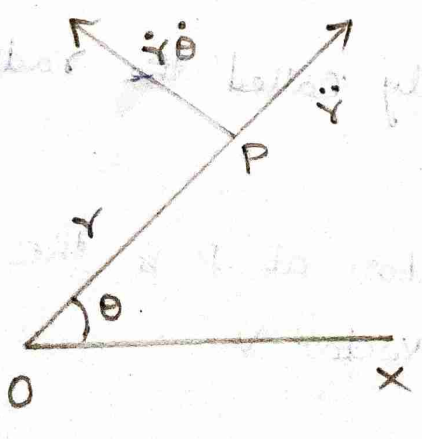
Hence the derivative of $r\dot{\theta}$ will have components

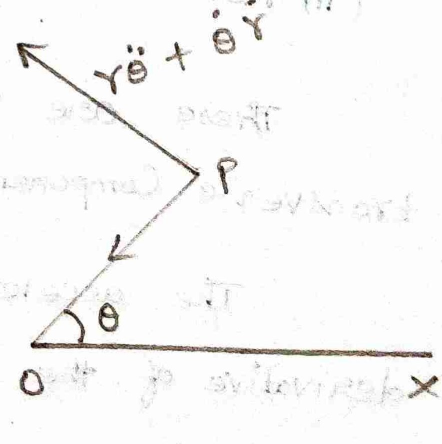
(i) $\frac{d}{dt}(r\dot{\theta}) = r\ddot{\theta} + \dot{\theta}\dot{r}$ along the line of $r\dot{\theta}$.

i.e., in the direction $\perp$ to OP

(ii) $r\dot{\theta}\frac{d}{dt}\left(\frac{\pi}{2}+\theta\right) = r\dot{\theta}^2$ in the direction $\perp$ to the line of $r\dot{\theta}$

i.e., in the direction PO





Hence the totals of the Components of acceleration are

$\ddot{r} - r\dot{\theta}^2$ in the direction OP

$r\ddot{\theta} + 2\dot{r}\dot{\theta}$ in the perpendicular direction

Now,

$$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{r}(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta})$$

$$= r\ddot{\theta} + 2\dot{r}\dot{\theta}$$

$\therefore$ Acceleration $\perp$ to OP is also $= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$

	Magnitude	Direction	Sense
1. Radial Component of Velocity	$\dot{r}$	Along the radius Vector	In the direction in which r increases
2. Transverse Component of Velocity	$r\dot{\theta}$	perpendicular to the radius Vector	In the direction in which θ increases
3. Radial Component of acceleration	$\ddot{r} - r\dot{\theta}^2$	Along the radius Vector	In the direction in which r increases
4. Transverse Component of Acceleration	$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$	Perpendicular to the radius Vector	In the direction in which θ increases.

Corollary-1 :

Suppose the particle P is describing a circle of radius 'a'.

Then $r = a$ throughout the motion.

Hence, $\ddot{r} = 0$

$$\text{radial acceleration} = \ddot{r} - r\dot{\theta}^2$$
$$= 0 - a\dot{\theta}^2$$
$$= -a\dot{\theta}^2$$

The acceleration $\perp$ to OP

$$= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$
$$= \frac{1}{a}\frac{d}{dt}(a^2\dot{\theta})$$
$$= \frac{a^2}{a}\frac{d}{dt}(\dot{\theta})$$
$$= a\ddot{\theta}$$

Hence for a particle describing a circle of radius 'a', the acceleration at any point P has the components $a\ddot{\theta}$ along the tangent at P and $a\dot{\theta}^2$ along the radius to the centre.

Corollary-2 :

The magnitude of the resultant velocity of P $= \sqrt{\dot{r}^2 + (r\dot{\theta})^2}$

$$= \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$$

and the magnitude of the resultant acceleration

$$= \sqrt{(\ddot{r} - r\dot{\theta}^2)^2 + \left[\frac{1}{r}\cdot\frac{d}{dt}(r^2\dot{\theta})\right]^2}$$

§11.3 Equations of motion in Polar Coordinates

If R and S are the Components of the external force acting on a particle of mass m in the radial and transverse directions, we have the equations

$$R = m(\ddot{r} - r\dot{\theta}^2) \rightarrow ①$$

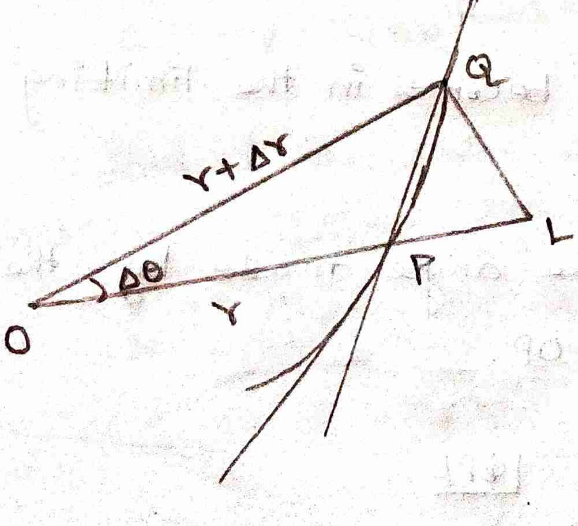
$$S = m \cdot \frac{1}{r} \cdot \frac{d}{dt}(r^2\dot{\theta}) \rightarrow ②$$

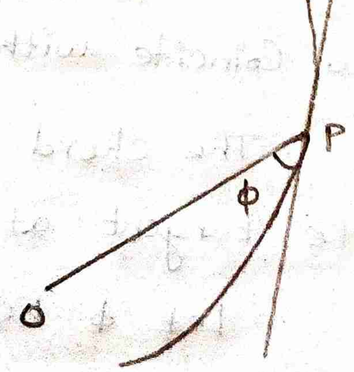
If R and S are known functions of the Coordinates r, θ and the time t, the differential equations ① & ② Can be solved to find r and θ as functions of t and by eliminating t, the polar equation to the path is got.

§.11.4 Note on the equiangular Spiral

Def :

The Curve has the property that the tangent at any point P on it makes a Constant angle with radius vector OP. This Curve is called equiangular Spiral.





Let $OP (= r)$

& $OQ (= r + \Delta r)$

be two consecutive radii vectors such that the included angle $POQ = \Delta\theta$

Draw $QL \perp$ to OP

Then $OL = (r + \Delta r) \cos \Delta\theta$

$= r + \Delta r$ app

Hence $PL = OL - OP$

$= \Delta r$

and $LQ = (r + \Delta r) \sin \Delta\theta$

$= (r + \Delta r) \Delta\theta$ nearly

$= r \Delta\theta$ to the first order of smallness.

Hence $\tan \angle QPL = \dfrac{QL}{PL}$

$= \dfrac{r \Delta\theta}{\Delta r}$

$= r \cdot \dfrac{\Delta\theta}{\Delta r}$

If $\lim\limits_{\Delta r \to 0}$ and $\lim\limits_{\Delta\theta \to 0}$, the point Q tends to coincide with P.

The chord QP becomes in the limiting position the tangent at P.

Let ϕ be the angle made by the tangent at P with OP.

Then,

$$\phi = \lim_{Q \to P} \angle QPL$$

Hence, $\tan\phi = \lim_{\Delta r \to 0} \tan \angle QPL$

$$= \lim_{\Delta r \to 0} r \frac{\Delta\theta}{\Delta r}$$

$$= r \cdot \frac{d\theta}{dr}$$

i.e., $\boxed{\tan\phi = r \frac{d\theta}{dr}}$.

This gives the angle between the radius vector and the tangent.

Now, for the equiangular spiral, at any point P on it the angle ϕ is constant.

Let $\phi = \alpha$.

Then $\tan\phi = \tan\alpha$.

i.e., $r \frac{d\theta}{dr} = \tan\alpha$

$$\Rightarrow \frac{dr}{r} = \frac{1}{\tan\alpha} d\theta \Rightarrow \frac{dr}{r} = \cot\alpha \, d\theta$$

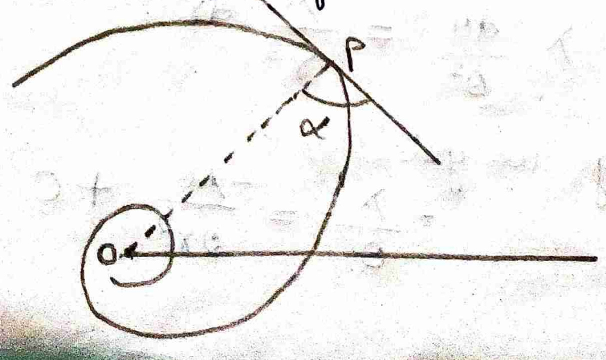
Integrating,

$$\log r = \theta \cot\alpha + \text{Constant}.$$

i.e., $r = ae^{\theta \cot\alpha}$

This is the polar equation to the equiangular spiral.

The shape of the curve is



Example-1 :

The velocities of a particle along and perpendicular to a radius vector from a fixed origin are λr^2 and $\mu\theta^2$ where μ and λ are constants.

Show that the equation to the path of the particle is $\frac{\lambda}{\theta} + c = \frac{\mu}{2r^2}$ where c is a constant.

Show also that the accelerations along and perpendicular to the radius vector are

$$2\lambda^2 r^3 - \frac{\mu^2\theta^4}{r} \quad \text{and} \quad \mu\left(\lambda r\theta^2 + \frac{2\mu\theta^3}{r}\right)$$

Solution :

Given that :

Radial velocity , $\frac{dr}{dt} = \lambda r^2 \rightarrow (1)$

Transverse velocity , $r \cdot \frac{d\theta}{dt} = \mu\theta^2 \rightarrow (2)$

(2) ÷ (1) , we have

$$\frac{r \cdot \frac{d\theta}{dt}}{\frac{dr}{dt}} = \frac{\mu\theta^2}{\lambda r^2}$$

$$r \cdot \frac{d\theta}{dr} = \frac{\mu\theta^2}{\lambda r^2}$$

$$\Rightarrow \quad \lambda \cdot \frac{d\theta}{\theta^2} = \frac{\mu}{r^3}\, dr$$

Integrating , we get

$$-\frac{\lambda}{\theta} = \frac{-\mu}{2r^2} + c$$

i.e., $\frac{\mu}{2r^2} = \frac{\lambda}{\theta} + c \rightarrow (3)$

(3) is the equation to the path.

Hence proved.

Now,

differentiating (1), we get

$$\frac{d^2r}{dt^2} = \lambda \cdot 2r \frac{dr}{dt}$$

$$= \lambda\, 2r\,(\lambda r^2) \quad [\because \text{by } (1)]$$

$$= 2\lambda^2 r^3 \rightarrow (4)$$

Radial acceleration $= \ddot{r} - r\dot{\theta}^2$

$$= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$= 2\lambda^2 r^3 - r\left(\frac{\mu\theta^2}{r}\right)^2$$

$[\because \text{by } (2) \;\&\; (4)]$

$$= 2\lambda^2 r^3 - \frac{r\mu^2\theta^4}{r^2}$$

$$= 2\lambda^2 r^3 - \frac{\mu^2\theta^4}{r}$$

Transverse acceleration $= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$

$$= \frac{1}{r}\frac{d}{dt}\left(r^2 \frac{\mu\theta^2}{r}\right) \quad [\because \text{by } (2)]$$

$$= \frac{1}{r}\frac{d}{dt}(\mu r\theta^2)$$

$$= \frac{\mu}{r}\left[\frac{d}{dt}(r\theta^2)\right]$$

$$= \frac{\mu}{r}\left[r\,2\theta\frac{d\theta}{dt} + \theta^2\frac{dr}{dt}\right]$$

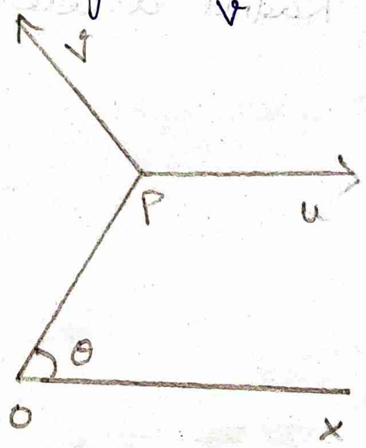
$$= \frac{\mu}{r}\left[2r\cdot\theta\cdot\frac{\mu\theta^2}{r} + \theta^2\cdot\lambda r^2\right]$$

$$= \mu\left[\frac{2\mu\theta^3}{r} + \theta^2\lambda r\right]$$

Hence proved.

Example-2 :

Show that the path of a point P which possesses two Constant Velocities u and v, the first of which is in a fixed direction and the Second of which is perpendicular to the radius OP drawn from a fixed point O, is a Conic whose focus is O and whose eccentricity is $\frac{u}{v}$



Solution :

Take

O - Pole

OX - initial line.

P has two Velocities

i) u parallel to OX

ii) v ⊥ to OP

Resolving the Velocities along and perpendicular to OP, we have

$$\frac{dr}{dt} = u\cos\theta \rightarrow (1)$$

$$r\cdot\frac{d\theta}{dt} = v - u\sin\theta \rightarrow (2)$$

To get the equation to the path, we have to eliminate t.

② ÷ ①, we have

$$r\frac{d\theta}{dr} = \frac{v - u\sin\theta}{u\cos\theta}$$

$$\Rightarrow \frac{u\cos\theta}{v - u\sin\theta}\, d\theta = \frac{dr}{r}$$

Integration, we get

$$-\log(v - u\sin\theta) + \log A = \log r$$

where A is the Constant of integration.

$$\Rightarrow \log r + \log(v - u\sin\theta) = \log A$$

$$\log[r(v - u\sin\theta)] = \log A$$

$$r(v - u\sin\theta) = A$$

$$\Rightarrow \frac{A}{r} = v - u\sin\theta$$

$$\frac{A}{r} = v\left(1 - \frac{u}{v}\sin\theta\right)$$

$$\left(\frac{A}{v}\right)\frac{1}{r} = 1 - \frac{u}{v}\sin\theta$$

$$\left(\frac{A}{v}\right)\frac{1}{r} = 1 + \frac{u}{v}\cos(90^\circ + \theta) \rightarrow ③$$

This is of the form $\frac{l}{r} = 1 + e\cos(\theta + \alpha) \rightarrow ④$

Comparing ③ & ④, we have

$$l = \frac{A}{v}, \quad e = \frac{u}{v} \text{ and } \alpha = 90^\circ$$

We know from Analytical geometry that ④ is the polar equation to a Conic whose focus is at the pole, semi-latus rectum is l, eccentricity is e and whose major axis makes an angle α with the initial line. Hence ③ is a conic whose focus is at O, semi-latus rectum is $\frac{A}{V}$, eccentricity is $\frac{u}{V}$ and whose major axis is perpendicular to the initial line

Example-3 :

A point P describes a curve with constant velocity and its angular velocity about a given fixed point O varies inversely as the distance from O.

Show that the curve is an equiangular spiral whose pole is O and that the acceleration of the point is along the normal at P and varies inversely as OP.

Solution:

Taking O as the pole

let P be (r,θ).

Given that :

Resultant velocity of P $= \sqrt{\dot{r}^2 + r^2\dot{\theta}^2} =$ Constant

$$\Rightarrow \sqrt{\dot{r}^2 + r^2\dot{\theta}^2} = k$$

$$\Rightarrow \dot{r}^2 + r^2\dot{\theta}^2 = k^2 \rightarrow (1)$$

Also,

Angular velocity about O $= \dot{\theta} = \frac{\lambda}{r} \rightarrow (2)$

From (1) & (2), we have

$$\dot{r}^2 + r^2\left[\frac{\lambda}{r}\right]^2 = k^2$$

$$\Rightarrow \dot{r}^2 + r^2 \cdot \frac{\lambda^2}{r^2} = k^2$$

$$\dot{r}^2 = k^2 - \lambda^2$$

$$\dot{r} = \sqrt{k^2 - \lambda^2} \rightarrow (3)$$

(3) ÷ (2), we get

$$\frac{dr}{d\theta} = \frac{\sqrt{k^2-\lambda^2}}{\lambda} r$$

$$\frac{dr}{r} = \frac{\sqrt{k^2-\lambda^2}}{\lambda} d\theta$$

Integrating, $\log r = \frac{\sqrt{k^2-\lambda^2}}{\lambda} \theta + B$

$$\Rightarrow r = e^{\frac{\sqrt{k^2-\lambda^2}}{\lambda}\theta + B}$$

$$r = e^{B} \cdot e^{\frac{\sqrt{k^2-\lambda^2}}{\lambda}}$$

put $e^B = a$ and $\frac{\sqrt{k^2-\lambda^2}}{\lambda} = \cot\alpha$,

it becomes,

$$r = a e^{\cot\alpha} \rightarrow (4)$$

Hence the path is an equiangular spiral, whose pole is O.

Differentiating (3), we get

$$\ddot{r} = 0$$

Radial acceleration $= \ddot{r} - r\dot{\theta}^2$

$$= 0 - r\dot{\theta}^2$$

$$= -r \frac{\lambda^2}{r^2}$$

$$= -\frac{\lambda^2}{r}$$

The negative sign shows that the radial acceleration at P is along PO.

Transverse acceleration

$$= \frac{1}{r} \cdot \frac{d}{dt}\left(r^2 \dot{\theta}\right)$$

$$= \frac{1}{r} \frac{d}{dt}\left(r^2 \frac{\lambda}{r}\right)$$

$$= \frac{1}{r} \cdot \frac{d}{dt}\left(r\lambda\right)$$

$$= \frac{\lambda}{r} \cdot \frac{dr}{dt}$$

$$= \frac{\lambda}{r} \sqrt{k^2 - \lambda^2} \quad \left[\because \text{by } 3\right]$$

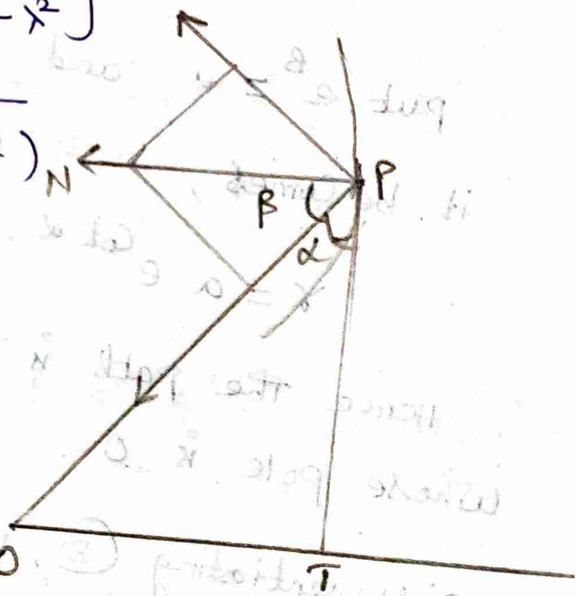
Resultant acceleration of P

$$= \sqrt{\left(\frac{-\lambda^2}{r}\right)^2 + \left[\frac{\lambda}{r}\sqrt{k^2-\lambda^2}\right]^2}$$

$$= \sqrt{\frac{\lambda^4}{r^2} + \frac{\lambda^2}{r^2}\left(k^2 - \lambda^2\right)}$$

$$= \sqrt{\frac{\lambda^2 k^2}{r^2}}$$

$$= \frac{\lambda k}{r}$$



Thus the resultant acceleration varies inversely as r.

i.e, as OP.

Let this acceleration be along PN making an angle with PO.

$$\tan\beta = \frac{\text{Component perpendicular to PO}}{\text{Component along PO}}$$

$$= \frac{\frac{\lambda}{r}\sqrt{k^2-\lambda^2}}{\frac{\lambda^2}{r}}$$

$$= \frac{\sqrt{k^2-\lambda^2}}{\lambda}$$

But, $\cot\alpha = \dfrac{\sqrt{k^2-\lambda^2}}{\lambda}$

Hence $\tan\beta = \cot\alpha$

$$\tan\beta = \tan(90^\circ - \alpha)$$

$$\beta = 90^\circ - \alpha$$

$$\Rightarrow \beta + \alpha = 90^\circ$$

Hence $\underline{|NPT} = 90^\circ$ where PT is the tangent at P.

Hence PN is the normal at P.

§11.5. Motion under a Central force:

Suppose a particle describes a path, acted on by an attractive force F towards a fixed point O. Such a force is called a Central force. and the Path described by the particle is called a Central orbit. The fixed point is known as the centre of the force.

Usually the magnitude of the Central attraction F is a function only of the distance r of the particle from O.

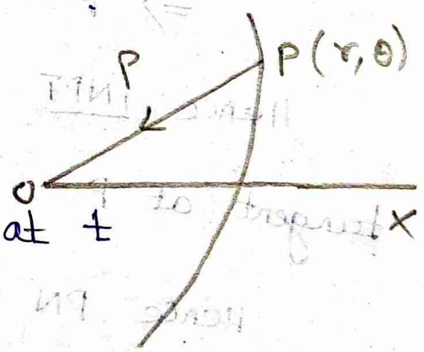
In such a motion, the particle must be always moving only in the plane Containing O and the tangent at any point of its path, since there is no Component of attraction $\perp$ to the above plane.

Hence a Central orbit must be a plane Curve.

§.11.6 Differential equation of Central orbits:

A particle moves in a plane with an acceleration which is always directed to a fixed point O in the plane; to obtain the differential equation of its path:

Take



O - Pole

$P(r,\theta)$ - Polar Coordinates at t

m - mass.

Also let P be the magnitude of the Central acceleration along PO.

The equation of motion of the particle are

$$m(\ddot{r} - r\dot{\theta}^2) = -mP$$

$$\Rightarrow \ddot{r} - r\dot{\theta}^2 = -P \longrightarrow (1)$$

and $\frac{m}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$

$$\Rightarrow \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) \rightarrow (2)$$

(2) follows from the fact that as there is no force at right angles to OP, the transverse Component of the acceleration is zero throughout the motion.

From (2), $r^2\dot{\theta} = \text{Constant} = h \text{ (say)} \rightarrow (3)$

To get the polar equation of the path, we have to eliminate the element of time between equations (1) & (3).

put $u = \frac{1}{r}$ and treat u as the dependent Variable.

From (3), $\dot{\theta} = \frac{h}{r^2} = hu^2 \rightarrow (A)$

Also, $\dot{r} = \frac{dr}{dt}$

$$= \frac{d}{dt}\left(\frac{1}{u}\right)$$

$$= \frac{-1}{u^2}\frac{du}{dt}$$

$$= \frac{-1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt}$$

$$= \frac{-1}{u^2}\frac{du}{d\theta}\cdot hu^2 \quad [\because \text{by } (A)]$$

$$\dot{r} = -h\frac{du}{d\theta}$$

$$\ddot{r} = \frac{d}{dt}\left(-h\frac{du}{d\theta}\right)$$

$$= -h \cdot \frac{d}{d\theta}\left(\frac{du}{d\theta}\right) \cdot \frac{d\theta}{dt}$$

$$= -h \cdot \frac{d^2u}{d\theta^2} \cdot hu^2$$

$$= -h^2u^2 \frac{d^2u}{d\theta^2}$$

Since $u = \frac{1}{r} \Rightarrow r = \frac{1}{u} \rightarrow$ (B)

Sub. (A) & (B) in (i), we get

$$-h^2u^2 \frac{d^2u}{d\theta^2} - \frac{1}{u} h^2u^4 = -P$$

$$\Rightarrow h^2u^2\left[\frac{d^2u}{d\theta^2} + u\right] = P$$

$$\Rightarrow u + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2}$$

This is the differential equation of a Central orbit in Polar Coordinates.

Note:

If the Central force is a repulsive one in a particular problem, the sign of P in (4) must be changed.

§. 11.7 Perpendicular from the pole on the tangent formulae in polar Coordinates:

Let ϕ be the angle made by the tangent at P with the Radius Vector OP.

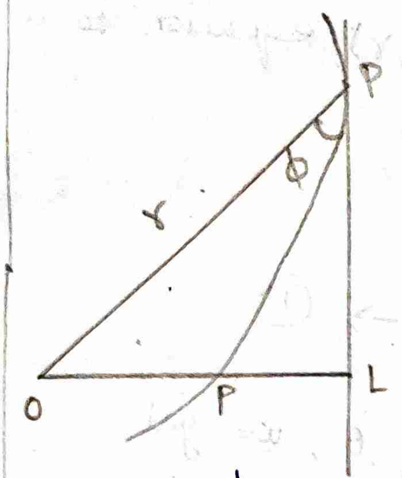
W.K.T, $\tan\phi = r\dfrac{d\theta}{dr} \rightarrow (1)$

from O draw OL perpendicular to the tangent at P and let $OL = P$

Then, $\sin\phi = \dfrac{OL}{OP} = \dfrac{P}{r}$

$$\Rightarrow P = r\sin\phi \rightarrow (2)$$

From (2), $\dfrac{1}{P^2} = \dfrac{1}{r^2\sin^2\phi}$



$$= \frac{1}{r^2}\operatorname{cosec}^2\phi$$

$$= \frac{1}{r^2}\left[1+\cot^2\phi\right]$$

$$= \frac{1}{r^2}\left[1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right]$$

[$\because$ by (1)]

$$\Rightarrow \frac{1}{P^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

Using $r = \dfrac{1}{u}$

$$\frac{dr}{d\theta} = \frac{dr}{du}\,\frac{du}{d\theta}$$

$$= -\frac{1}{u^2}\,\frac{du}{d\theta}.$$

Hence (3) becomes,

$$\frac{1}{P^2} = u^2 + u^4\cdot\frac{1}{u^4}\left(\frac{du}{d\theta}\right)^2$$

$$\Rightarrow \frac{1}{P^2} = u^2 + \left(\frac{du}{d\theta}\right)^2.$$

§ 11.8 Pedal equation of the Central orbit

Def - pedal equation :

In Certain Curves the relation between p (the perpendicular from the pole on the tangent) and r (the radius vector) is very simple. Such a relation is called the Pedal equation or the (p, r) equation to the Curve

We can get the (p, r) equation to a Central orbit as follows

W.K.T,

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \rightarrow (1)$$

Differentiating (1) w.r. to θ, we get

$$\frac{-2}{p^3} \frac{dp}{d\theta} = 2u \cdot \frac{du}{d\theta} + 2 \cdot \frac{du}{d\theta} \cdot \frac{d^2u}{d\theta^2}$$

$$= 2 \frac{du}{d\theta} \left[u + \frac{d^2u}{d\theta^2}\right] \rightarrow (2)$$

But the differential equation in polars is

$$u + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2}$$

Hence (2) becomes

$$\frac{-1}{p^3} \frac{dp}{d\theta} = \frac{du}{d\theta} \frac{P}{h^2u^2}$$

$$\Rightarrow \frac{-1}{p^3} dp = \frac{P}{h^2u^2} du$$

$$\Rightarrow \frac{-1}{p^3}\, dp = \frac{p}{h^2}\, r^2 .\, d\left(\frac{1}{r}\right)$$

$$= \frac{p r^2}{h^2}\left[\frac{-1}{r^2}\right] dr$$

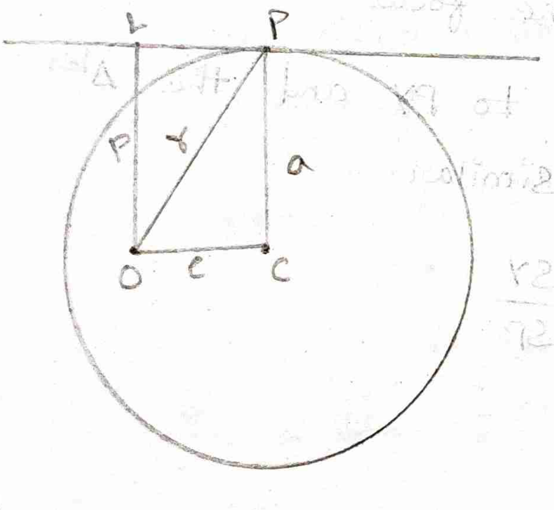
$$= \frac{-p}{h^2}\, dr$$

$$\Rightarrow \frac{h^2}{p^3}\frac{dp}{dr} = p$$

This is the (p, r) equation or the pedal equation to the Central orbit.

§.11.9 Pedal equations of Some of the Well-known Curves:

(1) Circle-pole at any point



Let

C - Centre

'a' - radius

O - Pole

OC = c

P - any point on the Circle

OL - ⊥ from O on the tangent at P

OP = r & OL = p

From Δ OPC,

$$c^2 = r^2 + a^2 - 2ra \cos \angle OPC$$

$$= r^2 + a^2 - 2ra \cos \angle POL$$

$$= r^2 + a^2 - 2ra \cdot \frac{p}{r}$$

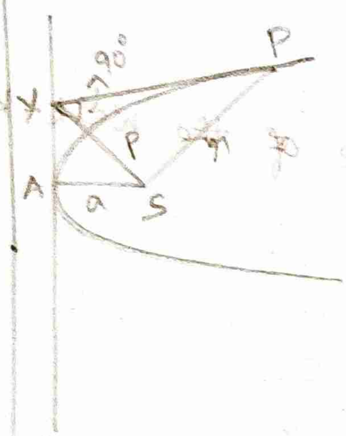
$$= r^2 + a^2 - 2ap$$

Hence the pedal equations of the Circle for a general position of the pole is

$$c^2 = r^2 + a^2 - 2ap$$

when $c = a$, the pole is on the circumference and the equation is $r^2 = 2ap$.

(2) Parabola – Pole at focus



To get the (p, r) equation to a parabola, we assume the geometrical property that if the tangent at P meets the tangent at the vertex A in Y and S is the focus,

then SY is the $\perp$ to PY and the Δles SAY and SYP are similar.

$$\text{Hence } \frac{SA}{SY} = \frac{SY}{SP}$$

$$\Rightarrow \frac{a}{p} = \frac{p}{r}$$

$$\Rightarrow p^2 = ar$$

(3) Ellipse (or) Hyperbola – pole at focus :

Let S and S' be the foci of the ellipse.

SY, $S'Y'$ be the perpendiculars to the tangent at P.

Taking S as the pole. Let $SP = r$, $S'P' = r'$.

$SY = p$; $S'Y' = p'$

Let 'a' and 'b' be the Semi-axes.

To find the (p, r) equation, we assume the following geometrical properties of the ellipse

(i) $SP + S'P = 2a \Rightarrow r + r' = 2a$

(ii) $SY \cdot S'Y' = b^2 \Rightarrow pp' = b^2$

(iii) The tangent at P is equally inclined to the focal distances so that SPY and $S'PY'$ are similar triangles.

So we have, $\dfrac{p}{r} = \dfrac{p'}{r'}$

Now,

$$\frac{b^2}{p^2} = \frac{pp'}{p^2} \quad \text{[using (ii)]}$$

$$= \frac{p'}{p} = \frac{r'}{r} \quad \text{[using (iii)]}$$

$$= \frac{2a - r}{r} \quad \text{[using (i)]}$$

$$= \frac{2a}{r} - 1$$

Hence $\dfrac{b^2}{p^2} = \dfrac{2a}{r} - 1$ is the (p, r) equation of ~~to~~ the ellipse.

By a Similar argument, the (p, r) equation of the branch of the hyperbola nearer to the focus is $\frac{b^2}{p^2} = \frac{2a}{r} + 1$

(4) Equiangular Spiral

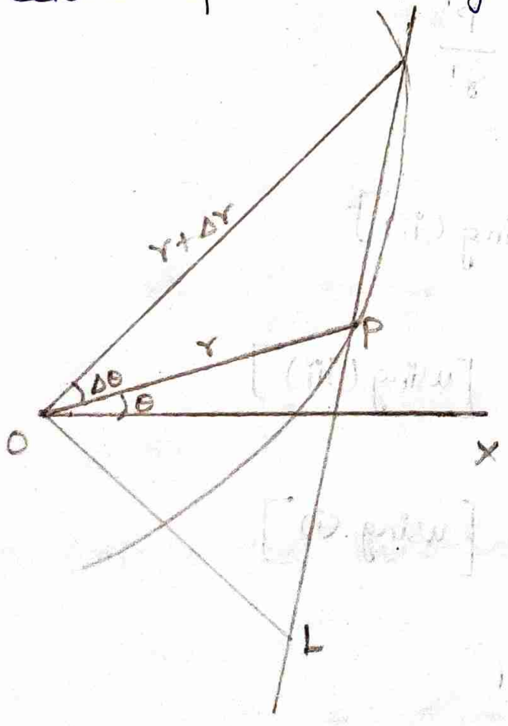
In any Curve $p = r \sin\phi$ is the usual notation.

In the equilateral Spiral, $\phi = \text{Constant} = \alpha$ (Say)

Hence $p = r \sin\alpha = kr$, is the (p, r) equation to the Spiral.

§.11.10 Velocities in a Central orbit

In every Central orbit the areal velocity is Constant and the linear velocity varies inversely as the perpendicular from the Centre upon the tangent to the path.



Let at time t the particle be at $P(r, \theta)$ and at time $t + \Delta t$, let it be at $Q(r+\Delta r, \theta + \Delta\theta)$.

Sectorial area OPQ described by the radius vector OP

= Area of ΔOPQ nearly

$= \frac{1}{2} OP \cdot OQ \sin \angle POQ$

$$= \frac{1}{2} r (r + \Delta r) \sin \Delta\theta$$

$$= \frac{1}{2} r^2 \Delta\theta, \text{ to the first order of Smallness}$$

The rate of description of the area traced out by the radius vector joining the particle to a fixed point is Called the areal Velocity of the particle.

Hence in the Central orbit, areal Velocity of P

$$= \lim_{\Delta t \to 0} \frac{1}{2} r^2 \frac{\Delta\theta}{\Delta t}$$

$$= \frac{1}{2} r^2 \frac{d\theta}{dt}$$

$$= \frac{1}{2} h \rightarrow (1)$$

Since $r^2 \dot{\theta} = \text{Constant} = h$

Hence h = twice the areal velocity and as h is a Constant, the areal velocity is Constant.

In otherwords, equal areas are described by the radius vector in equal times.

Another expression for the areal velocity:

Let Δs be the length of the arc PQ.

Draw $OL \perp PQ$.

Sectorial area $POQ = \Delta POQ$ nearly

$$= \frac{1}{2} PQ \cdot OL$$

As $\Delta t \to 0$, $Q \to P$ along the Curve and the chord QP becomes the tangent at P.

Length $PQ = \Delta s$ nearly and OL becomes the perpendicular from O on the tangent at P.

Let $OL = P$.

Hence,

$$\text{areal Velocity} = \lim_{\Delta t \to 0} \frac{1}{2} \frac{\Delta s}{\Delta t} \cdot P$$

$$= \frac{1}{2} P \cdot \frac{ds}{dt}$$

$$= \frac{1}{2} PV \rightarrow (2)$$

as $\frac{ds}{dt}$ is the rate of describing s and so is the linear Velocity of P.

Hence Combining (1) & (2), we get

$$\text{Areal Velocity} = \frac{1}{2} h$$

$$= \frac{1}{2} PV$$

$$\Rightarrow \frac{1}{2} h = \frac{1}{2} PV$$

$$\Rightarrow h = PV$$

$$\Rightarrow V = \frac{h}{P}$$

Hence linear Velocity Varies inversely as P

§.11.11. Two-fold problems in Central orbits :

It is clear that two types of problems arise in Connection with Central orbits. They are

(i) Given the orbit, to find the law of force to the pole

(ii) Given the law of force, to find the path.

We shall first take up (i)

The differential equation to the central orbit in polar coordinates is

$$u + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2}$$

Hence $P = h^2u^2\left[u + \frac{d^2u}{d\theta^2}\right]$

Since the orbit is given, u is known as a function of θ.

Hence by differentiation, P can be got from the above equation.

In a few cases we may know the (p, r) equation to the path.

To find P, we can use the equation

$$P = \frac{h^2}{p^3} \cdot \frac{dp}{dr}$$

Example - 5 :

(*) Find the law of force towards the pole under which the curve $r^n = a^n \cos n\theta$ can be described.

Solution :

$$r^n = a^n \cos n\theta$$

Since $r = \frac{1}{u}$, the equation is

$$u^n a^n \cos n\theta = 1 \rightarrow (1)$$

Taking logarithms,

$$\log [u^n a^n \cos n\theta] = \log 1$$

$$\log u^n + \log a^n + \log \cos n\theta = 0$$

$$n \log u + n \log a + \log \cos n\theta = 0 \rightarrow (2)$$

Differentiating (2) w.r. to θ, we have

$$n \cdot \frac{1}{u} \frac{du}{d\theta} + 0 - \frac{1}{\cos n\theta} n \sin n\theta = 0$$

$$\Rightarrow n \cdot \frac{1}{u} \frac{du}{d\theta} = n \tan n\theta$$

$$\Rightarrow \frac{1}{u} \frac{du}{d\theta} = \tan n\theta$$

$$\Rightarrow \frac{du}{d\theta} = u \tan n\theta \rightarrow (3)$$

Diff (3) w.r. to θ, we have

$$\frac{d^2u}{d\theta^2} = un \sec^2 n\theta + \tan n\theta \cdot \frac{du}{d\theta}$$

$$= nu \sec^2 n\theta + \tan n\theta \cdot [u \tan n\theta] \quad [\because \text{by } (3)]$$

$$= nu \sec^2 n\theta + u \tan^2 n\theta$$

$$u + \frac{d^2u}{d\theta^2} = u + nu \sec^2 n\theta + u \tan^2 n\theta$$

$$= nu \sec^2 n\theta + u(1 + \tan^2 n\theta)$$

$$= nu \sec^2 n\theta + u \sec^2 n\theta$$

$$= u \sec^2 n\theta [n+1]$$

$$(1) \Rightarrow u^n a^n \cos n\theta = 1$$

$$u^n a^n = \frac{1}{\cos n\theta}$$

$$u^n a^n = \sec n\theta$$

$$u + \frac{d^2u}{d\theta^2} = u \sec^2 n\theta \,(n+1)$$

$$= u\left[u^{2n} \cdot a^{2n}\right](n+1)$$

$$= (n+1)\, u^{2n+1} \cdot a^{2n}$$

$$P = h^2u^2\left[u + \frac{d^2u}{d\theta^2}\right]$$

$$= h^2u^2\left[(n+1)\, u^{2n+1} \cdot a^{2n}\right]$$

$$= (n+1)\, a^{2n} h^2 u^{2n+3}$$

$$= (n+1)\, a^{2n} \cdot h^2 \cdot \frac{1}{r^{2n+3}} \rightarrow (4)$$

i.e., $P \propto \frac{1}{r^{2n+3}}$ which means that the Central acceleration varies inversely as the $(2n+3)$rd power of the distance.

Note:

From eqn (4), P is +ve only when $n+1>0$

i.e., $n > -1$.

For values of $n < -1$, P will be negative and in such case the Central force will be a repulsive one.

The above case is a Comprehensive one, giving the law of force for describing the following well-known curves corresponding to particular values of n.

i) when $n=1$, the equation is $r = a\cos\theta$

The curve is a circle & $p \propto \frac{1}{r^5}$

ii) when $n=2$, the equation is $r^2 = a^2\cos 2\theta$

This is the Lemniscate of Bernoulli

and $p \propto \frac{1}{r^7}$

iii) when $n=\frac{1}{2}$, the equation is

$$r^{\frac{1}{2}} = a^{\frac{1}{2}}\cos\frac{\theta}{2}$$

i.e., $r = a\cos^2\frac{\theta}{2}$

$$= \frac{a}{2}\left[1+\cos\theta\right]$$

This is a Cardioid and $p \propto \frac{1}{r^4}$

iv) when $n = \frac{-1}{2}$, the equation is

$$r^{-\frac{1}{2}} = a^{-\frac{1}{2}}\cos\frac{\theta}{2}$$

$$\Rightarrow a^{+\frac{1}{2}} = r^{\frac{1}{2}}\cos\frac{\theta}{2}$$

So, $r = \frac{a}{\cos^2\frac{\theta}{2}}$

$$= \frac{2a}{1+\cos\theta}$$

$$\Rightarrow \frac{2a}{r} = 1+\cos\theta$$

This is a parabola and $p \propto \frac{1}{r^2}$

v) When $n = -2$, the equation is

$$r^{-2} = a^{-2} \cos 2\theta$$

$$\Rightarrow r^2 \cos 2\theta = a^2$$

This is a rectangular hyperbola.

In this case the actual value of $P = -a^{-4} h^2 r$.

The negative sign of P shows that the central force is a repulsive one.

Example - 6 :

Find the law of force to an internal point under which a body will describe a circle.

Solution :

W.K.T, the pedal equation of the circle for a general position of the pole is

$$c^2 = r^2 + a^2 - 2ap \rightarrow (1)$$

Diff w.r. to 'r', we get

$$0 = 2r + 0 - 2a \cdot \frac{dp}{dr}$$

$$\Rightarrow 2a \frac{dp}{dr} = 2r$$

$$\Rightarrow \frac{dp}{dr} = \frac{r}{a}$$

Now the central acceleration

$$P = \frac{h^2}{p^3} \frac{dp}{dr}$$

$$p = \frac{h^2}{p^3} \cdot \frac{r}{a} \rightarrow (2)$$

$$(1) \Rightarrow c^2 = r^2 + a^2 - 2ap$$

$$2ap = r^2 + a^2 - c^2$$

$$p = \frac{r^2 + a^2 - c^2}{2a} \rightarrow (3)$$

$$(2) \Rightarrow P = \frac{h^2 r}{a} \cdot \frac{8a^3}{(r^2+a^2-c^2)^3} \quad [\because \text{by } (3)]$$

$$P = \frac{8h^2 r a^2}{(r^2+a^2-c^2)^3}$$

Example-7 :

A particle moves in an ellipse under a force which is always directed towards its focus. Find the law of force, the velocity at any point of the path and its periodic time.

Solution :

W.K.T, the polar equation to the ellipse is

$$\frac{l}{r} = 1 + e\cos\theta \rightarrow (1)$$

where e is the eccentricity and l is the Semi-latus rectum.

put $\frac{1}{r} = u$

$$(1) \Rightarrow lu = 1 + e\cos\theta$$

$$u = \frac{1+e\cos\theta}{l}$$

Hence,

$$\frac{du}{d\theta} = \frac{-e\sin\theta}{l}$$

and $$\frac{d^2u}{d\theta^2} = \frac{-e\cos\theta}{l}$$

Now,

$$u + \frac{d^2u}{d\theta^2} = \frac{1+e\cos\theta}{l} - \frac{e\cos\theta}{l}$$

$$= \frac{1+e\cos\theta - e\cos\theta}{l}$$

$$= \frac{1}{l}$$

W.K.T,

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}$$

$$\Rightarrow \frac{P}{h^2u^2} = \frac{1}{l}$$

$$\Rightarrow P = \frac{h^2u^2}{l}$$

$$P = \frac{\mu}{r^2} \quad \left[\because \mu = \frac{h^2}{l} \ \& \ \frac{1}{r} = u\right]$$

i.e., The force varies inversely as the square of the distance from the pole.

Also, W.K.T,

$$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$$

$$\frac{1}{p^2} = \left(\frac{1 + e\cos\theta}{l}\right)^2 + \left(\frac{e\sin\theta}{l}\right)^2$$

$$= \frac{1 + e^2\cos^2\theta + 2e\cos\theta + e^2\sin^2\theta}{l^2}$$

$$= \frac{1 + 2e\cos\theta + e^2(\cos^2\theta + \sin^2\theta)}{l^2}$$

$$\frac{1}{p^2} = \frac{1 + 2e\cos\theta + e^2}{l^2} \rightarrow (*)$$

Also, $h = pv$, where v is the linear velocity.

$$\Rightarrow v = \frac{h}{p}$$

$$\Rightarrow v^2 = \frac{h^2}{p^2}$$

$$= \frac{h^2(1 + 2e\cos\theta + e^2)}{l^2} \quad [\because \text{by } (*)]$$

From (i),

$$\frac{l}{r} = 1 + e\cos\theta$$

$$\Rightarrow e\cos\theta = \frac{l}{r} - 1$$

$$v^2 = \frac{\mu}{l}\left(1 + 2\left(\frac{l}{r} - 1\right) + e^2\right)$$

$$= \frac{\mu}{l}\left[1 + \frac{2l}{r} - 2 + e^2\right]$$

$$= \frac{\mu}{l}\left[e^2 + \frac{2l}{r} - 1\right]$$

$$= \frac{\mu}{l}\left[\frac{2l}{r} - (1-e^2)\right]$$

$$v^2 = \mu\left[\frac{2}{r} - \frac{(1-e^2)}{l}\right] \rightarrow (2)$$

Now if a and b are the semi-axes of the ellipse.

W.k.T, $l = \frac{b^2}{a} = \frac{a^2(1-e^2)}{a}$

$$l = a(1-e^2) \rightarrow (3)$$

(3) in (2), we get

$$v^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right]$$

which gives the velocity v.

Areal velocity in the orbit $= \frac{1}{2}h$

and this is constant.

The total area of the ellipse $= \pi ab$

Periodic time, $T = \frac{\pi ab}{\left(\frac{1}{2}h\right)}$

$$= \frac{2\pi ab}{h}$$

$$= \frac{2\pi ab}{\sqrt{\mu l}} \quad \left[\because \mu = \frac{h^2}{l}\right]$$

$$= \frac{2\pi ab}{\sqrt{\mu}\cdot b}\cdot\sqrt{a} \quad \left[\because l = \frac{b^2}{a}\right]$$

$$= \frac{2\pi}{\sqrt{\mu}}\cdot a^{3/2}$$

§11.12 - Apses and apsidal distances :

Def :

If there is a point A on a Central Orbit at which the Velocity of the particle is perpendicular to the radius OA, then the point A is Called an apse and the length OA is the Corresponding aspidal distance.

Hence at an apse, the particle is moving at right angles to the radius vector

W.k.T, $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 \rightarrow (1)$

where $u = \frac{1}{r}$ and p is the perpendicular from the Centre of force upon the tangent.

At an apse, $p = r = \frac{1}{u} \cdot \rightarrow (2)$

(2) in (1), we get

$$\frac{1}{p^2} = \frac{1}{r^2} = u^2 = u^2 + \left(\frac{du}{d\theta}\right)^2 \quad \left[\because \text{by } (2)\right]$$

$$\Rightarrow u^2 = u^2 + \left(\frac{du}{d\theta}\right)^2$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2 = u^2 - u^2$$

$$\Rightarrow \left(\frac{du}{d\theta}\right)^2 = 0$$

$$\Rightarrow \frac{du}{d\theta} = 0 \quad \text{at an apse.}$$

§. 11.13 - Given the law of force to the pole, to find the orbit.

We now consider the second type of problems namely given the value of the central acceleration P, we will find the path.

We use the (u, θ) equation.

$$u + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2} \rightarrow (1)$$

To solve the differential equation (1), we multiply both sides by $2\frac{du}{d\theta}$ we then have

$$2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2} = 2\frac{P}{h^2u^2}\cdot\frac{du}{d\theta}$$

$$\Rightarrow \frac{d}{d\theta}(u^2) + \frac{d}{d\theta}\left(\frac{du}{d\theta}\right)^2 = \frac{2P}{h^2u^2}\cdot\frac{du}{d\theta}.$$

$\int$ing both sides w.r. to θ,

$$u^2 + \left(\frac{du}{d\theta}\right)^2 = \int \frac{2P}{h^2u^2}\,du + \text{Constant}.$$

[When the principle is understood, the solution (2) could be immediately written down.]

Example - 9 :

A particle moves with an acceleration $\mu[3au^4 - 2(a^2-b^2)u^5]$ and is projected from an apse at a distance $(a+b)$ with a velocity $\frac{\sqrt{\mu}}{a+b}$. Prove that the equation to its orbit is $r = a + b\cos\theta$.

Solution :

Here $P = \mu\left[3au^4 - 2(a^2 - b^2)u^5\right]$

The differential equation to the path is

$$u + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2}$$

$$= \frac{\mu}{h^2}\left[3au^2 - 2(a^2 - b^2)u^3\right] \rightarrow (1)$$

Multiplying (1) by $2\frac{du}{d\theta}$,

$$2u \cdot \frac{du}{d\theta} + 2\frac{du}{d\theta} \cdot \frac{d^2u}{d\theta^2} = 2\frac{P}{h^2u^2}\frac{du}{d\theta}$$

Ing w.r. to θ, we get

$$u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{2\mu}{h^2}\int\left[3au^2 - 2(a^2-b^2)u^3\right]du + c$$

$$= \frac{2\mu}{h^2}\left[au^3 - 2(a^2 - b^2)\frac{u^4}{2}\right] + c \rightarrow (2)$$

Now, $h = PV = \text{Constant} = P_0V_0$

where P_0 and V_0 are the initial values of P and V respectively.

The initial Conditions are

$$V_0 = \frac{\sqrt{\mu}}{a+b}$$

and $P_0 = a + b$ as the ~~principle~~ particle is projected from an apse.

Hence, $h = (a+b)\frac{\sqrt{\mu}}{a+b}$

$$h = \sqrt{\mu}$$

i.e., $\boxed{h^2 = \mu}$

(2) $\Rightarrow$ $u^2 + \left(\frac{du}{d\theta}\right)^2 = 2\left[au^3 - (a^2-b^2)\frac{u^4}{2}\right] + c$ $\rightarrow$ (3)

Initially at the apse, $\frac{du}{d\theta} = 0$

and $u = \frac{1}{a+b}$

(3) $\Rightarrow$ $\frac{1}{(a+b)^2} = 2\left[\frac{a}{(a+b)^3} - \frac{(a^2-b^2)}{2(a+b)^4}\right] + c$

$$= \frac{2a}{(a+b)^3} - \frac{(a-b)}{(a+b)^3} + c$$

$$= \frac{1}{(a+b)^2} + c$$

So, $c = 0$.

(3) $\Rightarrow$ $\left(\frac{du}{d\theta}\right)^2 = 2au^3 - (a^2-b^2)u^4 - u^2$

$$\frac{du}{d\theta} = \sqrt{2au^3 - (a^2-b^2)u^4 - u^2}$$

$$= u\sqrt{2au - (a^2-b^2)u^2 - 1} \rightarrow (4)$$

$$\frac{du}{u\sqrt{2au - (a^2-b^2)u^2 - 1}} = d\theta$$

Put $u = \frac{1}{r}$ Then $du = \frac{-1}{r^2} dr$

$$\frac{-1}{r^2} \cdot r \frac{dr}{\sqrt{\frac{2a}{r} - \frac{(a^2-b^2)}{r^2} - 1}} = d\theta$$

$$-\frac{dr}{\sqrt{2ar - (a^2-b^2) - r^2}} = d\theta$$

$$\Rightarrow \frac{-dr}{\sqrt{b^2 - (r-a)^2}} = d\theta$$

$\int$ing, we get

$$\cos^{-1}\left(\frac{r-a}{b}\right) = \theta + \alpha \rightarrow (5)$$

where α is the Constant

If θ is measured from the apse line, $r = a+b$ when $\theta = 0$.

Hence,

$$\cos^{-1}\left(\frac{a+b-a}{b}\right) = 0 + \alpha$$

$$\Rightarrow \cos^{-1}(1) = \alpha$$

$$\Rightarrow \alpha = 0$$

$$(5) \Rightarrow \cos^{-1}\left(\frac{r-a}{b}\right) = \theta$$

$$\Rightarrow \frac{r-a}{b} = \cos\theta$$

(or) $r = a + b\cos\theta$.

Note :

On taking the square root in (4) above, $\frac{du}{d\theta}$ can be taken either with the positive (or) negative sign. We will get the same equation for the path, with the negative sign for $\frac{du}{d\theta}$.

This is because a central orbit will be symmetrical about an apse line.

As θ increases, r can either increases (or) decrease. Hence $\frac{du}{d\theta}$ will have the positive (or) negative sign according as r decreases (or) increases.