unit - 3.

Solution of PDE of the standard forms :

Standard form I :-

If the given PDE contains p and q i.e., $F(p,q) = 0$.

A solution of this is $z = ax + by + c$.

Replacement of p, q by constants a, b respectively

solving this for b, $b = f(a)$.

Hence the complete integral is

$$z = ax + yf(a) + c \rightarrow (1)$$

Singular integral :

Eliminating a and c:

partially differentiating w.r. to a & c

$$0 = x + yf'(a)$$

$$0 = 0 + 0 + 1$$

$$0 = 1.$$

The last equation is absurd

$\therefore$ There is no singular integral.

General integral :

We assume that $c = \phi(a)$

$$z = ax + yf(a) + \phi(a).$$

partially differentiating w.r.to a,

$$0 = x + yf'(a) + \phi'(a) \longrightarrow (2)$$

The eliminate of a between these equations is the general integral.

1. solve $p^2 + q^2 = npq$.

solu: Given, $p^2 + q^2 = npq$

The solution is $z = ax + by + c \longrightarrow (1)$

complete integral:

sub $p = a, q = b$ in the given equ

$$a^2 + b^2 = nab$$

$$a^2 + b^2 - nab = 0$$

$$b^2 - nab + a^2 = 0$$

$$b = \frac{-(-na) \pm \sqrt{(-na)^2 - 4(1)(a^2)}}{2(1)}$$

$$= \frac{na \pm \sqrt{n^2a^2 - 4a^2}}{2}$$

$$= \frac{an \pm a\sqrt{n^2 - 4}}{2}$$

$$b = \frac{a(n \pm \sqrt{n^2 - 4})}{2} \longrightarrow (2)$$

The complete solution is $z = ax + \frac{a}{2}[n \pm \sqrt{n^2-4}]y + c \rightarrow (3)$

Singular integral:

There is no singular integral.

General integral:

Let $c = \phi(a)$

$(3) \Rightarrow z = ax + \frac{a}{2}[n + \sqrt{n^2-4}]y + \phi(a)$

Partially differentiating w.r.to a,

$0 = x + \frac{1}{2}[n + \sqrt{n^2-4}]y + \phi'(a) \rightarrow (4)$

The elimination of a between equ (3) & (4) gives the general solution.

2. Solve $pq + p + q = 0$.

Soln: Given, $pq + p + q = 0$

The solution is $z = ax + by + c \rightarrow (1)$

complete solution:-

Subs $p = a, q = b$ in the given equation

$ab + a + b = 0$

$(a+1)b = -a$

$b = \frac{-a}{a+1}$

Put $b = \frac{-a}{a+1}$ in (1)

The complete solution is $z = ax - \frac{a}{a+1}y + c \rightarrow (2)$

Singular solution:

There is no singular solution.

General solution:

Let $c = \phi(a)$

② $\Rightarrow z = ax - \frac{a}{a+1} y + \phi(a)$

partially differentiating w.r.to a,

$$0 = x - y\frac{(a+1)(1) - a(1)}{(a+1)^2} + \phi'(a)$$

$$0 = x - \frac{y}{(a+1)^2} + \phi'(a) \rightarrow ③$$

The elimination of a between equations ② & ③ gives the general solution.

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3. solve $p^2+q^2=4$.

solu: Given, $p^2+q^2=4$.

The solution is $z = ax+by+c \rightarrow ①$

complete solution:

Substitute $p=a$, $q=b$ in the given equation.

$$a^2+b^2=4.$$

$$b=\sqrt{4-a^2}$$

The complete solution is $z = ax+\sqrt{4-a^2}\,y+c \rightarrow ②$

singular solution:

There is no singular solution.

General solution:

Let $c = \phi(a)$

② $\Rightarrow z = ax + \sqrt{4-a^2}\,y + \phi(a)$

partially differentiating w.r.to a,

$$0 = x + (4-a^2)^{-1/2}(-2a)\,y + \phi'(a)$$

$$0 = x - \frac{2a}{\sqrt{4-a^2}}y + \phi'(a) \rightarrow ③$$

The elimination of a between equations ② & ③ gives the general solution.

Standard form II.

i) $F(x, p, q) = 0$.

ii) $F(y, p, q) = 0$.

iii) $F(z, p, q) = 0$.

Since z is a function of x & y.

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy.$$

$$\boxed{dz = p\,dx + q\,dy \longrightarrow ①}$$

case(i) : $F(x, p, q) = 0$

Let $q = a$

The equ becomes

$F(x, p, a) = 0$

solving this for p

we get $p = \phi(x, a)$

① ⇒ $dz = \phi(x,a)dx + a\,dy$

$$\boxed{z = \int \phi(x,a)dx + ay + b}$$

which is a complete ~~solution~~ integral

case(ii) → $F(y, p, q) = 0$

Let $p = a$

The equ becomes

$F(y, a, q) = 0$

Solving this for q

we get $q = \phi(y, a)$

① ⇒ $dz = a\,dx + \phi(y,a)dy$

~~z = a dx~~

$$\boxed{z = ax + \int \phi(y,a)dy + b}$$

which is a complete integral (CI)

case(iii) : $F(z, p, q) = 0$

Let $q = ap$.

The equ becomes

$F(z, p, ap) = 0$

Solving this for p

$p = \phi(z, a)$

① ⇒ $dz = \phi(z,a)dx + a\phi(z,a)dy$

$$\boxed{\frac{dz}{\phi(z,a)} = dx + a\,dy \qquad \int \frac{dz}{\phi(z,a)} = x + ay + b}$$

which is a complete integral.

1. Solve : $q = xp + p^2$.

Solu: Given $q = xp + p^2$

Let $q = a$

$a = xp + p^2$

$0 = xp + p^2 - a$

$p^2 + xp - a = 0.$

$a = 0, \quad b = x, \quad c = -a \Rightarrow p = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$\therefore p = \dfrac{-x \pm \sqrt{x^2 - 4(1)(-a)}}{2(1)}$

$p = \dfrac{-x \pm \sqrt{x^2 + 4a}}{2}$

Hence, $dz = p\,dx + q\,dy$

$dz = \dfrac{-x \pm \sqrt{x^2+4a}}{2}\, dx + a\,dy$

Integrating on both sides,

$z = \displaystyle\int \frac{-x \pm \sqrt{x^2+4a}}{2}\, dx + a\int dy.$

$= -\displaystyle\int \frac{x}{2}\, dx \pm \int \frac{\sqrt{x^2+4a}}{2}\, dx + ay + b.$

$z = \dfrac{-x^2}{4} \pm \left[\dfrac{x}{4}\sqrt{4a+x^2} + \dfrac{4a}{4}\sinh^{-1}\left(\dfrac{x}{2\sqrt{a}}\right)\right] + ay + b.$

$\left[\because \sqrt{a^2+x^2} = \dfrac{x}{2}\sqrt{a^2+x^2} + \dfrac{a^2}{2}\sinh^{-1}\left(\dfrac{x}{a}\right)\right]$

$\sqrt{4a+x^2} = \dfrac{x}{2}\sqrt{4a+x^2} + \dfrac{4a}{2}\sinh^{-1}\left(\dfrac{x}{2\sqrt{a}}\right)$

$a^2 = 4a$

$a = 2\sqrt{a}.$

$$z = -\frac{x^2}{4} \pm \left[\frac{x}{4}\sqrt{4a^2+x^2} + a\sinh^{-1}\left(\frac{x}{2\sqrt{a}}\right)\right] + ay + b.$$

2. Solve $pq = x$.

Solu: Given, $pq = x \Rightarrow f(p, q, x) = 0$.

Let $q = a$

$$p(a) = x$$

$$p = \frac{x}{a}.$$

$$\boxed{dz = p\,dx + q\,dy}$$

$$dz = \frac{x}{a}dx + a\,dy$$

Integrating on both sides,

$$\int dz = \frac{1}{a}\int x\,dx + a\int dy.$$

$$z = \frac{x^2}{2a} + ay + b.$$

3. Solve: $\sqrt{p} + \sqrt{q} = x$.

Sol: Given $\sqrt{p} + \sqrt{q} = x$

Let $q = a^2$

$$\sqrt{p} + \sqrt{a^2} = x$$

$$\sqrt{p} = x - a$$

$$p = (x-a)^2$$

$$dz = p\,dx + q\,dy.$$

$$dz = (x-a)^2 dx + a^2 dy$$

Integrating on both sides,

$$\int dz = \int (x-a)^2 dx + a^2 \int dy.$$

$$z = \frac{(x-a)^3}{3} + a^2 y + b.$$

4. solve : $q^2 = yp^4$.

Solu: Given $f(y,p,q) \Rightarrow q^2 = yp^4$

Let $p = a$

$q^2 = ya^4$

$q = \sqrt{y}\,a^2$

$\boxed{dz = pdx + qdy}$

$dz = adx + \sqrt{y}\,a^2 dy$

Integrating on both sides:

$\int dz = a\int dx + a^2 \int \sqrt{y}\,dy$

$z = ax + a^2 \left(\frac{y^{3/2}}{3/2}\right) + b$.

$z = ax + \frac{2a^2}{3} y^{3/2} + b$.

5. Solve : $p = y^2 q^2$.

Solu: Given $f(y,p,q) \Rightarrow p = y^2 q^2$.

Let $p = a^2$

$a^2 = y^2 q^2$

$\frac{a^2}{y^2} = q^2$

$\pm \frac{a^2}{y} = q$.

$dz = pdx + qdy$

$dz = a^2 dx \pm \frac{a}{y} dy$.

Integrating on both Sides

$\int dz = a^2 \int dx \pm a \int \frac{dy}{y}$.

$z = a^2 x \pm a \log y + b$.

6. Solve: $\sqrt{p}+\sqrt{q}=\sqrt{y}$.

Solu: Given, $f(y,p,q) \Rightarrow \sqrt{p}+\sqrt{q}=\sqrt{y}$.

Let $p=a^2$

$$\sqrt{a^2}+\sqrt{q}=\sqrt{y}.$$

$$\sqrt{q}=\sqrt{y}-\sqrt{a^2}$$

$$q=(\sqrt{y}-\sqrt{a})^2$$

$$q=(\sqrt{y})^2+a^2-2a\sqrt{y}.$$

$$q=(y+a^2)-2a\sqrt{y}.$$

$$dz=pdx+qdy.$$

$$dz=a^2dx+[(y+a^2)-2a\sqrt{y}]dy$$

$$dz=a^2dx+(y+a^2)dy-2a\sqrt{y}dy.$$

Integrating on both sides.

$$\int dz=a^2\int dx+\int(y+a^2)dy-2a\int\sqrt{y}dy.$$

$$z=a^2x+\frac{(y+a^2)^2}{2}-2a\frac{(y)^{3/2}}{3/2}+b.$$

$$z=a^2x+\frac{(y+a^2)^2}{2}-\frac{4a}{3}y^{3/2}+b.$$

7. Solve: $z=p^2+q^2$

Solu: Given, $f(z,p,q) \Rightarrow z=p^2+q^2$

Let $q=ap$.

$$z=p^2+(ap)^2$$

$$z=p^2(1+a^2)$$

$$\frac{z}{1+a^2}=p^2$$

$$p=\frac{\sqrt{z}}{\sqrt{1+a^2}}$$

$$dz=pdx+qdy.$$

$$dz=\frac{\sqrt{z}}{\sqrt{1+a^2}}dx+apdy.$$

$$dz = \frac{\sqrt{z}}{\sqrt{1+a^2}}dx + a\frac{\sqrt{z}}{\sqrt{1+a^2}}dy.$$

$$\sqrt{1+a^2}\,\frac{dz}{\sqrt{z}} = dx + a\,dy.$$

$$\sqrt{1+a^2}\,(z)^{-1/2}dz = dx + a\,dy.$$

Integrating on both sides.

$$\sqrt{1+a^2}\int (z)^{-1/2}\,dz = \int dx + a\int dy.$$

$$\sqrt{1+a^2}\left(\frac{z^{-1/2+1}}{-1/2+1}\right) = x + ay + b.$$

$$\sqrt{1+a^2}\,\frac{z^{1/2}}{1/2} = x+ay+b.$$

$$\sqrt{1+a^2}\;2\sqrt{z} = x+ay+b.$$

Squaring on both sides.

$$(1+a^2)4z = (x+ay+b)^2.$$

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8. $p(1+q^2) = q(z-1).$

solu: Let $q = ap$

Given $f(z,p,q) \Rightarrow p(1+q^2) = q(z-1)$

$$p[1+(ap)^2] = ap(z-1)$$

$$(1+a^2p^2) = a(z-1)$$

$$a^2p^2 = az-a-1$$

$$p^2 = \frac{az-a-1}{a^2}$$

$$p = \pm\frac{\sqrt{az-a-1}}{a}$$

$$dz = p\,dx + q\,dy.$$

$$dz = \pm\frac{\sqrt{az-a-1}}{a}dx + ap\,dy.$$

$$dz = \pm\frac{\sqrt{az-a-1}}{a}dx + a\frac{\sqrt{az-a-1}}{a}dy.$$

$$\pm \frac{a\,dz}{\sqrt{az-a-1}} = dx + a\,dy.$$

Integrating on both sides.

$$\pm \int \frac{a\,dz}{\sqrt{az-a-1}} = \int dx + a\int dy$$

$$\pm \int (az-a-1)^{-1/2}\,a\,dz = x + ay + b.$$

$$\pm \frac{(az-a-1)^{-1/2+1}}{-1/2+1} = x + ay + b.$$

$$\pm \frac{(az-a-1)^{1/2}}{1/2} = x + ay + b.$$

$$\pm\, 2\sqrt{az-a-1} = x + ay + b.$$

9. Solve : $z^2(p^2+q^2+1) = c^2$

Soln: Given $f(z,p,q) \Rightarrow z^2(p^2+q^2+1) = c^2$

Let $u = x + ay$, $\frac{\partial u}{\partial x} = 1$, $\frac{\partial u}{\partial y} = a$

so that

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du}.$$

$$q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = a\frac{dz}{du}.$$

$$z^2\left[\left(\frac{dz}{du}\right)^2 + \left(a\frac{dz}{du}\right)^2 + 1\right] = c^2$$

$$z^2\left[\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2\right] + z^2 = c^2$$

$$z^2(1+a^2)\left(\frac{dz}{du}\right)^2 = c^2 - z^2.$$

$$z(1+a^2)^{1/2}\frac{dz}{du} = \sqrt{c^2-z^2}$$

$$\sqrt{(1+a^2)}\,\frac{z}{\sqrt{c^2-z^2}}\,dz = du.$$

$$\sqrt{1+a^2}\,(c^2-z^2)^{-1/2} z\,dz = du.$$

Integrating on both sides,

$$\sqrt{1+a^2}\int (c^2-z^2)^{-1/2} z\,dz = \int du.$$

$$\sqrt{1+a^2}\; - \frac{(c^2-z^2)^{-1/2+1}}{-1/2+1} = u+b.$$

$$-\sqrt{1+a^2}\,\frac{(c^2-z^2)^{1/2}}{1/2} = x+ay+b$$

$$-2\sqrt{1+a^2}\,(c^2-z^2)^{1/2} = x+ay+b$$

$$4(1+a^2)(c^2-z^2) = (x+ay+b)^2.$$

Standard form - III :-

$$f_1(x,p) = f_2(y,q)$$

Solving $f_1(x,p) = a$, $p = \phi_1(a,x)$

Solving $f_2(y,q) = a$, $q = \phi_2(a,y)$.

$$dz = p\,dx + q\,dy.$$

$$dz = \phi_1(a,x)\,dx + \phi_2(a,y)\,dy$$

$$\therefore\; z = \int \phi_1(a,x)\,dx + \int \phi_2(a,y)\,dy.$$

1. Solve : $p+q = x+y$.

Soln: $p-x = y-q$

Let $p-x = a$, then $y-q = a$.

Hence, $p = x+a$, $q = y-a$

$$dz = p\,dx + q\,dy.$$

$$dz = (x+a)\,dx + (y-a)\,dy.$$

$\int$ing on both sides;

$$\int dz = \int (x+a)dx + \int (y-a)dy.$$

$$z = \frac{(x+a)^2}{2} + \frac{(y-a)^2}{2} + b.$$

2. Solve : $p^2+q^2 = x+y$.

Soln:

$$p^2 - x = y - q^2$$

Let $p^2 - x = a$ then $y - q^2 = a$.

Hence, $p^2 = x+a$ $\quad y - a = q^2$

$p = \pm\sqrt{x+a}$ $\quad q = \pm\sqrt{y-a}$

$$dz = pdx + qdy.$$

$$dz = \pm\sqrt{x+a}\,dx \pm \sqrt{y-a}\,dy$$

$\int$ing on both sides,

$$\int dz = \pm\int (x+a)^{1/2}dx \pm \int (y-a)^{1/2}dy.$$

$$z = \pm\frac{(x+a)^{3/2}}{3/2} \pm \frac{(y-a)^{3/2}}{3/2} + b.$$

$$z = \pm\frac{2}{3}(x+a)^{3/2} \pm \frac{2}{3}(y-a)^{3/2} + b$$

There is no singular integral.

General integral:

Assume $b = f(a)$.

$$z = \pm\frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y-a)^{3/2} + f(a) \rightarrow (1)$$

Partially differentiating w.r.to a,

$$0 = \pm\frac{2}{3}\frac{3}{2}(x+a)^{3/2-1}(1) \pm \frac{2}{3}\frac{3}{2}(y-a)^{3/2-1}(-1) + f'(a)$$

$$0 = \pm(x+a)^{1/2} \pm (y-a)^{1/2} + f'(a) \rightarrow (2)$$

The elimination of a between (1) & (2) represents the general integral.

3) Solve : $q(p-\sin x) = \cos y$.

Soln:

$$p-\sin x = \frac{\cos y}{q}$$

Let $p-\sin x = a$, $\frac{\cos y}{q} = a$

$$p = a+\sin x \qquad q = \frac{\cos y}{a}$$

$$dz = p\,dx + q\,dy$$

$$dz = (a+\sin x)dx + \frac{\cos y}{a}dy.$$

$\int$ing on both sides.

$$\int dz = a\int dx + \int \sin x\,dx + \frac{1}{a}\int \cos y\,dy.$$

$$z = ax - \cos x + \frac{1}{a}\sin y + b.$$

Obviously there is no singular integral.

General integral:

Assume $b = f(a)$

$$z = ax - \cos x + \frac{\sin y}{a} + f(a) \rightarrow (1)$$

partially differentiating w.r.to a,

$$0 = x - \frac{\sin y}{a^2} + f'(a) \rightarrow (2)$$

The elimination of a between (1) & (2) represents the general integral.

Standard form - IV - Clairaut's form: (15)

$$z = px + qy + f(p, q)$$

The solution is $z = ax + by + f(a,b)$ for $p = a$ & $q = b$.

1. Solve $z = px + qy + \sqrt{1+p^2+q^2}$.

Soln: Given $z = px + qy + \sqrt{1+p^2+q^2}$

complete integral:

Let $p = a$, $q = b$

$$\therefore z = ax + by + \sqrt{1+a^2+b^2} \longrightarrow (1)$$

Singular solution:

Partially Differentiating (1) w.r. to a & b.

$$0 = x + 0 + \frac{1}{2}(1+a^2+b^2)^{1/2-1}(2a)$$

$$0 = x + \frac{a}{\sqrt{a^2+b^2+1}}$$

$$\Rightarrow x = \frac{-a}{\sqrt{1+a^2+b^2}} \longrightarrow (2)$$

$$0 = 0 + y + \frac{1}{2}(1+a^2+b^2)^{1/2-1}(2b)$$

$$0 = y + \frac{b}{\sqrt{1+a^2+b^2}}$$

$$-y = \frac{b}{\sqrt{1+a^2+b^2}}$$

$$y = \frac{-b}{\sqrt{1+a^2+b^2}} \longrightarrow (3)$$

Squaring and adding (2) & (3)

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$$x^2+y^2 = \frac{a^2}{(1+a^2+b^2)} + \frac{b^2}{(1+a^2+b^2)}$$

$$x^2+y^2 = \frac{a^2+b^2}{1+a^2+b^2}$$

$$1-(x^2+y^2) = 1-\frac{a^2+b^2}{1+a^2+b^2}$$

$$= \frac{1+a^2+b^2-a^2-b^2}{1+a^2+b^2}$$

$$1-x^2-y^2 = \frac{1}{1+a^2+b^2}$$

$$1+a^2+b^2 = \frac{1}{(1-x^2-y^2)}$$

$$\sqrt{1+a^2+b^2} = \frac{1}{\sqrt{1-x^2-y^2}} \longrightarrow (4)$$

sub (4) in (2) & (3)

$$(2) \Rightarrow x = \frac{-a}{\frac{1}{\sqrt{1-x^2-y^2}}} \qquad (3) \Rightarrow y = \frac{-b}{\frac{1}{\sqrt{1-x^2-y^2}}}$$

$$x = -a\sqrt{1-x^2-y^2} \qquad y = -b\sqrt{1-x^2-y^2}$$

$$a = \frac{-x}{\sqrt{1-x^2-y^2}} \qquad \frac{-y}{\sqrt{1-x^2-y^2}} = b.$$

Substitute equ (4), a & b in (1)

$$(1) \Rightarrow z = \frac{-x}{\sqrt{1-x^2-y^2}}x - \frac{y}{\sqrt{1-x^2-y^2}} + \frac{1}{\sqrt{1-x^2-y^2}}$$

$$z = \frac{-x^2-y^2+1}{\sqrt{1-x^2-y^2}}.$$

$$z = \frac{1-x^2-y^2}{\sqrt{1-x^2-y^2}} = \frac{\sqrt{1-x^2-y^2}\sqrt{1-x^2-y^2}}{\sqrt{1-x^2-y^2}}$$ (17)

$$z = \sqrt{1-x^2-y^2}$$

$$z^2 = 1-x^2-y^2$$

$x^2+y^2+z^2 = 1$ which is the singular solution

General integral:

Take $b = \phi(a)$

$$① \Rightarrow z = ax + \phi(a)y + \sqrt{1+a^2+[\phi(a)]^2} \rightarrow ⑤$$

Partially differentiating w.r.to 'a'.

$$0 = x + y\phi'(a) + \frac{1}{2}[1+a^2+[\phi(a)]^2]^{\frac{1}{2}-1}[2a+2\phi(a)]$$

$$0 = x+y\phi'(a) + \frac{1}{2}\frac{1}{\sqrt{1+a^2+[\phi(a)]^2}}2[a+\phi(a)\phi'(a)]$$

$$0 = x+y\phi'(a) + \frac{a+\phi(a)+\phi'(a)}{\sqrt{1+a^2+[\phi(a)]^2}} \rightarrow ⑥$$

Eliminating a from ⑤ & ⑥, we get general integral

2. find complete integral a) $(p+q)(z-px-qy)=1$

b) $pqz = p^2(xq+p^2)+q^2(yp+q^2)$

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Solu.

a) $(p+q)(z-px-qy)=1$

$$(z-px-qy)=\frac{1}{p+q}$$

$$z = px+qy+\frac{1}{p+q}$$

put $p=a, q=b$.

$$z = ax+by+\frac{1}{a+b}$$

which is the complete integral.

b) $pqz = p^2(xq+p^2)+q^2(yp+q^2)$

$pqz = p^2qx + p^4 + q^2py + q^4$

$pqz = p^2qx + pq^2y + p^4+q^4$

$$z = \frac{p^2qx}{pq} + \frac{pq^2y}{pq} + \frac{p^4+q^4}{pq}.$$

$$z = px+qy+\frac{p^4+q^4}{pq}.$$

put $p=a, q=b$.

$z = ax+by+\frac{a^4+b^4}{ab}$ which is the complete integral

3. Solve $z = px + qy + ab$.

Solu:

(19)

Given, $z = px + qy + ab$.

complete integral:

Let $p = a$, $q = b$.

$$z = ax + by + ab \rightarrow (1)$$

Singular solution:

Partially differentiating (1) w.r.to a & b.

$0 = x + b$ $\qquad$ $0 = y + a$

$b = -x$ $\qquad$ $a = -y$.

Subs a & b in (1)

$$z = (-y)x + (-x)y + (-x)(-y)$$

$$z = -xy - xy + xy.$$

$$z = -xy.$$

$z + xy = 0$ which is the singular solution.

General integral:-

Take $b = \phi(a)$

$$(1) \Rightarrow z = ax + \phi(a)y + a\phi(a) \rightarrow (2)$$

Partially differentiating w.r.to 'a'

$$0 = x + \phi'(a)y + a\phi'(a) + \phi(a) \rightarrow (3)$$

Eliminating a form (2) & (3), we get general integral.

4. Solve $z = px + qy + p^2 + pq + q^2$

(20)

Soln. Given,

$$z = px + qy + p^2 + pq + q^2$$

complete integral:

put $p = a,\ q = b.$

$$z = ax + by + a^2 + ab + b^2 \rightarrow (*)$$

Singular solution:

Partially diff (*) w.r.to a & b.

$0 = x + 2a + b$ $\qquad$ $0 = y + 2b + a$

$2a + b = -x \rightarrow (1)$ $\qquad$ $a + 2b = -y \rightarrow (2)$

$(1) \Rightarrow 2a + b = -x$

$(2) \times 2 \Rightarrow 2a + 4b = -2y$

$\quad (-) \quad (-) \quad (+)$

$$-3b = 2y - x$$

$$b = \frac{2y - x}{-3}$$

$$b = \frac{x - 2y}{3}$$

$(1) \times 2 \Rightarrow 4a + 2b = -2x$

$(-)\ (2) \Rightarrow a + 2b = -y$

$\quad (-) \quad (-) \quad (+)$

$$3a = y - 2x$$

$$a = \frac{y - 2x}{3}$$

Subs a & b in (*)

$$(1) \Rightarrow z = \frac{y-2x}{3}x + \frac{x-2y}{3}y + \left(\frac{y-2x}{3}\right)^2 + \left(\frac{y-2x}{3}\right)\left(\frac{x-2y}{3}\right) + \left(\frac{x-2y}{3}\right)^2$$

$$z = \frac{xy - 2x^2}{3} + \frac{xy - 2y^2}{3} + \frac{y^2 + 4x^2 - 4xy}{9}$$

$$+ \frac{xy+2x^2-2y^2+4xy}{9} + \frac{x^2+4y^2-4xy}{9}$$

$$z = \frac{3xy-6x^2+3xy-6y^2+y^2+4x^2-4xy-2x^2-2y^2+5xy}{9}$$

$$+ \frac{x^2+4y^2-4xy}{9}$$

$$z = \frac{3xy-3x^2-3y^2}{9}$$

$$z = \frac{xy-x^2-y^2}{3}$$

$$3z = xy - x^2 - y^2$$

General Integral:

Take $b = \phi(a)$

$(1) \Rightarrow z = ax + \phi(a)y + a^2 + a\phi(a) + [\phi(a)]^2 \longrightarrow (2)$

partially diff. w.r.to 'a'

$$0 = x + \phi'(a)y + 2a + a\phi'(a) + \phi(a) + 2\phi(a)\phi'(a)$$

Eliminating a from (2) & (3), we get general integral.

Charpit's Method:

$$F(x,y,z,p,q) = 0.$$

$$\frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}}$$

which is the subsidiary equations.

1. using charpit's method find complete integral of $pxy + pq + qy = yz$.

(22)

Soln: Here $f(x,y,z,p,q) = pxy + pq + qy - yz = 0$ → (1)

By charpit's method, Auxiliary Equ is

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$$

$f_p = xy + q \qquad f_x = py \qquad f_z = -y.$

$f_q = p + y \qquad f_y = px + q - z$

$$\frac{dx}{-(xy+q)} = \frac{dy}{-(p+y)} = \frac{dz}{-p(xy+q) - q(p+y)} = \frac{dp}{py + p(-y)}$$

$$= \frac{dq}{px + q - z + q(-z)}$$

$$\frac{dx}{-(xy+q)} = \frac{dy}{-(p+y)} = \frac{dz}{-pxy - pq - py - qy} = \frac{dp}{0} = \frac{dq}{px + q - z - qy}$$

$dp = 0$

Integrating on both sides,

$p = a.$

putting $p = a$ in (1)

$$axy + aq + qy = yz$$

$$(a+y)\, q = yz - axy$$

$$q = \frac{y(z - ax)}{(a+y)}.$$

W.K.T

$$dz = p\,dx + q\,dy.$$

$$dz = a\,dx + \frac{y(z - ax)}{(a+y)}\,dy$$

$$dz - a\,dx = \frac{y(z-ax)}{a+y}\,dy$$

$$\frac{dz - a\,dx}{z - ax} = \frac{y\,dy}{a+y}$$

$$= \left(\frac{a-a+y}{a+y}\right)dy.$$

$$= \left(\frac{a+y-a}{a+y}\right)dy$$

$$= \left(1 - \frac{a}{a+y}\right)dy$$

$$\frac{dz - a\,dx}{z-ax} = dy - \frac{a\,dy}{a+y}$$

$\int$ing on both sides

$$\int \frac{dz - a\,dx}{z - ax} = \int dy - a\int \frac{dy}{a+y}.$$

$$\log(z-ax) = y - a\log(a+y) + \log b \rightarrow (2)$$

$$\log(z-ax) = y + \log(a+y)^{-a} + \log b.$$

(or)

$$(z-ax) = e^{y}.(a+y)^{-a}b.$$

$$z = ax + be^{y}(a+y)^{-a}.$$

Note:

There is no singular integral.

General Integral:

$$b = f(a).$$

(2) $\Rightarrow$ $\log(z-ax) = y - a\log(a+y) + \log(f(a))$

Differentiate with respect to 'a' & eliminating a the general integral is obtained.

(24)

② Solve : $p^2+q^2-2px-2qy+1=0$.

Soln: $f(x,y,z,p,q) = p^2+q^2-2px-2qy+1=0 \rightarrow ①$

Here $f_x = -2p \qquad f_p = 2p-2x \qquad f_z = 0$

$f_y = -2q \qquad f_q = 2q-2y$

Auxiliary Equ is

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p-qf_q} = \frac{dp}{f_x+pf_z} = \frac{dq}{f_y+qf_z}$$

$$\frac{dx}{-(2p-2x)} = \frac{dy}{-(2q-2y)} = \frac{dz}{-p(2p-2x)-q(2q-2y)}$$

$$= \frac{dp}{-2p+p(0)} = \frac{dq}{-2q+q(0)}$$

$$\frac{dx}{-2(p-x)} = \frac{dy}{-2(q-y)} = \frac{dz}{-2p^2+2px-2q^2+2qy} = \frac{dp}{-2p} = \frac{dq}{-2q}$$

The last 2 ratios

$$\frac{dp}{-2p} = \frac{dq}{-2q}.$$

$$\int \frac{dp}{p} = \int \frac{dq}{q}$$

$$\log p = \log q + \log a$$

$$p = aq.$$

Putting $p = aq$ in ①

$$(aq)^2+q^2-2(aq)x-2qy+1=0$$

$$(a^2+1)q^2-2(ax+y)q+1=0$$

$$q = \frac{+2(ax+y) \pm \sqrt{(2(ax+y))^2-4(a^2+1)(1)}}{2(a^2+1)}$$

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$$= \frac{2(ax+y) \pm \sqrt{4(ax+y)^2 - 4(a^2+1)}}{2(a^2+1)}$$

$$= \frac{2(ax+y) \pm 2\sqrt{(ax+y)^2 - (a^2+1)}}{2(a^2+1)}$$

$$q = \frac{ax+y \pm \sqrt{(ax+y)^2 - (a^2+1)}}{a^2+1}.$$

W.K.T

$$dz = p\,dx + q\,dy.$$

$$dz = (aq)dx + \frac{(ax+y) \pm \sqrt{(ax+y)^2 - (a^2+1)}\,dy}{a^2+1}.$$

$$dz = a\frac{(ax+y) \pm \sqrt{(ax+y)^2 - (a^2+1)}}{a^2+1}dx + \frac{(ax+y) \pm \sqrt{(ax+y) - (a^2+1)}}{a^2+1}dy$$

$$dz = \frac{(ax+y) \pm \sqrt{(ax+y)^2 - (a^2+1)}}{a^2+1}[a\,dx + dy]$$

$$(a^2+1)dz = \left[(ax+y) \pm \sqrt{(ax+y)^2 - (a^2+1)}\right]d(ax+y)$$

$$(a^2+1)dz = (ax+y)d(ax+y) \pm \sqrt{(ax+y)^2 - (a^2+1)}\,d(ax+y)$$

$\int$ing on both sides.

$$(a^2+1)\int dz = \int(ax+y)d(ax+y) \pm \int\sqrt{(ax+y)^2 - (a^2+1)}\,d(ax+y)$$

$$ax+y = l.$$

$$(a^2+1)\int dz = \int l\,dl \pm \int\sqrt{l^2 - (a^2+1)}\,dl$$

$$(a^2+1)z = \frac{l^2}{2} \pm \left[l\sqrt{l^2-(a^2+1)} - (a^2+1)\cosh^{-1}\frac{l}{\sqrt{a^2+1}}\right] + b$$

$$\left[\because \int\sqrt{x^2-y^2} = x\sqrt{x^2-y^2} - y^2\cosh^{-1}\frac{x}{\sqrt{y^2}}\right]$$ (26)

$$(a^2+1)z = \frac{l^2}{2} \pm \left[l\sqrt{l^2-(a^2+1)} - (a^2+1)\cosh^{-1}\left(\frac{l}{\sqrt{a^2+1}}\right)\right] + b$$

which is the complete integral.

There is no singular integral.

General integral:

We put $b = f(a)$ where f is arbitary and differentiate w.r. to 'a'. Eliminating a, the general integral is obtained.

---

3. Solve $q = 3p^2$

Soln: Given $f(x,y,z,p,q) = -3p^2+q = 0. \rightarrow (1)$

Here $f_x = 0$, $f_y = 0$, $f_z = 0$

$f_p = -6p$, $f_q = 1$.

Auxiliary equation is

$$\frac{dp}{f_x+pf_z} = \frac{dq}{f_y+qf_z} = \frac{dz}{-pf_p-qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}.$$

$$\frac{dp}{0+p(0)} = \frac{dq}{0+q(0)} = \frac{dz}{-p(-6p)-q(1)} = \frac{dx}{-(-6p)} = \frac{dy}{-1}.$$

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{6p^2-q} = \frac{dx}{6p} = -dy.$$

Taking 1^{st} ratio,

$$\frac{dp}{0} = 0$$

$$dp = 0$$

$$\boxed{p = a}$$

Putting $p = a$ in ①

$$-3a^2 + q = 0$$

$$\boxed{q = 3a^2}$$

W.K.T, $dz = p\,dx + q\,dy$.

$$dz = a\,dx + 3a^2\,dy.$$

Integrating on both sides,

$$\int dz = a\int dx + 3a^2\int dy$$

$$z = ax + 3a^2y + b \rightarrow ②$$

Which is the complete integral.

There is no singular integral.

General integral:

Take $b = f(a)$.

$$② \Rightarrow z = ax + 3a^2y + f(a) \rightarrow ③$$

Differentiating w.r. to 'a'.

$$0 = x + 6ay + f'(a) \rightarrow ④$$

Eliminating 'a' from ③ & ④, we get the general integral.

4. Solve $z = px + qy + p^2 + q^2$

Soln:

$$f(x,y,z,p,q) = z - px - qy - p^2 - q^2 = 0 \rightarrow (1)$$

Here,

$f_x = -p \qquad f_z = 1 \qquad f_q = -y - 2q.$

$f_y = -q \qquad f_p = -x - 2p$

Auxiliary Equ is

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{f_q}.$$

$$\frac{dp}{-p+p(1)} = \frac{dq}{-q-q(1)} = \frac{dz}{-p(-x-2p)-q(-y-2q)} = \frac{dx}{-(-x-2p)} = \frac{dy}{-(-y-2q)}.$$

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{p(x+2p)+q(y+2q)} = \frac{dx}{x+2p} = \frac{dy}{y+2q}.$$

Taking 1^{st} & 2^{nd} ratio

$dp = 0 \qquad dq = 0$

$p = a. \qquad q = b.$

putting $p = a$ & $q = b$ in (1)

$(1) \Rightarrow z - ax - by - a^2 - b^2 = 0$

$$z = ax + by + a^2 + b^2 \longrightarrow (2)$$

which is the complete integral.

Singular integral:

Partially differentiating (2) w.r.to 'a' & 'b'

$0 = x + 2a \qquad 0 = y + 2b$

$x = -2a \qquad y = -2b$

$a = -x/2 \qquad b = -y/2$

$② \Rightarrow z = \left(\frac{x}{-2}\right)x + \left(\frac{y}{-2}\right)y + \left(\frac{x}{-2}\right)^2 + \left(\frac{y}{-2}\right)^2$

$$z = -\frac{x^2}{2} - \frac{y^2}{2} + \frac{x^2}{4} + \frac{y^2}{4}$$

$$z = \frac{x^2 - 2x^2}{4} + \frac{y^2 - 2y^2}{4}$$

$$z = -\frac{x^2}{4} - \frac{y^2}{4}.$$

$$4z = -(x^2 + y^2).$$

$4z + x^2 + y^2 = 0$ which is the singular integral.

General integral:

Take $b = \phi(a)$

$② \Rightarrow z = ax + \phi(a)y + a^2 + (\phi(a))^2 \rightarrow ③$

partially differentiating w.r.to 'a'

$0 = x + y\phi'(a) + 2a + 2\phi(a)\phi'(a) \rightarrow ④$

Eliminating 'a' from ③ & ④, we get the general integral.

5. $z^2 = pqxy$. Find complete integral:

Soln: Given, $f(x, y, z, p, q) = z^2 - pqxy = 0$ $\phi\ pqxy - z^2 = 0$ $\rightarrow ①$

Here $f_x = pqy$, $f_y = pqx$, $f_z = -2z$

$f_p = qxy$, $f_q = pxy$

Auxiliary Equ is

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}.$$

$$\frac{dp}{pqy+p(-2z)} = \frac{dq}{pqx+q(-2z)} = \frac{dz}{-p(qxy)-q(pxy)} = \frac{dx}{-qxy}$$
$$= \frac{dy}{-pxy}.$$

$$\frac{dp}{p(qy-2z)} = \frac{dq}{q(px-2z)} = \frac{dz}{-2pqxy} = \frac{dx}{-qxy} = \frac{dy}{-pxy}.$$

$$\frac{dp/p}{qy-2z} = \frac{dq/q}{px-2z} = \frac{dx/x}{-qy} = \frac{dy/y}{-px}.$$

$$\frac{\frac{dp}{p} - \frac{dq}{q}}{qy-2z-px+2z} = \frac{\frac{dy}{y} - \frac{dx}{x}}{-px+qy}.$$

$$\frac{\frac{dp}{p} - \frac{dq}{q}}{qy-px} = \frac{\frac{dy}{y} - \frac{dx}{x}}{qy-px}$$

$$\frac{dp}{p} - \frac{dq}{q} = \frac{dy}{y} - \frac{dx}{x}$$

$\int$ing on both sides,

$$\int \frac{dp}{p} - \int \frac{dq}{q} = \int \frac{dy}{y} - \int \frac{dx}{x}$$

$$\log p - \log q = \log y - \log x + \log a^2$$

$$\log p + \log x = \log y - \log q + \log a^2$$

$$\log (px) = \log (qya^2)$$

Taking exponential on both sides

$$e^{\log(px)} = e^{\log(qya^2)}$$

$$px = qya^2$$

$$p = \frac{qya^2}{x}$$

putting $p = \frac{qya^2}{x}$ in ①

$$qxy\left(\frac{qya^2}{x}\right) - z^2 = 0$$

$$\frac{q^2a^2y^2x}{x} = z^2$$

$$q^2 = \frac{z^2}{a^2y^2}$$

$$q = \frac{z}{ay}$$

W.k.T

$$dz = p\,dx + q\,dy.$$

$$dz = q\frac{ya^2}{x}\,dx + \frac{z}{ay}\,dy.$$

$$dz = \frac{z}{ay}\cdot\frac{ya^2}{x}\,dx + \frac{z}{ay}\,dy$$

$$dz = z\left[a\frac{dx}{x} + \frac{1}{a}\frac{dy}{y}\right]$$

$$\frac{dz}{z} = a\frac{dx}{x} + \frac{1}{a}\frac{dy}{y}$$

∫ing on both sides,

$$\int\frac{dz}{z} = a\int\frac{dx}{x} + \frac{1}{a}\int\frac{dy}{y}.$$

$$\log z = a\log x + \frac{1}{a}\log y + \log b.$$

$$\log z = \log x^a + \log (y)^{1/a} + \log b.$$

$$\log z = \log (x^a y^{1/a}\cdot b)$$

Taking exponential on both sides

$$z = bx^a y^{1/a}.$$

6. Using charpit's method to find C.I of (32)

$p^2+q^2-2px-2qy+2xy=0$.

Soln: We have $f(x,y,z,p,q) = p^2+q^2-2px-2qy+2xy=0$ $\rightarrow (1)$

$f_x = -2p+2y \qquad f_y = -2q+2x \qquad f_z = 0$

$f_p = 2p-2x \qquad f_q = 2q-2y$

Auxiliary Equ is

$$\frac{dp}{f_x+pf_z} = \frac{dq}{f_y+qf_z} = \frac{dz}{-pf_p-qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}.$$

$$\frac{dp}{-2p+2x+p(0)} = \frac{dq}{-2q+2x+q(0)} = \frac{dz}{-p(2p-2x)-q(2q-2y)}$$

$$= \frac{dx}{-(2p-2x)} = \frac{dy}{2y-2q}.$$

$$\frac{dx+dy}{-2p+2y-2q+2x} = \frac{dp+dq}{2x-2p+2y-2q}$$

$$dx+dy = dp+dq$$

$\int$ing on both sides,

$$x+y = p+q$$

$$y-q = p-x$$

$$(p-x) = y-q \longrightarrow (2)$$

(1) can be written as

$$p^2-2px+x^2-x^2+q^2-2qy+y^2-y^2+2xy=0$$

$$p^2-2px+x^2+q^2-2qy+y^2 = x^2+y^2-2xy.$$

$$(p-x)^2+(q-y)^2=(x-y)^2$$

$$(y-q)^2+(q-y)^2=(x-y)^2 \quad (\text{from } ②)$$

$$(-(q-y))^2+(q-y)^2=(x-y)^2$$

$$(q-y)^2+(q-y)^2=(x-y)^2$$

$$2(q-y)^2=(x-y)^2$$

$$\sqrt{2}(q-y)=(x-y)$$

$$q-y=\frac{x-y}{\sqrt{2}}$$

$$q=\frac{x-y}{\sqrt{2}}+y.$$

② $\Rightarrow$

$$p-x=y-q$$

$$=y-\left(\frac{x-y}{\sqrt{2}}+y\right)$$

$$=y-\left(\frac{x-y}{\sqrt{2}}\right)-y.$$

$$p-x=\frac{-(x-y)}{\sqrt{2}}.$$

$$p=x-\frac{(x-y)}{\sqrt{2}}$$

W.K.T

$$dz=p\,dx+q\,dy$$

$$dz=\left(x-\frac{(x-y)}{\sqrt{2}}\right)dx+\left(\frac{x-y}{\sqrt{2}}+y\right)dy$$

$$=x\,dx-\frac{x\,dx}{\sqrt{2}}+\frac{y\,dx}{\sqrt{2}}+\frac{x}{\sqrt{2}}dy-\frac{y\,dy}{\sqrt{2}}+y\,dy$$

$$=x\,dx+y\,dy-\frac{x}{\sqrt{2}}dx-\frac{y\,dy}{\sqrt{2}}+\frac{y\,dx+x\,dy}{\sqrt{2}}$$

$$dz = x\,dx + y\,dy - \frac{1}{\sqrt{2}}x\,dx - \frac{1}{\sqrt{2}}y\,dy + \frac{d(xy)}{\sqrt{2}}.$$

$\int$ing on both sides.

$$\int dz = \int x\,dx + \int y\,dy - \frac{1}{\sqrt{2}}\int x\,dx - \frac{1}{\sqrt{2}}\int y\,dy + \frac{1}{\sqrt{2}}\int d(xy)$$

$$z = \frac{x^2}{2} + \frac{y^2}{2} - \frac{1}{\sqrt{2}}\left(\frac{x^2}{2}\right) - \frac{1}{\sqrt{2}}\left(\frac{y^2}{2}\right) + \frac{1}{\sqrt{2}}(xy) + b.$$

7\. using charpit's method to find complete integral of $z^2(p^2z^2+q^2)=1$.

Soln: We have, $f(x,y,z,p,q) = z^2(p^2z^2+q^2)-1=0 \quad \to (1)$

Then

$f_x = 0 \qquad f_y = 0 \qquad f_z = 4p^2z^3 + 2q^2z.$

$f_p = 2pz^4 \qquad f_q = 2qz^2$

Auxiliary Eqn is

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}.$$

$$\frac{dp}{0+p(4p^2z^3+2q^2z)} = \frac{dq}{0+q[4p^2z^3+2q^2z]}$$

$$= \frac{dz}{-p(2pz^4)-q(2qz^2)} = \frac{dx}{-2pz^4} = \frac{dy}{-2qz^2}$$

$$\frac{dp}{p(4p^2z^3+2q^2z)} = \frac{dq}{q(4p^2z^3+2q^2z)} = \frac{dz}{-2p^2z^4-2q^2z^2}$$

$$= \frac{dx}{-2pz^4} = \frac{dy}{-2qz^2}$$

Taking first 2 fractions,

$$\frac{dp}{p[4p^2z^3+2q^2z]} = \frac{dq}{q[4p^2z^3+2q^2z]}$$

$$\frac{dp}{p} = \frac{dq}{q}$$

$\int$ing on both sides,

$$\int \frac{dp}{p} = \int \frac{dq}{q}$$

$$\log p + \log a = \log q$$

$$\log(pa) = \log q$$

$$pa = q$$

putting $q = pa$ in (1)

$$z^2[p^2z^2 + (pa)^2] - 1 = 0$$

$$z^2p^2(z^2+a^2) = 1$$

$$p^2 = \frac{1}{z^2(z^2+a^2)}$$

$$p = \frac{1}{z\sqrt{z^2+a^2}}$$

W.K.T, $dz = p\,dx + q\,dy$.

$$dz = \frac{1}{z(\sqrt{z^2+a^2})}\,dx + ap\,dy.$$

$$dz = \frac{1}{z\sqrt{z^2+a^2}}\,dx + \frac{a}{z\sqrt{z^2+a^2}}\,dy.$$

$z\sqrt{a^2+z^2}\,dz = dx + a\,dy.$

$\int$ing on both sides,

$$\int z\sqrt{a^2+z^2}\,dz = \int dx + a\int dy.$$

put $a^2+z^2 = t^2$

$$2z\,dz = 2t\,dt$$

$$z\,dz = t\,dt$$

$$\int \sqrt{t^2}\,(t\,dt) = \int dx + a\int dy$$

$$\int t^2 dt = \int dx + a\int dy$$

$$\frac{t^3}{3} = x + ay + b$$

$$(\sqrt{a^2+z^2})^3 = 3(x+ay+b)$$

$$(a^2+z^2)^{3/2} = 3(x+ay+b)$$

$$(a^2+z^2)^3 = 9(x+ay+b)^2$$

8. Find the complete integral of $xy + 3yq = 2(z - x^2q^2)$.

Solu:

We have $f(x,y,z,p,q) = xp + 3yq - 2(z - x^2q^2) = 0 \to (1)$

$$\Rightarrow xp + 3yq - 2z + 2x^2q^2 = 0$$

Here,

$f_x = p + 4xq^2$ $\quad f_y = 3q$ $\quad f_z = -2.$

$f_p = x$ $\quad f_q = 3y + 4x^2q$

Auxiliary Equ is

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}.$$

$$\frac{dp}{p+4xq^2+p(-2)} = \frac{dq}{3q+q(-2)} = \frac{dz}{-p(x)-q(3y+4x^2q)} = \frac{dx}{-x} = \frac{dy}{-(3y+4x^2q)}$$

$$\frac{dp}{4xq^2-p} = \frac{dq}{q} = \frac{dz}{-[px+3qy+4x^2q]} = \frac{dx}{-x} = \frac{dy}{-(3y+4x^2q)}$$

$$\frac{dq}{q} = \frac{dx}{-x}$$

$\int$ing on both sides,

$$\int\frac{dq}{q} = -\int\frac{dx}{x}$$

$$\log q = -\log x + \log a$$

$$\log q + \log x = \log a$$

$$\log(qx) = \log a$$

$$qx = a$$

$$q = a/x$$

putting $q = a/x$ in (1)

$$xp + 3y\,(a/x) - 2\left(z - x^2\left(\frac{a}{x}\right)^2\right) = 0.$$

$$xp = 2(z-a^2) - \frac{3ay}{x}$$

$$p = \frac{2(z-a^2)}{x} - \frac{3ay}{x^2}$$

W.K.T, $dz = p\,dx + q\,dy$

$$dz = \frac{2(z-a^2)}{x}dx - \frac{3ay}{x^2}dx + \frac{a}{x}dy$$

Multiplying by x^2 on both sides.

$$x^2 dz = \frac{2x^2(z-a^2)}{x} dx - \frac{3ay x^2}{x^2} dx + \frac{ax^2}{x} dy.$$

$$x^2 dz = 2x(z-a^2)dx - 3ay dx + ax dy$$

$$x^2 dz - 2x(z-a^2)dx = -3ay dx + ax dy.$$

Dividing by x^4 on both sides

$$\frac{x^2 dz - 2x(z-a^2)dx}{x^4} = \frac{-3ay dx + ax dy}{x^4}$$

$$\frac{x^2 dz - (z-a^2)\,2x dx}{(x^2)^2} = \frac{(-3ay dx + ax dy)}{x^4 \cdot x^2} x^2$$

$$d\left[\frac{z-a^2}{x^2}\right] = \frac{ax^3 dy - 3ay\, x^2 dx}{x^6}$$

$$d\left[\frac{z-a^2}{x^2}\right] = \frac{a[x^3 dy - y(3x^2 dx)]}{(x^3)^2}$$

$$d\left[\frac{z-a^2}{x^2}\right] = a \cdot d\left(\frac{y}{x^3}\right)$$

$\int$ing on both sides,

$$\int d\left[\frac{z-a^2}{x^2}\right] = a \int d\left(y/x^3\right)$$

$$\frac{z-a^2}{x^2} = ay/x^3 + b$$

$$\frac{z}{x^2} = \frac{a^2}{x^2} + \frac{ay}{x^3} + b.$$

$$z = [a^2 + a/x] + bx^2.$$