UNIT-1

CLAIRAUT'S EQUATION

A differential equation of the form

$$y = px + f(p) \ldots\ldots\ldots.(1)$$

Where $f(p)$ is a function of p alone (not containing x or y) is called Clairaut's equation.

Differentiating (1) with respect to x we get

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx}$$

$$\text{(i.e)} \ \left(x + f'(p)\right)\frac{dp}{dx} = 0$$

$$\text{(i.e)} \ \frac{dp}{dx} = 0 \quad \text{(or)} \quad \left(x + f'(p)\right) = 0$$

Case 1:

Consider $\frac{dp}{dx} = 0$

Integrating we get, $\int \frac{dp}{dx} dx = 0$ implies $p = c$

Substituting $p = c$ in (1), we get

$$y = cx + f(c)$$

This is the general solution of (1).

Case 2:

Consider $x + f'(p) = 0$

By eliminating p between equations (1) and (4) we get a solution of equation (1) which is free from any arbitrary constant and this solution can be obtained from (3) by giving a particular value for c. This solution is called the singular solution of (1).

Note:

The general solution to clairaut's equaton is obtained by replacing c for p in the clairaut's equation (1)

Example 1:

Find the general solution of $y = xp + \frac{\alpha}{p}$

Solution:

This equation is in clairaut's form

∴ the solution is got by replacing p by c

∴ the general solution is $y = cx + \frac{\alpha}{c}$

Example 2:

Solve $p = \tan(y - xp)$

Solution:

$$p = \tan(y - xp)$$

$$\tan^{-1} p = y - xp$$

$$y = xp + \tan^{-1} p$$

This is of clairaut's form

∴the general solution is $y = cx + \tan^{-1} c$

Example 3:

Solve $(y - px)(p - 1) = p$

Solution:

$$(y - px)(p - 1) = p$$

$$(y - px) = \frac{p}{p - 1}$$

$$y = px + \frac{p}{p - 1}$$

This is of clairaut's form

$\therefore$ the general solution is $y = cx + \frac{c}{c-1}$

i.e., $(y - cx)(c - 1) = c$

Example 4:

Solve $e^{3x}(p - 1) + p^3 e^{2y} = 0$

Solution:

$$e^{3x}(p - 1) + p^3 e^{2y} = 0 \cdots\cdots\cdots (1)$$

Put $X = e^x$, $Y = e^y$

$$\frac{dY}{dX} = \frac{e^y}{e^x}\frac{dy}{dx}$$

$P = \frac{e^y}{e^x} p$

$p = \frac{X}{Y} P \cdots\cdots\cdots (2)$

$\therefore$ (1) becomes $X^3\left(\frac{X}{Y}P - 1\right) + \frac{X^3}{Y^3}P^3 Y^2 = 0$

i.e., $\frac{X^3(XP-Y)}{Y} + \frac{X^3P^3}{Y} = 0$

$$\frac{X^3(XP - Y)}{Y} + \frac{X^3P^3}{Y} = 0$$

$(XP - Y) + P^3 = 0$

$Y = PX + P^3$

This is of clairaut's form

$\therefore$ the solution is $Y = cX + c^3$

$i.e., e^y = ce^x + c^3$

Example 5: solve $(px - y)(py + x) = 2p$

Solution:

$$(px - y)(py + x) = 2p \cdots\cdots\cdots (1)$$

Let $= x^2$, $Y = y^2$

Then

$$\frac{dY}{dX} = \frac{y}{x}\frac{dy}{dx}$$

$$P = \frac{y}{x}p$$

(1) becomes $\left(x\frac{x}{y}p - y\right)\left(y\frac{x}{y}p + x\right) = 2\frac{x}{y}p$

$$\frac{x^2p - y^2}{y}x(p+1) = 2\frac{x}{y}p$$

$$(XP - Y)(P+1) = 2p$$

$$XP - Y = \frac{2P}{P+1}$$

$$Y = XP - \frac{2P}{P+1}$$

This is of clairaut's form

∴the general solution is

$$Y = cX - \frac{2c}{c+1}$$

$$i.e., y^2 = cx^2 - \frac{2c}{c+1}$$