ANALYTICAL GEOMETRY

UNIT- 5

1. Condition for the plane lx + my + nz = 0 to touch the quadric cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$.

(B.Sc.1989)

Let (x_1, y_1, z_1) be the point of contact.

The tangent plane at (x_1, y_1, z_1) is

$x(a x_1 + hy_1 + gz_1) + y (hx_1 + by_1 + fz_1) + z (gx_1 + fy_1 + cz_1) = 0$.

This is identical with the plane lx + my + nz = 0.

$$\frac{ax_1 + hy_1 + gz_1}{l} = \frac{hx_1 + by_1 + fz_1}{m} = \frac{gx_1 + fy_1 + cz_1}{n}$$

If each ratio is k.

$ax_1 + hy_1 + gz_1 - kl = 0$ …(1)

$hx_1 + by_1 + fz_1 - km = 0$ …(2)

$gx_1 + fy_1 + cz_1 - kn = 0$ ... (3)

Since (x_1, y_1, z_1) lies on lx + my + nz = 0

Therefore $lx_1 + my_1 + nz_1 = 0$ ………(4)

Eliminating x_1, y_1, z_1 from equations (1), (2), (3) and (4), we get

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix} = 0.$$

Simplifying, we get

$Al^2 + Bm^2 + Cn^2 + 2F mn + 2G nl + 2H lm = 0$ ... (5)

where A, B, C, F, G, H are the cofactors of a, b, c, f, g, h in the

determinant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

Multiplying (1) by A, (2) by H and (3) by G and adding, we get

$\Delta x_1 = k\,(Al + Hm + Gn)$

since $\Delta = Aa + Hh + Gg$, $0 = Ah + Hb + Gf$, $0 = Ag + Hf + Gc$.

Similarly, $\Delta y_1 = k\,(Hl + Bm + Fn)$.

$\Delta z_1 = k\,(Gl + Fm + Cn)$.

Hence the point of contact is given by the equations

$$\frac{x_1}{Al + Hm + Gn} = \frac{y_1}{Hl + Bm + Fn} = \frac{z_1}{Gl + Fm + Cn}$$

From condition (5), it can be seen that $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

which is perpendicular to the plane $lx + my + nz = 0$ at the origin, is a generator of the cone

$$Ax^2 + By^2 + Cz^2 + 2F\,yz + 2G\,zx + 2H\,xy = 0 \quad \ldots\ldots (6)$$

In the determinant $\Delta' = \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$, we get

$A' = BC - F^2 = a\,\Delta'$, $\quad F' = GH - AF = f\Delta'$

$B' = CA - G^2 = b\Delta'$, $\quad G' = HF - BG = g\,\Delta'$

$G' = AB - H^2 = c\Delta'$, $\quad H' = FG - CH = h\Delta'$

Hence the perpendiculars to the tangent planes to the cone (6) generate the cone

$$A'x^2 + B'y^2 + C'z^2 + 2F'yz + 2G'zx + 2H'xy = 0$$

i.e., $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

The cones (6) and (7) are said to be reciprocal.

Example 1. Find the equations of the tangent planes to the cone $9x^2 - 4y^2 + 16z^2 = 0$ which contain the line $\frac{x}{32} = \frac{y}{72} = \frac{z}{27}$

The line is the intersection of the planes

$72x - 32y = 0$, i.e., $9x - 4y = 0$.

and $27y - 72z = 0$, i.e., $3y - 8z = 0$.

Hence any plane passing through this line is of the form

9x-4y+ λ(3y- 8z) 0   9

ie., 9x+ y (3 λ -4)-8 λ z = 0. ... (1)

This line touches the cone

9x² - 4y²+ 16z² = 0 …. (2)

Hence the normal to the plane

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

$$\frac{x}{9} = \frac{y}{3\lambda - 4} = \frac{z}{-8\lambda}$$

is a generator of the reciprocal cone of the cone (2).

Equation of the reciprocal cone of (2) is

$$\frac{x^2}{9} - \frac{y^2}{4} + \frac{z^2}{16} = 0 \ldots\ldots (4)$$

(3) is a generator of cone (4).

$$\frac{9^2}{9} - \frac{(3\lambda - 4)^2}{4} + \frac{(-8\lambda)^2}{16} = 0$$

$$9 - \frac{9\lambda^2 - 24\lambda + 16}{4} + \frac{64\lambda^2}{16} = 0$$

$$9 - \frac{9\lambda^2 - 24\lambda + 16}{4} + 4\lambda^2 = 0$$

$$\frac{36 - 9\lambda^2 + 24\lambda - 16 + 16\lambda^2}{4} = 0$$

$$7\lambda^2 + 24\lambda + 20 = 0$$

Simplifying, we get $7\lambda^2 + 24\lambda + 20 = 0$

ie., $\lambda = -2 \ or \ {}^{-10}/_{7}$

Hence the equations of the planes are

$\lambda = -2$, (1) gives

9x-10y + 16z = 0

and $\lambda = {}^{-10}/_{7}$, (1) gives

$$63x - 58y + 80z = 0.$$

Example 2. Find the general equation to a cone which touches the co-ordinate planes.

If the co-ordinate planes touch a cone, the perpendiculars to co-ordinate planes touch the reciprocal cone.

Hence the cone touching the co-ordinate planes is reciprocal to the cone passing through the c0-ordinate axes.

The direction cosines of the co-ordinate axes are 1,0,0; 0,1,0; 0,0,1.

The equation of the cone passing through the axis is of the form 2fyz + 2gzx+ 2hxy = 0.

The required cone is the reciprocal cone of this cone and its equation is $f^2x^2+g^2y^2+h^2z^2-2ghyz- 2hfzx- 2fgxy= 0$.

This equation can be put in the form $\sqrt{fx}+ \sqrt{gy} + \sqrt{hz} = 0$.

2. The angle between the lines in which the plane ux + vy + wz = 0 cuts the cone.

$f(x,y,z)\equiv ax^2 +by^2+ cz^2+ 2fyz + 2gzx + 2hxy = 0.$

The plane meets the cone in two lines which pass through the origin and so the equations of the lines are of the form $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$.

The line lies in the plane and in the cone. Therefore,

$ul + vm + wn = 0$ ... (1) and

$al^2 +bm^2+ cn^2+ 2fyz + 2fmn + 2gnl + 2hlm = 0$ ... (2)

Eliminating n between (1) and (2), we get

$(1) \Rightarrow n = \frac{\text{ul + vm}}{w}$

$(2) \Rightarrow al^2 + bm^2 + c(-\frac{ul+vm}{w})^2 + 2fyz + 2fm\left(-\frac{ul+vm}{w}\right) +$
$2g\left(-\frac{ul+vm}{w}\right)l + 2hlm = 0$

$$\Rightarrow al^2 + bm^2 + c\left(\frac{u^2l^2 + 2uvlm + v^2m^2}{w^2}\right) - \left(\frac{2fulm + 2fvm^2}{w}\right) - \left(\frac{2gul^2 + 2gvlm}{w}\right) + 2hlm = 0$$

$$\Rightarrow al^2w^2 + bm^2w^2 + cu^2l^2 + 2cuvlm + cv^2m^2 + 2fyzw^2 - 2fuwlm - 2fvwm^2 - 2guwl^2 - 2gvwlm + 2hlmw^2 = 0$$

$$\Rightarrow l^2(cu^2 + aw^2 - 2guw) + 2lm(hw^2 + cuv - fuw - gvw) + m^2(cv^2 + bw^2 - 2fvw) = 0 \ldots (3)$$

The direction cosines of the two lines satisfy the equation (3) and if they are l_1, m_1, n_1 and l_2, m_2, n_2, we have

$$\frac{l}{m} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$\frac{l}{m} = \frac{-(hw^2 + cuv - fuw - gvw) \pm \sqrt{(hw^2 + cuv - fuw - gvw)^2 - 4(cu^2 + aw^2 - 2guw)(cv^2 + bw^2 - 2fvw)}}{2(cu^2 + aw^2 - 2guw)}$$

$$\frac{l_1}{m_1} = \frac{-(hw^2 + cuv - fuw - gvw) + \sqrt{\begin{array}{c}(hw^2 + cuv - fuw - gvw)^2 \\ -4(cu^2 + aw^2 - 2guw)(cv^2 + bw^2 - 2fvw)\end{array}}}{2(cu^2 + aw^2 - 2guw)}$$

$$\frac{l_2}{m_2} = \frac{-(hw^2 + cuv - fuw - gvw) - \sqrt{\begin{array}{c}(hw^2 + cuv - fuw - gvw)^2 \\ -4(cu^2 + aw^2 - 2guw)(cv^2 + bw^2 - 2fvw)\end{array}}}{2(cu^2 + aw^2 - 2guw)}$$

$$\frac{l_1}{m_1} - \frac{l_2}{m_2} = \frac{-(hw^2 + cuv - fuw - gvw) + \sqrt{\begin{array}{c}(hw^2 + cuv - fuw - gvw)^2 \\ -4(cu^2 + aw^2 - 2guw)(cv^2 + bw^2 - 2fvw)\end{array}}}{2(cu^2 + aw^2 - 2guw)} - \frac{-(hw^2 + cuv - fuw - gvw) - \sqrt{\begin{array}{c}(hw^2 + cuv - fuw - gvw)^2 \\ -4(cu^2 + aw^2 - 2guw)(cv^2 + bw^2 - 2fvw)\end{array}}}{2(cu^2 + aw^2 - 2guw)}$$

$$\frac{l_1 m_2 - m_1 l_2}{m_1 m_2} = \frac{2\sqrt{\begin{array}{c}(hw^2 + cuv - fuw - gvw)^2 \\ -4(cu^2 + aw^2 - 2guw)(cv^2 + bw^2 - 2fvw)\end{array}}}{2(cu^2 + aw^2 - 2guw)}$$

$$\frac{l_1 m_2 - m_1 l_2}{m_1 m_2} = \frac{\sqrt{\begin{array}{c}(hw^2 + cuv - fuw - gvw)^2 \\ -4(cu^2 + aw^2 - 2guw)(cv^2 + bw^2 - 2fvw)\end{array}}}{(cu^2 + aw^2 - 2guw)}$$

$$\frac{l_1m_2 - m_1l_2}{\sqrt{\begin{matrix}(hw^2 + cuv - fuw - gvw)^2 \\ -4(cu^2 + aw^2 - 2guw)(cv^2 + bw^2 - 2fvw)\end{matrix}}} = \frac{m_1m_2}{(cu^2 + aw^2 - 2guw)}$$

Sum of the roots

$$\frac{l_1}{m_1} + \frac{l_2}{m_2} = -\frac{2(hw^2 + cuv - fwv - gvw)}{cu^2 + aw^2 - 2guw}$$

$$\frac{l_1m_2 + l_2m_1}{m_1m_2} = -\frac{2(hw^2 + cuv - fwv - gvw)}{cu^2 + aw^2 - 2guw}$$

$$\frac{l_1m_2 + l_2m_1}{-2(hw^2 + cuv - fwv - gvw)} = \frac{m_1m_2}{cu^2 + aw^2 - 2guw}$$

Product of the roots

$$\frac{l_1}{m_1}\frac{l_2}{m_2} = \frac{cv^2 + bw^2 - 2fvw}{cu^2 + aw^2 - 2guw}$$

$$\frac{l_1l_2}{cv^2 + bw^2 - 2fvw} = \frac{m_1m_2}{cu^2 + aw^2 - 2guw} = \frac{l_1m_2 + l_2m_1}{-2(hw^2 + cuv - fwv - gvw)}$$

$$= \frac{l_1m_2 - l_2m_1}{\sqrt{\begin{matrix}(hw^2 + cuv - fuw - gvw)^2 \\ -4(cu^2 + aw^2 - 2guw)(cv^2 + bw^2 - 2fvw)\end{matrix}}}$$

$$= \frac{l_1m_2 - l_2m_1}{\sqrt{\pm 2w(-Au^2 - Bv^2 - Cw^2 - 2Fvw - 2Gwu - 2Huv)}}$$

$$\frac{l_1l_2}{cv^2 + bw^2 - 2fvw} = \frac{m_1m_2}{cu^2 + aw^2 - 2guw} = \frac{l_1m_2 - l_2m_1}{\pm 2wP} \quad \dots\dots (4)$$

Where $P = -(Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv)$ and A, B, C, F, G, H are the cofactors of a, b, c, f, g, h in the determinant $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

From the symmetry, we get the expression in (4) is equal to

$$\frac{n_1n_2}{av^2 + bu^2 - 2huv} = \frac{m_1n_2 - m_2n_1}{\pm 2uP} = \frac{n_1l_2 - l_1n_2}{\pm 2vP} \ldots\ldots (5)$$

Each expression in (4) and (5) is

$$\frac{l_1l_2 + m_1m_2 + n_1n_2}{cv^2 + bw^2 - 2fvw + cu^2 + aw^2 - 2guw + av^2 + bu^2 - 2huv}$$

$$= \frac{\sqrt{\sum(m_1n_2 - m_2n_1)^2}}{\pm 2\sqrt{(u^2 + v^2 + w^2)P}}$$

If θ is the angle between the lines

$$\frac{cos\theta}{l_1l_2 + m_1m_2 + n_1n_2} = \frac{sin\theta}{\sqrt{\sum(m_1n_2 - m_2n_1)^2}}$$

$$\frac{cos\theta}{(a + b + c)(u^2 + v^2 + w^2) - f(u, v, w)} = \frac{sin\theta}{\pm 2\sqrt{(u^2 + v^2 + w^2)P}} \quad \ldots (6)$$

3. Condition that the cone has three mutually perpendicular generators.

The condition that the plane should cut the cone in perpendicular generators is that θ = 90°. In that case by (6) of the previous article

$$(a + b + c)(u^2 + v^2 + w^2) = f(u, v, w).$$

The third generator is perpendicular to these two generators. Hence it is normal to the plane containing these perpendicular generators. If the normal to the plane ux + vy + wz = 0 lies on the cone, we have f (u, v, w) = 0.

∴ a+ b+ c = 0.

Example. ***Find the equation to the cone through the co-ordinate axes and the lines in which the plane lx + my+ nz = 0 cuts***
$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$.

Let the equation of the cone passing through the co-ordinates axes by

Fyz + Gzx+ Hxy = 0.

Eliminating z between lx+ my + nz = 0 and

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$, we get

$$z = \frac{-(lx + my)}{n}$$

$\Rightarrow ax^2 + by^2 + c\left(\frac{-(lx+my)}{n}\right)^2 + 2fy\left(\frac{-(lx+my)}{n}\right) + 2g\left(\frac{-(lx+my)}{n}\right)x + 2hxy = 0$

$\Rightarrow ax^2 + by^2 + c\left(\frac{l^2x^2+m^2y^2+2lmxy}{n^2}\right) - \frac{2flxy}{n} - \frac{2fmy^2}{n} - \frac{2glx^2}{n} - \frac{2gmxy}{n} + 2hxy = 0$

$\Rightarrow an^2x^2 + bn^2y^2 + c(l^2x^2 + m^2y^2 + 2lmxy) - 2flnxy - 2fmny^2 - 2glnx^2 - 2gmnxy + 2hnxy = 0$

$\Rightarrow \quad x^2(an^2 + cl^2 - 2gln) + xy(2clmxy - 2fln - 2gmn + 2hn) + y^2(bn^2 + cm^2 - 2fmn) = 0$

Similarly eliminate z between lx+ my + nz = 0 and Fyz + Gzx+ Hxy = 0.

$Fy\left(\frac{-(lx+my)}{n}\right) + G\left(\frac{-(lx+my)}{n}\right)x + Hxy = 0.$

$-Flxy - Fmy^2 - Glx^2 - Gmxy + Hnxy = 0$

$Flxy + Fmy^2 + Glx^2 + Gmxy - Hnxy = 0$

$Glx^2 + (Fl + Gm - Hn)xy + Fmy^2 = 0$

Since the two cones have common generators, we get

$$\frac{an^2 + cl^2 - 2gln}{Gl} = \frac{bn^2 + cm^2 - 2fmn}{Fm}$$

Similarly eliminating x, we get the condition

$$\frac{an^2 + cl^2 - 2gnl}{Gn} = \frac{bl^2 + am^2 - 2hml}{Hm}$$

Dividing by l on both sides,

$$\frac{an^2 + cl^2 - 2gln}{Gnl} = \frac{bl^2 + am^2 - 2hml}{Hlm}$$

$$= \frac{bn^2 + cm^2 - 2fmn}{Fmn}$$

Hence the equation of the required cone is

$$l(bn^2 + cm^2 - 2fmn)yz + m(an^2 + cl^2 - 2gln)zx + n(bl^2 + am^2 - 2hml)xy = 0.$$