KINEMATICS

3.1. Speed:

The speed of a moving point is the rate at which it describes its path without any reference to its direction of motion. Thus, speed is a quantity having magnitude only but no direction. Hence it is a scalar.

The speed of a particle is said to be *uniform* when it describes equal lengths of its path in equal intervals of time, however small these equal time intervals may be. When a particle is moving with uniform speed, its speed can be got by measuring the distance travelled in one unit of time.

The *average speed* of a particle in any time interval is got by dividing the distance travelled in that time interval by the time interval. For instance, when we say that the speed of a train is 30 km/h, it means that it would describe 30 kms. in an hour, if its speed remained constant during that hour. In other words, the average speed of the train is 30 kms. per hour..

The speed of a particle at any instant is given by the ratio of the distance described by it in a very short interval of time including that instant to the interval, when the interval is made sufficiently small.

Using the notation of Differential Calculus, let s be the distance travelled in time t and $s + \Delta s$ be the distance travelled in time $t + \Delta t$. The distance travelled in the time interval $\Delta t = \Delta s$. Hence the average speed of the particle during time $\Delta t = \frac{\Delta s}{\Delta t}$. By making Δt sufficiently small, the fraction $\frac{\Delta s}{\Delta t}$ will give approximately the speed at the instant t.

Hence the speed at time t is given by $\underset{\Delta t \to 0}{Lt} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$.

In the F.P.S. system, the unit of speed is 1 foot per second, abbreviated to 1 ft/sec or 1 ft. per sec. In the C.G.S. system, the unit of speed is 1 centimetre per second, abbreviated to 1 cm/sec or 1 cm.per sec. In the M.K.S. system, the unit of speed is 1 metre per second abbreviated to 1 m/sec.

Speeds are often expressed in other units, such as miles per hour (abbreviated to m.p.h.) and kilometres per hour (abbreviated to km/h). The unit of speed used in navigation is the knot which means the speed of 1 nautical mile (6080 ft.) per hour.

3.2. Displacement:

The displacement of a moving point in any interval of time is its change of position. If O and P are the initial and final positions of a particle in its path in a certain time interval, then its displacement is represented by $\overrightarrow{OP}$. Thus displacement of a moving point is a vector.

§ 3.3. Velocity:

The *velocity of a moving point is the rate of its displacement.* A velocity therefore has both magnitude and direction and is a vector quantity.

A point is said to be moving with *uniform velocity*, if it moves always in the same direction and describes equal distances in equal intervals of time, however small these intervals may be. When uniform, the velocity, of a moving point is measured by its displacement per unit of time.

When there is a change in the magnitude or in the direction of a moving particle, its velocity is said to be variable. The velocity of a particle at any instant may be defined as follows:

The velocity of a particle at any instant is given by the ratio of the displacement described by it in a very short interval of time including that instant, to the interval when the interval is made sufficiently small.

Let a point describe a length s (measured from a fixed point on its path) in time t and $s + \Delta s$ in time $t + \Delta t$. The average velocity during Δt is $\frac{\Delta s}{\Delta t}$. The magnitude of the velocity at time t

$$= \underset{\Delta t \to 0}{Lt} \frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$ and its direction is along the tangent to the path.

When a point is moving in a straight line, its velocity is the same as its speed. But in the case of all other paths, the velocity is not the same as the speed. For instance, suppose a point to be describing a circle uniformly, so that it describes equal lengths of arc in equal intervals of time however small. Clearly its speed is constant. But its direction of motion, namely the tangent to the circle is different at different points of the circumference. Hence the velocity of the point is not constant. It is variable.

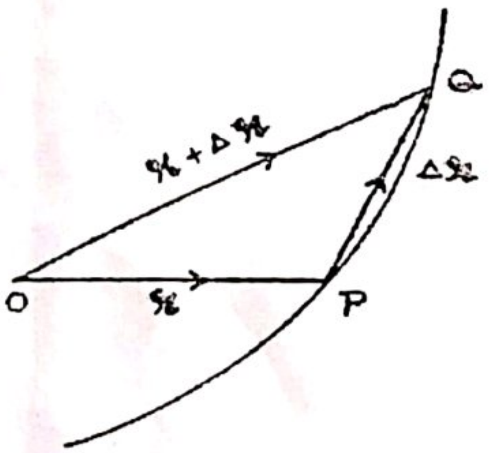


Fig. 11

Consider a particle moving on a curve. Suppose in time t it is at a point P whose position vector is r and at time $t + \Delta t$ let it be at Q whose position vector is $\mathbf{r} + \Delta\mathbf{r}$.

Since $\vec{OP} + \vec{PQ} = \vec{OQ}$,

$$\vec{PQ} = \vec{OQ} - \vec{OP}$$

$$= r + \Delta r - r = \Delta \mathbf{r}$$

$\therefore$ Velocity at P $= \underset{\Delta t \to 0}{Lt} \frac{\Delta r}{\Delta t}$

$$= \frac{dr}{dt} = v \text{ (say)}$$

As $\Delta t \to 0$, $\Delta\mathbf{r}$ also $\to 0$ and the chord QP of the curve becomes the tangent at P. Hence *the velocity vector v of a moving point P is the time derivative of the position vector r and it is tangential to the curve at P.*

A particle may possess simultaneously more than one velocity. A simplest example of this is when a man walks on the deck of a moving ship from one point of the deck to another. The man shares the motion of the ship and so has the velocity of the ship. In addition, he has his own velocity. So his motion in space will be entirely different from what it would have been either in the case when the ship had remained at rest or in the case if he had stayed at the same original position of the ship.

Thus we can think of a number of simultaneous motions of the same particle. In such cases, we can always find a single velocity which produces the same effect on the particle as the different velocities. This single velocity is called the *resultant* of the given simultaneous velocities which, in their turn, are called the *components* of the single resultant. The process of finding the resultant velocity is called *composition of velocities.*

§ 3.4. Composition of velocities: Parallelogram Law:

Since velocity is a vector quantity, the method for composition of two or more velocities of a particle is the same as the rule for addition of vectors. Thus if a particle has simultaneously two velocities in directions inclined to each other, the resultant velocity is obtained by applying the Parallelogram Law of Velocities:

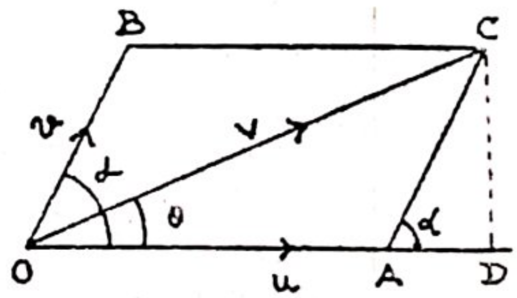


Fig. 12

If a moving point has simultaneously two velocities which are represented in magnitude and direction by the two sides of a parallelogram drawn from a point, the resultant velocity is represented in magnitude and direction by the diagonal of the parallelogram drawn from that point.

Let $\vec{OA}$ and $\vec{OB}$ represent the two velocities u and v. Complete the parallelogram AOBC. By the vector law of addition, $\vec{OA} + \vec{OB} = \vec{OC}$. Hence $\vec{OC}$ is the resultant velocity.

Analytical results:

Let $\angle AOB = \alpha$ and V the magnitude of the resultant velocity.

Draw CD $\perp$ to OA.

From rt $\angle d\Delta CAD$, $AD = AC.\cos\alpha = v\cos\alpha$ and

$CD = AC.\sin\alpha = v\sin\alpha$

From rt. $\angle d\ \Delta$ COD,

$$
\begin{aligned}
OC^2 &= OD^2 + CD^2 \\
&= (OA+AD)^2 + CD^2 \\
&= OA^2 + 2OA.AD + (AD^2+CD^2) \\
&= OA^2 + 2OA.AD + AC^2 \\
&= u^2 + 2uv\cos\alpha + v^2
\end{aligned}
$$

i.e. $V = \sqrt{u^2 + 2uv\cos\alpha + v^2}$ (1)

Also $\tan\theta = \dfrac{CD}{OD} = \dfrac{CD}{OA + AD} = \dfrac{v\sin\alpha}{u + v\cos\alpha}$ . . . (2)

(1) gives the magnitude V and (2) the direction θ of the resultant velocity.

Particular Cases:

(i) If $\alpha = 0$, $V = \sqrt{u^2 + 2uv + v^2} = u + v$ and $\theta = 0$

i.e. *The resultant of two simultaneous velocities along the same line and in the same directions is their sum.*

(ii) If $\alpha = \pi$, $V = \sqrt{u^2 - 2uv + v^2} = u \sim v$ and $\theta = 0$

i.e. *The resultant of two simultaneous velocities along the same line but in opposite directions is their algebraic sum.*

(iii) If $\alpha = \dfrac{\pi}{2}$, $V = \sqrt{u^2 + v^2}$ and $\theta = \tan^{-1}\dfrac{v}{u}$

(iv) When $v = u$, $V = \sqrt{u^2 + 2u^2\cos\alpha + u^2}$

$$= \sqrt{2u^2(1 + \cos\alpha)}$$

$$= \sqrt{(2u^2 \,.\, 2\cos^2\frac{\alpha}{2})} = 2u\cos\frac{\alpha}{2}$$

$$\tan\theta = \frac{u\sin\alpha}{u + u\cos\alpha} = \frac{\sin\alpha}{1+\cos\alpha} = \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}} = \tan\frac{\alpha}{2}$$

i.e. $\theta = \frac{\alpha}{2}$

So the resultant of two equal velocities, u, u at an angle α *is a velocity* $2u\cos\frac{\alpha}{2}$ *in a direction bisecting the angle between them.*

§ 3.5. Resolution of Velocities:

We can use the parallelogram law to resolve a given velocity into two component velocities. It is clear that this can be done in an infinite number of ways, for an infinite number of parallelograms can be described having a given line OC as diagonal; (see fig.12 page 17) If AOBC is any one of these, the velocity OC is equivalent to the two component velocities OA and OB.

The most important case of resolution of a velocity occurs when a given velocity is to be resolved in two directions at right angles, one of these directions being given. In this case, the magnitudes of the component velocities are easily got as follows:

Let OC represent the given velocity u and OX be a line inclined at an angle θ to OC. Let OY be perpendicular to OX. Draw CA ⊥ to OX and complete the parallelogram OACB. Then the velocity OC is equivalent to the two component velocities OA and OB.

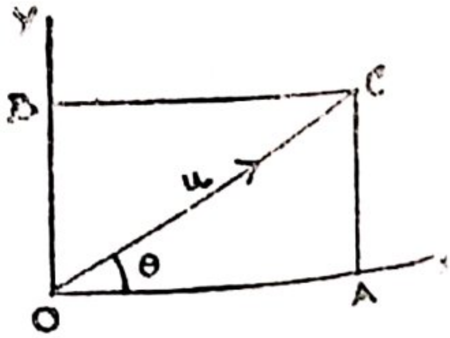


Fig. 13

Also OA = OC.cos θ = $u\cos\theta$

and OB = AC = OC.sin θ = $u\sin\theta$

Hence we have the following important proposition:

A velocity u is equivalent to a velocity u cos θ along a line making an angle θ with its own direction, together with a velocity u sin θ perpendicular to the direction of the first component.

When a given velocity is resolved into two components in two mutually perpendicular directions, the components are *referred* to as the *resolved parts* in the corresponding directions.

§ 3.6. Components of a velocity along two given directions:

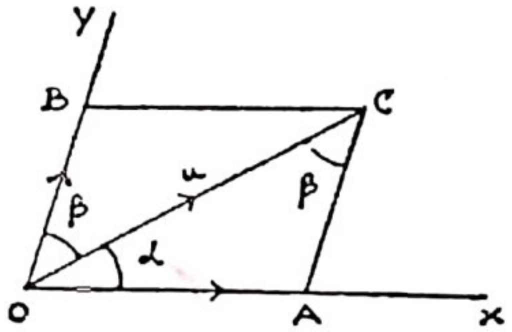


Fig. 14

Let OC represent a given velocity u and OX, OY be two lines making angles α and β with OC. Draw CA parallel to OY and CB parallel to OX, making the parallelogram OACB as shown in fig.14. Then OA and OB are the components of the velocity OC along OX and OY respectively.

From Δ OAC,

$$\frac{OA}{\sin \angle OCA} = \frac{AC}{\angle \sin AOC} = \frac{OC}{\angle \sin OAC}$$

i.e. $$\frac{OA}{\sin \beta} = \frac{AC}{\sin \alpha} = \frac{OC}{\sin \{180^\circ - (\alpha + \beta)\}}$$

i.e. $$\frac{OA}{\sin \beta} = \frac{AC}{\sin \alpha} = \frac{u}{\sin (\alpha + \beta)}$$

$$\therefore OA = \frac{u \sin \beta}{\sin (\alpha + \beta)}$$

and $$OB = AC = \frac{u \sin \alpha}{\sin (\alpha + \beta)}$$

§ 3.7. Triangle of Velocities: Theorem.

If a moving point possesses simultaneously two velocities represented in magnitude and direction successively by the two *sides of a triangle taken in order, their resultant will be represented in magnitude and direction by the third side taken in the reverse order.*

Let AB and BC represent the two velocities u and v in magnitude and direction. Complete the parallelogram ABCD.

Since AD is equal and parallel to BC, it represents the same velocity as BC. Hence the two simultaneous velocities u and v of the moving point are represented in magnitude and direction by AB and AD.

So, by parallelogram law, their resultant velocity is represented by the diagonal AC.

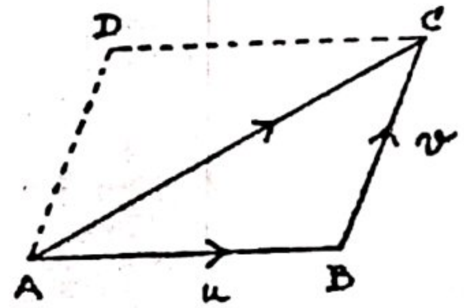


Fig. 15

In vector notation, we write this as $\vec{AB} + \vec{BC} = \vec{AC}$

Corollary 1. If a moving point simultaneously three velocities which are represented in magnitude and direction by the three sides of a triangle taken in order, it will be at rest.

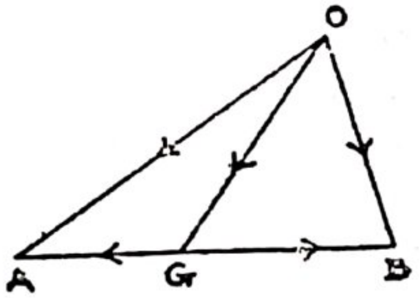


Fig. 16

Corollary 2. If a moving point possesses two velocities represented by λ. OA and μ. OB (λ, μ being constants), then their resultant velocity is represented by (λ + μ). OG, where G is a point on AB such that λAG = μGB.

From Δ OAG, $\vec{OG} + \vec{GA} = \vec{OA}$ ... (1)

Multiplying (1) by λ,

$\lambda . \vec{OG} + \lambda . \vec{GA} = \lambda . \vec{OA}$ ... (2)

From Δ OBG, $\vec{OG} + GB = \vec{OB}$ ... (3)

Multiplying (3) by μ,

$\mu . \vec{OG} + \mu . \vec{GB} = \mu . \vec{OB}$ ... (4)

Adding (2) and (4)

$(\lambda + \mu)\vec{OG} + \lambda \vec{GA} + \mu \vec{GB} = \lambda \vec{OA} + \mu \vec{OB}$ ... (5)

But the velocities $\lambda\,\vec{GA}$ and $\mu\,\vec{GB}$ are equal in magnitude but opposite in direction. They destroy one another.

$\therefore$ (5) becomes $(\lambda + \mu)\,\vec{OG} = \lambda\,\vec{OA} + \mu\,\vec{OB}$ ... (6)

Corollary 3. Put $\lambda = 1 = \mu$ in the above result.

then AG = GB and G is the midpoint of AB.

$\therefore \vec{OA} + \vec{OB} = 2\vec{OG}$

3.8. Polygon of Velocities : Theorem

If a moving point possesses simultaneously velocities represented by the sides AB, BC, CD. . .MN of a polygon ABCD. . . MN taken in order, the resultant velocity is represented by AN.

$\vec{AB} + \vec{BC} = \vec{AC}$ from ΔABC

$\therefore \vec{AB} + \vec{BC} + \vec{CD} = \vec{AC} + \vec{CD}$

$= \vec{AD}$ from Δ ACD

Thus $\vec{AB} + \vec{BC} + \vec{CD} + ... + \vec{MN} = \vec{AN}$

It is obvious that this result also holds if the sides of the polygon are not in one plane.

Corollary: If a moving point has simultaneously several velocities which are represented in magnitude and direction by the sides of a closed polygon taken in order, it will be at rest.

§ 3.9. Resultant of several simultaneous coplanar velocities of a particle:

Let a point O have several simultaneous velocities represented by vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$...etc. in directions inclined at angles θ_1, θ_2, θ_3 ... to a fixed line OX and let OY be $\perp$ to OX.

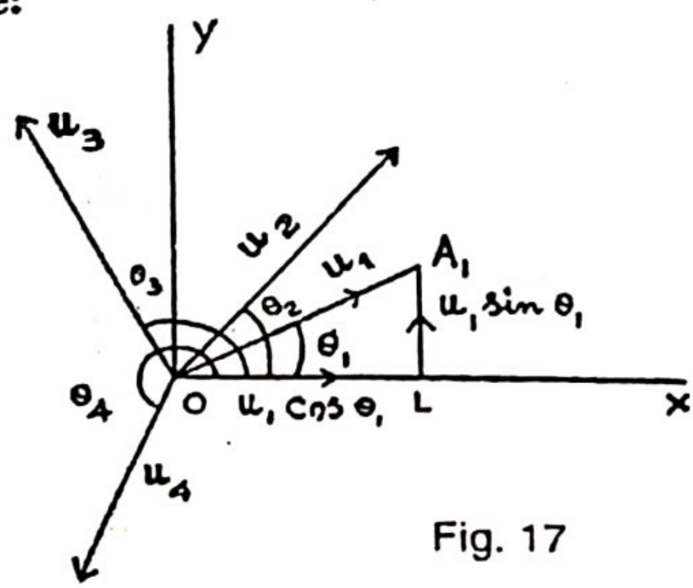


Fig. 17

Let **i** and **j** be unit vectors along OX and OY and $\vec{OA_1} = \mathbf{u}_1$

From A_1, draw $A_1L \perp$ to OX.

$\mathbf{u}_1 = \vec{OA_1} = \vec{OL} + \vec{LA_1} = \cos\theta_1 \mathbf{i} + u_1 \sin\theta_1 \mathbf{j}$.

Similarly $\mathbf{u}_2 = u_2 \cos\theta_2 \mathbf{i} + u_2 \sin\theta_2 \mathbf{j}$ and so on.

Let V be the vector representing the resultant velocity.

$$\mathbf{V} = \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 + \ldots..$$

$$= (u_1 \cos\theta_1 \mathbf{i} + u_1 \sin\theta_1 \mathbf{j}) + (u_2 \cos\theta_2 \mathbf{i} + u_2 \sin\theta_2 \mathbf{j}) + \ldots.$$

$$= (u_1 \cos\theta_1 + u_2 \cos\theta_2 + \ldots)\mathbf{i} + (u_1 \sin\theta_1 + u_2 \sin\theta_2 \ldots)\mathbf{j}$$

Magnitude of the resultant velocity

$$V = \sqrt{(u_1 \cos\theta_1 + u_2 \cos\theta_2 + \ldots)^2 + (u_1 \sin\theta_1 + u_2 \sin\theta_2 \ldots)^2} \quad \ldots (1)$$

If the vector V makes and angle θ with OX,

$$\tan\theta = \frac{\mathbf{j} - \text{component of } \mathbf{V}}{\mathbf{i} - \text{component of } \mathbf{V}} = \frac{u_1 \sin\theta_1 + u_2 \sin\theta_2 + \ldots}{u_1 \cos\theta_1 + u_2 \cos\theta_2 + \ldots} \quad \ldots (2)$$

Equations (1) and (2) give the magnitude and direction of the resultant.

WORKED EXAMPLES

Ex.1. *A boat capable of moving in still water with a speed of 5½kms. an hour, crosses a river, ½ km broad, flowing with a velocity of 3/2 kms. an hour. Find (i) the time of crossing by the shortest route (ii) the minimum time of crossing.*

Let O be the position of the boat on one bank and E the directly opposite point on the other bank. OE is the shortest route. To cross by the shortest route, the resultant velocity of the boat and the current must be along OE.

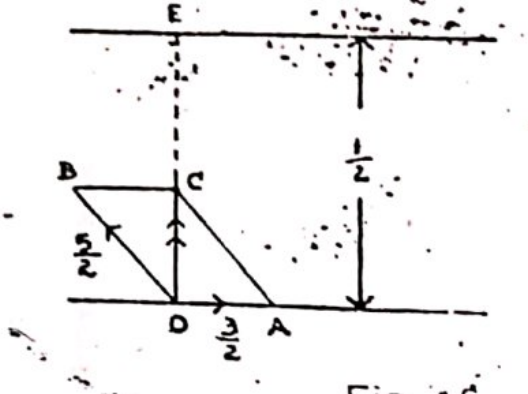


Fig. 18

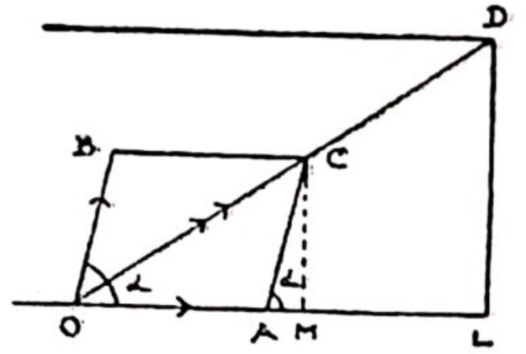


Fig. 19

In fig.18, let $\vec{OA}$ = velocity of the current = $\frac{3}{2}$

$\vec{OB}$ = velocity of the boat = $\frac{5}{2}$

The resultant velocity is $\vec{OC}$, the diagonal of the parallelogram AOB and it is perpendicular to OA.

From rightangled ΔCOA,

$$OC^2 = CA^2 - OA^2 = \left(\frac{5}{2}\right)^2 - \left(\frac{3}{2}\right)^2 = 4; \quad \therefore\ OC = 2$$

i.e. The resultant velocity of the boat is 2 km/h. Hence the time of crossing by the shortest route,

$$= \frac{OE}{2} = \frac{(1/2)}{2} = \frac{1}{4} \text{ hour} = 15 \text{ minutes.}$$

In fig.19, let $\vec{OA}$ = velocity of current = $\frac{3}{2}$ and the boat be steered in the direction OB making an angle α with OA.

Let $\vec{OB}$ = velocity of boat = $\frac{5}{2}$

Complete the parallelogram AOB and the resultant velocity is along the diagonal OC.

Produce OC to meet the opposite bank at D.

Then OD is the length of the path described by the boat with the resultant velocity represented by OC.

Draw DL and CM $\perp$ to OA.

$$\text{Time of crossing} = \frac{\text{Length of the path}}{\text{Resultant velocity}} = \frac{OD}{OC}$$

$$= \frac{DL}{CM} \quad (|||\ \Delta s\ ODL\ \&\ OCM)$$

$$= \frac{(1/2)}{AC \cdot \sin\alpha} = \frac{(1/2)}{(5/2)\sin\alpha} = \frac{1}{5\sin\alpha}$$

Clearly this time of crossing is least, when $\sin\alpha$ is greatest. i.e. when $\sin\alpha = 1$ or $\alpha = 90^\circ$.

Minimum time of crossing $= \frac{1}{5}$ hour $= 12$ minutes.

Ex.2. *A particle has two simultaneous velocities of equal magnitudes in two directions. If one of them is halved in magnitude, the angle which the resultant velocity makes with the other is halved also. Find the angle between the directions. (B.Sc. 52 Madras Uty.)*

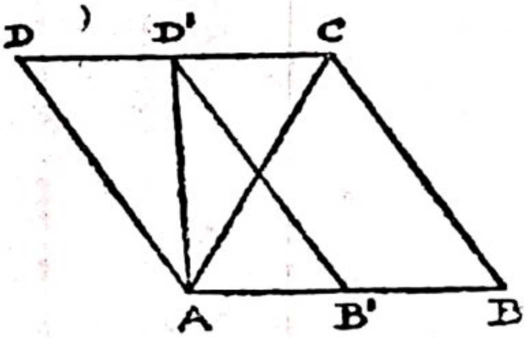


Fig. 20

Let the velocitites of equal magnitudes but in different directions be represented by the sides AB, AD of || gm. BADC. Then AB = AD and the resultant velocity will be along the diagonal AC, bisecting ∠BAD. (Refer particular case iv on page 18)

In the second case, one of the velocities is halved in magnitude. Let B' be the midpoint of AB. Now the particle has two velocities represented by AB' and AD. Complete the || gm $B'AD$. The diagonal AD' will be the new resultant. Clearly D' is the midpoint of DC. It is given that the angle between AD' and AD $= \frac{1}{2} \times$ angle between AC and AD.

$\therefore$ AD' bisects ∠CAD.

go from A to another point C of its own enbankment and return to A if $AC = a\sqrt{1 - \frac{v^2}{u^2}}$

§ 3.10. Relative Velocity:

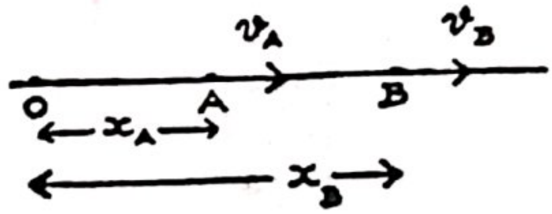


Fig. 22

Let two particles A and B move along the same straight line and at time t their displacements measured from some fixed origin O on the line be x_A and x_B respectively. The velocities of A and B are

$$v_A = \frac{dx_A}{dt} \text{ and } v_B = \frac{dx_B}{dt}$$

The displacement of B relative to A (i.e. displacement of B as measured from A)

$= x_B - x_A$ and the rate of this displacement is called the velocity of B relative to A.

∴ The velocity of B relative to A

$$= \frac{d}{dt}\left(x_B - x_A\right) = \frac{dx_B}{dt} - \frac{dx_A}{dt} = v_B - v_A$$

Clearly this is the velocity which B appears to have as seen from A.

The above idea of one-dimensional relative motion can be easily extended to motion in two dimensions.

Let r_A and r_B be the position vectors at time t of two moving particles with respect to a fixed origin O. The velocities v_A and v_B are then given by

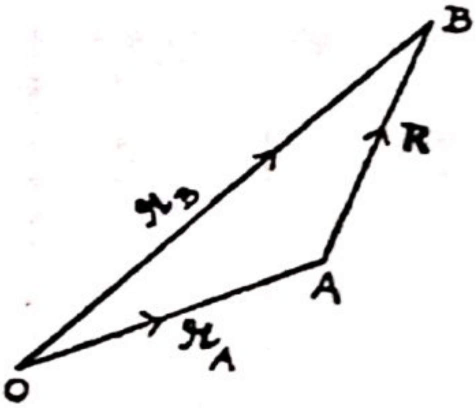


Fig. 23

$$v_A = \frac{dr_A}{dt} \text{ and } v_B = \frac{dr_B}{dt}$$

By the triangle law,

$\vec{OA} + \vec{AB} = \vec{OB}$

$\therefore \vec{AB} = \vec{OB} - \vec{OA} = r_B - r_A = R$ (say)

$\vec{AB}$ is the displacement of B relative to A and R is the position vector of B relative to A. The velocity v of B relative to A is defined as the time derivative of R.

$$\therefore\ v = \frac{dR}{dt} = \frac{dr_B}{dt} - \frac{dr_A}{dt} = v_B - v_A = v_B + (-v_A) \quad ...(1)$$

(1) shows that the relative velocity of two moving points is the vector difference of their absolute velocities.

Let $v_{B/A}$ denote the velocity of B relative to A.

Then $v_{B/A} = v_B - v_A$ ... (2)

Similarly $v_{A/B} = v_A - v_B$ ... (3)

From (2) and (3), we find that the relative velocity vector of one moving point B with respect to another moving point A is obtained by compounding the velocity vector of B with a vector velocity equal and opposite to that of a.

From (2), $v_B = v_{B/A} + v_A$

$\therefore$ The true velocity of B is got by compounding the relative velocity of B and the velocity of A.

Analytical Results:

Let two points A and B be moving with velocities u and v along OC and O_1D inclined at an angle α. The velocity v can be resolved into two components (i) $v\cos\alpha$ $\parallel$ to OC and (ii) $v\sin\alpha$ $\perp$ to OC.

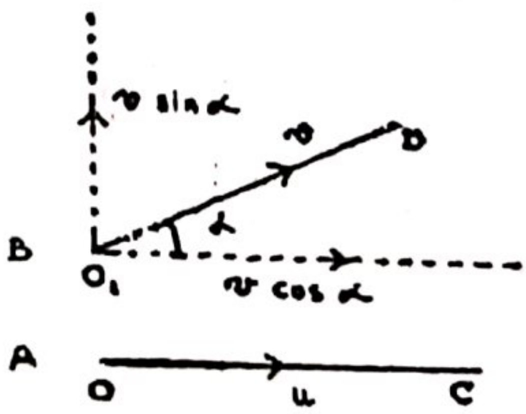


Fig. 24

The velocity of B relative to A, to $\parallel$ OC is $= v\cos\alpha - u$. Since A has no velocity perpendicular to OC, the velocity of B relative to A, $\perp$ to OC is $= v\sin\alpha - 0$.

= v sin α. Let V be the resultant of these two components, at an angle θ to OC.

$$V^2 = (v\cos\alpha - u)^2 + v^2\sin^2\alpha = u^2 + v^2 - 2uv\cos\alpha \quad \ldots (1)$$

$$\tan\theta = \frac{\text{Component} \perp \text{to OC}}{\text{Component} \parallel \text{to OC}} \quad \frac{v\sin\alpha}{v\cos\alpha - u} \quad \ldots (2)$$

(1) and (2) give respectively the magnitude V and direction θ of the relative velocity of A with respect to B.

Ex.4. *A ship P is sailing due east at a speed of 16 km/h when another ship Q which is due north of P at a distance of 10 km. from it, starts at a speed of 12 km/h in a southern direction. Find the velocity of Q relative to P. What is the least distance apart that Q will attain from P and how long after starting will it attain it?*

(B.A. 44 Madras Uty.)

The relative velocity of Q with respect to P is got by compounding with the velocity of Q (12 km/h due south) a velocity equal and opposite to that of P (16 km/h due west).

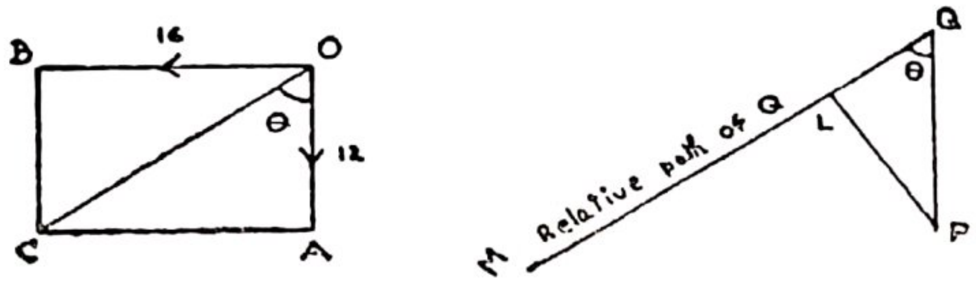


Fig. 25

In the figure, $\vec{OA} = 12$ and $\vec{OB} = 16$

Complete the rectangle AOB. Then $\vec{OC}$ represents the relative velocity of Q with respect to P.

$$OC^2 = OA^2 + AC^2 = 12^2 + 16^2 = 400$$

$\therefore$ OC = 20

Let $\angle AOC = \theta$

$$\tan\theta = \frac{AC}{OA} = \frac{16}{12} = \frac{4}{3}$$

Hence the velocity of Q relative to P is 20 km/h at an angle $\tan^{-1}\frac{4}{3}$ west of south.

Now keeping P at rest, allow Q to move along the relative path QM, as shown in the figure.

Q is due north of P and PQ = 10.

From P, draw PL perpendicular to QM, the relative path. The ships are nearest to each other when Q comes to L.

Least distance between P and Q

$= PL = PQ.\sin\theta = 10 \times \frac{4}{5} = 8$ km.

$QL = PQ.\cos\theta = 10 \times \frac{3}{5} = 6$ km.

Time taken by Q to travel 6km. along the relative path

$= \frac{QL}{\text{Relative velocity}} = \frac{6}{20}$ hour = 18 minutes

Aliter: The least distance between P and Q can be got as follows:

At a certain instant, Q is due north of P and PQ = 10km. At the end of time t hours, let P_1 and Q_1 be their actual positions and $P_1Q_1 = x$

Then $PP_1 = 16t$ and $QQ_1 = 12t$

$\therefore PQ_1 = PQ - QQ_1 = 10 - 12t$

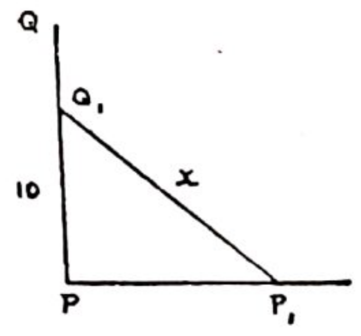


Fig. 26

Hence, $x^2 = PP_1^2 + PQ_1^2$

$= (16t)^2 + (10 - 12t)^2$

$= 256t^2 + 100 - 240t + 144t^2$

$= 400t^2 - 240t + 100$ ... (1)

Let $x^2 = y$

$\therefore y = 400t^2 - 240t + 100$ ... (2)

Differentiating (2) with respect to t,

$$\frac{dy}{dt} = 800t - 240 \text{ and } \frac{d^2y}{dt^2} = 800 = \text{positive}$$

$\therefore$ y is minimum when $\frac{dy}{dt} = 0$

i.e. when $800t - 240 = 0$

$$\text{or } t = \frac{240}{800} = \frac{3}{10} \text{ hours.}$$

i.e. after 18 minutes, the ships are nearest to each other.

Putting $t = \frac{3}{10}$ in (1),

$$x^2 = \left(400 \times \frac{9}{100}\right) - \left(240 \times \frac{3}{10}\right) + 100$$

$$= 36 - 72 + 100 = 64$$

$\therefore$ x = 8km. and this is the least distance between them.

Ex.5. *To a man walking along a level road at 5km/h, the rain appears to be beating into his face at 8km/h at an angle 60^o with the vertical. Find the true direction and velocity of the rain.*

(B.A. 30 Madras Uty)

Let OB represent the actual velocity v of the rain at an angle θ to the vertical. Draw OA westwards to represent the velocity of the man in the opposite driection. i.e. OA represents the reversed velocity of the man. Complete the parallelogram AOB. The diagonal OC gives the relative velocity of the rain and this is given to be at an angle 60^o to the vertical.

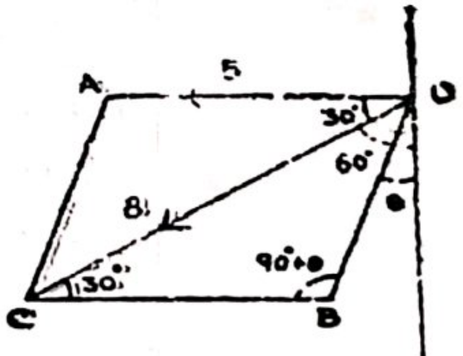


Fig. 27

In the figure, OA = 5, OC = 8,
OB = v, $\angle AOC = 30^o$
$\angle COL = 60^o$ $\angle BOL = \theta$

$\frac{a\,v}{V}$ and that the time that elapses before they arrive at their nearest distance is $\frac{a\,u}{V^2}$

(B.Sc. 53 Madras Uty)

11. Two particles move with uniform velocities u and v respectively along perpendicular lines XO and YO intersecting at O, the particles moving towards O. If they are initially at distances a and b from O, show that they will be nearest to each other at the time t given by $t = \frac{bv + au}{u^2 + v^2}$

(B.Sc. 75 Applied Science Madras Uty)
(B.Sc 94 Bharathidasan Uty)

12. Two cars A and B are moving due north and due east at 40 and 30km per hour respectively. At noon B is west of A, at a distance of 20km. When are the cars closest to each other and what is the distance between them at that time? (B.Sc. 71 Madurai Uty)

13. To a cyclist riding due west at 10 km. per hour, the wind appears to him to blow from south. When he doubles his speed, it appears to him to blow from southwest. Show that the speed of the wind is $10\sqrt{2}$ km. per hour and it is from southeast.

(B.Sc.94 Bharathidasan Uty)

§ 3.11. Angular Velocity: Definition:

If a particle P be moving along any path in a plane and if O be a fixed point in the plane and OA a fixed straight line through O, the rate at which the angle AOP increases is called the angular velocity of P about O.

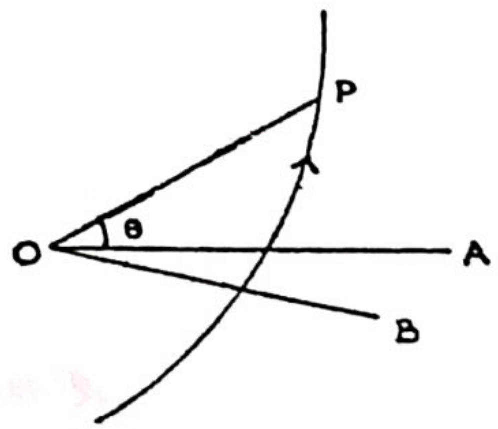


Fig. 29

If we take any other fixed line OB through O as the initial line instead of OA, the angular velocity of P about O is the rate at which the angle BOP increases. But this is the same as the rate at which the angle AOP increases, since the angle AOB is the same for all positions of the moving point P. Hence the angular velocity about O is independent of the line through O, taken as the initial line.

Let the particle P be moving with uniform angular velocity and θ be the angle in radian measure described by OP in t seconds. Then the angular velocity about O is given by $\omega = \dfrac{\theta}{t}$ radians/sec.

To get a measure for variable angular velocity, we proceed as follows: Let θ and $\theta + \Delta\theta$ be the angles made by OP with OA in times t and $t + \Delta t$ respectively. The average angular velocity of P in the short interval of time $\Delta t = \dfrac{\Delta\theta}{\Delta t}$. The limiting value of this angular velocity at $\Delta t \to 0$ is the angular velocity of P about O and so it is given by

$$\omega = \underset{\Delta t \to 0}{\mathrm{Lt}} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

Angular velocity is always expressed in *radians per second.*

Examples:

(i) If the line OP turns through 2 right angles in one second, the angular velocity of P $= 2 \times \dfrac{\pi}{2} = \pi$ radians per sec.

(ii) If OP makes 4 revolutions in one second, the angular velocity of P $= 4 \times 2\pi = 8\pi$ radians per sec.

§ 3.12 Angular velocity of a particle moving along a circle with uniform speed:

Let a point move with uniform speed v along a circle centre O and radius r. Let P be its position at time t secs and s be the arc AP measured from a fixed point A on the circle.

OA is a fixed direction and let $\angle AOP = \theta$.

At time $t + \Delta t$ secs, let the point be at Q such that $\angle POQ = \Delta \theta$ and are $PQ = \Delta s$.

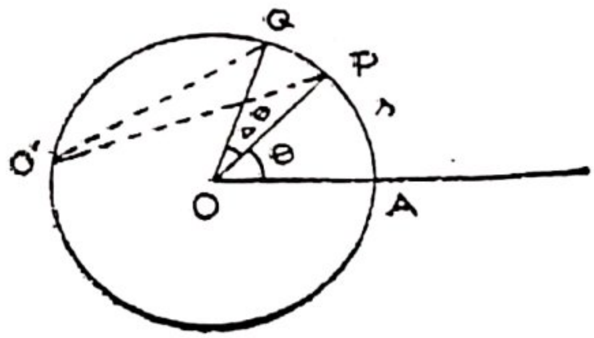


Fig. 30

Then we know that $\Delta s = r. \Delta \theta$

$$\therefore \quad \frac{\Delta s}{\Delta t} = r. \frac{\Delta \theta}{\Delta t}$$

Taking limits,

$$\underset{\Delta t \to 0}{Lt} \frac{\Delta s}{\Delta t} = r. \underset{\Delta t \to 0}{Lt} \frac{\Delta \theta}{\Delta t}$$

$$\text{i.e.} \quad \frac{ds}{dt} = r \frac{d\theta}{dt} \qquad \text{... (1)}$$

Now $\frac{ds}{dt}$ is the rate at which the length of the path is described and so it is the linear velocity v of the particle.

$\frac{d\theta}{dt}$ is the angular velocity ω.

So (1) becomes, $v = r\,\omega$

or $\omega = \frac{v}{r}$

Corolloary: Let O' be any point on the circumference.

We know that $\angle POQ = 2 \angle PO'Q$

$\therefore$ Rate of change of $\angle POQ = 2$ x rate of change of $\angle PO'Q$

$\therefore$ Angular velocity about the centre O

$= 2 \times$ angular velocity about O'

$\therefore$ Angular velocity about O'

$= \frac{1}{2} \times$ angular velocity about O

$= \frac{1}{2} \omega = \frac{v}{2r}$

§ 3.13. Angular velocity of a particle moving along any curve:

Let a particle move along any curve and P and Q be its two consecutive positions at times t and t + Δt + Δt respectively. OX is a fixed line.

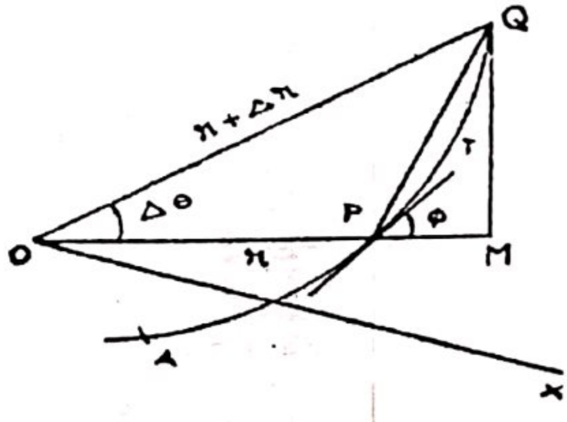


Fig. 31

$OP = r$, $OQ = r + \Delta r$

$\angle XOP = \theta$ and $\angle XOQ = \theta + \Delta\theta$

Let v be the linear velocity of the point at time t.

This velocity is along PT, the tangent at P.

From Q, draw QM ⊥ to OP. Let $\angle TPM = \varphi$

Arcual distance described along the curve during the short time $\Delta t = v.\Delta t$ nearly.

As Q is close to P, arc QP = chord PQ nearly.

$\therefore PQ = v.\Delta t$

Also $\angle QPM = \angle TPM$ nearly $= \varphi$

From right angled ΔQPM,

$$QM = PQ.\sin \angle QOM = v.\Delta t.\sin\varphi \qquad \ldots (1)$$

From right angled ΔQOM,

$$QM = OQ.\sin\Delta\theta$$

$$= (r + \Delta r)\sin\Delta\theta$$

$$= (r + \Delta r)\Delta\theta \quad (\because \Delta\theta \text{ is small}) \qquad \ldots (2)$$

Equating (1) and (2),

$(r + \Delta r)\Delta\theta = v.\Delta t.\sin\varphi$

$$\frac{\Delta\theta}{\Delta t} = \frac{v\sin\varphi}{r + \Delta r}$$

10. A and B describe concentric circles of radii a and b with uniform speeds, u, v the motion being the same way round. Prove that the angular velocity of either with respect to the other is zero when the line joining them subtends at the centre an angle whose cosine is $\frac{au + bv}{av + bu}$.

11. Two planets describe circles of radii a and b round the sun as centre, with speeds varying inversely as the square roots of the radii; show that their relative angular velocity vanishes when the angle θ between the radii to those planets is given by $\cos\theta = \frac{\sqrt{ab}}{a - \sqrt{ab} + b}$ (B.Sc. 82 Madras Uty.)

§ 3.16. Change of velocity:

Since a velocity has both magnitude and direction, it will be changed if one of these changes or both change.

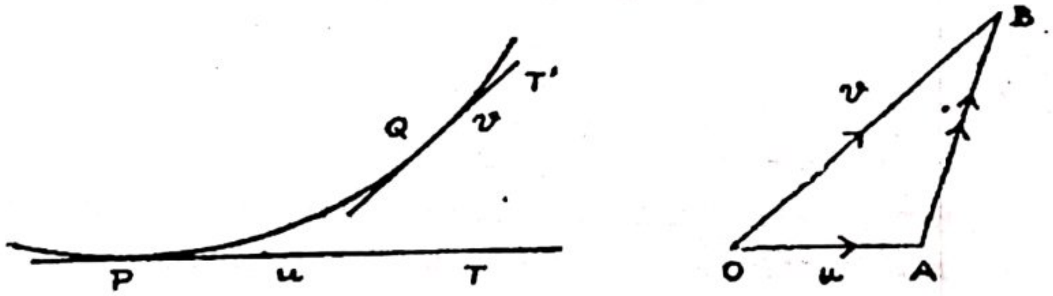


Fig. 35

Suppose a particle is moving along a curve. Let P be its position at a certain instant and Q its position after an interval. Let u and v be the velocities at P and Q. These are along the tangents PT and QT′. From any point O, draw OA and OB to represent the velocities u and v respectively.

By the triangle of velocities, we have $\vec{OA} + \vec{AB} = \vec{OB}$

Hence a velocity represented by $\vec{AB}$ has been added to the velocity at P, to give the velocity at Q. Thus the change in velocity in that interval is represented in magnitude and direction by AB.

PROJECTILES

§ 6.1. In the present chapter, we shall consider motion of a particle projected into the air in any direction and with any velocity. Such a particle is called a projectile. The two forces that act on the projectile are its weight and the resistance of air. For simplicity, we suppose the motion to take place within such a moderate distance from the surface of the earth that we can neglect the variations in the acceleration due to gravity. This means that g may be considered to be constant in magnitude throughout the motion of the projectile. Secondly, we shall neglect the resistance of the air and consider the motion to take place in vaccum.

§ 6.2. Definitions:

The following terms are used in connection with projectiles:

The angle of projection is the angle that the direction in which the particle is initially projected makes with the horizontal plane through the point of projection.

The velocity of projection is the velocity with which the paricle is projected.

The *trajectory* is the path which the particle describes.

The *range on a plane* through the point of projection is the distance between the point of projection and the point where the trajectory meets that plane.

The *time of flight* is the interval of time that elapses from the instant of projection till the instant when the particle again meets the horizontal plane through the point of projection.

§ 6.3. Two fundamental principles:

To discuss the motion of a projectile, we consider the horizontal and vertical components of the motion separately. The only force acting on the projectile is gravity and this acts vertically downwards. Hence by the Physical Independence of forces, it has no effect on the horizontal motion of the particle. So the *horizontal* *velocity remains constant throughout the motion*, as there is no force to cause any acceleration in that direction. On the other hand, the weight of the particle acting vertically downwards, will have its full effect on the vertical motion of the particle. The weight mg acting vertically downwards on a particle of mass m will produce an acceleration g vertically downwards. Hence *the vertical component of the velocity will be subject to a retardation g*. These two main principles will help us to study the motion of a projectile.

§ 6.4. To show that the path of a projectile (in vacuo) is a parabola:

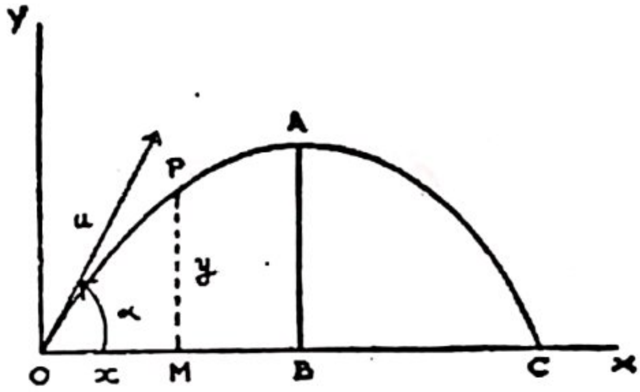


Fig. 70

Let a particle be projected from O, with a velocity u at an angle α to the horizon. Take O as the origin, the horizontal and the upward vertical through O as axes of x and y respectively. The initial velocity u can be split into two components, which are $u \cos \alpha$ in the horizontal direction and $u \sin \alpha$ in the vertical direction. The horizontal component $u \cos \alpha$ is constant throughout the motion as there is no horizontal acceleration. The vertical component $u \sin \alpha$ is subject to an acceleration g downwards.

Let P(x, y) be the position of the particle at time t secs. after projection. Then

x = horizontal distance described in t secs. $= (u \cos \alpha) \cdot t$ ... (1)

y = vertical distance described in t secs. $= (u \sin \alpha) t - \frac{1}{2} g t^2$ ... (2)

(1) and (2) can be taken as the parametric equations of the trajectory. The equation to the path is got by eliminating t between them.

From (1), $t = \dfrac{x}{u \cos \alpha}$ and putting this in (2) we get

$$y = u \sin \alpha \,.\, \frac{x}{u \cos \alpha} - \frac{1}{2} g \,.\, \left(\frac{x}{u \cos \alpha}\right)^2$$

i.e. $$\boxed{y = x \tan \alpha - \frac{g x^2}{2 u^2 \cos^2 \alpha}} \quad \dots \quad (3)$$

Multiplying (3) by $2 u^2 \cos^2 \alpha$,

$$2 u^2 \cos^2 \alpha \,.\, y = 2u^2 \cos^2 \alpha \,.\, x \frac{\sin \alpha}{\cos \alpha} - gx^2$$

i.e. $$x^2 - \frac{2u^2 \sin \alpha \cos \alpha}{g} x = - \frac{2u^2 \cos^2 \alpha}{g} y$$

or $$\left(x - \frac{u^2 \sin\alpha \cos\alpha}{g}\right)^2 = \frac{u^4 \sin^2 \alpha \cos^2 \alpha}{g^2} - \frac{2u^2 \cos^2 \alpha}{g} y$$

$$= - \frac{2u^2 \cos^2 \alpha}{g} \left(y - \frac{u^2 \sin^2 \alpha}{2g}\right)$$

Transfer the origin to the point

$$\left(\frac{u^2 \sin\alpha \cos\alpha}{g}, \frac{u^2 \sin^2 \alpha}{2g}\right)$$

The above equation then becomes

$$X^2 = - \frac{2u^2 \cos^2 \alpha}{g} \,.\, Y \quad \dots \quad (4)$$

(4) is clearly the equation to a parabola of latus rectum $\dfrac{2 u^2 \cos^2 \alpha}{g}$, whose axis is vertical and downwards and whose vertex is the point $\left(\dfrac{u^2 \sin\alpha \cos\alpha}{g}, \dfrac{u^2 \sin^2 \alpha}{2g}\right)$

Note: The latus rectum of the above parabola is

$$= \frac{2 u^2 \cos^2 \alpha}{g} = \frac{2}{g} \,.\, (u \cos \alpha)^2$$

$$= \frac{2}{g} \times \text{ square of the horizontal velocity}$$

So the latus rectum (i.e. the size of the parabola) is independent of the initial vertical velocity and depends only on the horizontal velocity.

§ 6.5. Characteristics of the motion of a projectile:

Refer to fig. 70 on page 140. Let a particle be projected from O with velocity u at an angle α to the horizontal OX. Let A be the highest point of the path and C the point where it again meets the horizontal plane through O. Using the two fundamental principles given in § 6.3. page 139, we can derive the following results relating to the motion of a projectile.

(1) Greatest height attained by a projectile.

At A, the highest point, the particle will be moving only horizontally, having lost all its vertical velocity. Let AB = h = the greatest height reached. Considering vertical motion separately, initial upward vertical velocity $= u \sin\alpha$ and the acceleration in this direction is -g. The final vertical velocity at A is = 0.

Hence $0 = (u \sin \alpha)^2 - 2g \cdot h$ i.e. $\boxed{h = \dfrac{u^2 \sin^2 \alpha}{2g}}$

i.e. the vertex of the parabola is the highest point of the path.

(2) Time taken to reach the greatest height.

Let T be the time from O to A. Then, in time T, the initial vertical velocity $u \sin \alpha$ is reduced to zero, acted on by an acceleration -g. Hence $0 = u \sin \alpha - gT$.

$$\therefore \boxed{T = \frac{u \sin \alpha}{g}}$$

(3) Time of flight i.e. the time taken to return to the same horizontal level as O.

When the particle arrives at O, the effective vertical distance it has described is zero. Hence if t is the time of flight, considering vertical motion, we have $0 = u \sin \alpha\, t - \frac{1}{2} gt^2$.

i.e. $t = 0$ or $t = \dfrac{2u \sin \alpha}{g}$

$t = 0$ is the instant of projection when also the vertical distance travelled is zero.

$$\therefore \boxed{\text{The time of flight} = \frac{2u \sin \alpha}{g}}$$

We find that the time of flight is twice the time taken to reach the highest point, as we should expect from symmetry.

(4) The range on the horizontal plane through the point of projection.

The time of flight is $t = \dfrac{2u \sin \alpha}{g}$. During this time, the horizontal velocity remains constant and is equal to $u \cos \alpha$

Hence OC = horizontal distance described in time t

$$= u \cos\alpha \,.\, t = u \cos\alpha \,.\, \frac{2u \sin \alpha}{g} = \frac{2u^2 \sin\alpha \cos \alpha}{g}$$

Hence

$$\boxed{\text{the horizontal range } R = \frac{2u^2 \sin\alpha \cos\alpha}{g} = \frac{u^2 \sin 2\alpha}{g}}$$

Note: (1) The horizontal range can also be found thus: The equation to the path is $y = x \tan\alpha - \dfrac{gx^2}{2u^2 \cos^2 \alpha}$ ... (1)

The equation to the x axis is $y = 0$.

Putting $y = 0$ in (1), we have $x \tan \alpha - \dfrac{gx^2}{2u^2 \cos^2 \alpha} = 0$

i.e. $x = 0$ or $x = \dfrac{2u^2 \cos^2 \alpha \tan\alpha}{g} = \dfrac{2u^2 \sin \alpha \cos \alpha}{g}$

$x = 0$ corresponds to the point of projection and so the other value $\dfrac{2u^2 \sin \alpha \cos\alpha}{g}$ gives the horizontal range.

(2) Horizontal range $= \frac{2u^2 \sin\alpha \cos\alpha}{g} = \frac{2(u\cos\alpha).(u\sin\alpha)}{g}$

$= 2\frac{UV}{g}$ where U and V are the initial horizontal and vertical velocities.

§ 6.6. A particle is projected horizontally from a point at a certain height above the ground; to show that the path described by it is a parabola.

Let a particle be projected horizontally with a velocity u from a point A at a height h above the ground level. Let it strike the ground at M. Take A as origin, the horizontal through A as x axis and the downward vertical through A as y axis. Let P(x,y) be the position of the particle at time t. As there is no horizontal acceleration, the horizontal velocity remains constant throughout the motion.

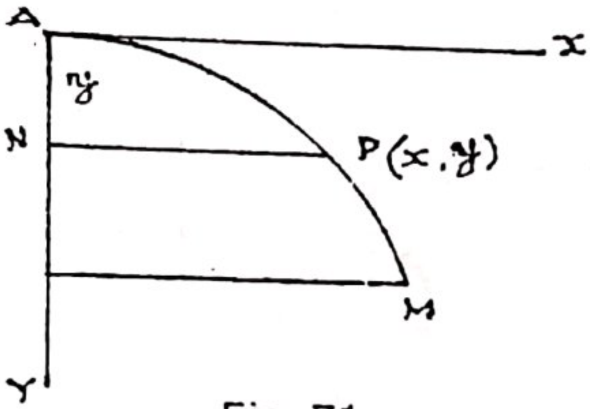


Fig. 71

So x = horizontal distance described in time t = ut ... (1)

Due to gravity, the vertical acceleration during the motion is g downwards.

y = vertical distance described in time $t = \frac{1}{2}gt^2$ ... (2)

Eliminate t between (1) and (2).

We have $y = \frac{1}{2}g.\frac{x^2}{u^2}$ i.e. $x^2 = \frac{2u^2}{g}y$ ... (3)

(3) shows that y is a quadratic function of x. So it represents a parabola with vertex at A and axis AN.

WORKED EXAMPLES

Ex.1. *A body is projected with a velocity of 98 metres per sec. in a direction making an angle $\tan^{-1}3$ with the horizon; show that it rises to a vertical height of 441 metres and that its time of flight is* *about 19 secs. Find also horizontal range through the point of projection ($g = 9.8$ metres/sec^2)*

Here $u = 98$; $\alpha = \tan^{-1} 3$ i.e. $\tan \alpha = 3$.

$$\therefore \sin \alpha = \frac{\sin \alpha}{\cos \alpha} \cdot \cos \alpha = \frac{\tan \alpha}{\sec \alpha} = \frac{\tan\alpha}{\sqrt{1 + \tan^2\alpha}} = \frac{3}{\sqrt{10}}$$

$$\cos \alpha = \frac{\sin \alpha}{\tan \alpha} = \frac{1}{\sqrt{10}}$$

Greatest height reached $= \dfrac{u^2 \sin^2 \alpha}{2g} = \dfrac{98 \times 98 \times 9}{10 \times 2 \times 9.8}$

$= 441$ metres.

Time of flight $= \dfrac{2u \sin \alpha}{g} = \dfrac{2 \times 98 \times 3}{\sqrt{10} \times 9.8} = 6\sqrt{10}$

$= 6 \times 3.162 = 18.972 = 19$ secs. nearly

Horizontal range $= \dfrac{2u^2 \sin\alpha \cos\alpha}{g}$

$$= \frac{2 \times 98 \times 98}{9.8} \times \frac{3}{\sqrt{10}} \times \frac{1}{\sqrt{10}} = 588 \text{ metres}.$$

Ex.2. *If the greatest height attained by the particle is a quarter of its range on the horizontal plane through the point of projection, find the angle of projection* (B.Sc. 67 Madras Uty.)

Let u be the initial velocity and α the angle of projection.

Then, the greatest height $= \dfrac{u^2 \sin^2 \alpha}{2g}$.

and horizontal range $= \dfrac{2u^2 \sin \alpha \cos\alpha}{g}$

It is given that $\dfrac{u^2 \sin^2 \alpha}{2g} = \dfrac{1}{4} \times \dfrac{2u^2 \sin\alpha \cos\alpha}{g}$

i.e. $\dfrac{u^2 \sin^2 \alpha}{2g} = \dfrac{u^2 \sin\alpha \cos\alpha}{2g}$

i.e. $\sin \alpha = \cos \alpha$ or $\tan \alpha = 1$ $\therefore \alpha = 45°$

Ex. 3. *A stone is thrown with a velocity of 39.2 m/sec. at 30° to the horizontal. Find at what times it will be at a height of 14.7m ($g = 9.8 m/sec^2$)*

Initial vertical velocity $= 39.2 \times \sin 30° = 19.6$ m/sec..

This is subject to an acceleration -g.

Let the particle be at a height 14.7 m after time t sec.

Applying the formula " $s = ut + \frac{1}{2}at^2$ ", we have

$$14.7 = 19.6\,t - \frac{1}{2}gt^2 = 19.6\,t - 4.9\,t^2$$

i.e. $3 = 4t - t^2$ or $t^2 - 4t + 3 = 0$

i.e. $(t - 3)\ (t - 1) = 0$; $t = 1$ or $t = 3$.

Hence at the end of 1 sec. and again at the end of 3 secs. it will be at a height of 14.7m

Ex.4. *A bomb was released from an aeroplane when it was at a height of 1960 m. above a point A on the ground and was moving horizontally with a speed of 100m per sec. Find the distance from A of the point where the bomb strikes the ground. ($g = 9.8 m/sec^2$)*

Let us consider the motion of the bomb in the horizontal and the vertical directions separately. The dynamical details of each motion may be presented as follows:

Horizontal Motion	**Vertical Motion**
	(upwards +ve)
Initial velocity = 100 m/sec.	Initial velocity = 0
	Acceleration = $-9.8 m/sec^2$
Acceleration = 0	(Distance = -1960m (∵ it is downwards)

Let t be the time taken by the bomb to strike the point on the ground.

§ 6.7. To determine when the horizontal range of a projectile is maximum, given the magnitude u of the velocity of projection.

If u is the initial velocity and α is the angle of projection, the range R on the horizontal plane through the point of projection is given by

$$R = \frac{2u^2 \sin \alpha \cos \alpha}{g} = \frac{u^2 \sin 2\alpha}{g} \qquad \dots (1)$$

Now, g being a constant, for a given value of u, the value of R is greatest when $\sin 2\alpha$ is greatest.

i.e. when $\sin 2\alpha = 1$.

This happens when $2\alpha = 90^0$ i.e. $\alpha = 45^0$. Hence *for a given velocity of projection, the horizontal range is a maximum when the particle is projected at an angle of 45^0 to the horizontal.* i.e. the direction of projection for maximum horizontal range bisects the angle between the horizontal and the vertical. Also when

$$\alpha = 45^\circ, \ (1), \ R = \frac{u^2}{g}.$$

i.e. *The maximum horizontal range is* $\frac{u^2}{g}$

§ 6.8. To show that, for a given initial velocity of projection there are, in general two possible directions of projections so as to obtain a given horizontal range.

Let u be the velocity of projection of a particle, and α the necessary angle of projection so as to get a given horizontal range equal to k.

$$\text{Then } \frac{k}{1} = \frac{u^2 \sin 2\alpha}{g} \quad \therefore \ \sin 2\alpha = \frac{gk}{u^2} \qquad \dots (1)$$

Since u and k are given and g is a constant, the R.H.S. of (1) is a known positive quantity. If $gk < u^2$, we can determine an acute angle θ whose sine is exactly equal to $\frac{gk}{u^2}$

Then (1) becomes $\sin 2\alpha = \sin \theta$ ... (2)

$\therefore 2\alpha = \theta$ or $\alpha = \frac{\theta}{2}$ ... (3)

Since $\sin(180^0 - \theta) = \sin \theta$, (2) can also be written as

$\sin 2\alpha = \sin(180^0 - \theta)$; $\therefore 2\alpha = 180^0 - \theta$

i.e. $\alpha = 90^0 - \frac{\theta}{2}$ ... (4)

From (3) and (4) we find that there are two values of α and so two directions of projection, each giving the same range k. Let α_1 and α_2 be these two values of α.

Then $\alpha_1 = \frac{\theta}{2}$ and $\alpha_2 = 90^0 - \frac{\theta}{2}$; $\therefore \alpha_1 + \alpha_2 = 90^0$

As $\theta < 90^o$, $\alpha_1 < 45^o$, and so $\alpha_2 > 45^o$.

Now $45^o - \alpha_1 = 45^o - \frac{\theta}{2}$

and $\alpha_2 - 45^o = 90^o - \frac{\theta}{2} - 45^o = 45^o - \frac{\theta}{2}$

i.e. $45^o - \alpha_1 = \alpha_2 - 45^o$ ... (5)

But 45^0 is the angle of projection to get maximum horizontal range with the same initial velocity. So (5) shows that the two directions α_1 and α_2 are equally inclined to the direction of maximum range. This is shown in fig.73.

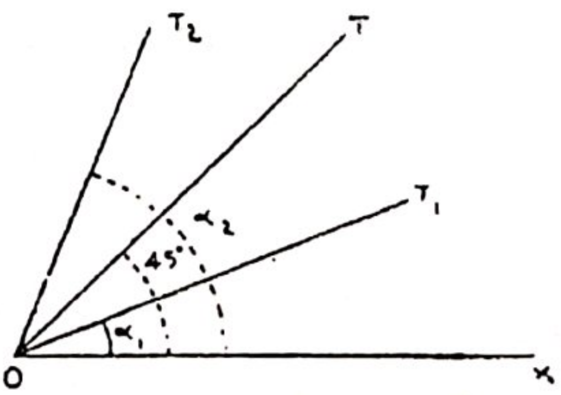


Fig. 73

In fig.73, OT_1 and OT_2 are the directions α_1 and α_2 necessary to get a given range k.

OT is the direction giving maximum horizontal range.

$\angle T_1 OT = \angle XOT - \angle XOT_1 = 45^0 - \alpha_1$

$\angle T_2 OT = \angle XOT_2 - \angle XOT = \alpha_2 - 45^0$

and $\angle T_1 OT = \angle T_2 OT$

In other words, OT bisects the angle between OT_1 and OT_2.

If $u^2 = gk$, from (1), $\sin 2\alpha = 1$.

Then $2\alpha = 90^0$ or $\alpha = 45^0$. Only one value of α is possible and this corresponds to the case of maximum range.

If $u^2 < gk$, the R.H.S. of (1) is greater than 1 and so we cannot get a real value for α i.e. There is no angle of projection to get a range greater than $\frac{u^2}{g}$, which is really the maximum range possible.

Note: To get a given horizontal range k, we find that $u^2 \geq gk$. So the minimum value of $u = \sqrt{gk}$.

2 mark.

Ex.8. *If h and h' be the greatest heights in the two paths of a projectile with a given velocity for a given rangeR, prove that* $R = 4\sqrt{(hh')}$

(B.Sc. 51,75,80 Madras Uty; B.Sc. 85 Madurai Uty.)

Let α and α' be the two angles of projection with a given velocity u to get a given range R.

Then we know that $\alpha + \alpha' = 90^0$ i.e. $\alpha' = 90^0 - \alpha$ ... (1)

Also $R = \frac{2u^2 \sin\alpha \cos\alpha}{g}$ ... (2)

$h = \frac{u^2 \sin^2\alpha}{2g}$ ... (3) and $h' = \frac{u^2 \sin^2\alpha'}{2g}$ ... (4)

$\therefore hh' = \frac{u^4 \sin^2\alpha \cdot \sin^2\alpha'}{4g^2}$

Hence $\sqrt{hh'} = \frac{u^2 \sin^2\alpha . \sin^2\alpha'}{2g} = \frac{u^2 \sin^2\alpha \sin(90^0 - \alpha)}{2g}$ using (1)

$$= \frac{u^2 \sin\alpha \cos\alpha}{2g} = \frac{R}{4} \text{ using (2)}$$

i.e. $R = 4\sqrt{hh'}$ //.

Ex.9. *A shell bursts on contact with the ground and pieces from it fly in all directions with all velocities upto 30 metres per second. Show that a man 30m away is in danger for 5secs. nearly.*

Here the given velocity u = 30m, and the given range R = 30m.

Since $R = \frac{u^2 \sin 2\alpha}{g}$, we have

$$\sin 2\alpha = \frac{gR}{u^2} = \frac{9.81 \times 30}{30 \times 30} = .327$$

From the tables, $2\alpha = 19^0 5'$ or $160^0 55'$

i.e. $\alpha = 9^0 32\frac{1}{2}'$ or $80^0 27\frac{1}{2}'$

These are the two angles of projection to get the given range. Let t_1 and t_2 be the two times of flight for the two particular pieces which fly in the directions $9^0 32\frac{1}{2}'$ and $80^0 27\frac{1}{2}'$ respectively. These two pieces will strike the man and so he is in danger for the interval $t_2 - t_1$ secs.

$$\text{Now } t_1 = \frac{2u \sin 9^0 32\frac{1}{2}'}{g} = \frac{2 \times 30 \sin 9^0 32\frac{1}{2}'}{9.81} = 1.014$$

$$t_2 = \frac{2u \sin 80^0 27\frac{1}{2}'}{g} = \frac{2 \times 30 \times \sin 80^0 27\frac{1}{2}'}{9.81} = 6.032$$

Period of danger = $t_2 - t_1$ = 6.032 - 1.014
= 5.018 = 5 secs. (nearly)

Ex.10. *The range of a rifle bullet is 1000m. when α is the angle of projection. Show that if the bullet is fired with the same* *elevation from a car travelling 36km/h towards the target, the range will be increased by* $\frac{1000\sqrt{\tan\alpha}}{1}$ *m.* $(g = 9.8 m/sec^2)$

Let u m/sec be the velocity of projection. The horizontal range

$$R = \frac{2u^2 \sin\alpha \cos\alpha}{g} = 1000 \text{ (given)} \quad \text{... (1)}$$

$$\text{Also } R = \frac{2}{g}(u\cos\alpha).\ (u\sin\alpha)$$

$$= \frac{2}{g}(\text{horizontal velocity}) \times (\text{initial vertical velocity}) \ldots (2)$$

When the bullet is fired from a moving car, the horizontal velocity is increased and the increase

$$= 36 \text{ km/h} = \frac{36 \times 1000}{60 \times 60} = 10 \text{ m/sec.}$$

New horizontal velocity $= u\cos\alpha + 10$

As there is no change in the vertical motion, new initial vertical velocity $= u\sin\alpha$.

Hence in the second case, horizontal range

$$R' = \frac{2}{g}(u\cos\alpha + 10)(u\sin\alpha) \text{ using the form given in (2).}$$

$$R' - R = \frac{2}{g}(u\cos\alpha + 10)(u\sin\alpha) - \frac{2u^2\sin\alpha\cos\alpha}{g}$$

$$= \frac{20\,u\sin\alpha}{g} \quad \text{... (3)}$$

$$\text{From (1), } u^2 = \frac{g \times 1000}{2\sin\alpha\cos\alpha} = \frac{500\,g}{\sin\alpha\cos\alpha} \quad \text{... (4)}$$

Putting this value of u in (3),

$$R' - R = \frac{20\sin\alpha}{g} \times \sqrt{\frac{500\,g}{\sin\alpha\cos\alpha}} = \frac{20 \times 10 \times \sqrt{\tan\alpha} \times 5}{\sqrt{g}}$$

$$= \frac{200\sqrt{\tan\alpha}\sqrt{5}}{\sqrt{9.8}} = \frac{200\sqrt{\tan\alpha}\sqrt{5} \times \sqrt{10}}{\sqrt{98}}$$

shot is projected at the same elevation, the velocity of projection must be increased to $\dfrac{V^2}{(V^2 - gh)^{1/2}}$. (B.Sc. 73, Madras Uty.)

§ 6.9. To find the velocity of the projectile in magnitude and direction at the end of time t:

Let a particle be projected from O with a velocity 'u' at an angle α to the horizon. After time 't', let it be at P and 'v' be its velocity inclined at an angle θ to the horizontal. We know that the horizontal component of the velocity is constant and equal to $u \cos \alpha$, throughout the motion. The horizontal component of the velocity at P = $v \cos \theta$.

$$\therefore v \cos \theta = u \cos \alpha \quad \text{.... (1)}$$

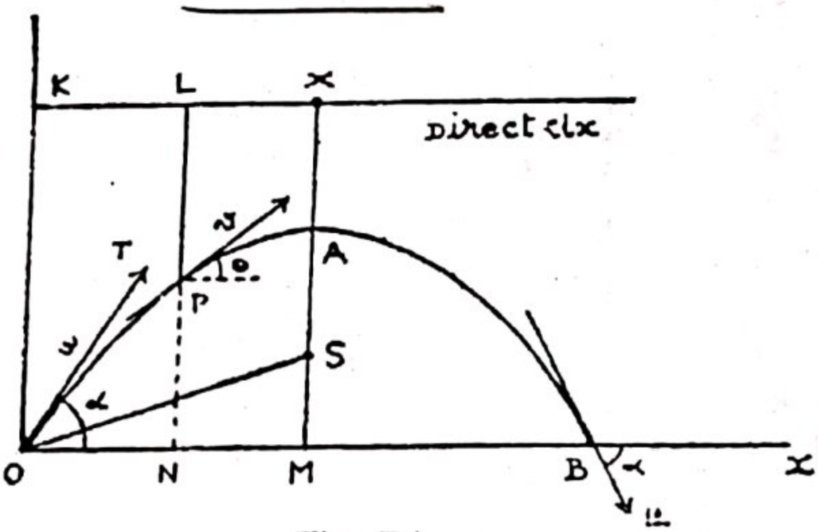


Fig. 74

The vertical component of the initial velocity of projection = $u \sin \alpha$ and this is subject to a retardation g.

The vertical component of the velocity at P = $v \sin \theta$.

v = u - gt

$$\therefore v \sin \theta = u \sin \alpha - gt \quad \text{.... (2)}$$

Squaring (1) and (2) and adding, we have

$$v^2 = u^2 \cos^2 \alpha + (u \sin \alpha - gt)^2 = u^2 - 2u \sin \alpha\, gt + g^2 t^2$$

$$\therefore v = \sqrt{u^2 - 2u \sin \alpha\, gt + g^2 t^2} \quad \text{... (3)}$$

Dividing (2) by (1), $\tan \theta = \dfrac{u \sin \alpha - gt}{u \cos \alpha}$ ... (4)

Equations (3) and (4) give ' v' and θ i.e. the velocity at P in magnitude and direction.

Note: (i) If $t < \frac{u \sin \alpha}{g}$ which is the time taken to reach the highest point A, $u \sin \alpha - gt$ is +ve. So $\tan \theta$ is +ve and θ is also +ve. After the time taken to reach A, $t > \frac{u \sin \alpha}{g}$. In that case θ is -ve. If $t = \frac{u \sin \alpha}{g}$, $\tan \theta = 0$ and so $\theta = 0$.Hence at the highest point A, the direction of the velocity is horizontal.

(ii) Putting $t = \frac{2u \sin \alpha}{g}$ which is the time of flight, in (3) and (4) we have

$$v = \sqrt{u^2 - 2u \sin \alpha \, g . \frac{2u \sin \alpha}{g} + g^2 . \frac{4u^2 \sin^2 \alpha}{g^2}}$$

$$= \sqrt{u^2 - 4u^2 \sin^2 \alpha + 4u^2 \sin^2 \alpha} = u$$

$$\tan \theta = \frac{u \sin \alpha - g . \frac{2 u \sin \alpha}{g}}{u \cos \alpha} = - \tan \alpha \text{ . i.e. } \theta = - \alpha$$

Hence the particle strikes the horizontal plane downwards at B with the same velocity as the initial velocity and at the same angle as that with which it was projected.

(iii) Equation (3) can be deduced from the Principle of Energy. Change in kinetic energy of the particle when it moves from O to $P = \frac{1}{2} m v^2 - \frac{1}{2} m u^2$, 'm' being its mass.

Work done by the external force (gravity) against the particle during its motion from O to P = mg.y where 'y' is the vertical height of P above O.

But y = vertical distance travelled in time t

$$= u \sin \alpha . t - \frac{1}{2} g t^2 \text{ (considering vertical motion)}$$

7. If the focus of a trajectory lies as much below the horizontal plane as the vertex is above, show that the angle of projection is given by $\sin\alpha = \frac{1}{\sqrt{3}}$.

(Hint: In a parabola, the tangent at any point P bisects the angle between SP and the perpendicular from P to the directrix.)

8. If S is the focus and P any point of the path of a projectile, show that the components of the velocity at P along and perpendicular to SP are respectively equal to its vertical and horizontal components. (Andhra Uty.)

9. A particle is to pass through a given point whose horizontal and vertical distances from the point of projection are x and y and to travel in a direction making an angle θ with the horizontal. Find the velocity and the direction of projection.

§ 6.11. Given the magnitude of the velocity of projection, to show that there are two directions of projection for the particle so as to reach a given point:

Let V be the velocity and α the angle of projection. The equation to the path of the particle is

$$y = x\tan\alpha - \frac{g x^2}{2V^2\cos^2\alpha} \qquad \dots (1)$$

Let the given point be (a, b).

This will lie on (1), if

$$b = a\tan\alpha - \frac{ga^2}{2V^2\cos^2\alpha} = a\tan\alpha - \frac{ga^2}{2V^2}(1+\tan^2\alpha)$$

i.e. $ga^2\tan^2\alpha - 2aV^2\tan\alpha + (ga^2 + 2V^2 b) = 0 \qquad \dots (2)$

Since a,b, u are given, equation (2) is a quadratic in $\tan\alpha$ and hence has two roots. The corresponding values of α are the two possible directions of projection to hit the point (a.b).

Ex.13. *P is a point at a horizontal distance a and a vertical distance b from the point of projection. It is required to project a* *particle to pass through P, with an initial velocity V. Show that this is impossible if* $V^2 < g\,(b + \sqrt{a^2 + b^2})$ *and that, if* $V^2 > g\,(b + \sqrt{a^2 + b^2})$ *there are two possible directions of projection.* (B.Sc. 61 Madras Uty. B.Sc. 71 Calicut Uty.)

Prove also that, if α *and* β *are the inclinations of the two directions,* $b \tan(\alpha + \beta) + a = 0$.

Referring to § 6.11 if α is the angle of projection the quadratic in tan α is

$$ga^2 \tan^2 \alpha - 2a V^2 \tan \alpha + (ga^2 + 2V^2 b) = 0 \qquad \ldots (1)$$

The two roots of (1) will be real, only when the discriminant is positive or zero.

Discriminant

$$= (2aV^2)^2 - 4ga^2(ga^2 + 2V^2 b) = 4a^2(V^4 - 2V^2gb - g^2a^2)$$

$\therefore$ This should be positive.

i.e. $V^4 - 2V^2 gb - g^2 a^2 > 0$

i.e. $(V^2 - gb)^2 - g^2 b^2 - g^2 a^2 > 0$ or $(V^2 - gb)^2 > g^2(a^2 + b^2)$

i.e. $V^2 - gb > g\sqrt{a^2 + b^2}$ or $V^2 > g(b + \sqrt{a^2 + b^2})$

If this condition is satisfied, the quadratic (1), will give two real values of tan α and so, there are two angles of projection.

If $V^2 < g(b + \sqrt{a^2 + b^2})$, the discriminant of the above quadratic will be negative and so we cannot get real values for α the angle of projection. In that case, it will be impossible to hit the particular point, with the given velocity.

If $V^2 = g(b + \sqrt{a^2 + b^2})$, the two roots are equal giving us only one direction of projection.

If α and β are the two angles of projection to hit the point, tan α and tan β will be the roots of (1).

$$\therefore \tan \alpha + \tan \beta = \frac{2aV^2}{ga^2} = \frac{2V^2}{ga}; \quad \tan \alpha \tan \beta = \frac{ga^2 + 2V^2 b}{ga^2}$$

EXERCISES

1. At what angle should a ball be thrown with a velocity of $14\sqrt{6}$ m./sec to reach the top of a cliff 40m. high and $40\sqrt{3}$ away from the point of projection. ($g = 9.8$m./sec^2)

2. The angular elevation of an enemy's position on a hill h' high is β. Show that, in order to shell it, the velocity of projection must not be less than $\sqrt{gh(1 + \operatorname{cosec}\beta)}$ (B.Sc. 57 Madras Uty.)

3. A particle is to the projected from a point P so as to pass through another point Q. Show that the product of the two times of flight from P to Q with a given velocity of projection is $2\frac{PQ}{g}$

(B.Sc. 82 Madras Uty.)

4. Show that the least speed with which a particle can be projected from a point on the ground so that it may pass over a vertical wall of height 50m. at a distance of $50\sqrt{3}$ m. from the point of projection, is that due to a fall of 75m. and find the direction of projection.

5. A particle is projected from a point A so as to pass through a second point B which is at a depth of 50m. below A and at a horizontal distance of 100m. from A. Show that the two possible directions of projection are at right angles if the velocity of projection is that due to a fall from a height of 100m.

(B.Sc. 72 Calicut Uty.)

§ 6.12. Range on an inclined plane:

From a point on a plane, which is inclined at an angle β to the horizon, a particle is projected with a velocity u at an angle α with the horizontal, in a plane passing through the normal to the inclined plane and the line of greatest slope. To find the range on the inclined plane.

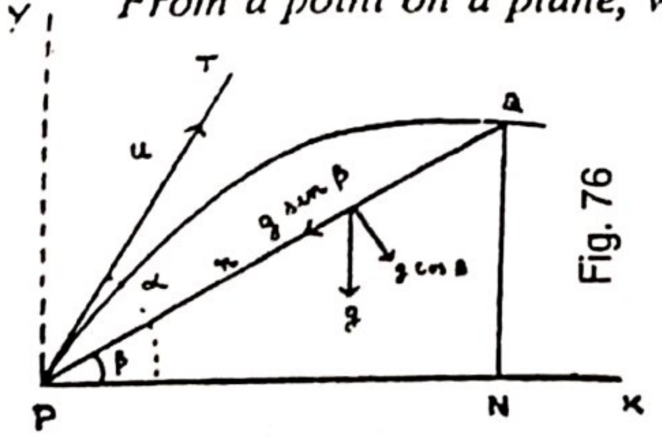


Fig. 76

Let P be the point of projection and the particle strike the inclined plane at Q. Then PQ is the range on the inclined plane. Let PQ = r. Taking P as the origin and the horizontal and the vertical through P as the axes of x and y respectively the equation to the path is,

$$y = x \tan\alpha - \frac{g x^2}{2 u^2 \cos^2\alpha} \qquad \ldots \quad (1)$$

Draw QN ⊥ to the horizontal plane through P. The co-ordinates of Q are $(r\cos\beta, r\sin\beta)$. Substituting these in (1),

$$r\sin\beta = r\cos\beta \cdot \tan\alpha - \frac{gr^2\cos^2\beta}{2u^2\cos^2\alpha}$$

Multiplying by $2u^2\cos^2\alpha$ and canceling r throughout, we have

$$2u^2\cos^2\alpha\sin\beta = 2u^2\cos\beta\sin\alpha\cos\alpha - gr\cos^2\beta$$

$$\therefore\ r = \frac{2u^2\cos\beta\sin\alpha\cos\alpha - 2u^2\cos^2\alpha\sin\beta}{g\cos^2\beta}$$

$$= \frac{2u^2\cos\alpha(\sin\alpha\cos\beta - \cos\alpha\sin\beta)}{g\cos^2\beta}$$

$$\text{i.e.}\quad r = \boxed{\frac{2u^2\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta}}$$

Aliter: We can study separately the motion of the particle along the inclined plane and the motion perpendicular to the plane. The initial velocity u can be resolved into two components (i) $u\cos(\alpha-\beta)$ along PQ, the inclined plane and (ii) $u\sin(\alpha-\beta)$, perpendicular to the inclined plane. The acceleration g can be resolved into two components (i) $g\cos\beta$ perpendicular to the inclined plane in the downward direction and (ii) $g\sin\beta$ along the inclined plane towards P. This resolution is shown in the figure. Let T be the time which the particle takes to go from P to Q. After time T, the particle is again on the inclined plane and so, during time T, the distance travelled perpendicular to the inclined plane is = 0.

$$\therefore \quad 0 = u\sin(\alpha - \beta) \, . \, T - \frac{1}{2} g \cos\beta \, . \, T^2$$

$$\boxed{\text{i.e.} \quad T = \frac{2u \sin(\alpha - \beta)}{g \cos \beta}}$$

This is the time of flight on the inclined plane. During this time, the horizontal velocity remains constant and $= u \cos\alpha$. So horizontal distance described in time $T = PN = u \cos\alpha \, T$. But $PN = PQ \cos\beta \; \therefore \; PQ \, . \cos\beta = u \cos\alpha \, T$

$$\text{i.e.} \quad PQ = \frac{u \cos \alpha}{\cos \beta} \, . \, T = \frac{u \cos\alpha}{\cos \beta} \; \frac{2u \sin(\alpha - \beta)}{g \cos\beta}$$

$$= \frac{2u^2 \sin(\alpha - \beta) \cos \alpha}{b \cos^2 \beta}$$

§ 6.13. To find the greatest distance of the projectile from the inclined plane and show that is attained in half the total time of flight:

Let us consider the motion perpendicular to the inclined plane. As explained in § 6.12., the initial velocity in this direction is $u \sin(\alpha - \beta)$ and this is subject to an acceleration $g \cos \beta$ in the same direction but acting downwards. Let y be the distance travelled by the particle in this direction in time t. Then

$$y = u \sin(\alpha - \beta) \, . \, t - \frac{1}{2} g \cos\beta \, . \, t^2 \quad \cdots \quad (1)$$

Differentiating with respect to t,

$$\frac{dy}{dt} = u \sin(\alpha - \beta) - g \cos\beta \, . \, t \quad \cdots \quad (2)$$

$$\text{and} \; \frac{d^2 y}{dt^2} = - g \cos\beta = \text{negative.}$$

So y is maximum when $\frac{dy}{dt} = 0$

i.e. when $u \sin(\alpha - \beta) - g \cos\beta \, . \, t = 0$

$$\text{i.e.} \quad t = \frac{u \sin(\alpha - \beta)}{g \cos \beta} \quad \cdots \quad (3)$$

Substituting (3) in (1), maximum value of y

$$= u\sin(\alpha-\beta) \cdot \frac{u\sin(\alpha-\beta)}{g\cos\beta} - \frac{1}{2} g\cos\beta \cdot \frac{u^2\sin^2(\alpha-\beta)}{g^2\cos^2\beta}$$

$$= \frac{u^2\sin^2(\alpha-\beta)}{g\cos\beta} - \frac{u^2\sin^2(\alpha-\beta)}{2g\cos\beta} = \frac{u^2\sin^2(\alpha-\beta)}{2g\cos\beta} \quad \ldots \quad (4)$$

(4) is the greatest distance of the projectile from the inclined plane.

Also, from (3), time to this greatest distance $= \frac{u\sin(\alpha-\beta)}{g\cos\beta}$ and this is clearly half of the time of flight.

Aliter: When the particle is at the greatest distance from the inclined plane, it will have all its velocity only parallel to the inclined plane. Hence the component velocity perpendicular to the inclined plane is zero. So, if s is the greatest distance, we have

$$0 = [u\sin(\alpha-\beta)]^2 - 2g\cos\beta \cdot s$$

$$\text{i.e. } s = \frac{u^2\sin^2(\alpha-\beta)}{2g\cos\beta}$$

Also if t is the corresponding time,

$$0 = u\sin(\alpha-\beta) - g\cos\beta t \quad \text{or} \quad t = \frac{u\sin(\alpha-\beta)}{g\cos\beta}$$

§ 6.14. To determine when the range on the inclined plane is maximum, given the magnitude u of the velocity of projection:

From § 6.12 the range R on the inclined plane is given by

$$R = \frac{2u^2\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta} = \frac{u^2}{g\cos^2\beta}[\sin(2\alpha-\beta) - \sin\beta] \ldots (1)$$

Now u and β are given. The quantity outside the bracket, $\frac{u^2}{g\cos^2\beta}$ is constant. So R is maximum, when the value of the expression inside the bracket is a maximum.

i.e. when $\sin(2\alpha-\beta)$ is greatest.

i.e. when $2\alpha-\beta = \frac{\pi}{2}$.

i.e. $\boxed{\alpha = \frac{\pi}{4} + \frac{\beta}{2}}$ for maximum range.

When α takes this value,

$$\alpha - \beta = (2\alpha - \beta) - \alpha = 90^\circ - \alpha \qquad \dots (2)$$

Referring to fig. 76,

$$\alpha - \beta = \angle TPN - \angle QPN = \angle TPQ \text{ and}$$

$$90^\circ - \alpha = \angle YPT$$

Hence from (2), $\angle TPQ = \angle YPT$.

i.e. PT, the direction of projection for maximum range bisects the angle between the vertical and the inclined plane.

From (1), the value of maximum range

$$= \frac{u^2}{g\cos^2\beta}(1 - \sin\beta) = \frac{u^2}{g(1+\sin\beta)}$$

§ 6.15. To show that, for a given initial velocity of projection, there are, in general, two possible directions of projection so as to obtain a given range on an inclined plane:

Let u be the velocity of projection of a particle and α the necessary angle of projection so as to get a given range k on an inclined plane of inclination β to the horizontal.

$$\text{Then } k = \frac{2u^2\sin(\alpha - \beta)\cos\alpha}{g\cos^2\beta} = \frac{u^2(\sin(2\alpha - \beta) - \sin\beta)}{g\cos^2\beta} \qquad \dots (1)$$

$$\text{From (1), } \sin(2\alpha - \beta) = \frac{gk\cos^2\beta}{u^2} + \sin\beta \qquad \dots (2)$$

Since k,u, β are given, the R.H.S. of (2) is a known positive quantity. So we can determine an acute angle θ whose sine is exactly to $\frac{gk\cos^2\beta}{u^2} + \sin\beta$

$$\text{Then (2) becomes, } \sin(2\alpha - \beta) = \sin\theta \qquad \dots (3)$$

$$\text{i.e. } 2\alpha - \beta = \theta \text{ or } \alpha = \frac{\theta}{2} + \frac{\beta}{2} \qquad \dots (4)$$

Since $\sin(180^\circ - \theta) = \sin\theta$, (3) can also be written as $\sin(2\alpha - \beta) = \sin(180^\circ - \theta)$. Then $2\alpha - \beta = 180^\circ - \theta$ ... (5)

i.e. $\alpha = 90^\circ - \frac{\theta}{2} + \frac{\beta}{2}$

From (4) and (5), we find that there are two value of α and so two directions of projection, each giving the same range k.

Let α_1 and α_2 these two values of α.

Then $\alpha_1 = \frac{\theta}{2} + \frac{\beta}{2}$ and $\alpha_2 = 90^\circ - \frac{\theta}{2} + \frac{\beta}{2}$

Now $(45^\circ + \frac{\beta}{2}) - \alpha_1 = 45^\circ + \frac{\beta}{2} - \frac{\theta}{2} - \frac{\beta}{2} = 45^\circ - \frac{\theta}{2}$

and $\alpha_2 - (45^\circ + \frac{\beta}{2}) = 90^\circ - \frac{\theta}{2} + \frac{\beta}{2} - 45^\circ - \frac{\beta}{2} = 45^\circ - \frac{\theta}{2}$

$\therefore\ (45^\circ + \frac{\beta}{2}) - \alpha_1 = \alpha_2 - (45^\circ + \frac{\beta}{2})$ ... (6)

But $45^\circ + \beta/2$ is the angle of projection for maximum range on the inclined plane. So (6) shows that the two directions α_1 and α_2 are equally inclined to the direction of maximum range.

Important Note: In articles § 6.12 to § 6.15 the direction of projection is expressed as an elevation to the horizontal. We can also take the elevation *relative to the inclined plane*. In problems, it should be carefully found out which of these angles is given.

§ 6.16 Motion on the surface of a smooth inclined plane:

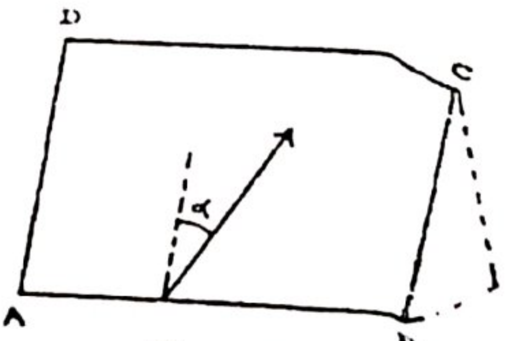


Fig. 77

Let a particle be projected with velocity u on the surface of a smooth inclined plane ABCD of slope β in a direction inclined at an angle α to the line of greatest slope of the plane. The acceleration due to gravity can be resolved into two components:

$$\text{i.e.} \quad \frac{U}{\sin(\alpha - \beta)} = \frac{V}{\cos\alpha} = \frac{u}{\cos\beta}$$

$$\therefore \quad U = \frac{u \sin(\alpha - \beta)}{\cos\beta} \quad \text{and} \quad V = \frac{u\cos\alpha}{\cos\beta}$$

$$\therefore \quad 2\frac{UV}{g} = \frac{2u\sin(\alpha-\beta)}{g\cos\beta} \times \frac{u\cos\alpha}{\cos\beta} = \frac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta}$$

= range on the inclined plane.

Ex. 16. *Show that, for a given velocity of projection the maximum range down an inclined plane of inclination* α *bears to the maximum range up the inclined plane the ratio* $\dfrac{1 + \sin\alpha}{1 - \sin\alpha}$

(B.Sc. 69 Madurai Uty. B.Sc. 92 Madras Uty.)

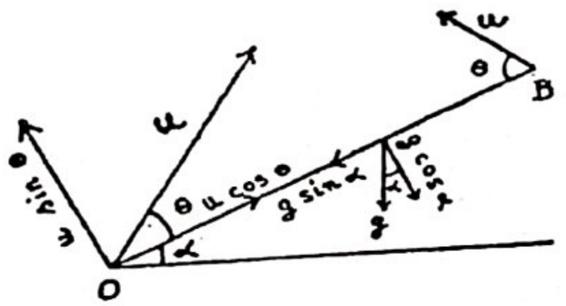


Fig. 79

Let u be the given velocity of projection and θ the inclination of the direction of projection with the plane. The velocity u can be resolved into two components u cos θ along the upward inclined plane and u sin θ perpendicular to the inclined plane. The acceleration g can be resolved into two components, g sin α along the downward inclined plane and g cos α perpendicular to the inclined plane and downwards.

Consider the motion perpendicular to the inclined plane. Let T be the time of flight. Distance travelled perpendicular to the inclined plane in time T is = 0.

$$\therefore \quad 0 = u\sin\theta \,.\, T - \frac{1}{2} g\cos\alpha \,.\, T^2$$

$$\text{i.e.} \quad T = \frac{2u\sin\theta}{g\cos\alpha}.$$

During this time, the distance travelled along the plane

$$= u\cos\theta \,.\, T - \frac{1}{2} g\sin\alpha \,.\, T^2$$

$$= u\cos\theta \,.\, \frac{2u\sin\theta}{g\cos\alpha} - \frac{1}{2} g\sin\alpha \,.\, \frac{4u^2\sin^2\theta}{g^2\cos^2\alpha}$$

$$= \frac{2u^2\sin\theta\cos\theta}{g\cos\alpha} - \frac{2u^2\sin\alpha\,\sin^2\theta}{g\cos^2\alpha}$$

$$= \frac{2u^2\sin\theta}{g\cos^2\alpha}\,(\cos\alpha\cos\theta - \sin\alpha\sin\theta).$$

$$= \frac{2u^2\sin\theta}{g\cos^2\alpha}\cos(\theta+\alpha) = \frac{u^2}{g\cos^2\alpha} \,.\, 2\cos(\theta+\alpha)\sin\theta$$

$$= \frac{u^2}{g\cos^2\alpha}\,[\sin(2\theta+\alpha) - \sin\alpha]$$

This is the range R_1 up the inclined plane.

R_1 is maximum, when $\sin(2\theta+\alpha) = 1$

∴ Maximum range up the plane

$$= \frac{u^2}{g\cos^2\alpha}\,(1 - \sin\alpha) = \frac{u^2}{g\,(1+\sin\alpha)}$$

When the particle is projected down the plane from B at the same angle to the plane, the time of flight has the same value $\frac{2u\sin\theta}{g\cos\alpha}$. But the component of the initial velocity along the inclined plane is $u\cos\theta$ downwards and the component acceleration $g\sin\alpha$ is also downwards.

Hence range down the plane

R_2 = distance travelled along the plane in time T

$$= u\cos\theta \,.\, T + \frac{1}{2} g\sin\alpha \,.\, T^2$$

$$= \frac{2u^2\sin\theta}{g\cos^2\alpha}\,(\cos\alpha\cos\theta + \sin\alpha\sin\theta)$$

$$= \frac{2u^2 \sin\theta}{g\cos^2\alpha}\cos(\theta-\alpha) = \frac{u^2}{g\cos^2\alpha}[\sin(2\theta-\alpha)-\sin\alpha]$$

R_2 is maximum, when $\sin(2\theta-\alpha) = 1.$

So maximum range down the plane

$$= \frac{u^2}{g\cos^2\alpha}(1+\sin\alpha) = \frac{u^2}{g(1-\sin\alpha)}$$

$$\therefore \frac{\text{Max. range down the plane}}{\text{Max. range up the plane}}$$

$$= \frac{u^2}{g(1-\sin\alpha)} \cdot \frac{g(1+\sin\alpha)}{u^2} = \frac{1+\sin\alpha}{1-\sin\alpha}$$

Note: The range R_2 down the plane can be got from the range R_1 up the plane, by changing α into $-\alpha$

Ex.17. *A particle is projected at an angle* α *with a velocity* u *and it strikes up an inclined plane of inclination* β *at right angles to the plane. Prove that (i)* $\cot\beta = 2\tan(\alpha-\beta)$

(B.Sc. 84, 87 Madras Uty; B.Sc. 79,80; Madurai Uty and B.Sc. 71 Calicut Uty;

(ii) $\cot\beta = \tan\alpha - 2\tan\beta$ *(B.Sc. 77 Madras Uty; B.Sc. 71 Calicut Uty)*

If the plane is struck horizontally, show that $\tan\alpha = 2\tan\beta$

Refer to fig. 76 § 6.12 page 172. The initial velocity and acceleration are split into components along the plane and perpendicular to the plane as explained. We have shown that the time of flight is $T = \dfrac{2u\sin(\alpha-\beta)}{g\cos\beta}$ ... (1)

Since the particle strikes the inclined plane normally, its velocity parallel to the inclined plane at the end of time T is = 0.

i.e. $0 = u\cos(\alpha-\beta) - g\sin\beta \,.\, T$

or $T = \dfrac{u\cos(\alpha-\beta)}{g\sin\beta}$ ... (2)

Equating (1) and (2), we have

$$\frac{2u \sin(\alpha - \beta)}{g \cos \beta} = \frac{u \cos(\alpha - \beta)}{g \sin \beta}$$

i.e. $\cot \beta = 2 \tan(\alpha - \beta)$ ... (i)

i.e. $\cot \beta = \dfrac{2(\tan \alpha - \tan \beta)}{1 + \tan \alpha \tan \beta}$

Cross-multiplying,

$\cot \beta + \tan \alpha = 2 \tan \alpha - 2 \tan \beta$ or

$\cot \beta = \tan \alpha - 2 \tan \beta$ ... (ii)

If the plane is struck horizontally, the vertical velocity of the projectile at the end of time T is = 0. Initial vertical velocity = $u \sin\alpha$, and acceleration in this direction = g downwards.

Vertical velocity in time $T = u \sin \alpha - gT$

$\therefore u \sin \alpha - gT = 0$ or $T = u \sin \alpha / g$ ... (3)

Equating (1) and (3), we have

$$\frac{2u \sin(\alpha - \beta)}{g \cos \beta} = \frac{u \sin \alpha}{g}$$

or $2 \sin(\alpha - \beta) = \sin\alpha \cos\beta$

i.e. $2(\sin \alpha \cos \beta - \cos \alpha \sin\beta) = \sin \alpha \cos \beta$.

i.e. $\sin \alpha \cos \beta = 2\cos \alpha \sin\beta$ or $\tan \alpha = 2\tan \beta$.

EXERCISES

1. The greatest range with a given velocity of projection on a horizontal plane is 3000 metres. Find the greatest ranges up and down a plane inclined at 30° to the horizon.

2.(a) An inclined plane is inclined at an angle of 30° to the horizon. Show that, for a given velocity of projection, the maximum range up the plane is 1/3 of the maximum range down the plane.

(b) If the greatest range down an inclined plane is three times its greatest range up the plane, show that the plane is inclined at 30° to the horizon. (B.Sc. 76, 83 Madras Uty.)

3. A particle is projected from the top of a plane inclined at 60° to the horizontal. If the direction of projection is (i) 30° above the

COLLISION OF ELASTIC BODIES

§ 8.1. Introduction:

A solid body has a definite shape. When a force is applied at any point of it tending to change its shape, in general, all solids which we meet with in nature yields slightly and get more or less deformed near the point. Immediately, internal forces come into play tending to restore the body to its original form and as soon as the disturbing force is removed, the body regains its original shape. The internal force which acts, when a body tends to recover its original shape after a deformation or compression is called the *force of restitution*. Also, the property which causes a solid body to recover its shape is called *elasticity*. If a body does not tend to recover its shape, it will cause no force of restitution and such a body is said to be *inelastic*.

Suppose a ball is dropped from any height h upon a hard floor. It strikes the floor with a velocity $u = \sqrt{2gh}$ and makes an impact. Soon it rebounds and moves vertically upwards with a velocity v. The height h_1 to which it rebounds is given by $h_1 = \frac{v^2}{2g}$ i.e. $v = \sqrt{2gh_1}$. Generally we find that $h_1 < h$. So $v<u$. As soon as the ball strikes the floor, the impulsive action of the floor rapidly stops the downward velocity of the ball and at the same time causes a temporary compression near the point of contact. Due to the elastic property of the solid, the ball tends to regain its original form quickly. It presses the floor and receives an equal and opposite impulsive reaction from it and with a new upward velocity, it rebounds.

Now, bodies made of various materials are elastic in different degrees. If balls of different materials (like iron, glass, lead etc) are dropped from the same height upon a floor or if the same ball is dropped upon floors of different constitution (like wooden floor, marble floor etc), it will be found that the heights to which they rebound after striking the floor will be different. In all these cases, the velocity of the ball on reaching the floor is the same, as it is dropped from the same height. But the velocity of the ball after impact is not the same in each case, as the height to which it rebounds is different. Thus due to the elastic property of solid bodies, a change in velocity takes place when they strike each other.

If v = u, the velocity with which the ball leaves the floor is the same as that with which it strikes it. In this case, the ball is said to be *perfectly elastic*. If v = 0, the ball does not rebound at all. It is said to be *inelastic*. More generally, when a body completely regains its shape after a collision, it is said to be *perfectly elastic*. If it does not come to its original shape, it is said to be *perfectly inelastic*. These two cases of bodies are only ideal.

In this chapter, we shall study some simple cases of the impact of elastic bodies. We shall consider the cases of particles in collision with particles, or planes and of spheres in collision with planes or spheres. In all cases, we consider the impinging bodies to be smooth, so that the only mutual action they can have on each other will be along the common normal at the point where they touch.

§ 8.2. Definitions:

Two bodies are said to *impinge directly* when the direction of motion of each before impact is along the common normal at the point where they touch.

They are said to *impinge obliquely*, if the direction of motion of either body or both is not along the common normal at the point where they touch.

The common normal at the point of contact is called the *line of impact*. Thus, in the case of two spheres, the line of impact is the line joining their centres.

§ 8.3. Fundamental Laws of Impact:

The following three general principles hold good when two smooth moving bodies make an impact.

1. Newton's Experimental Law:

Newton studied the rebound of elastic bodies experimentally and the result of his experiments is embodied in the following law:

When two bodies impinge directly, their relative velocity after impact bears a constant ratio to their relative velocity before impact and is in the opposite direction. If two bodies impinge obliquely, their relative velocity resolved along their common normal after impact bears a constant ratio to their relative velocity before impact, resolved in the same direction, and is of opposite sign.

The constant ratio depends on the material of which the bodies are made and is independent of their masses. It is generally denoted by e, and is called the *coefficient (or modulus) of elasticity (or restitution or resilience)*.

This law can be put symbolically as follows: If u_1, u_2 are the components of the velocities of two impinging bodies along their common normal before impact and v_1, v_2 their component velocities along the same line after impact, all components being measured in the same direction and e is the coefficient of restitution, then

$$\boxed{\frac{v_2 - v_1}{u_2 - u_1} = -e}$$

The quantity e, which is a positive number, is never greater than unity. It lies between 0 and 1. Its value differs widely for different bodies; for two glass balls it is about 0.9; for ivory 0.8; while for lead it is 0.2. For two balls, one of lead and the other of iron, its value is about 0.13. Thus, when one or both the bodies are altered, e becomes different but so long as both the bodies remain the same, e is constant. Bodies for which $e = 0$ are said to be *inelastic* while for *perfectly elastic* bodies, $e = 1$. Probably, there are no bodies in nature coming strictly under either of these headings. Newton's law is purely empirical and is true only approximately, like many experimental laws.

2. Motion of two smooth bodies perpendicular to the line of Impact:

When two smooth bodies impinge, the only force between them at the time of impact is the mutual reaction which acts along the common normal. There is no force acting along the common tangent and hence there is no change of velocity in that direction. Hence *the velocity of either body resolved in a direction perpendicular to the line of impact is not altered by impact.*

3. Principle of Conservation of Momentum:

We can apply the law of conservation of momentum in the case of two impinging bodies. *The algebraic sum of the momenta of the impinging bodies after impact is equal to the algebraic sum of their momenta before impact, all momenta being measured along the common normal.*

The above three principles are sufficient to study the changes in the motion of two impinging elastic bodies.

We shall now proceed to discuss particular cases.

§ 8.4. Impact of a smooth sphere on a fixed smooth plane:

A smooth sphere, or particle whose mass is m and whose coefficient of restitution is e, impinges obliquely on a smooth fixed plane; to find its velocity and direction of motion after impact.

Let AB be the plane and P the point at which the sphere strikes it. The common normal at P is the vertical line at P passing through the centre of the sphere. Let it be PC. This is the line of impact. Let the velocity of the sphere before impact be u at an angle α with CP and v its velocity after impact at an angle θ with CN as shown in the figure.

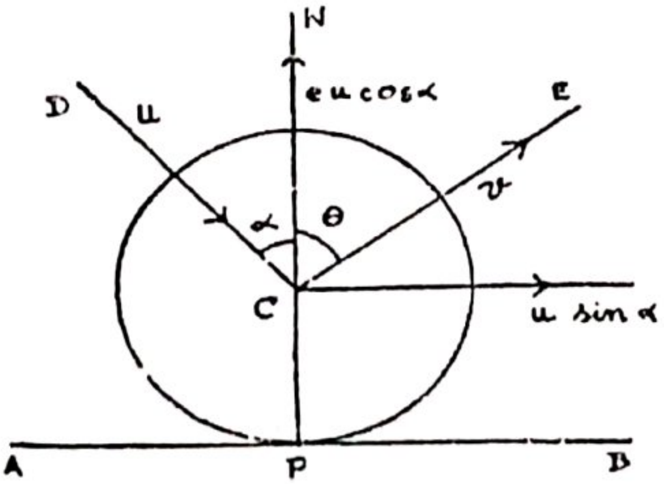


Fig. 86

Since the plane and the sphere are smooth, the only force acting during impact is the impulsive reaction and this is along the common normal. There is no force parallel to the plane during impact. Hence the velocity of the sphere, resolved in a direction parallel to the plane is unaltered by the impact.

Hence $v \sin\theta = u \sin\alpha$ ... (1)

By Newton's experimental law, the relative velocity of the sphere along the common normal after impact is (-e) time its relative velocity along the common normal before impact. Hence

$$v\cos\theta - 0 = -e(-u\cos\alpha - 0)$$

i.e. $v\cos\theta = eu\cos\alpha$ ... (2)

Squaring (1) and (2), and adding, we have

$v^2 = u^2(\sin^2\alpha + e^2\cos^2\alpha)$ i.e. $v = u\sqrt{\sin^2\alpha + e^2\cos^2\alpha}$ ... (3)

Dividing (2) by (1), we have $\cot\theta = e\cot\alpha$ ... (4)

Hence (3) and (4) give the velocity and direction of motion after impact.

Corollary 1: If $e = 1$, we find that from (3) $v = u$ and from (4) $\theta = \alpha$ Hence if a perfectly elastic sphere impinges on a fixed smooth plane, its velocity is not altered by impact and the angle of reflection is equal to the angle of incidence.

Cor. 2: If $e = 0$, then from (2), $v\cos\theta = 0$ and from (3), $v = u\sin\alpha$. Hence $\cos\theta = 0$ i.e. $\theta = 90°$. Hence the inelastic sphere slides along the plane with velocity $u\sin\alpha$.

Cor. 3: If the impact is direct we have $\alpha = 0$. Then $\theta = 0$ and from (3) $v = eu$. Hence *if an elastic sphere strikes a plane normally with velocity u, it will rebound in the same direction with velocity eu.*

Cor 4: The impulse of the pressure on the plane is equal and opposite to the impulse of the pressure on the sphere. The impulse I on the sphere is measured by the change in momentum of the sphere along the common normal.

$$I = mv\cos\theta - (-mu\cos\alpha) = m(v\cos\theta + u\cos\alpha)$$

$$= m(eu\cos\alpha + u\cos\alpha) = mu\cos\alpha(1+e)$$

Cor. 5 : Loss of kinetic energy due to the impact

$$= \frac{1}{2}mu^2 - \frac{1}{2}mv^2 = \frac{1}{2}mu^2 - \frac{1}{2}mu^2(\sin^2\alpha + e^2\cos^2\alpha)$$

$$= \frac{1}{2}mu^2(1 - \sin^2\alpha - e^2\cos^2\alpha) = \frac{1}{2}mu^2(\cos^2\alpha - e^2\cos^2\alpha)$$

$$= \frac{1}{2}(1-e^2)\ mu^2\cos^2\alpha$$

If the sphere is perfectly elastic e = 1 and the loss of kinetic energy is zero.

WORKED EXAMPLES

Ex.1. *A smooth circular table is surrounded by a smooth rim whose interior surface is vertical. Show that a ball projected along the table from a point A on the rim in a direction making an angle* α *with the radius through A will return to the point of projection after two impacts if* $\tan\alpha = \dfrac{e^{(3/2)}}{\sqrt{1+e+e^2}}$

(B.Sc. 59,62 Madras Uty. B.Sc. 73, 87 Madurai Uty.)

Prove also that when the ball returns to the point of projection, its velocity is to its original velocity as e (3/2):1

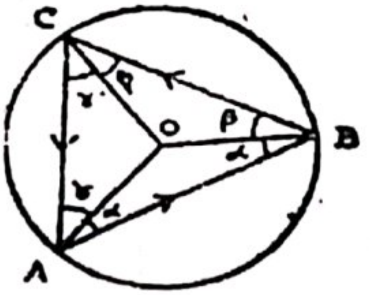


Fig. 87

Let the ball starting from A return to it after two reflections at B and C. At B, the point of the first impact, the common normal is the radius OB and at C, the point of the second impact, the common normal is OC.

Let $\angle OBC = \beta$ and $\angle OCA = \gamma$. Then $\angle OCB = \beta$ and $\angle OAC = \gamma$.

Considering the impact at B, and applying equation (4) of § 8.4, we have $\cot\beta = e\cot\alpha$ i.e. $\tan\beta = \frac{1}{e}\tan\alpha$ ... (1)

Similarly, considering the impact at C, $\cot\gamma = e\cot\beta$

i.e. $\tan\gamma = \frac{1}{e}\tan\beta = \frac{1}{e^2}\tan\alpha$ ... (2)

Now, in ΔABC, $\angle A + \angle B + \angle C = 2\alpha + 2\beta + 2\gamma = 180°$

$\therefore \alpha + \beta + \gamma = 90°$ or $\alpha = 90° - (\beta + \gamma)$

$\therefore \tan\alpha = \tan(90° - \overline{\beta + \gamma}) = \cot(\beta + \gamma)$

$$= \frac{1}{\tan(\beta + \gamma)} = \frac{1 - \tan\beta\tan\gamma}{\tan\beta + \tan\gamma}$$

i.e. $\tan\alpha(\tan\beta + \tan\gamma) = 1 - \tan\beta\tan\gamma$

i.e. $\tan\alpha\left(\frac{1}{e}\tan\alpha + \frac{1}{e^2}\tan\alpha\right) = 1 - \frac{1}{e}\tan\alpha \cdot \frac{1}{e^2}\tan\alpha$

uaing (1) and (2)

or $\tan^2\alpha\left(\frac{1}{e} + \frac{1}{e^2}\right) = 1 - \frac{\tan^2\alpha}{e^3}$

i.e. $\tan^2\alpha\left(\frac{1}{e} + \frac{1}{e^2} + \frac{1}{e^3}\right) = 1$ or $\tan^2\alpha\left(\frac{1 + e + e^2}{e^3}\right) = 1$

i.e. $\tan^2\alpha = \frac{e^3}{1 + e + e^2}$ ... (3)

or $\tan\alpha = \frac{e^{(3/2)}}{\sqrt{1 + e + e^2}}$

Let u be the velocity of projection from A, v be the velocity of the ball after the first impact at B and w be the velocity after the second impact at C.

Applying equation (3) of § 8.4, we have

$v^2 = u^2(\sin^2\alpha + e^2\cos^2\alpha)$ and $w^2 = v^2(\sin^2\beta + e^2\cos^2\beta)$

$\therefore w^2 = u^2(\sin^2\alpha + e^2\cos^2\alpha)(\sin^2\beta + e^2\cos^2\beta)$

$= u^2\cos^2\alpha(\tan^2\alpha + e^2) \cdot \cos^2\beta(\tan^2\beta + e^2)$

$$= \frac{u^2(\tan^2\alpha + e^2)(\tan^2\beta + e^2)}{(1 + \tan^2\alpha)(1 + \tan^2\beta)}$$

$$= \frac{u^2 (\tan^2 \alpha + e^2)\ (\frac{1}{e^2} \tan^2 \alpha + e^2)}{(1 + \tan^2 \alpha)\ (1 + \frac{1}{e^2} \tan^2 \alpha)} \quad \text{using (1)}$$

$$= \frac{u^2 (\tan^2 \alpha + e^2)\ (\tan^2 \alpha + e^4)}{(1 + \tan^2 \alpha)\ (e^2 + \tan^2 \alpha)} = \frac{u^2 (\tan^2 \alpha + e^4)}{(1 + \tan^2 \alpha)}$$

$$= \frac{u^2 [\frac{e^4}{1 + e + e^2} + e^4]}{[1 + \frac{e^4}{1 + e + e^2}]} \quad \text{substituting from (3)}$$

$$= \frac{u^2 (e^3 + e^4 + e^5 + e^6)}{(1 + e + e^2 + e^3)} = u^2 e^3$$

$\therefore \quad w = u \,.\, e^{(3/2)} \qquad \text{or} \qquad w : u = e^{(3/2)} : 1$

Ex.2. *A particle falls from a height h upon a fixed horizontal plane: if e be the coefficient of restitution, show that the whole distance described before the particle has finished rebounding is* $h\left(\frac{1 + e^2}{1 - e^2}\right)$. (B.Sc. 71 Madras Uty; B.Sc. 75, 81, 85 Madurai Uty.)

Show also that the whole time taken is $\frac{1 + e}{1 - e}\sqrt{\frac{2h}{g}}$

(B.Sc. 71 Calicut Uty.)

Let u the velocity of the particle on first hitting the plane. Then $u^2 = 2gh$. After the first impact, the particle rebounds with a velocity eu and ascends a certain height, retraces its path and makes a second impact with the plane with velocity eu. After the second impact, it rebounds with a velocity e^2u and the process is repeated a number of times. The velocities after the third, fourth etc. impacts are e^3u, e^4u etc.

The height ascended after the first impact with velocity eu

$$= \frac{(\text{velocity})^2}{2g} = \frac{e^2 u^2}{2g}$$

The height ascended after the second impact with velocity $e^2 u = e^4 u^2/2g$ and so on.

$\therefore$ Total distance travelled before the particle stops rebounding

$$= h + 2\left(\frac{e^2 u^2}{2g} + \frac{e^4 u^2}{2g} + \frac{e^6 u^2}{2g} + \dots\dots\dots\right)$$

$$= h + \frac{2 . e^2 u^2}{2g}(1 + e^2 + e^4 + \dots\dots\dots \text{ to } \infty)$$

$$= h + \frac{e^2 u^2}{g} . \frac{1}{1 - e^2} = h + \frac{e^2 . 2gh}{h} . \frac{1}{1 - e^2}$$

$$= h\ 1 + \left(\frac{2e^2}{1 - e^2}\right) = h . \frac{(1 + e^2)}{(1 - e^2)}$$

Considering the motion before the first impact, we have the initial velocity = 0, acceleration = g, final velocity = u and so if t is the time taken, u = 0 + gt.

$$\therefore \quad t = \frac{u}{g} = \frac{\text{velocity}}{g}$$

Time interval between the first and second impacts is

= 2 × time taken for gravity to reduce the velocity eu to 0.

= 2 . velocity / g = 2 eu / g.

Similarly time interval between the second and third impacts

= $2 e^2 u/g$ and so on.

So total time taken

$$= \frac{u}{g} + 2\left(\frac{eu}{g} + \frac{e^2 u}{g} + \frac{e^3 u}{g} + \dots\dots \infty\right)$$

$$= \frac{u}{g} + \frac{2 e u}{g}(1 + e + e^2 + \dots\dots\dots \text{ to } \infty)$$

$$= \frac{u}{g} + \frac{2eu}{g} . \frac{1}{1 - e} = \frac{u}{g}\left[1 + \frac{2e}{1 - e}\right]$$

$$= \frac{u}{g}\left(\frac{1+e}{1-e}\right) = \frac{\sqrt{2gh}}{g}\left(\frac{1+e}{1-e}\right) = \left(\frac{1+e}{1-e}\right)\sqrt{\frac{2h}{g}}$$

University Question

Ex.3. *A particle of elasticity 'e' is dropped from a vertical height 'a' upon the highest point of a plane which is of length b and is inclined at an angle* α *to the horizon and descends to the bottom in three jumps. Show that*

$$b = 4ae(1+e)(1+e^2)(1+e+e^2)\sin\alpha$$

The downward vertical velocity at A before striking $= \sqrt{2ga}$ and let this be $= u$. This can be resolved into two components as $u\cos\alpha$ perpendicular to the inclined plane and $u\sin\alpha$ parallel to the plane. At the impact at A, there is no force parallel to the plane and there is only the impulsive reaction normal to the plane. So the component $u\sin\alpha$ is not affected by impact while the component $u\cos\alpha$ is reversed as $eu\cos\alpha$. Hence the particle describes a parabola and strikes the plane at B with a velocity $eu\cos\alpha$ perpendicular to it. After impact, this is reversed as $e^2u\cos\alpha$. The particle strikes the inclined plane at C with a velocity $e^2u\cos\alpha$ normal to it, which is reversed as $e^3u\cos\alpha$. Let t_1, t_2, t_3 be the times taken to describe the paths AB, BC and CD respectively.

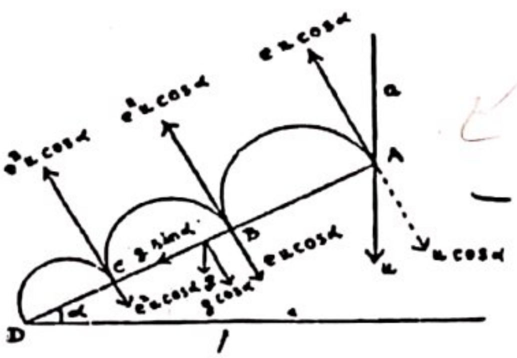


Fig. 88

Consider the motion perpendicular to the inclined plane. Distance travelled in that direction in the time $t_1 = 0$. Applying the formula "$s = ut + \frac{1}{2}ft^2$" we have

$$0 = eu\cos\alpha \,.\, t_1 - \frac{1}{2}g\cos\alpha \,.\, t_1^2$$

$$\therefore\ t_1 = \frac{2eu\cos\alpha}{g\cos\alpha} = \frac{2eu}{g}$$

Similarly, $t_2 = \dfrac{2e^2u}{g}$ and $t_3 = \dfrac{2e^3u}{g}$

Hence the total time taken from A to D

$$= t_1 + t_2 + t_3 + = \frac{2eu}{g}(1 + e + e^2)$$

In this period the particle has described a distance b down the inclined plane, starting with an initial velocity $u \sin \alpha$ and acted on by an acceleration $g \sin \alpha$.

$$\therefore \; b = u \sin \alpha (t_1 + t_2 + t_3) + \frac{1}{2} g \sin \alpha (t_1 + t_2 + t_3)^2$$

$$\text{i.e. } b = u \sin \alpha \,.\, \frac{2eu}{g}(1 + e + e^2) + \frac{1}{2} g \sin \alpha \,.\, \frac{4e^2u^2}{g^2} \times (1 + e + e^2)^2$$

$$= \frac{2eu^2 \sin \alpha}{g}(1 + e + e^2) + \frac{2e^2u^2 \sin \alpha}{g}(1 + e + e^2)^2$$

$$= \frac{2eu^2 \sin \alpha}{g}(1 + e + e^2)\,[\,1 + e(1 + e + e^2)\,]$$

$$= \frac{(2eu^2 \sin \alpha)}{g}(1 + e + e^2)(1 + e + e^2 + e^3)$$

$$= \frac{2e \,.\, 2ga \,.\, \sin \alpha}{g}(1 + e + e^2)(1 + e)(1 + e^2)$$

$$= 4ae \sin \alpha (1 + e)(1 + e^2)(1 + e + e^2)$$

Ex.4. *A particle is projected from a point on an inclined plane and at the rth impact it strikes the plane perpendicularly and at the nth impact is at the point of projection. Show that* $e^n - 2e^r + 1 = 0$.

(B.Sc. 49 Madras Uty.)

Let α be the inclination of the plane to the horizontal and u the velocity of projection at an angle θ to the inclined plane. This velocity can be resolved into two components, $u \cos \theta$ along the upward inclined plane and $u \sin \theta$ perpendicular to the inclined plane. The acceleration g can be resolved into two components. $g \sin \alpha$ along the downward inclined plane and $g \cos \alpha$

perpendicular to the inclined plane and downwards. Consider motion perpendicular to the plane.

Let t_1 be the time upto the first impact.

Distance travelled perpendicular to the plane in time t_1 is 0.

i.e. $0 = u \sin\theta \,.\, t_1 - \frac{1}{2} g \cos\alpha \,.\, t_1^{\,2}$

$$t_1 = \frac{2u \sin\theta}{g \cos\alpha}.$$

The particle strikes the plane the first time with a velocity $u \sin\theta$ perpendicular to it and after this impact, this component is reversed as $eu \sin\theta$. Hence time interval between the first and second impacts = $2eu \sin\theta / g \cos\alpha$. The particle strikes the plane a second time with a velocity $eu \sin\theta$ perpendicular to it and after the second impact, this component is reversed as $e^2 u \sin\theta$. Hence time interval between the second and third impacts = $2e^2 u \sin\theta / g \cos\alpha$ and so on.

Time till the rth impact

$$= \frac{2u \sin\theta}{g \cos\alpha} + \frac{2eu \sin\theta}{g \cos\alpha} + \frac{2e^2 u \sin\theta}{g \cos\alpha} + \dots\dots\dots \text{ to } r \text{ terms}$$

$$= \frac{2u \sin\theta}{g \cos\alpha} (1 + e + e^2 + \dots\dots \text{ to } r \text{ terms})$$

$$= \frac{2u \sin\theta}{g \cos\alpha} \,.\, \left(\frac{1 - e^r}{1 - e}\right) \qquad \dots (1)$$

At the end of this time, the particle strikes the plane perpendicularly. So the velocity parallel to the plane at that instant = 0.

Hence $u \cos\theta - g \sin\alpha \,.\, \frac{2u \sin\theta}{g \cos\alpha} \,.\, \left(\frac{1 - e^r}{1 - e}\right) = 0$

i.e. $\cos\theta \cos\alpha (1 - e) = 2 \sin\alpha \sin\theta (1 - e^r) \qquad \dots (2)$

Putting $r = n$ in (1).

13. A small elastic sphere is projected with a given velocity V from the foot of a vertical wall, in the vertical plane normal to the wall. It strikes a second parallel wall at a distance 'a' and after rebounding, strikes the first wall at P. Show that the greatest height of P above the point of projection is $\frac{1}{2g}\left[V^2 - \frac{(1+e)^2 g^2 a^2}{e^2 V^2}\right]$

(B.Sc. 38 Madras Uty.)

14. A billiard ball of coefficient of elasticity e is projected from the centre of a billiard table ABCD where AB = CD = 2a and BC = AD = 2b so as to return to the centre after three impacts, first with AB, then with BC and afterwards with CD. Show that if α is the angle the direction of projection makes with AB.

$\tan\alpha = \frac{b(1+e)}{ae}$ (B.Sc. 35 Madras Uty.)

15. A smooth ring is fixed horizontally on a smooth table and from a point of the ring a particle is projected along the surface of the table. If e be the coefficient of restitution between the ring and the particle, show that, the latter will after three rebounds return to the point of projection, if its initial direction of projection makes an angle $\tan^{-1}(e^{(3/2)})$ with the normal to the ring.

16. A smooth elliptical tray is surrounded by a smooth vertical rim. Prove that a perfectly elastic particle projected from a focus along the tray in any direction will after two impacts return to the focus. (B.Sc. 72 Madras Uty.)

§ 8.5. Direct impact of two smooth spheres:

A smooth sphere of mass m_1 impinges directly with velocity u_1 on another smooth sphere of mass m_2, moving in the same direction with velocity u_2; if the coefficient of restitution is e, to find their velocities after the impact:

AB is the line of impact, i.e. the common normal. Due to the impact there is no tangential force and hence, for either sphere the velocity along the tangent is not altered by impact. But before impact, the spheres had been moving only along the line AB (as this is a case of direct impact). Hence for either sphere tangential velocity after impact = its tangent velocity before impact = 0. So, after impact, the spheres will move only in the direction AB. Let their velocities be v_1 and v_2.

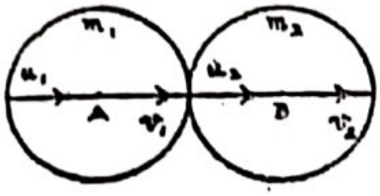


Fig. 90

By Newton's experimental law, the relative velocity of m_2 with respect to m_1 after impact is (-e) times the corresponding relative velocity before impact.

$$\therefore \quad v_2 - v_1 = -e\,(u_2 - u_1) \qquad \text{... (1)}$$

By the principle of conservation of momentum, the total momentum along the common normal after impact is equal to the total momentum in the same direction before impact.

$$\therefore \quad m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2 \qquad \text{... (2)}$$

(2) - (1) × m_2 gives

$$v_1(m_1 + m_2) = m_1 u_1 + m_2 u_2 + em_2(u_2 - u_1)$$

$$= m_2 u_2(1 + e) + (m_1 - em_2)u_1$$

$$\therefore \quad v_1 = \frac{m_2 u_2(1+e) + (m_1 - em_2)u_1}{m_1 + m_2} \qquad \text{... (3)}$$

(1) × m_1 + (2) gives

$$v_2(m_1 + m_2) = -em_1(u_2 - u_1) + m_1 u_1 + m_2 u_2$$

$$= m_1 u_1(1 + e) + (m_2 - em_1)u_2$$

$$\therefore \quad v_2 = \frac{m_1 u_1(1+e) + (m_2 - em_1)u_2}{m_1 + m_2} \qquad \text{... (4)}$$

Equations (3) and (4) give the velocities of the spheres after impact.

Note: If one sphere say m_2 is moving originally in a direction opposite to that of m_1, the sign of u_2 will be negative. Also it is most important that the directions of v_1 and v_2 must be specified clearly. Usually we take the positive direction as from left to right and then assume that both v_1 and v_2 are in this direction. If either of them is actually in the opposite direction, the value obtained for it will turn to be negative.

In writing equation (1) corresponding to Newton's law, the velocities must be subtracted in the same order on both sides. In all problems it is better to draw a diagram showing clearly the positive direction and the directions of the velocities of the bodies.

Corollary 1. If the two spheres are perfectly elastic and of equal mass, then $e = 1$ and $m_1 = m_2$. Then, from equations (3) and (4), we have

$$v_1 = \frac{m_1 u_2 \,.\, 2 + 0}{2m_1} = u_2 \text{ and } v_2 = \frac{m_1 u_1 \,.\, 2 + 0}{2m_1} = u_1.$$

i.e. *If two equal perfectly elastic spheres impinge directly, they interchange their velocities.*

Cor. 2. The impulse of the blow on the sphere A of mass m_1 = change of momentum of A = $m_1 (v_1 - u_1)$.

$$= m_1 \left[\frac{m_2 u_2 (1+e) + (m_1 - em_2) u_1}{m_1 + m_2} - u_1 \right]$$

$$= m_1 \left[\frac{m_2 u_2 (1+e) + m_1 u_1 - em_2 u_1 - m_1 u_1 - m_2 u_1}{m_1 + m_2} \right]$$

$$= \frac{m_1 [m_2 u_2 (1+e) - m_2 u_1 (1+e)]}{m_1 + m_2}$$

$$= \frac{m_1 m_2 (1+e)(u_2 - u_1)}{m_1 + m_2}$$

The impulsive blow on m_2 will be equal and opposite to the impulsive blow on m_1.

8.6. Loss of kinetic energy due to direct impact of two smooth spheres:

Two spheres of given masses with given velocities impinge directly; to show that there is a loss of kinetic energy and to find the amount:

Let m_1 m_2 be the masses of the spheres, u_1 and u_2, v_1 and v_2 be their velocities before and after impact and e the coefficient of restitution.

By Newton's law, $v_2 - v_1 = -e(u_2 - u_1)$ ... (1)

By the principle of conservation of momentum,

$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2$ ...(2)

Total kinetic energy before impact

$= \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2$ and total kinetic energy after impact

$= \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2$

Change in K.E. = initial K.E. - final K.E.

$$= \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 - \frac{1}{2}m_1 v_1^2 - \frac{1}{2}m_2 v_2^2$$

$$= \frac{1}{2} m_1 (u_1 - v_1)(u_1 + v_1) + \frac{1}{2} m_2 (u_2 - v_2)(u_2 + v_2)$$

$$= \frac{1}{2} m_1 (u_1 - v_1)(u_1 + v_1) + \frac{1}{2} m_1 (v_1 - u_1)(u_2 + v_2)$$

$[\because \quad m_2 (u_2 - v_2) = m_1 (v_1 - u_1)$ from (2) $]$

$$= \frac{1}{2} m_1 (u_1 - v_1)[u_1 + v_1 - (u_2 + v_2)]$$

$$= \frac{1}{2} m_1 (u_1 - v_1)[u_1 - u_2 - (v_2 - v_1)]$$

$$= \frac{1}{2} m_1 (u_1 - v_1)[u_1 - u_2 + e(u_2 - u_1)] \text{ using (1)}$$

$$= \frac{1}{2} m_1 (u_1 - v_1)(u_1 - u_2)(1 - e) \quad .... (3)$$

Now, from (2), $m_1 (u_1 - v_1) = m_2 (v_2 - u_2)$

$$\therefore \frac{u_1 - v_1}{m_2} = \frac{v_2 - u_2}{m_1} \text{ and each } = \frac{u_1 - v_1 + v_2 - u_2}{m_1 + m_2}$$

$$\text{i.e. each} = \frac{(u_1 - u_2) + (v_2 - v_1)}{m_1 + m_2}$$

$$= \frac{(u_1 - u_2) - e\,(u_2 - u_1)}{m_1 + m_2} \quad \text{using (1)}$$

$$= \frac{(u_1 - u_2)\ (1 + e)}{m_1 + m_2}$$

$$\therefore\ u_1 - v_1 = \frac{m_2\ (u_1 - u_2)\ (1 + e)}{m_1 + m_2} \quad \text{and substituting this in (3),}$$

$$\text{Change in K.E.} = \frac{1}{2}\ \frac{m_1 m_2 (u_1 - u_2)(1 + e)(u_1 - u_2)(1 - e)}{m_1 + m_2}$$

$$= \frac{1}{2}\ \frac{m_1 m_2 (u_1 - u_2)^2\ (1 - e^2)}{(m_1 + m_2)} \qquad \dots (4)$$

As e < 1, the expression (4) is always positive and so the initial K.E. of the system is greater than the final K.E. So there is actually a loss of total K.E. by a collision. Only in the case, when e=1, i.e. only when the bodies are perfectly elastic, the expression (4) becomes zero and hence the total K.E. is unchanged by impact.

Ex.7. A *ball of mass 8gm. moving with a velocity of 10 cm. per sec. impinges directly on another of mass 24 gm., moving at 2cm per sec. in the same direction. If e = ½, find the velocities after impact. Also calculate the loss in kinetic energy.*

(B.Sc. 71 Calicut Uty.)

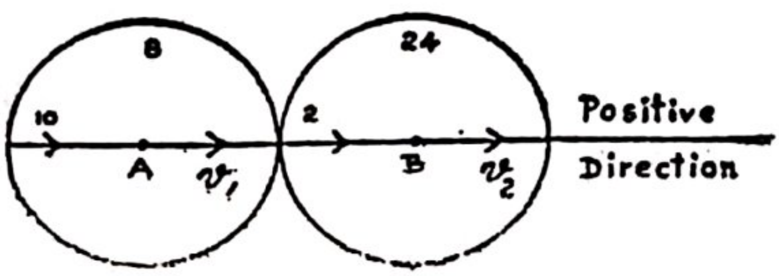


Fig. 91

Let v_1 and v_2 cm. per sec. be the velocities of the masses 8gm and 24gm respectively after impact.

By Newton's Law, $v_2 - v_1 = -\frac{1}{2}(2 - 10) = 4$ ... (1)

By the principle of momentum,

$24v_2 + 8v_1 = 24 \times 2 + 8 \times 10 = 128$

i.e. $3v_2 + v_1 = 16$ ... (2)

Solving (1) and (2), $v_1 = 1$ cm. / sec., $v_2 = 5$ cm./ sec.

The K.E. before impact $= \frac{1}{2} . 8.10^2 + \frac{1}{2} . 24.2^2$

$= 448$ dynes

The K.E. after impact $= \frac{1}{2} . 8.1^2 + \frac{1}{2} . 24.5^2 = 304$ dynes.

$\therefore$ Loss in K.E. = 144 dynes

Ex.8. *If the 24gm. mass in the previous question be moving in a direction opposite to that of the 8gm. mass, find the velocities after impact.*

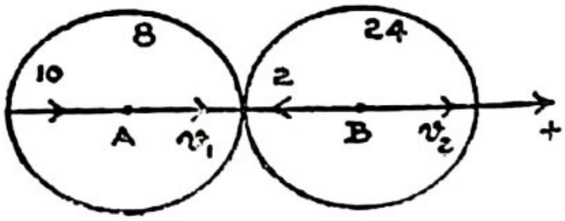


Fig. 92

Let v_1 and v_2 cm/sec. be the velocities of the 8gms and 24 gms mass respectively after impact.

By Newton's law,

$v_2 - v_1 = -\frac{1}{2}(-2 - 10) = 6$ ... (1)

By conservation of momentum,

$24v_2 + 8v_1 = 24 \times (-2) + 8 \times 10 = 32$ i.e. $3v_2 + v_1 = 4$ ... (2)

Solving (1) and (2), $v_1 = -\frac{1}{2}$ cm/sec $v_2 = \frac{5}{2}$ cm/sec.

The negative sign of v_1 shows that the direction of motion of the 8gm. mass is reversed, as we had taken the direction left to right as positive and assumed v_1 to be in this direction. Since v_2 is positive, the 24gm. ball moves from left to right after impact, so that its direction of motion is also reversed.

Ex. 9. *A ball overtakes another ball of m times its mass, which is moving with* $\frac{1}{n}$ *th of its velocity in the same direction. If the impact reduces the first ball to rest, prove that the coefficient of elasticity is* $\frac{m+n}{m(n-1)}$.

(B.Sc. 52 Madras; B.Sc 83 Madurai; B.Sc. 71 Kerala Uty.)

Deduce that $m > \frac{n}{n-2}$. *(B.Sc. 7? Madras Uty; B.Sc. 94 Bharathidasan Uty.)*

Taking AB in fig. 91 as the positive direction, let the mass of the first ball be k and u its velocity along AB before impact. Then, for the second ball, the mass is mk and $\frac{u}{n}$ is the velocity before impact. After impact, the first ball is reduced to rest and let v be the velocity of the second ball.

By Newton's law of impact, we have

$$v - 0 = -e.\left(\frac{u}{n} - u\right) \text{ i.e. } v = \frac{eu(n-1)}{n} \quad \text{... (1)}$$

By principle of conservation of momentum along AB,

$$k \times 0 + mk.\, v = ku + mk.\, \frac{1}{n} u$$

$$\text{i.e. } mv = u + \frac{m}{n} u = \frac{u(m+n)}{n} \quad \text{... (2)}$$

Substituting the value of v from (1) in (2), we have

$$\frac{meu(n-1)}{n} = \frac{u(m+n)}{n} \text{ or } e = \frac{(m+n)}{m(n-1)}$$

Now e is positive and less than 1.

$$\therefore m(n-1) > m+n \text{ i.e. } mn - 2m > n$$

$\therefore\ m(n-2) > n \quad \text{or} \quad m > \dfrac{n}{n-2}$

Ex.10. *Two equal spheres A and B, of masses 2 gm. and 30 gm. respectively lie on a smooth floor, so that their line of centres is perpendicular to a fixed vertical wall. A being nearer to the wall. A is projected towards B. Show that if the coefficient of restitution between the two spheres and that between the first sphere and the wall is $\frac{3}{5}$, then A will be reduced to rest after its second impact with B.*

(B.Sc. 62 Madras, B.Sc. 86 Calicut
B.Sc. 70 Kerala Uty., B.Sc. 76 Madurai Uty.)

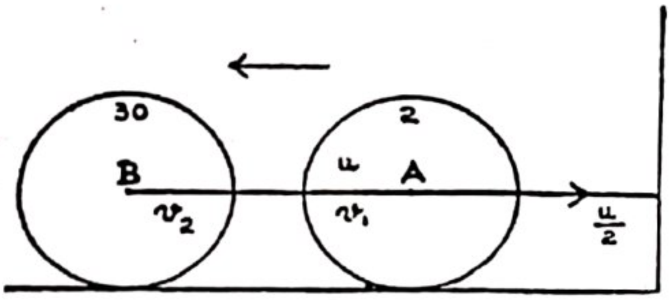


Fig. 93

Consider the impact between A and B. Taking AB as the positive direction, let the velocity of A before impact be u. B is at rest. After the impact, let the velocities of A and B be v_1 and v_2 respectively in the same direction.

By Newton's rule,

$$v_2 - v_1 = -e(0 - u) = \frac{3}{5}u \qquad \text{... (1)}$$

By conservation of momentum along AB,

$$30v_2 + 2v_1 = 30 \times 0 + 2 \times u \text{ i.e. } 15v_2 + v_1 = u \qquad \text{... (2)}$$

Solving (1) and (2), we get $v_1 = -\frac{u}{2}$ and $v_2 = \frac{u}{10}$.

Since v_1 is negative, the velocity of A after the impact towards the wall and $= \frac{u}{2}$ while the velocity of B is $\frac{u}{10}$ away from the wall.

Now A strikes the wall with a velocity $\frac{u}{2}$. After this impact, its velocity will be reversed as $e\left(\frac{u}{2}\right) = \frac{3}{5} \cdot \frac{u}{2} = \frac{3u}{10}$. With this velocity, A moves in the direction AB, away from the wall and strikes B a second time. Let the velocities of A and B be v_3 and v_4 after this impact, in the direction AB. For convenience, the velocity distribution can be noted as follows:

	A (2)	B (30)
Before impact	$\frac{3u}{10}$	$\frac{u}{10}$
After impact	v_3	v_4

By Newton's rule,

$$v_4 - v_3 = -e\left(\frac{u}{10} - \frac{3u}{10}\right) = \frac{3u}{25} \quad \text{... (3)}$$

By conservation of momentum,

$$30v_4 + 2v_3 = 30.\frac{u}{10} + 2.\frac{3u}{10} = \frac{18u}{5}$$

$$\text{i.e. } 15v_4 + v_3 = \frac{9u}{5} \quad \text{... (4)}$$

Multiplying (3) by 15, we have

$$15v_4 - 15v_3 = \frac{9u}{5} \quad \text{... (5)}$$

Subtracting (5) from (4), $16\ v_3 = 0$ or $v_3 = 0$.

i.e. A is reduced to rest after its second impact with B.

Ex. 11. *Two equal marble balls A, B lie in a horizontal circular groove at the opposite ends of a diameter; A is projected along the groove and after time t, impinges on B; show that a second impact takes place after a further interval* $\frac{2t}{e}$

(B.Sc. 95 Bharathidasan Uty.; B.Sc. 63 Madras Uty; B.Sc. 85 Calicut)

Let the ball A move with velocity u. As there is no tangential force acting on A at any point of its path, its speed remains the same throughout. Hence it impinges on B with a velocity u.

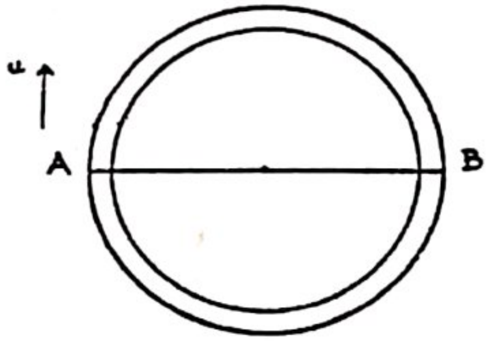


Fig. 94

Since the time from A to B is = t, we get $ut = \pi r$ or $u = \frac{\pi r}{t}$ ... (1)

Let v and v′ be the velocities of A and B respectively after impact.

Then, by the principle of momentum,

$mv + mv' = mu$ (m being the mass of each ball)

i.e. $v + v' = u$ ... (2)

Also, by Newton's law, $v - v' = -e(u - 0)$

i.e. $v - v' = -eu$ ... (3)

Solving (2) and (3), we get $v = \frac{u}{2}(1 - e)$; $v' = \frac{u}{2}(1 + e)$

Clearly v′ is greater than v. Hence B will move in advance of A. Let it strike A again t_1 secs. after the first impact.

The velocity of B relative to A, after the first impact = $v' - v$ = eu from (3)

Before striking again, B should cover a distance equal length to the circumference relative to A.

$\therefore (v' - v) . t_1 = 2\pi r$ i.e. $eu . t_1 = 2\pi r$

$$t_1 = \frac{2\pi r}{eu} = \frac{2\pi r}{e.\left(\frac{\pi r}{t}\right)} \text{ using (1)}$$

$$= \frac{2t}{e}$$

The second impact occurs $\frac{2t}{e}$ secs. after the first.

the balls is e, show that, the coefficient of restitution between the ball and the cushion is $\frac{1-e}{3e-1}$ (B.A. 51 Andhra Uty.)

§ 8.7. Oblique impact of two smooth spheres:

A smooth sphere of mass m_1 impinges obliquely with velocity u_1 on another smooth sphere of mass m_2 moving with velocity u_2. If the directions of motion before impact make angles α_1 and α_2 respectively with the line joining the centres of the spheres and if the coefficient of restitution be e, to find the velocities and directions of motion after impact.

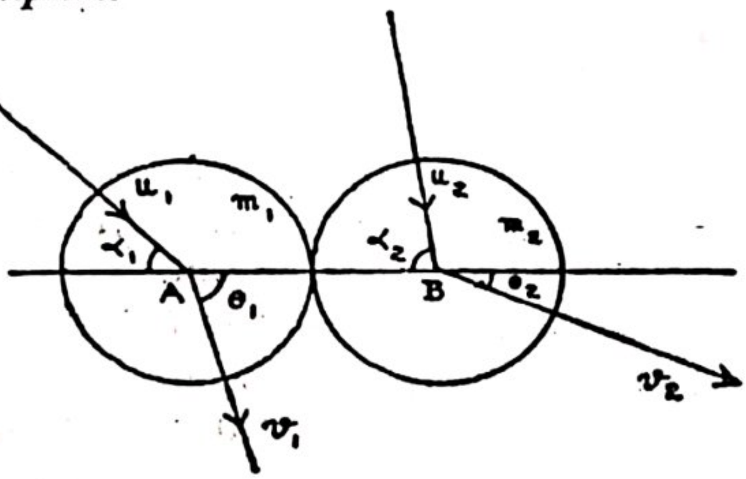


Fig. 95

Let the velocities of the spheres after impact be v_1 and v_2 in directions inclined at angles θ_1 and θ_2 respectively to the line of centres. Since the spheres are smooth, there is no force perpendicular to the line of centres and therefore, for each sphere the velocities in the tangential direction are not affected by impact.

$\therefore\ v_1 \sin\theta_1 = u_1 \sin\alpha_1$ ... (1) and

$v_2 \sin\theta_2 = u_2 \sin\alpha_2$ ... (2)

By Newton's law concerning velocities along the common normal AB,

$v_2 \cos\theta_2 - v_1 \cos\theta_1 = -e(u_2 \cos\alpha_2 - u_1 \cos\alpha_1)$ ... (3)

By the principle of conservation of momentum along AB,

$m_2 . v_2 \cos\theta_2 + m_1 . v_1 \cos\theta_1 = m_2 . u_2 \cos\alpha_2 + m_1 . u_1 \cos\alpha_1$ (4)

(4) - (3) x m_2 gives

$$v_1 \cos\theta_1 . (m_1 + m_2) = m_2 u_2 \cos\alpha_2 + m_1 u_1 \cos\alpha_1 + em_2 (u_2 \cos\alpha_2 - u_1 \cos\alpha_1)$$

$$\text{i.e. } v_1 \cos\theta_1 = \frac{u_1 \cos\alpha_1 (m_1 - em_2) + m_2 u_2 \cos\alpha_2 (1 + e)}{m_1 + m_2} \quad \dots (5)$$

(4) + (3) × m_1 gives

$$v_2 \cos\theta_2 = \frac{u_2 \cos\alpha_2 (m_2 - em_1) + m_1 u_1 \cos\alpha_1 (1 + e)}{m_1 + m_2} \quad \dots (6)$$

From (1) and (5), by squaring and adding, we obtain $v_1^{\,2}$ and by division, we have tan θ_1. Similarly from (2) and (6) we get $v_2^{\,2}$ and tan θ_2. Hence the motion after impact is completely determined.

Corollary 1. If the two spheres are perfectly elastic and of equal mass, then e = 1 and $m_1 = m_2$.

Then from equations (5) and (6) we have

$$v_1 \cos\theta_1 = \frac{0 + m_1 u_2 \cos\alpha_2 .2}{2m_1} = u_2 \cos\alpha_2$$

$$\text{and } v_2 \cos\theta_2 = \frac{0 + m_1 u_1 \cos\alpha_1 .2}{2m_1} = u_1 \cos\alpha_1$$

Hence if two equal perfectly elastic spheres impinge, they interchange their velocities in the direction of the line of centres.

Corollary 2. Usually, in most problems on oblique impact, one of the spheres is at rest. Suppose m_2 is at rest i.e. $u_2 = 0$.

From equation (2), $v_2 \sin\theta_2 = 0$ i.e. $\theta_2 = 0$. Hence m_2 moves along AB after impact. This is seen independently, since the only force on m_2 during impact is along the line of centres.

Corollary 3:

The impulse of the blow on the sphere A of mass m_1

= change of momentum of A along the common normal

$$= m_1 (v_1 \cos \theta_1 - u_1 \cos \alpha_1)$$

$$= m_1 \left[\frac{u_1 \cos \alpha_1 (m_1 - em_2) + m_2 u_2 \cos \alpha_2 (1 + e)}{m_1 + m_2} - u_1 \cos \alpha_1 \right]$$

$$= \frac{m_1 [m_1 u_1 \cos \alpha_1 - em_2 u_1 \cos \alpha_1 + m_2 u_2 \cos \alpha_2 + em_2 u_2 \cos \alpha_2 - m_1 u_1 \cos \alpha_1 - m_2 u_1 \cos \alpha_1]}{m_1 + m_2}$$

$$= \frac{m_1 [m_2 u_2 \cos \alpha_2 (1 + e) - m_2 u_1 \cos \alpha_1 (1 + e)]}{m_1 + m_2}$$

$$= \frac{m_1 m_2 (1 + e)}{m_1 + m_2} (u_2 \cos \alpha_2 - u_1 \cos \alpha_1)$$

The impulsive blow on m_2 will be equal and opposite to the impulsive blow on m_1.

§ 8.8. Loss of kinetic energy due to oblique impact of two smooth spheres:

Two spheres of masses m_1 and m_2, moving with velocities u_1 and u_2 at angles α_1 and α_2 with their line of centres, come into collision. To find an expression for the loss of kinetic energy:

The velocities perpendicular to the line of centres are not altered by impact. Hence the loss of kinetic energy in the case of oblique impact is therefore the same as in the case of direct impact if we replace in the expression (4) on page 236, the quantities u_1 and u_2 by $u_1 \cos \alpha_1$ and $u_2 \cos \alpha_2$ respectively. Therefore the loss is $\frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (1 - e^2) (u_1 \cos \alpha_1 - u_2 \cos \alpha_2)^2$.

We shall now derive this independently.

Let v_1 and v_2 be the velocities of the spheres after impact, in directions inclined at angles θ_1 and θ_2 respectively to the line of centres. As explained in § 8.7 the tangential velocity of each sphere is not altered by impact.

$\therefore$ $v_1 \sin \theta_1 = u_1 \sin \alpha_1$ .... (1) and $v_2 \sin \theta_2 = u_2 \sin \alpha_2$ ... (2)

By Newton's of rule,

$$v_2 \cos\theta_2 - v_1 \cos\theta_1 = -e\,(u_2 \cos\alpha_2 - u_1 \cos\alpha_1) \quad \ldots (3)$$

By conservation of momenta,

$$m_2 v_2 \cos\theta_2 + m_1 v_1 \cos\theta_1 = m_2 u_2 \cos\alpha_2 + m_1 u_1 \cos\alpha_1$$

i.e. $m_1 (u_1 \cos\alpha_1 - v_1 \cos\theta_1) = m_2 (v_2 \cos\theta_2 - u_2 \cos\alpha_2) \quad \ldots. (4)$

Change in K.E.

$$= \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2 - \frac{1}{2} m_1 v_1{}^2 - \frac{1}{2} m_2 v_2{}^2$$

$$= \frac{1}{2} m_1 u_1^2 (\cos^2\alpha_1 + \sin^2\alpha_1) + \frac{1}{2} m_2 u_2{}^2 (\cos^2\alpha_2 + \sin^2\alpha_2)$$

$$- \frac{1}{2} m_1 v_1{}^2 (\cos^2\theta_1 + \sin^2\theta_1) - \frac{1}{2} m_2 v_2^2 (\cos^2\theta_2 + \sin^2\theta_2)$$

$$= \frac{1}{2} m_1 u_1^2 \cos^2\alpha_1 + \frac{1}{2} m_2 u_2^2 \cos^2\alpha_2 - \frac{1}{2} m_1 v_1^2 \cos^2\theta_1$$

$$- \frac{1}{2} m_2 v_2^2 \cos^2\theta_2 \quad \text{using (1) and (2)}$$

$$= \frac{1}{2} m_1 (u_1^2 \cos\alpha_1 - v_1^2 \cos^2\theta_1) + \frac{1}{2} m_2 (u_2^2 \cos^2\alpha_2 - v_2^2 \cos^2\theta_2)$$

$$= \frac{1}{2} m_1 (u_1 \cos\alpha_1 + v_1 \cos\theta_1)(u_1 \cos\alpha_1 - v_1 \cos\theta_1)$$

$$+ \frac{1}{2} m_2 (u_2 \cos\alpha_2 + v_2 \cos\theta_2)(u_2 \cos\alpha_2 - v_2 \cos\theta_2)$$

$$= \frac{1}{2} m_1 (u_1 \cos\alpha_1 + v_1 \cos\theta_1)(u_1 \cos\alpha_1 - v_1 \cos\theta_1)$$

$$- \frac{1}{2} (u_2 \cos\alpha_2 + v_2 \cos\theta_2) . m_1 (u_1 \cos\alpha_1 - v_1 \cos\theta_1)$$

using (4)

$$= \frac{1}{2} m_1 (u_1 \cos\alpha_1 - v_1 \cos\theta_1)(u_1 \cos\alpha_1 + v_1 \cos\theta_1$$

$$- u_2 \cos\alpha_2 - v_2 \cos\theta_2)$$

$$= \frac{1}{2} m_1 (u_1 \cos\alpha_1 - v_1 \cos\theta_1) [u_1 \cos\alpha_1 - u_2 \cos\alpha_2$$

$$+ e (u_2 \cos\alpha_2 - u_1 \cos\alpha_1)] \text{ using (3)}$$

$$= \frac{1}{2} m_1 (u_1 \cos \alpha_1 - v_1 \cos \theta_1) (u_1 \cos \alpha_1 - u_2 \cos \alpha_2) (1 - e) \quad \ldots (5)$$

Now from (4),

$$\frac{u_1 \cos \alpha_1 - v_1 \cos \theta_1}{m_2} = \frac{v_2 \cos \theta_2 - u_2 \cos \alpha_2}{m_1}$$

and each $= \dfrac{u_1 \cos \alpha_1 - v_1 \cos \theta_1 + v_2 \cos \theta_2 - u_2 \cos \alpha_2}{m_1 + m_2}$

$$= \frac{(u_1 \cos \alpha_1 - u_2 \cos \alpha_2) + (v_2 \cos \theta_2 - v_1 \cos \theta_1)}{m_1 + m_2}$$

$$= \frac{u_1 \cos \alpha_1 - u_2 \cos \alpha_2 - e (u_2 \cos \alpha_2 - u_1 \cos \alpha_1)}{m_1 + m_2} \quad \text{using (3)}$$

$$= \frac{(u_1 \cos \alpha_1 - u_2 \cos \alpha_2) (1 + e)}{m_1 + m_2}$$

$$\therefore u_1 \cos \alpha_1 - v_1 \cos \theta_1 = \frac{m_2 (1+ e)}{m_1 + m_2} (u_1 \cos \alpha_1 - u_2 \cos \alpha_2)$$

Substituting in (5),

chsnge in K.E. $= \dfrac{1}{2} \dfrac{m_1 m_2 (1+ e)}{m_1 + m_2} (u_1 \cos \alpha_1 - u_2 \cos \alpha_2)$

$$\times (u_1 \cos \alpha_1 - u_2 \cos \alpha_2) (1 - e)$$

$$= \frac{1}{2} \frac{m_1 m_2}{m_1 + m_2} (1 - e^2) (u_1 \cos \alpha_1 - u_2 \cos \alpha_2)^2$$

If the spheres are perfectly elastic, $e = 1$ and the loss of kinetic energy is zero.

§ 8.9. Dissipation of energy due to impact:

We have found that in any impact, except where the coefficient of restitution is unity, some kinetic energy is lost. This missing kinetic energy is converted into other forms of energy and chiefly reappears in the shape of heat. Hence the Principle of Conservation of Energy will not hold good in problems of impact.

Ex.12. *A ball of mass 8 gms. moving with velocity 4cms. per sec. impinges on a ball of mass 4 gms. moving with velocity 2 cm. per sec. If their velocities before impact be inclined at angle 30^0 and 60^0 to the line joining their centres at the moment of impact, find their velocities after impact when $e = \frac{1}{2}$* (B.Sc. 50 Madras Uty.)

Refer to fig. 95 on page 244. $m_1 = 8$; $u_1 = 4$; $\alpha_1 = 30^0$; $m_2 = 4$; $u_2 = 2$; $\alpha_2 = 60^0$ Let v_1 and v_2 be the velocities after impact in directions making θ_1 and θ_2 respectively with AB. The tangential velocity of each sphere is not affected by impact.

$$\therefore\ v_1 \sin\theta_1 = 4\sin 30^0 = 2 \quad \text{... (1)}$$

$$\text{and } v_2 \sin\theta_2 = 2\sin 60^\circ = \sqrt{3} \quad \text{... (2)}$$

By Newton's law,

$$v_2\cos\theta_2 - v_1\cos\theta_1 = -e\ (2\cos 60^\circ - 4\cos 30^\circ)$$

$$= -\frac{1}{2}\left(2.\frac{1}{2} - 4.\frac{\sqrt{3}}{2}\right)$$

$$= \frac{1}{2}(2\sqrt{3} - 1) \quad \text{... (3)}$$

By conservation of momenta along AB,

$$4v_2\cos\theta_2 + 8v_1\cos\theta_1 = 4.2\cos 60^\circ + 8.4\cos 30^\circ$$

$$= 4 + 16\sqrt{3}$$

$$\text{i.e. } v_2\cos\theta_2 + 2v_1\cos\theta_1 = 1 + 4\sqrt{3} \quad \text{... (4)}$$

$$\therefore\ 3v_1\cos\theta_1 = 1 + 4\sqrt{3} - \frac{1}{2}(2\sqrt{3} - 1) = \frac{3 + 6\sqrt{3}}{2}$$

$$\text{i.e. } v_1\cos\theta_1 = \frac{1 + 2\sqrt{3}}{2} \quad \text{... (5)}$$

$$\text{From (4), } v_2\cos\theta_2 = 1 + 4\sqrt{3} - 1 - 2\sqrt{3} = 2\sqrt{3} \quad \text{... (6)}$$

From (1) and (5), $v_1^2 = 2^2 + \left(\frac{1 + 2\sqrt{3}}{2}\right)^2$

$$= 4 + \frac{1 + 4\sqrt{3} + 12}{4} = \frac{29 + 4\sqrt{3}}{4}$$

$\therefore\ v_1 = \frac{\sqrt{29 = 4\sqrt{3}}}{2}$ cm. per sec.

Dividing (1) by (5), $\tan\theta_1 = \frac{4}{1 + 2\sqrt{3}}$

From (2) and (6).

$v_2{}^2 = 3 + 12 = 15$ and $\therefore\ v_2 = \sqrt{15}$ cm /sec

Dividing (2) by (6), $\tan\theta_2 = \frac{1}{2}$.

Ex.13. *A smooth sphere of mass m impinges obliquely on a smooth sphere of mass M which is at rest. Show that if m = eM, the directions of motion after impact are at right angles. (e is the coefficient of restitution)*

(B.Sc. 76 Madurai Uty. B.Sc. 68 Madras Uty.)

Considering the sphere M, its tangential velocity before impact is zero and hence after impact also, its tangential velocity is zero. ($\because$ During impact, there is no force acting along the common tangent). Hence, after impact, M will move along AB. Let its velocity be v_2. Let the velocity of m be v_1 at an angle θ to AB, after impact.

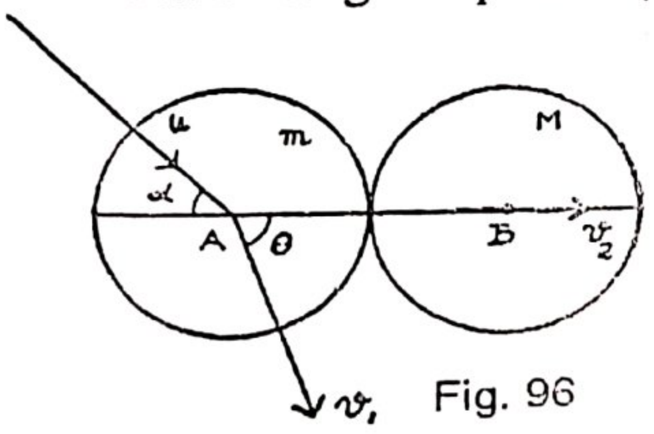


Fig. 96

By Newton's rule $v_2 - v_1\cos\theta = -e(0 - u\cos\alpha)$

i.e. $v_2 - v_1\cos\theta = eu\cos\alpha$ .... (1)

By conservation of momenta along AB,

$M.v_2 + m\,v_1\cos\theta = M.0 + m.u\cos\alpha$ ... (2)

Multiplying (1) by M and subtracting from (2), Find also the velocity with which each of the other two balls moves. Also, find the kinetic energy lost by the impact.

§ 8.10. Compression and Restitution:

When two elastic bodies impinge, the time during which the impact lasts may be divided into two stages. During the first stage, the bodies are slightly compressing one another and during the second stage, they are recovering their shape. We can experimentally show that bodies are compre ed during impact. Suppose we drop a billiard ball on a floor, whi has been already covered with fine coloured powder. At the spot where the ball comes into contact with the floor, it will be seen that the powder is removed not merely from a geometrical point but from a small circle. This shows that, near the point of contact, the ball actually meets the floor in a small circle. Hence at that time the ball must have undergone a slight deformation and subsequently recovered its shape.

The first portion of the impact where bodies get compressed lasts until they are instantaneously moving with the same velocity. Forces then come into play tending to make the bodies recover their shape. The mutual action between with bodies during the first portion of the impact is often called *"the force of compression"* and that during the second portion *the force of restitution.*

Ex.16. *Prove that the ratio of the impulses of the forces of restitution and compression is equal to the coefficient of restitution.*

(B.Sc. 57 Madras Uty.)

Let a sphere of mass m impinge directly with velocity u_1 on another sphere of mass m_2, moving in the same direction with velocity u_2. Let v be the common velocity of the spheres at the instant when compression is over. Also let v_1, v_2 be their final velocities after impact. During compression, $m_1 (u_1 - v)$ is the loss of momentum by the first ball and $m_2(v-u_2)$ is the gain of momentum by the second ball. So if I is the impulse of the force of compression, we have

$$I = m_1 (u_1 - v) = m_2 (v - u_2) \qquad \dots (1)$$

SIMPLE HARMONIC MOTION

§ 10.1. Introduction:

A very common and important type of motion occurring in nature is that which involves oscillations backwards and forwards about some fixed point. For instance, suppose one end of an elastic string is tied to a fixed point and a heavy particle is attached to the other end. If the particle is disturbed vertically from its position of equilibrium, it is found that it oscillates to and fro about this position. Clearly the particle cannot be moving under constant acceleration. It is found that it has an acceleration which is always directed towards the equilibrium position and varies in magnitude as the distance of the particle from that position. This kind of motion occurs frequently in nature and since it is of the type which produces all musical notes, it is called *Simple Harmonic Motion* (Shortly written as S.H.M). The oscillations of a simple pendulum and the transverse vibrations of a plucked violin string are examples of simple harmonic motion.

§ 10.2. Simple Harmonic Motion in a Straight line:

Definition: *When a particle moves in a straight line so that its acceleration is always directed towards a fixed point in the line and proportional to the distance from that point, its motion is called Simple Harmonic Motion.*

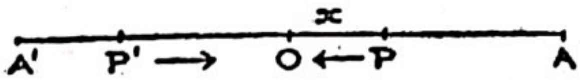


Fig. 120

Let O be a fixed point on the straight line $A^1 OA$ on which a particle is having simple harmonic motion. Take O as the origin and OA as the X axis. Let P be the position of the particle at time t such that OP = x. The magnitude of the acceleration at P = μx where μ is a positive constant. As this acceleration acts towards O, the acceleration at P in the positive direction of the X axis is $-\mu x$.

Hence the equation of motion of P is $\frac{d^2 x}{dt^2} = -\mu x$ ... (1)

Here it must be noted that for a position of P to the right of O, the x-coordinate x is positive and so the acceleration $\frac{d^2x}{dt^2}$ is negative, directed towards O. If P_1 is a position of the particle to the left of O, x is negative and so the acceleration $\frac{d^2x}{dt^2}$ is positive again towards O. Hence the same equation of motion (1) holds good for all positions of P on the line.

Equation (1) is the fundamental differential equation representing a S.H.M. We now proceed to solve it.

If v is the velocity of the particle at time t, (1) can be written as

$$v\frac{dv}{dx} = -\mu x \quad \text{i.e.} \quad vdv = -\mu x\, dx \qquad \ldots \quad (2)$$

Integrating (2), we have $\frac{v^2}{2} = -\frac{\mu x^2}{2} + c \qquad \ldots \quad (3)$

where c is the constant of integration.

Initially let the particle start from rest at the point A where OA = a and let us measure time also from this instant.

Hence when x = a, v = 0.

Putting these in (3), $0 = -\frac{\mu a^2}{2} + c$ or $c = \frac{\mu a^2}{2}$

$$\therefore \quad v^2 = -\mu x^2 + \mu a^2 = \mu(a^2 - x^2)$$

$$\therefore \quad v = \pm\sqrt{\mu(a^2 - x^2)}$$

Equation (4) gives the velocity v corresponding to any displacement x.

Now as t increases, x decreases. So $\frac{dx}{dt}$ is negative.

Hence taking the negative sign in (4),

$$\frac{dx}{dt} = v = -\sqrt{\mu(a^2 - x^2)} \qquad \ldots \quad (5)$$

$$\text{or} \quad -\frac{dx}{\sqrt{(a^2 - x^2)}} = \sqrt{\mu}\, dt$$

Integrating, $\cos^{-1}\frac{x}{a} = \sqrt{\mu}\, t + A$

Initially when $t = 0, x = a$.

$\therefore \quad \cos^{-1} 1 = 0 + A \quad$ i.e. $A = 0$.

Hence $\cos^{-1}\frac{x}{a} = \sqrt{\mu}\, t$

i.e. $\frac{x}{a} = \cos\sqrt{\mu}\, t \quad$ or $\quad x = a\cos\sqrt{\mu}\, t \qquad \ldots \quad (6)$

Equation (6) gives the displacement x in terms of time t.

When the particle comes to O, $x = 0$ and by (5), its velocity then $= -a\sqrt{\mu}$. So the particle passes through O and immediately the acceleration alters its direction and tends to decrease the velocity. From (5), $v = 0$ when $x = -a$. So the particle comes to rest at a point A' to the left of O such that $OA = OA'$. It then retraces its path, passes through O, and again is instantaneously at rest at A. The whole motion of the particle is an oscillation from A to A' and back.

To get the time from A to A , put $x = -a$ in (6).

We have $\cos\sqrt{\mu}\, t = -1 = \cos\pi \qquad t = \frac{\pi}{\sqrt{\mu}}$

The time from A to A' and back $= \frac{2\pi}{\sqrt{\mu}}$

Equation (6) can be written as

$$x = a\cos\sqrt{\mu}\, t = a\cos(\sqrt{\mu}\, t + 2\pi) = a\cos(\sqrt{\mu}\, t + 4\pi) \text{ etc.}$$

$$= a\cos\sqrt{\mu}\left(t + \frac{2\pi}{\sqrt{\mu}}\right)$$

$$= a\cos\sqrt{\mu}\left(t + \frac{4\pi}{\sqrt{\mu}}\right) \text{ etc.}$$

This shows that the displacement of the particle at any particular time t_1 is repeated at times

$$t_1 + \frac{2\pi}{\sqrt{\mu}}, \quad t_1 + \frac{4\pi}{\sqrt{\mu}} \text{ etc.}$$

Differentiating (6),

$$\frac{dx}{dt} = -a\sqrt{\mu} \,.\, \sin\sqrt{\mu}\, t$$

$$= -a\sqrt{\mu}\sin(\sqrt{\mu}\, t + 2\pi) = -a\sqrt{\mu}\sin(\sqrt{\mu}\, t + 4\pi) \text{ etc.}$$

$$= -a\sqrt{\mu}\sin\sqrt{\mu}\,(t + \frac{2\pi}{\sqrt{\mu}}) = -a\sqrt{\mu}\sin\sqrt{\mu}\,(t + \frac{4\pi}{\sqrt{\mu}}) \text{ etc.}$$

This shows that the values of $\frac{dx}{dt}$ are the same if t is increased by $\frac{2\pi}{\sqrt{\mu}}$ or by any multiple of $\frac{2\pi}{\sqrt{\mu}}$. Hence after a time $\frac{2\pi}{\sqrt{\mu}}$ the particle is again at the same particular point moving with the same velocity in the same direction as before, having covered the whole path of the motion just once. The particle is said to have the period $\frac{2\pi}{\sqrt{\mu}}$.

Definitions: *The period or the periodic time* of a simple harmonic motion is the interval of time that elapses from any instant till a subsequent instant when the particle is again moving through the same position with the same velocity in the same direction. The *frequency* of the oscillation is the number of complete oscillations that the particle makes in one second. So frequency is the reciprocal of the period and is equal to $\frac{\sqrt{\mu}}{2\pi}$.

The distance through which the particle moves away from the centre of motion on either side of it is called the *amplitude* of the oscillation.

Thus in the above case, amplitude = OA = OA$'$ = a.

We notice that the periodic time being = $\frac{2\pi}{\sqrt{\mu}}$, is independent of the amplitude which is the distance from the centre at which the particle started. It depends only on the constant μ which is the acceleration at unit distance from the centre.

Note: (1) Since $\frac{d^2x}{dt^2} = -\mu x$, maximum acceleration corresponds to the greatest value of x and so it is numerically $= \mu \cdot a = \mu \cdot$ (amplitude)

(2) Since $v = \sqrt{\mu(a^2 - x^2)}$, the greatest value of v is got at x = 0 and it is $= a\sqrt{\mu} = \sqrt{\mu} \cdot$ (amplitude).

§ 10.3. General solution of the S.H.M. equation:

The S.H.M. equation is $\frac{d^2x}{dt^2} = -\mu x$

i.e. $\frac{d^2x}{dt^2} + \mu x = 0$ ... (1)

(1) is a linear differential equation of the second order with constant coefficients. Its most general solution is of the form

$x = A\cos\sqrt{\mu}\,t + B\sin\sqrt{\mu}\,t$ ... (2)

where A and B are arbitrary constants.

Other forms of the solution equivalent to (2) are

$x = C\cos(\sqrt{\mu}\,t + \varepsilon)$ ... (3) and $x = D\sin(\sqrt{\mu}\,t + \alpha)$ ... (4)

The constants A and B in (2), C and ε in (3) and D and α in (4) are known if we know the values of x and $\frac{dx}{dt}$ corresponding to a given time t.

From (3) and (4), the maximum value of x = C or D.

Hence if a is the amplitude of the motion, the forms (3) and (4) can be respectively put as

$x = a\cos(\sqrt{\mu}\,t + \varepsilon)$ ... (5) and $x = a\sin(\sqrt{\mu}\,t + \alpha)$ ... (6)

When the solution of the S.H.M. equation is expressed as $x = a\cos(\sqrt{\mu}\,t + \varepsilon)$, the quantity ε is called the *epoch*. The *phase* of a S.H.M. at any instant is the time that has elapsed since the particle was at its maximum distance in the positive direction.

From equation (5), x is maximum when $\cos(\sqrt{\mu} + \varepsilon) = 1$.

If t_0 is the then value of t, $\sqrt{\mu}\, t_o + \varepsilon = 0$.

$$\text{i.e.} \quad t_o = -\frac{\varepsilon}{\sqrt{\mu}}$$

Hence phase at time $t = t - t_o = t + \dfrac{\varepsilon}{\sqrt{\mu}} = \dfrac{\sqrt{\mu}\, t + \varepsilon}{\sqrt{\mu}}$

Note: Two simple harmonic motions of the same period can be represented by

$$x_1 = a_1 \cos(\sqrt{\mu}\, t + \varepsilon_1) \quad \text{and} \quad x_2 = a_2 \cos(\sqrt{\mu}\, t + \varepsilon_2).$$

The difference in phase $= \dfrac{\varepsilon_1 - \varepsilon_2}{\sqrt{\mu}}$

If $\varepsilon_1 = \varepsilon_2$ the motions are in the same phase.

If $\varepsilon_1 = \varepsilon_2 = \pi$, they are in opposite phases.

§ 10.4. Geometrical Representation of a Simple Harmonic Motion:

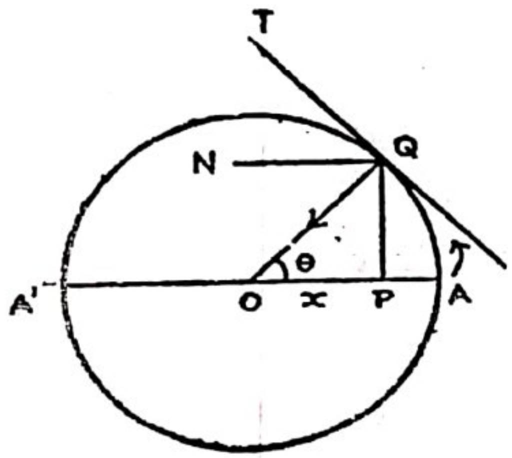


Fig. 121

Let a point Q describe, with uniform angular velocity ω, a circle of radius 'a' and centre O of which $A'OA$ is a fixed diameter. Let P be the foot of the perpendicular from Q on AA'. As Q moves round the circle. P will move to and fro on the diameter. We can show that the motion of P along AA' is simple harmonic, with O as centre.

Let Q move in the direction AQA' as shown in the figure. As Q moves uniformly in a circle, its only acceleration is $\omega^2 . QO$ along QO. The velocity of Q in the circle is $\omega . QO$ along the tangent QT. The velocity and acceleration of P must be the same as the resolved parts along AA', of the velocity and acceleration of Q.

Hence acceleration of P = ω^2 QO . cos ∠POQ

$$= \omega^2 \, QO \cdot \frac{PO}{QO} = \omega^2 \, . \, PO \text{ towards O.}$$

i.e. the acceleration of P is always directed towards O and proportional to its distance from O. Hence the motion of P is simple harmonic.

The various formulae of a S.H.M. derived in § 10.2, can be deduced by considering the motion of Q along the circle.

Taking O as the origin and OA as the positive direction of measuring displacement, let OP = x and ∠QOP = θ .

Velocity of Q = a ω along QT.

Hence velocity of P = resolved part of velocity of Q along AA′.

$$= a\,\omega \, . \cos \angle TQN = a\,\omega \, . \sin \angle OQN$$

$$= a\,\omega \, . \sin \theta = a\,\omega \, . \frac{PQ}{a}$$

$$= \omega \sqrt{OQ^2 - OP^2}$$

$$= \omega \sqrt{a^2 - x^2} \text{ (in magnitude)} \quad \dots \quad (1)$$

and this velocity of P is along AO towards O.

As Q moves round the circle from A to A′ and back to A, P moves from A to A′ through O and back to A.

Hence the periodic time of the S.H.M. described by P

$$= \text{Time taken for Q to describe the circle} = \frac{2\pi}{\omega} \quad \dots \quad (2)$$

Also, if t is the time from A to Q, $t = \frac{\theta}{\omega} = \frac{\cos^{-1}\left(\frac{x}{a}\right)}{\omega}$

$$\omega t = \cos^{-1}\left(\frac{x}{a}\right) \text{ or } x = a \cos \omega t \quad \dots \quad (3)$$

The acceleration of P towards O = ω^2 . PO and putting $\omega^2 = \mu$.

i.e. $\omega = \sqrt{\mu}$ in (1), (2), (3) we get

(i) the velocity of P $= \sqrt{\mu} \,.\, \sqrt{a^2 - x^2}$ in magnitude

(ii) the periodic time of P $= \dfrac{2\pi}{\sqrt{\mu}}$

(iii) the displacement $x = a \cos \sqrt{\mu}\, t$.

These are the formulae derived in § 10.2.

§ 10.5. Change of origin:

A differential equation of the form $\dfrac{d^2 x}{dt^2} = \mu x$ where μ is a positive number, always represents a simple harmonic motion of period $\dfrac{2\pi}{\sqrt{\mu}}$ which is independent of the amplitude. The centre of the S.H.M. is the origin from where the displacement x is measured.

Consider now the equation.

$$\frac{d^2 x}{dt^2} = -\mu x + \alpha \quad \ldots \quad (1), \text{ This can be written as}$$

$$\frac{d^2 x}{dt^2} = -\mu\left(x - \frac{\alpha}{\mu}\right) \quad \ldots \quad (2)$$

Put $x - \dfrac{\alpha}{\mu} = X \quad \ldots \quad (3)$ When $x = \dfrac{\alpha}{\mu}$, $X = 0$.

So this means that we are transferring the origin for measuring displacement to the point distant $\dfrac{\alpha}{\mu}$ from the original origin.

Differentiating (3) twice,

$\dfrac{d^2 x}{dt^2} = \dfrac{d^2 X}{dt^2}$ and hence (2) becomes $\dfrac{d^2 X}{dt^2} = -\mu X \quad \ldots \quad (4)$

(4) clearly represents a simple harmonic motion about the new origin.

WORKED EXAMPLES

Ex.1. *A particle is moving with S.H.M. and while making an oscillation from one extreme position to the other, its distances from the centre of oscillation at 3 consecutive seconds are x_1, x_2, x_3. Prove that the period of oscillation is* $\dfrac{2\pi}{\cos^{-1}\left(\dfrac{x_1 + x_3}{2\pi}\right)}$

(B.E. 65, Andhra Uty; B.Sc. 71 Calicut Uty.)

If a is the amplitude, μ the constant of the S.H.M. and x is the displacement at time t, we know that $x = a \cos \sqrt{\mu}\, t$ ... (1)

Let at three consecutive seconds t_1, $t_1 + 1$, $t_1 + 2$ the corresponding displacements be x_1, x_2, x_3.

Then $x_1 = a \cos \sqrt{\mu}\, t_1$ ... (2)

$$x_2 = a \cos \sqrt{\mu}\,(t_1 + 1) = a \cos (\sqrt{\mu}\, t_1 + \sqrt{\mu}) \quad \dots (3)$$

and $x_3 = a \cos \sqrt{\mu}\,(t_1 + 2) = a \cos (\sqrt{\mu}\, t_1 + 2\sqrt{\mu})$ ... (4)

$$\therefore\ x_1 + x_3 = a\,[\cos (\sqrt{\mu}\, t_1 + 2\sqrt{\mu}) + \cos \sqrt{\mu}\, t_1]$$

$$= a \cdot 2 \cos \frac{\sqrt{\mu}\, t_1 + 2\sqrt{\mu} + \sqrt{\mu}\, t_1}{2} \cdot \cos \frac{\sqrt{\mu}\, t_1 + 2\sqrt{\mu} - \sqrt{\mu}\, t_1}{2}$$

$$= 2a \cos (\sqrt{\mu}\, t_1 + \sqrt{\mu}) \cdot \cos \sqrt{\mu} = 2x_2 \cdot \cos \sqrt{\mu}$$

$$\therefore\ \frac{x_1 + x_3}{2x_2} = \cos \sqrt{\mu} \quad \text{or} \quad \sqrt{\mu} = \cos^{-1}\left(\frac{x_1 + x_3}{2x_2}\right)$$

$$\text{Period} = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\cos^{-1}\left(\dfrac{x_1 + x_3}{2x_2}\right)}$$

Ex. 2. *If the displacement of a moving point at any time be given by an equation of the form $x = a \cos \omega t + b \sin \omega t$, show that the motion is a simple harmonic motion.* **(B.Sc. 71 Madras Uty.)**

If a = 3, b = 4, ω = 2 determine the period, amplitude, maximum velocity and maximum acceleration of the motion.

(B.Sc. 50 Madras Uty.)

$$x = a \cos \omega t + b \sin \omega t \qquad \dots (1)$$

We have to show that the acceleration varies directly as the displacement. Differentiating (1) with respect to t,

$$\frac{dx}{dt} = -a\omega \sin \omega t + b\omega \cos \omega t \qquad \dots (2)$$

$$\frac{d^2x}{dt^2} = -a\omega^2 \cos \omega t - b\omega^2 \sin \omega t$$

$$= -\omega^2 (a \cos \omega t + b \sin \omega t) = -\omega^2 x \qquad \dots (3)$$

(3) shows that the motion is simple harmonic.

The constant μ of the S.H.M. = ω^2.

$$\therefore \quad \text{Period} = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi \text{ secs.}$$

Amplitude is the greatest value of x.

When x is maximum, $\frac{dx}{dt} = 0$.

$$\therefore \quad -a\omega \sin \omega t + b\omega \cos \omega t = 0 \quad \text{i.e.} \quad a \sin \omega t = b \cos \omega t.$$

or $\tan \omega t = \frac{b}{a} = \frac{4}{3}$ using the given values.

When $\tan \omega t = \frac{4}{3}$, $\sin \omega t = \frac{4}{5}$ and $\cos \omega t = \frac{3}{5}$.

Putting these values in (1), greatest value of x

$$= a \times \frac{3}{5} + b \times \frac{4}{5} = \frac{3a + 4b}{5} = \frac{3.3 + 4.4}{5} = 5$$

Hence amplitude = 5.

Using the formulae of § 10.2,

Max. acceleration = μ . amplitude = $4 \times 5 = 20$

Max. velocity = $\sqrt{\mu}$. amplitude = $2 \times 5 = 10$.

Ex.3. *A horizontal shelf moves vertically with S.H.M whose complete period is one second; find the greatest amplitude in centimeters, it can have, so that an object resting on the shelf may always remain in contact.*

Let m be the mass of an object lying on the shelf, O the centre of the S.H.M. and P the position of m at time t. Let OP = x. The forces acting on the mass at P are: (i) its weight mg acting vertically downwards and (ii) the normal reaction R due to the shelf acting upwards. Resultant force on the mass = mg - R and so the acceleration on it $= \dfrac{mg - R}{m}$ and this acts towards O. Since the particle is moving with S.H.M. towards O,

acceleration at P $= \mu \, . \, PO = \mu x$.

$$\therefore \quad \mu x = \frac{mg - R}{m}$$

i.e. $R = mg - m\mu x = m(g - \mu x)$ ... (1)

Period of the S.H.M. $= \dfrac{2\pi}{\sqrt{\mu}} = 1$ (given)

$\therefore \quad \sqrt{\mu} = 2\pi$ or $\mu = 4\pi^2$

$\therefore$ From (1), $R = m(g - 4\pi^2 x)$

For the mass to remain always in contact with the shelf, reaction R must not be negative.

$$\therefore \quad m(g - 4\pi^2 x) \geq 0$$

i.e. $g - 4\pi^2 x \geq 0$ or $x \leq \dfrac{g}{4\pi^2}$

$\therefore$ Greatest value of $x = \dfrac{g}{4\pi^2} = \dfrac{981}{4\pi^2} = 24.8$ cms.

Ex.4. *A particle P, of mass m, moves in a straight line OX under a force m μ (distance) directed towards a point A which moves in the straight line OX with constant acceleration α. Show that the motion of P is simple harmonic, of period $2\pi/\sqrt{\mu}$ about a moving centre which is always at a distance a/μ behind A.*

(B.Sc. 55 Madras; B.Sc. 82 Madurai Uty.)

Let at time t, the particle be at P where OP = x and A be such that OA = y. The equation of motion of P is

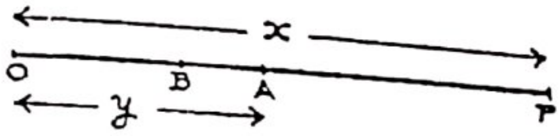


Fig. 122

$$\frac{d^2x}{dt^2} = -\mu \, . \, PA \quad \text{(since the acceleration is towards A)}$$

$$= -\mu(x-y) \qquad \dots (1)$$

The equation of motion of A is $\frac{d^2y}{dt^2} = \alpha$ ... (2)

Subtracting (2) from (1) we have

$$\frac{d^2x}{dt^2} - \frac{d^2y}{dt^2} = -\mu(x-y) - \alpha = -\mu\left(x - y + \frac{\alpha}{\mu}\right) \qquad \dots (3)$$

Put $x - y + \frac{\alpha}{\mu} = z$ ... (4)

Differentiating (4), $\frac{d^2x}{dt^2} - \frac{d^2y}{dt^2} = \frac{d^2z}{dt^2}$

Hence (3) becomes $\frac{d^2z}{dt^2} = -\mu z$ ... (5)

∴ The displacement z is simple harmonic and the period is $= \frac{2\pi}{\sqrt{\mu}}$.

The centre of the S.H.M. represented by (5) is clearly the new origin from where z is measured.

Now $z = x - y + \frac{\alpha}{\mu} = AP + \frac{\alpha}{\mu}$.

If B is a point behind A such that $BA = \frac{\alpha}{\mu}$,

We have z = AP + BA = BP.

i.e. z denotes the displacement of P measured from B.

of mass m resting on the plate will not leave it, provided $n^2 \le \frac{g}{a}$.

In the case, when it leaves, find its velocity then.

(B.Sc. 82 Madras Uty.)

20. A particle is moving in a straight line with simple harmonic motion of amplitude 'a'. At a distance s from the centre of motion, the particle receives a blow in the direction of motion which instantaneously doubles the velocity. Find the new amplitude.

(B.A. 48 Madras Uty.)

§ 10.6. Composition of two Simple Harmonic Motions of the same period and in the same straight line:

Since the period is dependent only on the cosntant μ, the two separate simple harmonic motions are expressed by the same differential equation $\frac{d^2 x}{d t^2} = -\mu x$.

Let x_1 and x_2 be the displacements for the separate motions. Then we can take

$$x_1 = a_1 \cos(\sqrt{\mu}\, t + \varepsilon_1) \text{ and } x_2 = a_2 \cos(\sqrt{\mu}\, t + \varepsilon_2).$$

Let x be the resultant displacement.

$$\begin{aligned} \text{Then } x &= x_1 + x_2 \\ &= a_1 \cos(\sqrt{\mu}\, t + \varepsilon_1) + a_2 \cos(\sqrt{\mu}\, t + \varepsilon_2) \\ &= \cos\sqrt{\mu}\, t\,(a_1 \cos\varepsilon_1 + a_2 \cos\varepsilon_2) \\ &\qquad - \sin\sqrt{\mu}\, t\,(a_1 \sin\varepsilon_1 + a_2 \sin\varepsilon_2) \\ &= \cos\sqrt{\mu}\, t \,.\, A\cos\varepsilon - \sin\sqrt{\mu}\, t \,.\, A\sin\varepsilon \qquad \ldots (1) \end{aligned}$$

$$\text{where } A\cos\varepsilon = a_1 \cos\varepsilon_1 + a_2 \cos\varepsilon_2 \qquad \ldots (2)$$

$$\text{and } A\sin\varepsilon = a_1 \sin\varepsilon_1 + a_2 \sin\varepsilon_2 \qquad \ldots (3)$$

We can find the new constants A and ε.

Squaring (2) and (3) and adding,

$$A^2 = a_1^2 + a_2^2 + 2a_1a_2 \cos(\varepsilon_1 - \varepsilon_2) \qquad \ldots (4)$$

Dividing (3) by (2), $\tan \varepsilon = \dfrac{a_1 \sin \varepsilon_1 + a_2 \sin \varepsilon_2}{a_1 \cos \varepsilon_1 + a_2 \cos \varepsilon_2}$ ... (5)

Now (1) becomes $x = A(\cos\sqrt{\mu}\, t \cos \varepsilon - \sin\sqrt{\mu}\, t \sin \varepsilon)$

$$= A \cos(\sqrt{\mu}\, t + \varepsilon) \quad \text{... (6)}$$

The resultant displacement given by (6) also represents a simple harmonic motion of the same period as the individual motions. A, the new amplitude, is the diagonal of the parallelogram whose sides are the original amplitudes a_1 and a_2 inclined to one another at an angle $\varepsilon_1 - \varepsilon_2$, the difference of the epochs.

§ 10.7. Composition of two Simple Harmonic Motions of the same period in two perpendicular directions:

If a particle possesses two simple harmonic motions in perpendicular directions and of the same period, we can prove that its path is an ellipse. Take the two perpendicular lines as the axes of x and y. The displacements of the particle due to the separate motions can be taken as

$$x = a_1 \cos\sqrt{\mu}\, t \quad \text{... (1)} \qquad y = a_2 \cos(\sqrt{\mu}\, t + \varepsilon) \quad \text{... (2)}$$

The path of the particle is obtained by eliminating t between (1) and (2). From (2),

$$y = a_2 \cos\sqrt{\mu}\, t \,.\, \cos \varepsilon - a_2 \sin\sqrt{\mu}\, t \,.\, \sin \varepsilon$$

$$= a_2 \cos \varepsilon \,.\, \frac{x}{a_1} - a_2 \sin \varepsilon \,.\, \sqrt{1 - \frac{x^2}{a_1^2}}$$

i.e. $$\frac{y}{a_2} - \frac{x \cos \varepsilon}{a_1} = -\sin \varepsilon \,.\, \sqrt{1 - \frac{x^2}{a_1^2}}$$

Squaring,

$$\frac{y^2}{a_2^2} + \frac{x^2 \cos^2 \varepsilon}{a_1^2} - \frac{2xy \cos \varepsilon}{a_1 a_2} = \sin^2 \varepsilon - \frac{x^2}{a_1^2} \sin^2 \varepsilon$$

i.e. $\frac{x^2}{a_1^2} - \frac{2xy}{a_1 a_2}\cos\varepsilon + \frac{y^2}{a_2^2} = \sin^2\varepsilon$ ... (3)

This is of the form $ax^2 + 2hxy + by^2 = \lambda$ ... (4)

where $a = \frac{1}{a_1^2}$, $h = -\frac{\cos\varepsilon}{a_1 a_2}$, $b = \frac{1}{a_2^2}$

Clearly (4) represents a conic with centre at the origin.

Also, $ab - h^2 = \frac{1}{a_1^2 a_2^2} - \frac{\cos^2\varepsilon}{a_1^2 a_2^2} = \frac{\sin^2\varepsilon}{a_1^2 a_2^2} = +\text{ve}$

Hence (4) represents an ellipse.

If $\varepsilon = 0$, equation (3) gives $\frac{x}{a_1} - \frac{y}{a_2} = 0$ which is a straight line.

If $\varepsilon = \pi$, (3) gives $\frac{x}{a_1} + \frac{y}{a_2} = 0$ which is also a straight line.

If $\varepsilon = \frac{\pi}{2}$, (3) gives $\frac{x^2}{a_1^2} + \frac{y^2}{a_2^2} = 1$ which is an ellipse whose principal axes are along the axes of x and y.

If $\varepsilon = \frac{\pi}{2}$ and $a_1 = a_2$, the path is the circle $x^2 + y^2 = a_1^2$

Ex.7. *Show that the resultant of two simple harmonic motions in the same direction and of equal periodic time, the amplitude of one being twice that of the other and its phase a quarter of a period in advance, is a simple harmonic motion of amplitude $\sqrt{5}$ times that of the first and whose phase is in advance of the first by $\frac{\tan^{-1} 2}{2\pi}$ of a period.*

Referring to § 10.6. let the separate displacements be

$x_1 = a_1 \cos(\sqrt{\mu}\, t + \varepsilon_1)$ ... (1) and

$x_2 = a_2 \cos(\sqrt{\mu}\, t + \varepsilon_2)$ ... (2)

Here $a_2 = 2a_1$ and $\frac{\varepsilon_2 - \varepsilon_1}{\sqrt{\mu}}$ = phase difference

$$= \frac{1}{4} \times \frac{2\pi}{\sqrt{\mu}}$$

$$\therefore \quad \varepsilon_2 - \varepsilon_1 = \frac{\pi}{2} \quad \text{or} \quad \varepsilon_2 = \frac{\pi}{2} + \varepsilon_1.$$

We know that the resultant displacement is

$$x = A\cos(\sqrt{\mu}\, t + \varepsilon) \qquad .. \quad (3)$$

where $A^2 = a_1^2 + a_2^2 + 2a_1 a_2 \cos(\varepsilon_1 - \varepsilon_2)$

$$= a_1^2 + 4a_1^2 + 4a_1^2 \cos(-90^\circ) = 5a_1^2$$

$\therefore$ Amplitude of the resultant motion $= A = a_1\sqrt{5}$

$$\text{Also } \tan\varepsilon = \frac{a_1 \sin\varepsilon_1 + a_2 \sin\varepsilon_2}{a_1\cos\varepsilon_1 + a_2\cos\varepsilon_2}$$

$$= \frac{a_1\sin\varepsilon_1 + 2a_1\sin(90^\circ + \varepsilon_1)}{a_1\cos\varepsilon_1 + 2a_1\cos(90^\circ + \varepsilon_1)}$$

$$\text{i.e. } \frac{\sin\varepsilon}{\cos\varepsilon} = \frac{\sin\varepsilon_1 + 2\cos\varepsilon_1}{\cos\varepsilon_1 - 2\sin\varepsilon_1}$$

$$\sin\varepsilon\cos\varepsilon_1 - 2\sin\varepsilon\sin\varepsilon_1 = \sin\varepsilon_1\cos\varepsilon + 2\cos\varepsilon_1\cos\varepsilon$$

or $\sin(\varepsilon - \varepsilon_1) = 2\cos(\varepsilon - \varepsilon_1)$ i.e. $\tan(\varepsilon - \varepsilon_1) = 2$

or $\varepsilon - \varepsilon_1 = \tan^{-1}2$

$$\therefore \frac{\varepsilon - \varepsilon_1}{\sqrt{\mu}} = \frac{\tan^{-1}2}{\sqrt{\mu}} = \frac{\tan^{-1}2}{2\pi} \quad \left(\frac{2\pi}{\sqrt{\mu}}\right)$$

$$= \frac{\tan^{-1}2}{2\pi} \text{ of a period.}$$

This is the phase difference of the resultant simple harmonic motion.

EXERCISES

1. Two simple harmonic motions in the same straight line of equal periods and differing in phase by $\frac{\pi}{2}$ are impressed simultaneously on a particle. If the amplitudes are 4 and 6, find the amplitude and phase of the resulting motion.

(B.A. 38 Physics Madras Uty.)

2. A particle possesses two simple harmonic motions of the same period with amplitudes a and b and phase difference $\frac{\pi}{2}$ in two perpendicular directions. Show that the particle traces an ellipse whose semi-major and minor axes are a and b. (B.E. 67, S.V. Uty.)

§ 10.8. Force necessary to produce Simple Harmonic Motion:

If F is the force required to produce an acceleration f in a particle of mass m, then by Newton's second law of motion. F = mf. If m is constant, F must obey the same law as f. Hence to produce a simple harmonic motion, the force must be always directed towards a fixed centre which is usually the equilibrium position of the particle and its magnitude must be proportional to the displacement from that position. The force tending to restore an elastic body to its natural shape or size is found to be of the above nature and a practical instance is the force exerted by an elastic string or spiral spring.

§ 10.9 Motion of a particle suspended by a spiral spring:

A particle is suspended from a fixed point by a spiral spring of natural length a and modulus λ. If it is displaced slightly in the vertical direction, to discuss the subsequent motion:

Let A be the fixed point and AB = a, the unstretched length of the spring. Let m be the mass of the particle.

The particle will pull the string further and come to rest. Let C be the equilibrium position of the particle and BC = l.

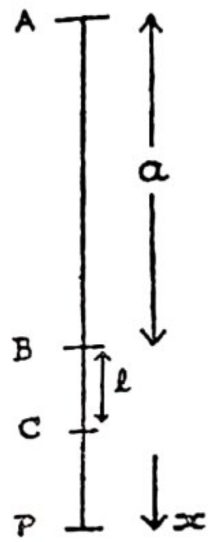


Fig. 123

The forces acting at C are: (i) mg, the weight of the particle acting vertically downwards and (ii) the upward tension. These two must be equal.

By Hooke's law, tension

$$= \frac{\lambda}{a}(AC - AB) = \frac{\lambda l}{a}$$

Hence $\frac{\lambda l}{a} = mg$ ... (1)

Let the particle be slightly displaced vertically downwards through a certain distance and then released. Clearly it will begin to move upwards. Let P be the subsequent position of the particle so that CP = x (x being measured in the direction CP).

The forces acting at P are the weight and the upward tension.

Hence the equation of motion is

$$m\frac{d^2x}{dt^2} = \text{Resultant downward force}$$

$$= mg - \text{upward tension} = mg - \frac{\lambda}{a}(AP - AB)$$

$$= mg - \frac{\lambda}{a}(BP) = mg - \frac{\lambda}{a}(l + x)$$

$$= -\frac{\lambda x}{a}, \text{ since } mg = \frac{\lambda l}{a} \ (1)$$

... (2)

i.e. $\frac{d^2x}{dt^2} = -\frac{\lambda}{am}x$

MOTION UNDER THE ACTION OF CENTRAL FORCES

§ 11.1. Introduction:

In the previous chapters, we have considered some particular cases of motion of a particle in two dimensions. To fix the position of a particle in a plane, we require two coordinates and to study the motion of the particle, we require its component velocities and accelerations in two mutually perpendicular directions. We had previously used cartesian coordinates. In this chapter we shall use polar coordinates.

§ 11.2. Velocity and Acceleration in Polar Coordinates:

Let P be the position of a moving particle at time t. Taking O as the pole and OX as the initial line, let the polar coordinates of P be (r, θ). $\overline{OP} = r$ is the position vector of P. Hence the velocity of P $= \frac{d}{dt}(r)$. Since **r** has modulus r and amplitude θ, $\frac{d}{dt}(r)$ will have components $\dot{r}$ along OP and $r\dot{\theta}$ to OP. (Refer rule of differentiation of a vector given in § 9.1). Hence the velocity vector v at P has components $\dot{r}$ along OP in the direction in which r increases and $r\dot{\theta} \perp$ to OP in the direction in which θ increases. These are respectively called the *radial* and *transverse* components of v.

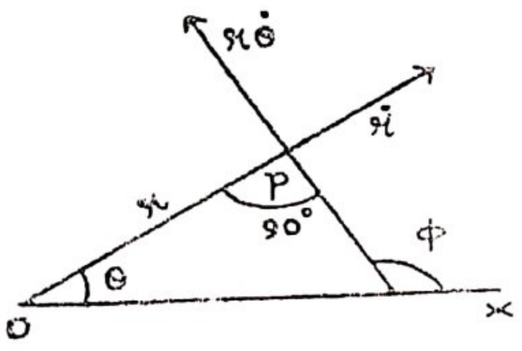
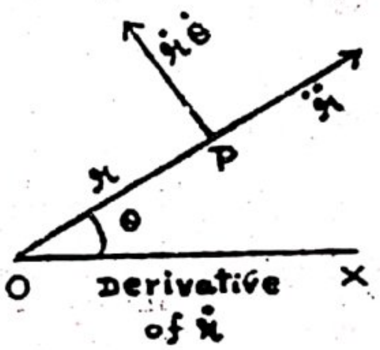


Fig. 131

The acceleration vector at P is the derivative of the velocity vector v. The radial component of v is a vector with modulus r and amplitude θ. Hence the derivative of $\dot{r}$ will have components (i) $\frac{d}{dt}(\dot{r}) = \ddot{r}$ along OP in the direction in which r increases and (ii) $\dot{r}\frac{d}{dt}(\theta) = \dot{r}\dot{\theta} \perp$ to OP in the direction in which θ increases. This is shown in fig. 132.

The transverse component of v is a vector with modulus $r\dot{\theta}$ and amplitude $\varphi = \frac{\pi}{2} + \theta$. Hence the derivative of $r\dot{\theta}$ will have components (i) $\frac{d}{dt}(r\dot{\theta}) = r\ddot{\theta} + \dot{\theta}\dot{r}$ along the line of $r\dot{\theta}$ i.e. in the direction ⊥ to OP and (ii) $r\dot{\theta}\frac{d}{dt}\left(\frac{\pi}{2} + \theta\right) = r\dot{\theta}^2$ in the direction ⊥ to the line of $r\dot{\theta}$ i.e. in the direction PO. (This component is towards O, as it is in the direction in which φ increases). This is shown in fig. 133.



· Fig. 132

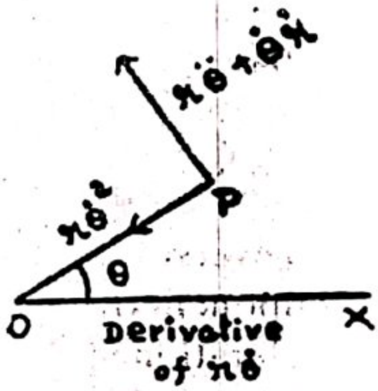


Fig. 133

Hence the totals of the components of acceleration are $\ddot{r} - r\dot{\theta}^2$ in the direction OP and $r\ddot{\theta} + 2\dot{r}\dot{\theta}$ in the perpendicular direction.

Now $\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{r}(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}) = r\ddot{\theta} + 2\dot{r}\dot{\theta}$

∴ Acceleration ⊥ to OP is also $= \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$

The above results are collected in a table of reference.

		Magnitude	Direction	Sense
1.	Radial component of velocity	$\dot{r}$	Along the radius vector	In the direction in which r increases
2.	Transverse component of velocity	$r\dot{\theta}$	Perpendicular to the radius vector	In the direction in which θ increases
3.	Radial component of acceleration	$\ddot{r} - r\dot{\theta}^2$	Along the radius vector.	In the direction in which r increases
4.	Transverse component of acceleration	$\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$	Perpendicular to the radius vector	In the direction in which θ increases.

Corollary: (1) Suppose the particle P is describing a circle of radius 'a'. Then $r = a$ throughout the motion.

Hence $\ddot{r} = 0$ and the radial acceleration $= \ddot{r} - r\dot{\theta}^2$

$$= 0 - a\dot{\theta}^2 = -a\dot{\theta}^2$$

The acceleration ⊥ to OP

$$= \frac{1}{r} \cdot \frac{d}{dt}(r^2\dot{\theta}) = \frac{1}{a}a^2\ddot{\theta} = a\ddot{\theta}$$

Hence for a particle describing a circle of radius a, the acceleration at any point P has the components $a\ddot{\theta}$ along the tangent at P and $a\dot{\theta}^2$ along the radius to the centre.

(2) The magnitude of the resultant velocity of P

$$= \sqrt{\dot{r}^2 + (r\dot{\theta})^2} = \sqrt{\dot{r}^2 + r^2\dot{\theta}^2}$$

and the magnitude of the resultant acceleration

WORKED EXAMPLES

Ex. 1. *The velocities of a particle along and perpendicular to a radius vector from a fixed origin are* λr^2 *and* $\mu\,\theta^2$ *where* μ and λ *are constants. Show that the equation to the path of the particle is* $\frac{\lambda}{\theta} + C = \frac{\mu}{2\,r^2}$ *where C is a constant.*

(B.Sc. 76 Applied Sciences, Madras; B.Sc. 69 Madurai Uty.)

Show also that the accelerations along and perpendicular to the radius vector are

$$2\,\lambda^2 r^3 - \frac{\mu^2\,\theta^4}{r} \text{ and } \mu\,(\,\lambda\, r\,\theta^2 + \frac{2\,\mu\,\theta^3}{r}\,)$$

Radial velocity $= \frac{dr}{dt} = \lambda\, r^2$ ... (1)

Transverse velocity $= r\,\frac{d\,\theta}{dt} = \mu\,\theta^2$ ... (2)

Dividing (2) by (1), we have

$$r\,\frac{d\,\theta}{dr} = \frac{\mu\,\theta^2}{\lambda\,r^2} \text{ i.e. } \lambda\,\frac{d\,\theta}{\theta^2} = \frac{\mu}{r^3}\,dr$$

Integrating, $-\frac{\lambda}{\theta} = -\frac{\mu}{2r^2} + C$

i.e. $\frac{\mu}{2r^2} = \frac{\lambda}{\theta} + C$ .... (3)

(3) is the equation to the path,

Differentiating (1), $\frac{d^2r}{dt^2} = \lambda\,.\,2r\,\frac{dr}{dt}$

$$= 2\,\lambda^2\, r^3 \text{ using (1)}$$

Radial acceleration

$$= \ddot{r} - r\dot{\theta}^2 = \frac{d^2r}{dt^2} - r\,(\frac{d\,\theta}{d\,t})^2$$

$$= 2\lambda^2 r^3 - r(\frac{\mu\theta^2}{r})^2 = 2\lambda^2 r^3 - \frac{\mu^2\theta^4}{r} \text{ using (2)}$$

Transverse acceleration

$$= \frac{1}{r}\cdot\frac{d}{dt}(r^2\dot\theta) = \frac{1}{r}\cdot\frac{d}{dt}(r^2\frac{\mu\theta^2}{r})$$

$$= \frac{1}{r}\cdot\frac{d}{dt}(\mu r\theta^2) = \frac{\mu}{r}[r^2\theta\frac{d\theta}{dt} + \theta^2\frac{dr}{dt}]$$

$$= \frac{\mu}{r}[2r.\theta\frac{\mu\theta^2}{r} + \theta^2 . \lambda r^2] = \mu[\frac{2\mu\theta^3}{r} + \lambda r\theta^2]$$

Ex. 2 *Show that the path of a point P which possesses two constant velocities u and v, the first of which is in a fixed direction and the second of which is perpendicular to the radius OP drawn from a fixed point O, is a conic whose focus is O and whose eccentricity is* $\frac{u}{v}$. (B.Sc. 82 Madras; B.Sc. 81, 84 Madurai Uty.)

Take O as the pole and the line OX parallel to the given direction as the initial line. P has two velocities u parallel to OX and v perpendicular to OP.

Resolving the velocities along and perpendicular to OP, we have

$$\frac{dr}{dt} = u\cos\theta \quad \text{... (1)}$$

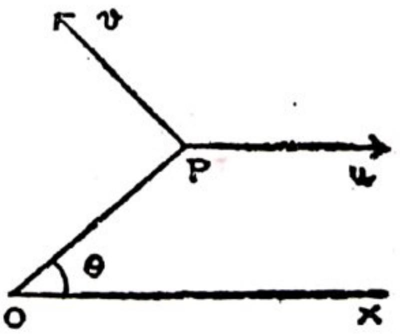


Fig. 136

$$r\frac{d\theta}{dt} = v - u\sin\theta \quad \text{... (2)}$$

To get the equation to the path, we have to eliminate t.

Dividing (2) by (1), we have

$$r\frac{d\theta}{dr} = \frac{v - u\sin\theta}{u\cos\theta}$$

Putting this is (1), $\frac{h^2}{p^2} = P\,r$ ... (2)

We know that $P = \frac{h^2}{p^3} \cdot \frac{dp}{dr}$ ... (3)

Substituting (3) in (2),

$\frac{h^2}{p^2} = \frac{h^2}{p^3} \cdot \frac{dp}{dr} \cdot r$ i.e. $\frac{dp}{p} = \frac{dr}{r}$

Integrating, $\log p = \log r + \log A$ i.e. $p = Ar$ ... (4)

(4) is clearly the (p,r) equation to an equiangular spiral.

From (4), $\frac{dp}{dr} = A$. Substituting this in (3),

$P = \frac{h^2}{p^3} \cdot A = \frac{A\,h^2}{A^3\,r^3}$ using (4)

$= \frac{h^2}{A^2}\left(\frac{1}{r^3}\right)$ i.e. $P \propto \frac{1}{r^3}$

EXERCISES

1. Find the law of force towards the pole under which the following curves can be described:

(i) $r^2 = a^2 \cos 2\theta$ (ii) $r^{1/2} = a^{1/2} \cos \frac{\theta}{2}$

(iii) $r^n \cos n\theta = a^n$

(iv) $r^n = A \cos n\theta + B \sin n\theta$.

[Hint: The equation can be taken as $r^n = \lambda \cos(n\theta + \alpha)$]

(v) $a = r \sin n\theta$ (vi) $r = a \sin n\theta$ (vii) $\frac{a}{r} = e^{n\theta}$

(viii) $r = ae^{\theta \cot \alpha}$ (ix) $r = a \cosh n\theta$

called an *apse* and the length OA is the corresponding apsidal distance. Hence at an *apse*, the particle is moving at right angles to the radius vector.

We know that $\frac{1}{p^2} = u^2 + (\frac{du}{d\theta})^2$ where $u = \frac{1}{r}$ and p is the perpendicular from the centre of force upon the tangent. At an apse, $p = r = \frac{1}{u}$. Hence from the above relation. we get $\frac{du}{d\theta} = 0$ at an apse.

§ 11.13. Given the law of force to the pole, to find the orbit:

We now consider the second type of problems namely given the value of the central acceleration P, we will find the path. We use the (u, θ) equation

$$u + \frac{d^2 u}{d\theta^2} = \frac{P}{h^2 u^2} \qquad \dots (1)$$

To solve the differential equation (1), we multiply both sides by $2\frac{du}{d\theta}$ We then have

$$2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\cdot\frac{d^2 u}{d\theta^2} = 2\frac{P}{h^2 u^2}\cdot\frac{du}{d\theta}$$

$$\text{i.e.} \frac{d}{d\theta}(u^2) + \frac{d}{d\theta}(\frac{du}{d\theta})^2 = \frac{2P}{h^2 u^2}\cdot\frac{du}{d\theta}$$

Integrating both sides with respect to θ,

$$u^2 + (\frac{du}{d\theta})^2 = \int \frac{2P}{h^2 u^2}\, du + \text{constant} \qquad \dots (2)$$

When the principle is understood, the solution (2) could be immediately written down. The method of procedure is illustrated by the following worked examples.

Ex. 9. *A particle moves with an acceleration* $\mu\,[\,3au^4 - 2(a^2 - b^2)u^5\,]$ *and is projected from an apse at a* *distance (a+b) with a velocity* $\frac{\sqrt{u}}{a+b}$. *Prove that the equation to its orbit is* $r = a + b\cos\theta$ (B.Sc. 71 Calicut Uty.)

Here $P = \mu\,[\,3au^4 - 2(a^2 - b^2)u^5\,]$

The differential equation to the path is

$$u + \frac{d^2u}{d\theta^2} = \frac{P}{h^2u^2} = \frac{\mu}{h^2}[\,3au^2 - 2(a^2-b^2)u^3\,] \qquad \dots (1)$$

Multiplying (1) by $2\frac{du}{d\theta}$ and integrating with respect to θ we get

$$u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{2\mu}{h^2}\int[\,3au^2 - 2(a^2-b^2)u^3\,]\,du + C$$

$$= \frac{2\mu}{h^2}\left[\,au^3 - 2(a^2-b^2)\frac{u^4}{2}\,\right] + C \qquad \dots (2)$$

Now $h = pv$ = constant = $p_o v_o$ where p_o and v_o are the initial values of p and v respectively.

The initial conditions are

$v_o = \frac{\sqrt{\mu}}{a+b}$ and $p_o = a + b$ as the particle is projected from an apse.

Hence $h = (a+b)\frac{\sqrt{\mu}}{a+b} = \sqrt{\mu}$ i.e. $h^2 = \mu$

So (2) becomes

$$u^2 + \left(\frac{du}{d\theta}\right)^2 = 2\left[\,au^3 - (a^2-b^2)\frac{u^4}{2}\,\right] + C \qquad \dots (3)$$

Initially at the apse, $\frac{du}{d\theta} = 0$ and $u = \frac{1}{a+b}$.

Hence substituting these in (3), we have

$$\frac{1}{(a+b)^2} = 2\left[\frac{a}{(a+b)^3} - \frac{(a^2-b^2)}{2(a+b)^4}\right] + C$$

$$= \frac{2a}{(a+b)^3} - \frac{(a-b)}{(a+b)^3} + C = \frac{1}{(a+b)^2} + C$$

So C = 0 and (3) reduces to

$$\left(\frac{du}{d\theta}\right)^2 = 2au^3 - (a^2 - b^2)u^4 - u^2$$

$$\frac{du}{d\theta} = \sqrt{2au^3 - (a^2-b^2)u^4 - u^2} = u\sqrt{2au - (a^2-b^2)u^2 - 1} \quad \text{... (4)}$$

i.e. $$\frac{du}{u\sqrt{2au - (a^2-b^2)u^2 - 1}} = d\theta$$

To integrate, put $u = \frac{1}{r}$ Then $du = -\frac{1}{r^2}\,dr$

$$-\frac{1}{r^2} \cdot r \frac{dr}{\sqrt{\frac{2a}{r} - \frac{(a^2-b^2)}{r^2} - 1}} = d\theta$$

$$-\frac{dr}{\sqrt{2ar - (a^2-b^2) - r^2}} = d\theta \text{ i.e. } \frac{-dr}{\sqrt{b^2 - (r-a)^2}} = d\theta$$

Integrating, $\cos^{-1}\left(\frac{r-a}{b}\right) = \theta + \alpha$ ... (5)

where α is the constant of integration.

If θ is measured from the apse line, $r = a + b$ when $\theta = 0$. Hence

$$\cos^{-1}\left(\frac{a+b-a}{b}\right) = 0 + \alpha \quad \text{i.e. } \cos^{-1} 1 = \alpha \text{ or } \alpha = 0.$$

Hence (5) becomes $\cos^{-1}\left(\frac{r-a}{b}\right) = \theta$ i.e. $\frac{r-a}{b} = \cos\theta$ or $r = a + b\cos\theta$.

Note: On taking the squre root in equation (4) above, $\frac{du}{d\theta}$ can be taken either with the positive or negative sign. We will get the