CORE COURSE - III

DIFFERENTIAL EQUATIONS AND LAPLACE TRANSFORM

UNIT - I

First Order Higher degree differential equations Solvable for x, Solvable for y. Solvable for $\frac{dy}{dx}$ Clairaut's form - Conditions of integrability of $Mdx + Ndy = 0$.

UNIT - II

Particular Integrals of Second order differential Equations with Constant Coefficient linear equations with Variable Co-efficient method of Variation of parameter.

UNIT - III

Formation of Partial differential equation General, particular and Complete integrals - Solution of PDE of the standard forms - Lagranges method - solving of Charpits method and a few standard.

UNIT - IV

PDE of second order Homogeneous equation with Constant Co-efficient particular integrals of the forms e^{ax+by}, $\sin(ax+by)$, $\cos(ax+by)$, $x^r y^n$ and $e^{ax+by} f(x,y)$

UNIT - V

Laplace Transforms - Standard formulae - Basic theorems and simple applications. Inverse Laplace Transforms - Use of Laplace Transforms in Solving PDE with Constant Co-efficient.

Text Book:

1.) T. K. Manickavachagam pillay & S. Narayanan, differential Equations S. Viswanathan publishers Pvt Ltd 1996.

2) Arumugam & Isaac, Differential Equation, New gamma publishing House palayamkottai - 2003

UNIT - I Chap IV - Sec 1, 2 & 3 chap II - Sec 6 (1)

UNIT - II Chap V - Sec 1, 2, 3, 4 & 5 chap VIII Sec 4 (1)

UNIT - III Chap XII Sec 1 - 6 (1)

UNIT - IV Chap V (2)

UNIT - V Chap IX - Sec 1 - 8 (1)

Reference book:-

1. M. D. Raisinghania, Ordinary and partial Differential Eqns. S. chand & Co.

2. M. K. Venketraman, Engineering Mathematics, S. V. publication, 1985 Raised Edwin

1. Solve the differential equation $\left(\frac{dy}{dx}\right)^2 = 5\left(\frac{dy}{dx}\right) + 6 = 0$

Given $\left(\frac{dy}{dx}\right)^2 = 5\left(\frac{dy}{dx}\right) + 6 = 0$

$\therefore P = \frac{dy}{dx}$

$P^2 = 5P + 6 = 0 \Rightarrow (P6 - 3)(P - 2) = 0$

$P = 3, 2$

$P = 3$

$\frac{dy}{dx} = 3$

$dy = 3dx$

$\int$ on B.S.

$\int dy = 3\int \cancel{dx}\, dx$

$y = 3x + C$

$y - 3x - c = 0$

$P = 2$

$\frac{dy}{dx} = 2$

$dy = 2dx$

$\int dy = 2\int dx$

$y = 2x + C$

$y - 2x - c = 0$

The required eqn is

$\therefore (y - 3x - c)(y - 2x - c) = 0$ //

2. Solve the DE $P^2 - 7P + 12 = 0$

Given $P^2 - 7P + 12 = 0$ $\quad \therefore \frac{dy}{dx} = P$

$(P-4)(P-3) = 0$

$P = 4, 3.$

$\frac{dy}{dx} = 4$

$dy = 4xdx$

$\int dy = 4\int dx$

$y = 4x + c$

$y - 4x - c = 0$

$\frac{dy}{dx} = 3$

$dy = 3dx$

$\int dy = 3\int dx$

$y = 3x + C$

$y - 3x - C = 0$

$\therefore$ The required eqn is

$(y - 4x + c)(y - 3x - c) = 0$ //

Solve the eqn $\left(\frac{dy}{dx}\right)^2 - ax^3 = 0$

$p^2 - ax^3 = 0$

$p^2 = ax^3 \Rightarrow p = a^{1/2} x^{3/2}$

$\frac{dy}{dx} = a^{1/2} x^{3/2}$

$dy = a^{1/2} x^{3/2} dx$

$\int$ing on B. S.

$\int dy = a^{1/2} \int x^{3/2} dx$

$y = a^{1/2} \frac{x^{5/2}}{5/2} + c$

$y = a^{1/2} \frac{2}{5} x^{5/2} + c$

$5y = 2a^{1/2} x^{5/2} + c$

$5y^2 = 2ax^5 + c$

$5y - 2a^{1/2} x^{5/2} - c = 0.$

Hence the required equation is $(5y - 2a^{1/2}x^{5/2} - c) = 0$

2. Solve the eqn $p^2 - (\cos x + \sec x) p + 1 = 0$

$p^2 - (\cos x + \sec x) p + 1 = 0$

$p^2 - p\cos x - \frac{1}{\cos x} p + \frac{\cos x}{\cos x} = 0.$

$p(p - \cos x) - \frac{1}{\cos x} (p - \cos x) = 0.$

$(P - \cos x)$; $\left(P - \frac{1}{\cos x}\right)$

$P = \cos x$; $P = \frac{1}{\cos x}$

$\frac{dy}{dx} = \cos x$; $\frac{dy}{dx} = \frac{1}{\cos x}$

$dy = \cos x \, dx$; $dy = \frac{dx}{\cos x}$

$\int$ing on B. S

$\int dy = \int \cos x \, dx$; $\int dy = \int \frac{1}{\cos x} dx$

$y = \sin x + c$; $\int dy = \int \sec x \, dx$

$\therefore y - \sin x + c = 0$; $y = \log(\sec x + \tan x) + c$

$\therefore y - \log(\sec x + \tan x) - c = 0$

I.I is an required equation.

Method : 2.

2) Equation solvable for x.

Let the given differential equations of the form. $F(x, y, z) = 0$ —— (1)

From it is possible to find x in terms to y and p - the equation (1) can be written as.

$$x = f(y, p) \quad \text{—— (2)}$$

Now, differentiating equation (2) with respect to y

$$\frac{dx}{dy} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial y}$$

$$\frac{1}{p} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial p} \cdot \frac{\partial p}{\partial y}.$$

Let the Solution of the terms (2) & (3).

The Solution of the given different equation $(p^2+1)^{1/2}$

$\frac{1}{2}(p^2+1)^{-1/2}\, 2p$

(1) Solve the D.E $x = 1 - \frac{p}{\sqrt{p^2+1}}$

Given $x = 1 - \frac{p}{\sqrt{p^2+1}}$ —— (1) $\therefore \frac{u}{v} = \frac{vu' - vu'}{v^2}$

Now, equation diff w. r. to y

$$\frac{dx}{dy} = \left[0 - \frac{\left[\sqrt{p^2+1}\,\frac{dp}{dy} - p\left(\frac{p\,dp}{\sqrt{p^2+1}\,dy}\right)\right]}{\left(\sqrt{p^2+1}\right)^2} \right]$$

$$= -\left[\frac{\sqrt{p^2+1}\,\frac{dp}{dy} - \frac{p^2\,dp}{\sqrt{p^2+1}\,dy}}{\left(\sqrt{p^2+1}\right)^2} \right]$$

$$= \frac{\left(\sqrt{p^2+1} - \frac{p^2}{\sqrt{p^2+1}}\right)\frac{dp}{dy}}{p^2+1}$$

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$$= -\left[\frac{\left(\frac{p^2+1-p^2}{\sqrt{p^2+1}}\right)}{p^2+1}\right]\frac{dx}{dy}$$

$$\frac{dx}{dy} = \frac{-1}{(p^2+1)^{1/2}} \cdot \frac{dp}{dy} \cdot \frac{1}{p^2+1}$$

$$= \frac{-1}{(p^2+1)^{3/2}} \cdot \frac{dp}{dy}$$

$$\frac{1}{p} = \frac{-1}{(p^2+1)^{3/2}} \cdot \frac{dp}{dy}.$$

$$dy = \frac{-p}{(p^2+1)^{3/2}}\, dp \quad \text{—— (2)}$$

Let $p^2+1 = t^2$

$$\frac{2p\,dp}{dt} = 2t$$

$$2p\,dp = 2t\,dt$$

$$p\,dp = t\,dt \quad \text{—— (3)}$$

From eq. (2)

$$dy = \frac{-t\,dt}{(t^2)^{3/2}}$$

$$= \frac{-t\,dt}{t^3}$$

$$dy = \frac{-1}{t^2}\,dt \quad \text{—— (4)}$$

Eqn (4) $\int$ing on B. S

$$\int dy = -\int \frac{dt}{t^2}$$

$$y = -\left[\frac{-1}{t}\right] + c$$

$$y - c = \frac{1}{6}$$

$$y - c = \frac{1}{\sqrt{(p^2+1)}} \quad \text{—— (5)}$$

From eqn (1)

$$x = 1 - \frac{p}{\sqrt{p^2+1}}$$

$$x - 1 = \frac{-p}{\sqrt{p^2+1}} \quad \text{—— (6)}$$

Eqn (6) Square on B.S.

$$(x - 1)^2 = \frac{p^2}{p^2+1} = 1 - \frac{1}{p^2+1}$$

$$(x - 1)^2 = 1 - (y - e)^2$$

$$(x - 1)^2 + (y - e)^2 = 1 //$$

3\. Solve $x = p^2 + y$.

Soln

$x = p^2 + y$.

Diff w.r.t. to y.

$$\frac{dx}{dy} = 2p\frac{dp}{dy} + 1$$

$$\frac{1}{p} - 1 = 2p\frac{dp}{dy}$$

$$2p\frac{dp}{dy} = \frac{1-p}{p}$$

$$\frac{2p \cdot dp \cdot p}{1-p} = dy.$$

$$dy = \frac{2p^2 dp}{1-p}$$

$$dy = 2\left(-1 - p + \frac{1}{1-p}\right)dp.$$

$$dy = -2dp - 2p\,dp + \frac{2}{1-p}dp$$

$\int$ing on B.S.

$$\int dy = -2\int dp - 2\int p\,dp + 2\int\frac{dp}{1-p}$$

$$y = -2p - \frac{2p^2}{2} - 2\log(p-1) + c$$

$$x = p^2 + y$$

$$x = p^2 - 2p + p^2 - 2\log(p-1) + c$$

Hence the required eqn is

$$(p^2 - 2p + p^2 - 2\log(p-1) + c)$$

$$(-2p - p^2 - 2\log(p-1) + c) = 0$$

$$\begin{array}{r|l} & -p-1 \\ \hline 1-p & p^2 \\ & p^2-p \\ & (-)\ (+) \\ \hline & p \\ & p-1 \\ \hline & 1 \end{array}$$

$$\frac{p^2}{-p} = -p \qquad \frac{p}{-p} = -1$$

5) Solve $x = \tan^{-1} p + \frac{p}{1+p^2}$

Soln

$$x = \tan^{-1} p + \frac{p}{1+p^2}$$

Diff w. to. x

$$\frac{dx}{dy} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{(1+p^2)\frac{dp}{dy} - p\,2p\,\frac{dp}{dy}}{(1+p^2)^2}$$

$$\frac{dx}{dy} = \frac{1}{1+p^2}\frac{dp}{dy} + \frac{(1+p^2)\frac{dp}{dy} - p2p\cdot\frac{dp}{dy}}{(1+p^2)^2}$$

$$= \frac{1}{1+p^2}\frac{dp}{dy} + \frac{\left(\frac{dp}{dy} + p^2\cdot\frac{dp}{dy} - 2p^2\frac{dp}{dy}\right)}{(1+p^2)^2}$$

$$\frac{1}{p} = \frac{1}{1+p^2}\frac{dp}{dy}\left[1 + \frac{1+p^2-2p}{(1+p^2)}\right]$$

$$= \frac{1}{1+p^2}\frac{dp}{dy}\left[\frac{1+p^2+1+p^2-2p^2}{1+p^2}\right]$$

$$= \frac{1}{1+p^2}\frac{dp}{dy}\left(\frac{2}{1+p^2}\right)$$

$$= \frac{2}{(1+p^2)^2}\frac{dp}{dy}$$

$$dy = \frac{2dp\cdot p}{(1+p^2)^2}$$

Let $1+p^2 = t$

$$2p\,dp = dt$$

$$\int dy = \int \frac{dt}{t^2}$$

$$y = -\frac{1}{t} + c$$

$$y - c = -\frac{1}{t}$$

$$y - c = -\frac{1}{1+p^2}$$

$$1 + p^2 = \frac{-1}{y-c}$$

$$p^2 = \frac{-1}{(y-c)} - 1$$

$$p^2 = \frac{-(1+y-c)}{-(c-y)} = \frac{1+y-c}{c-y}$$

$$p = \sqrt{\frac{1+y-c}{c-y}}.$$

$$x = \tan^{-1}(p) + \frac{p}{1+p^2}$$

$$x = \tan^{-1}\left(\frac{\sqrt{1+y-c}}{c-y}\right) + \frac{\sqrt{\frac{1+y-c}{c-y}}}{1 + \frac{1+y-c}{c-y}}.$$

$$= \tan^{-1}\left(\frac{\sqrt{1+y-c}}{c-y} + \frac{\sqrt{\frac{1+y-c}{c-y}}}{\frac{c-y+1+y-c}{c-y}}\right)$$

$$= \tan^{-1}\left(\sqrt{\frac{1+y-c}{c-y}} + \frac{\sqrt{\frac{1+y-c}{c-y}}}{1/c-y}\right.$$

$$= \tan^{-1}\sqrt{\frac{1+y-c}{c-y}} + \sqrt{\frac{1+y-c}{c-y}} \times \frac{c-y}{1}$$

$$= \tan^{-1}\sqrt{\frac{1+y-c}{c-y}} + c-y\sqrt{\frac{1+y-c}{c-y}}$$

$$x = \tan^{-1}\sqrt{\frac{1+y-c}{c-y}} + \sqrt{c-y}\sqrt{1+y-c}$$

$$x - \sqrt{c-y}\sqrt{1+y-c} = \tan^{-1}\sqrt{\frac{1+y-c}{c-y}}$$

$$\tan(x - \sqrt{x-y}\sqrt{1+y-c} = \sqrt{\frac{1+y-c}{c-y}}$$

4. Solve $4y = x^2 + p^2$

Soln

$$4y = x^2 + p^2$$

Diff. w. r. to x

$$4\frac{dy}{dx} = 2x + 2p\frac{dp}{dx}$$

$$2p = x + p\frac{dp}{dx}$$

$$\frac{2p - x}{p} = \frac{dp}{dx}$$

Put $p = vx$

$$\frac{dp}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{2p - x}{p}$$

$$v + x\frac{dv}{dx} = \frac{2vx - x}{vx}$$

$$x\frac{dv}{dx} = \frac{2v-1}{v} - v$$

$$x\frac{dv}{dx} = \frac{2v - 1 - v^2}{v}$$

$$= \frac{-(v^2 + 1 - 2v)}{v}$$

$$x\frac{dv}{dx} = \frac{-(v-1)^2}{v}$$

$$\frac{v}{(v-1)^2}\,dv = \frac{dx}{x}$$

$\int$ing on the B.S

$$\int\frac{v-1+1}{(v-1)^2}\,dv = \int\frac{dx}{x}$$

$$\int\frac{(v-1)}{(v-1)^2}\,dv + \int\frac{dv}{(v-1)^2} = \int\frac{dx}{x}$$

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$$\int \frac{1}{(v-1)}\, dv + \int \frac{1}{(v-1)^2}\, dv = -\int \frac{dx}{x}$$

$$\log(v-1) - \frac{1}{v-1} = -\log x + \log c$$

$$\log(v-1) + \log x - \log c = \frac{1}{v-1}$$

$$\log \frac{(v-1)x}{c} = \frac{1}{v-1}$$

$$\log \frac{\left(\frac{p}{x} - 1\right)x}{c} = \frac{1}{p/x - 1}$$

$$\log \frac{\left(\frac{p-x}{x}\right)x}{c} = \frac{1}{\frac{p-x}{2}}$$

$$\log \left(\frac{\frac{p-x}{2} \times x}{c} \right) = \frac{x}{p-x}$$

$$\log \left(\frac{p-x}{c} \right) = \frac{x}{p-x}$$

4. $x - yp = ap^2$

$$x - yp = ap^2 \quad (1)$$

$$-yp = ap^2 - x$$

$$yp = x - ap^2$$

$$y = \frac{x}{p} - ap \quad (2)$$

Diff w. r. to. x

$$\frac{dy}{dx} = \left(\frac{p(1) - x\left(\frac{dp}{dx}\right)}{p^2}\right) - a\left(\frac{dp}{dx}\right)$$

$$p = \frac{p}{p^2} - \frac{x}{p^2}\left(\frac{dp}{dx}\right) - a\frac{dp}{dx}$$

$$p = \frac{1}{p} - \left(\frac{x}{p^2} + a\right)\frac{dp}{dx}$$

$$\frac{1}{p} - p = \left(\frac{x}{p^2} + a\right)\frac{dp}{dx}$$

$$\frac{1-p^2}{p} = \left(\frac{x+ap^2}{p^2}\right)\frac{dp}{dx}$$

$$p(1-p^2) = (x+ap^2)\frac{dp}{dx}$$

$$\frac{dp}{dx} = \frac{p(1-p^2)}{(x+ap^2)}$$

$$\frac{dx}{dp} = \frac{x+ap^2}{p(1-p^2)}$$

$$= \frac{x}{p(1-p^2)} + \frac{ap^2}{p(1-p^2)}$$

$$= \frac{x}{p(1-p^2)} + \frac{ap}{(1-p^2)}$$

$$\frac{dx}{dp} = -\frac{x}{p(p^2-1)} - \frac{ap}{(p^2-1)}$$

$$\frac{dx}{dp} + \frac{x}{p(p^2-1)} = \frac{-ap}{(p^2-1)}$$

This is of the form $\frac{dy}{dx} + Py = Q$. Then the Solution of the equation is

$$ye^{\int P dx} = \int Q e^{\int P dx} dx + c$$

$\because x = p ; dx = dp$

$y = x ; dy = dx$

Now, $xe^{\int P dp} = \int Q e^{\int P dp} dp + c$ — (3)

To find.

$$\frac{1}{p(p^2-1)} = \frac{A}{p} + \frac{Bp+C}{(p^2-1)}$$

$$1 = A(p^2-1) + p(Bp+C) \longrightarrow (4)$$

$p = 0 \Rightarrow 1 = A(-1)$

$A = -1$

$p = 1 \Rightarrow 1 = B + C \longrightarrow (5)$

$p = -1 \Rightarrow 1 = -B + C \longrightarrow (6)$

Solve (5) & (6)

$$B + C = 1$$
$$-B + C = 1$$
$$2C = 2$$
$$C = \frac{2}{2} \Rightarrow C = 1$$

$C = 1$ in eqn (5)

$$(5) \Rightarrow B + C = 1$$
$$B + 1 = 1$$
$$B = 0$$

Sub eq. (4)

$$1 = -\left(1\right)/p + \frac{p}{p^2-1}$$

$$e^{\int \frac{1}{p(p^2-1)} dp} = e^{\int -dp/p + \int \frac{p}{p^2-1} dp}$$

$$= e^{-\log p + \frac{1}{2}\log(p^2-1)}$$

$$= e^{\log\left(\sqrt{p^2-1}/p\right)}$$

$$= \frac{\sqrt{p^2-1}}{p}$$

$$\therefore x e^{\int P dp} = \int Q e^{\int P dp} dp + c$$

$$x \cdot \frac{\sqrt{p^2-1}}{p} = \int \frac{-ap}{p^2-1} \cdot \frac{\sqrt{p^2-1}}{p} dp + c$$

$$\frac{x\sqrt{p^2-1}}{p} = -a \int \frac{1}{\sqrt{p^2-1}} dp + c$$

$$\frac{x\sqrt{p^2-1}}{p} = -a \cosh^{-1} p + c$$

$$x\sqrt{p^2-1} = -ap \cosh^{-1} p + c$$

$$x = \frac{-ap \cosh^{-1} p}{\sqrt{p^2-1}} + c \rightarrow (7)$$

The elimination p is very difficult from equation (2) & (7)

(3) Chairaut's equation . $y = px + ap^{-1}$

Soln. Given ;

$$y = px + ap^{-1} \quad \text{--- (1)}$$

Step : 1

which is of the form $y = xp + \phi(p)$

Put $P = c$ in equation (1)

$$y = (x + ac^{-1}) \quad \text{--- (2)}$$

Step : 2

Differentiate with respect to c .

$$0 = x + a\left[(-1)(c)^{-2}\right]$$

$$0 = x - ac^{-2}$$

$$x = ac^{-2}$$

$$x = a \cdot 1/c^2$$

$$c^2 = a/x$$

$$c = \sqrt{a/x} \quad \text{--- (3)}$$

Sub eqn (3) in (2)

$$y = x\sqrt{a/x} + a\sqrt{x/a}$$

$$= \sqrt{ax} + \sqrt{ax}$$

$$y = 2\sqrt{ax}$$

Squaring on B. S.

$$y^2 = 4ax$$

$$y^2 - 4ax = 0$$

Hence $y = cx + ac^{-1}$ is a general soln and $y^2 - 4ax = 0$ is the singular soln.

(6) $y = (x-a)p - p^2$

Soln:

$y = (x-a) \cdot p - p^2$

$y = (x-a)p - p^2 \quad \text{--- (1)}$

which is of the form $y = xp + \phi(p)$

Step 1 Put, $p = c$ in eq. (1)

$y = xc - ac - c^2 \quad \text{--- (2)}$

Step: 2 Diff w.r.to c we get

$0 = x - a - 2c$

$2c = x - a$

Step: 3

$c = \frac{x-a}{2} \quad \text{--- (3)}$

Sub equation (3) in (1)

$y = x\left(\frac{x-a}{2}\right) - a\left(\frac{x-a}{2}\right) - \left(\frac{x-a}{2}\right)^2$

$y = \frac{x^2 - xa}{2} - \frac{ax + a^2}{2} - \left(\frac{x^2 + a^2 - 2ac}{4}\right)$

$= \frac{2(x^2 - xa) + 2(-ax + a^2) - x^2 - a^2 + 2ax}{4}$

$y = \frac{2x^2 - 2xa - 2ax + 2a^2 - x^2 - x^2 + 2ax}{4}$

$4y = 2x^2 - 2xa - 2ax + 2a^2 - x^2 - x^2 + 2ax.$

$4y = x^2 + a^2 - 2ax$

$4y = (x - a^2$

$4y - (x - a^2) = 0$ is the singular Solution.

Hence it is an singular ...

(10) $\sin px \cos y = \cos px \sin y + p$

$$\sin px \cos y = \cos px \sin y + p \quad (1)$$

$$\sin px \cos y - \cos px \sin y = p$$

$$\sin(px - y) = p$$

$$(px - y) = \sin^{-1}(p)$$

$$y = px - \sin^{-1}(p) \quad (2)$$

This is from $y = xp + \phi(p)$ — (3)

Put $p = c$

Step: 1

$$y = cx - \sin^{-1} c$$

Step: 2

Diff w. r. to. c

$$0 = x - \frac{1}{\sqrt{1-c^2}}$$

$$x = \frac{1}{\sqrt{1-c^2}}$$

Step: 3

(2) $\Rightarrow$ $y = c\left(\frac{1}{\sqrt{1-c^2}}\right) - \sin^{-1} c$

The elimination of c is very difficult

Hence it is general Solution

to c is the solution.

Solve $(x^2 - 2xy + 3y^2\, dx) + (y^2 + 6xy - x^2)\, dy = 0$.

This eqn of the form.

$$M dx + N dy = 0$$

$M = x^2 - 2xy + 3y^2$, $N = y^2 + 6xy - x^2$

$\frac{\partial M}{\partial y} = -2x + 6y$, $\frac{\partial N}{\partial x} = 6y - 2x$.

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

This the given diff. eqn is exact the Solution is

$\int M dx + \int N dy = c$.

[y constant] [Not contains x]

$\Rightarrow \int (x^2 - 2xy + 3y^2) dx + \int y^2 dy = c$

$\frac{x^3}{3} - \frac{2x^2y}{2} + 3y^2 x + \frac{y^3}{3} = c$

$\frac{x^3}{3} - x^2y + 3y^2x + \frac{y^3}{3} = c$

$x^3 - 3x^2y + 3y^2x + y^3 = c$

where 'c' is constant.

(2). $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$

Soln Given,

$$\frac{dy}{dx} = -\left(\frac{y\cos x + \sin y + y}{\sin x + x\cos y + x}\right)$$

$$(\sin x + x\cos y + x)\,dy = -(y\cos x + \sin y + y)\,dx$$

$$(y\cos x + \sin y + y)\,dx + (\sin x + x\cos y + x)\,dy = 0$$

This eqn of the form.

$$M dx + N dy = 0$$

$M = y\cos x + \sin y + y$, $N = \sin x + x\cos y + x$.

$\frac{\partial M}{\partial y}$ $\cos x + \cos y + 1$, $\frac{\partial N}{\partial x} = \cos x + \cos y + 1$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

There the given diff. is exact.

$$\int M dx + \int N dy = c$$

(y constant (Not containg x)

$$\int (y\cos x + \sin y + y\;dx) + 0 = c$$

$$y\sin x + x\sin y + xy = c$$

where c is a Constant

(3). Solve : $(1 + e^{x/y})\,d$

1) Solve $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$

$$(x^2y - 2xy^2)dx - (x^3 - 3x^2y)dy = 0$$

This is the equation of the form.

$$Mdx + Ndy = 0$$

$M = x^2y - 2xy^2$, $N = x^3 - 3x^2y = 3x^2y - x^3$

$\frac{\partial M}{\partial y} = x^2 - 4xy$; $\frac{\partial N}{\partial x} = 6xy - 3x^2$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

The given differential eqn. is not exact

Now.

$$Mx + Ny = (x^2y - 2xy^2)x + (3x^2y - x^3)y$$

$$= x^3y - 2x^2y^2 + 3x^2y^2 - x^3y$$

$$= x^2y^2$$

$Mx + Ny \neq 0$ ⟵ 2

$\frac{1}{Mx+Ny} = \frac{1}{x^2y^2}$ is an integrating factor.

$$① \times \frac{1}{x^2y^2} = \frac{(x^2y - 2xy^2)}{x^2y^2}\,dx - \frac{(x^3 - 3x^2y)}{x^2y^2}\,dy = 0$$

$$\Rightarrow \left(1/y - 2/x\right)dx - \left(x/y^2\right) - \left(3/y\right)dy = 0 .$$

$$\int M_1 dx + \int N_1\, dy = c$$

$$M_1 = \frac{1}{y} - \frac{2}{x} \quad ; \quad N_1 = \frac{3}{y} - \frac{x}{y^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{-1}{y^2} \quad ; \quad \frac{\partial N_1}{\partial x} = \frac{-1}{y^2}$$

$$\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$$

$\therefore$ eqn ③ is an exact

$$\int \left(\frac{1}{y} - \frac{2}{x}\right)dx - \int \left(\frac{-3}{y}\right)dy = c .$$

(y Constant) (Not Containing x)

$$\frac{x}{y} - 2\log x + 3\log y = c$$

$$\frac{x}{y} - \log x^2 + \log y^3 = c$$

$$x - y\log x^2 + y\log^3 = c$$

Where 'c' is a Constant .

(1). Solve $(D^5 - D)\, y = 12e^x$

The given eqn is

$(D^5 - D)\, y = 12e^x$

The auxillary eqn is

$(m^2)^2 - 1^2 = 0$

$m^5 - m = 0$

$m(m^4 - 1) = 0$

$m = 0 \;,\; m^4 - 1 = 0$

$m = 0 \;,\; m^4 - 1 = 0$

$(m^2+1)(m^2-1) = 0$

$m = 0 \;,\; m^2 = -1 \;,\; m^2 = 1$

$m = 0 \;,\; m = \sqrt{-1} \;,\; m = \pm 1$

$m = \pm i$

$0 \pm i$

$\alpha = 0 \; \beta = 1$

Complementary function (C.F).

$y_c = Ae^{ax} + e^{0x}(B\cos x + C\sin x) + De^{x} + Ee^{-x}$ —— (1)

$P.I = y_p = \frac{1}{\phi(D)} f(x)$

$= \frac{1}{D^5 - D} 12e^{x}$

$= \frac{12}{1-1} e^{x}$

$= \frac{12}{0} e^{x}$ $(\because Dr$ is $0)$

$P.I \Rightarrow y_p = \frac{x}{\phi'(D)} f(x) = \frac{12x}{5D^4 - 1} e^{x}$

$= \frac{12x}{5-1} e^{x}$

$= \frac{12x}{4} e^{x}$

$= 3x\,e^{x}$ —— (2)

Complete Soln is

$y = C.F + P.I.$

$y = [Ae^{0x} + e^{0x}(B\cos x + \sin x) + De^{x} + Ee^{-x}] + 3xe^{x}$

(1)

Type - II :

If $f(x) = \sin ax$ or $\cos ax$, then particular integral is given by .

$$P.I = \frac{1}{\phi(D)} \sin ax \text{ (or) } \cos ax .$$

In $\phi(D)$ replace D^2 by $-a^2$ provided $\phi(D) = 0$

If $\phi(D) = 0$ when replace D^2 by $-a^2$

Then $P.I \quad x \cdot \frac{1}{\phi'(D)} \sin ax \text{ (or) } \cos ax$ and

this process may be repeated $\phi'(D) = 0$.

and so on. Again replace D^2 by $-a^2$ in $\phi'(D)$

Provided $\phi'(D) \neq 0$, then .

$$P.I = \frac{x^2}{\phi''(D)} \sin ax \text{ (or) } \cos ax .$$

1). 2 $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = \sin 3x$.

$D^2 = \frac{d^2}{dx^2}$. $D = \frac{d}{dx}$

Soln

$(D^2 + 3D + 2)y = \sin 3x$

The auxillary eqn is $m^2 + 3m + 2 = 0$

$(m+1)(m+2) = 0$

$m = -1 \quad m = -2$

Complementary function $y_c = Ae^{M_1 x} + Be^{M_2 x}$.

$$y_c = Ae^{-x} + Be^{-2x}$$

$$y_p = \frac{1}{\phi(D)} \sin ax$$

$$y_p = \frac{1}{D^2+3D+2} \sin 3x$$

$$= \cancel{\frac{1}{D^2+3D+2} \sin 3x}$$

$$= \frac{1}{-9+3D+2} \sin 3x$$

$$= \frac{1}{3D-7} \sin 3x$$

$$= \frac{1}{3D-7} \times \frac{3D+7}{3D+7} \sin 3x$$

Mathematical Statistics.

$$= \frac{(3D+7)\sin 3x}{(3D)^2 - 7^2}$$

$$= \frac{3D(\sin 3x) + 7\sin 3x}{9D^2 - 49}$$

$$= \frac{3(\cos 3x)3 + 7\sin 3x}{(9 \times -9) - 49}$$

$$y_p = \frac{9\cos 3x + 7\sin 3x}{-81 - 49}$$

Complete Soln.

$$y = y_c + y_p$$

$$y = Ae^{-x} + Be^{-2x} + \frac{9\cos 3x + 7\sin 3x}{130}$$

1. Solve $(D^2+5D+4)y = x^2+7x+9$.

Sol:

(A) Given : $(D^2+5D+4)y = x^2+7x+9$

The auxillary eqn is $m^2+5m+4=0$

$(m+1)(m+4)=0$

$m=-1, m=-4$

The Complementry function $y_c = Ae^{m_1 x}+Be^{m_2 x}$

$y_c = Ae^{-x}+Be^{-4x}$

$\cancel{y_2}=0$

$P.I = \frac{1}{\phi(D)} f(x)$

$= \frac{1}{D^2+5D+4}(x^2+7x+9)$

$= \frac{1}{4\left(\frac{D^2}{4}+\frac{5D}{4}+1\right)}(x^2+7x+9)$.

$= \frac{1}{4} \frac{1}{1+\left(\frac{D^2}{4}+\frac{5D}{4}\right)}(x^2+7x+9$

$= \frac{1}{4}\left[1+\left(\frac{D^2}{4}+\frac{5D}{4}\right)\right]^{-1}(x^2+7x+9)$

$= \frac{1}{4}\left[1-\frac{D^2}{4}-\frac{5D}{4}+\left(\frac{D^2}{4}+\frac{5D}{4}\right)^2 \dots\right]$.

(x^2+7x+9)

$= \frac{1}{4}\left[1-\frac{D^2}{4}-\frac{5D}{4}+\frac{25D^2}{16}\right](x^2+7x+9)$

$= \frac{1}{4}\left[x^2+7x+9-\frac{D^2}{4}(x^2+7x+9)-\frac{5D}{4}(x^2+7x+9)\right.$

$\left.+\frac{25D^2}{16}(x^2+7x+9)\right]$

$= \frac{1}{4}\left[x^2+7x+9-\frac{1}{4}(2)-\frac{5}{4}(2x+7)+\frac{25}{16}(2)\right]$

$$= \frac{1}{4}\,\frac{\left[8x^2+56x+72-4-10(2x+7)+25\right]}{8}$$

$$= \frac{8x^2+56x+72-4-20x-70+25}{32}$$

$$P.I = \frac{8x^2+36x+23}{32}$$

the Complete soln $y = y_c + y_p$

$$y = Ae^{-x} + Be^{-4x} + \frac{1}{32}\left(8x^2+36x+23\right)$$

Type IV

If $f(x) = e^{ax} X$, where X is some function of x

Then $P.I = \frac{1}{\phi(D)} e^{ax} X$

$$= e^{ax} \frac{1}{\phi(D+a)} X$$

(1) Solve $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 4y = e^{-x}\sin 2x$

Soln Given $(D^2+4D+4)y = e^{-x}\sin 2x$.

The Auxillary eqn $m^2+4m+4 = 0$

$(m+2)(m+2) = 0$

$m = -2$ (twice

The Compl. fn $= y_c = (Ax+B)e^{-2x}$.

~~$P.I = y_p = \frac{1}{(D-1)^2+4(D-1)+4} e^{-x}\sin 2x$~~

$P.I = y_p = \frac{1}{(D-1)^2+4(D-1)+4} e^{-x}\sin 2x$.

$$= \frac{1}{D^2+1-2D+4D-\cancel{4}+\cancel{4}} e^{-x}\sin 2x$$

$$= \frac{1}{D^2+2D+1} e^{-x}\sin 2x$$

$$= e^{-x}\frac{1}{D^2+2D+1}\sin 2x$$

$\boxed{D^2=-4}$

$$= e^{-x}\frac{1}{-4+2D+1}\sin 2x$$

$$= e^{-x} \frac{1}{2D-3} \sin 2x$$

$$= e^{-x} \frac{1}{2D-3} \times \frac{2D+3}{2D+3} \sin 2x$$

$$= e^{-x} \frac{2D+3}{(2D-3)(2D+3)} \sin 2x$$

$$= e^{-x} \frac{2D(\sin 2x) + 3(\sin 2x)}{4D^2 - 9}$$

$$= e^{-x} \frac{2(\cos 2x) \cdot 2 + 3\sin 2x}{4(-4) - 9}$$

$$= e^{-x} \frac{4\cos 2x + 3\sin 2x}{-16 - 9}$$

$$y_p = e^{-x} \frac{4\cos 2x + 3\sin 2x}{-25}$$

$$\text{P.I} = y_c + y_c$$

$$y = (Ax+B)e^{-2x} + e^{-x}\left(\frac{4\cos 2x + 3\sin 2x}{}\right)$$

Laplace Transforms

Define :-

f (t) be a function of a Variable t which is defined for all positive values of (t)

Let S be the real constant

If $\int_0^\infty e^{-st} f(t)\, dt$ exist and this equation to F(s) that F(s) is called the Laplace transform of f(t).

It is denoted by .

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)$$

(1) Prove that $L[e^{at}] = \frac{1}{s-a}$, $s-a > 0$

Proof :

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$L(f(t)) = L(e^{at})$$

$$L[e^{at}] = \int_0^\infty e^{-st} e^{at}\, dt$$

$$\int_0^\infty e^{-st} e^{at}\, dt$$

$$= \int_0^\infty e^{-t(s-a)}\, dt$$

$$\int_0^\infty e^{-t(s-a)}\, dt$$

$$= \left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^\infty$$

$$= \left[\frac{e^{-t(s-a)}}{-(s-a)}\right]_0^\infty$$

$$= \frac{-1}{s-a}\left[e^{-t(s-a)}\right]_0^\infty \quad \frac{1}{s-a}$$

$$= \frac{-1}{s-a}[0-1]$$

$$L[e^{at}] = \frac{1}{s-a}$$

2). Find $L[3e^{5t} + 5\cos t]$

Soln. $L[3e^{5t} + 5\cos t] = 3(L(e^{5t})) + 5\, L[\cos t]$

$L(\cos at) = \frac{s}{s^2+a^2}$

$a = 1$

$= 3\left(\frac{1}{s-5}\right) + 5\, \frac{s}{s^2+1^2}$

$= \frac{3}{s-5} + \frac{5s}{s^2+1}$

$= \frac{3(s^2+1) + 5s(s-5)}{(s-5)(s^2+1)}$

$= \frac{3s^2 + 3 + 5s^2 - 25s}{s^3 + s - 5s^2 - 5}$

$= \frac{8s - 25s + 3}{s^3 - 5s^2 + s - 5}$

$$= \frac{15}{8} \cdot \frac{\sqrt{\pi}}{s^{7/2}}$$

Type . III

First shifting theorem.

If the Laplace transform of $f(t)$ is $F(s)$ then the Laplace transform of $e^{-at} f(t)$ is $f(s+a)$.

(ie) $L(e^{-at} f(t)) = f(s)$, $s \to s+a$

Proof:

Given $L(e^{-at} f(t)) = \int_0^\infty e^{-st} e^{-at} f(t)\, dt$

$= \int_0^\infty e^{-t(s+a)} f(t)\, dt$

$= F(s+a) \Rightarrow F(s)$, $s \to s+a$

① Find $L[e^{-3t} \sin^2 t]$

Soln

Given: $L[e^{-3t} \sin^2 t]$

$L(f t) = \int_0^\infty e^{-st} f(t)\, dt = F(s)$

$\therefore F(t) = \sin^2 t$

$$L(\sin^2 t) = L\left(\frac{1-\cos 2t}{2}\right)$$

$$= \frac{1}{2}\left[L(1) - L(\cos 2t)\right]$$

$$= 1/2\left(\frac{1}{s} - \frac{s}{s^2+4}\right)$$

$$= 1/2\left[\frac{s^2+4-s^2}{s(s^2+4)}\right]$$

$$= 1/2\left[\frac{4}{s^3+4s}\right]$$

$$= 1/2\left[\frac{s^2+4-s^2}{s(s^2+4)}\right]$$

$$= 1/2\left[\frac{4}{s^3+4s}\right]$$

- Replace s by s+3

$(a+b)^3 = a^3+3a^2b+3b^2a+b^3$

$$= 1/2\left(\frac{4}{(s+3)^3+4(s+3)}\right)$$

$$= \frac{1}{2}\left[\frac{4}{s^3+3s^2(3)+3(3^2)(s)+(3)^3+4s+12}\right]$$

$$= \frac{1}{2}\left[\frac{4}{s^3+9s^2+27s+27+4s+12}\right]$$

$$= \frac{1}{2}\left[\frac{4}{s^3+9s^2+31s+39}\right]$$

$$= \frac{\cancel{4}^{2}}{\cancel{2}_{1}}\left[\frac{1}{s^3+9s^2+31s+39}\right]$$

$$= \frac{2}{s^3+9s^2+31s+39}$$

Type - IV

Derivatives and Integrals of transform :-

$$L[f(t)] = F(s)$$

then $L[t f(t)] = -F'(s)$.

w.k.T.

$$L[f(t)] = \int_0^\infty e^{-st} f(t) dt$$

$$= F(s).$$

$$F'(s) \; F(\tfrac{1}{s}) = \frac{d}{ds}\left[\int_0^\infty e^{-st} f(t) dT\right]$$

$$= \int_0^\infty \frac{d}{ds}(e^{-st}) f(t) dt$$

$$F'(s) = \int_0^\infty -te^{-st} f(t) dt$$

$$= -\int_0^\infty e^{-st}[t f(t)] dt.$$

$$F'(s) = -L[t f(t)]$$

$$L[t f(t)] = -F'(s)$$

1) Find $L[t \cos 3t]$

$$L[f(t)] = F(s).$$

$$f(t) = \cos 3t$$

$$L[f(t)] = L[\cos 3t] \; (a=3)$$

$$= \frac{s}{s^2+3^2}$$

$$F(s) = \frac{s}{s^2+9}$$

$$F'(s) = \frac{(s^2+a)(1) - (s)(2s)}{(s^2+a)^2}$$

$$= \frac{s^2+9-2s^2}{(s^2+9)^2}$$

$$= \frac{-s^2+9}{(s^2+9)^2}$$

$$-F'(s) = \frac{s^2-9}{(s^2+9)^2}$$

(s^2+4s)

Type V

1. Find $L[t e^{-t} \sin t]$

$f(t) = e^{-t} \sin t$

$L(\sin t) = \frac{1}{s^2+1}$

Replace s by $s+1$.

$f(t) = \frac{1}{(s+1)^2+1}$

$= \frac{1}{s^2+2s+1+1}$

$F(s) = \frac{1}{s^2+2s+2}$

$F'(s) = \frac{(s^2+2s+2)(0)-(1)(2s+1)}{(s^2+2s+2)^2}$

$= \frac{-(2s+2)}{(s^2+2s+2)^2}$

$-F'(s) = \frac{2s+2}{(s^2+2s+2)^2}$

) Using Laplace Transform evaluate the following integral

$\int_0^\infty t e^{2t} \sin 3t \, dt$.

Soln

$\int_0^\infty t e^{2t} \sin 3t \, dt$:

$f(t) = \int e^{2t} \sin 3t \, dt$

$L[\sin t] = \frac{a}{s^2 + a^2}$

$L(\sin 3t) = \frac{3}{s^2 + 3^2}$ $(a=3)$

$= \frac{3}{s^2 + 9}$.

Replace s by $s-2$.

$= \frac{3}{(s-2)^2 + 9}$.

$= \frac{3}{s^2 - 4s + 4 + 9}$

$F(s) = \frac{3}{s^2 - 4s + 13}$.

$F's = \cancel{\frac{s^2}{}} = \frac{(s^2 - 4s + 13)(0) - (3)(2s - 4)}{(s^2 - 4s + 13)^2}$.

$= \frac{-6s + 12}{(s^2 - 4s + 13)}$

$-F(s) = \frac{6s - 12}{s^2 - 4s + 13)}$

Type - vi

$$L f(t) = \int_0^{\alpha} F(s)\, ds.$$

1) Find the Laplace transform of $L\left(\frac{e^{-at} - e^{-bt}}{t}\right)$

Soln

Given $L\left(\frac{e^{-at} - e^{-bt}}{t}\right) = L\left(\frac{f(t)}{t}\right) = \int_0^{\alpha} F(s)\, ds$

$$f(t) = e^{-at} - e^{-bt}$$

$$L[f(t)] = L[e^{-at} - e^{-bt}]$$

$$= L[e^{-at}] - L[e^{-bt}]$$

$$= \frac{1}{s+a} - \frac{1}{s-b} = F(s)$$

$$= \int_s^\alpha \left(\frac{1}{s+a} - \frac{1}{s+b} \right) ds$$

$$= \int_s^\alpha \frac{ds}{s+a} - \frac{ds}{s+b}$$

$$= \left[\log(s+a) - \log(s+b) \right]_s^\alpha$$

$$L[f(t)] = \left[\log\left(\frac{s+a}{s+b} \right) \right]_s^\alpha$$

$$= 0 - \left(\log \frac{s+a}{s+b} \right)$$

$$= \log \left[\frac{s+b}{s+a} \right]$$

Laplace transform of special & Function.

Unit step function (or) heavy sides function.

1. Prove that $L[u_a(t)] = \frac{e^{-as}}{s}$.

$$L[u_a(t)] = \frac{e^{-as}}{s}$$

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t)\, dt$$

$$L[u_a(t)] = \int_0^{\infty} e^{-st} u_a(t)\, dt$$

$$= \int_0^{a} e^{-st} u_a(t)\, dt + \int_0^{\infty} e^{-st} u_a(t)\, dt$$

$$= 0 + \int_a^{\infty} e^{-st} (1)\, dt$$

$$= \left[\frac{-e^{-st}}{s}\right]_a^{\infty}$$

$$= \frac{e^{-\infty}}{s} + e^{-as}$$

$$L[u_a(t)] = \frac{e^{-as}}{s}$$

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Initial Value theorem [I.V.T]

Statement :

If $f(t)$ and $f'(t)$ are laplace transformable and $L[f(t)] = F(s)$ then $\underset{t\to 0}{Lt}\, f(t) = \underset{s\to\infty}{Lt}\, sF(s)$

Proof :

$$L[f'(t)] = sL[f(t)] - f(0)$$

$$\underset{s\to\infty}{Lt}\, L[f'(t)] = \underset{s\to\infty}{Lt}\, [sL\, f(t)] - f(0)].$$

$$[\because e^{-\infty} = 0] \qquad = \int_0^\infty \underset{s\to\infty}{Lt}\, e^{-st} f'(t)\, dt$$

$$\underset{s\to\infty}{Lt}\, L[f'(t)] = 0$$

$$0 = \underset{s\to\infty}{Lt}\, [sL\, f(t)] - f(0)$$

$$\underset{s\to\infty}{Lt}\, sL[f(t)] = \underset{s\to\infty}{Lt}\, f(0) = 0.$$

$$\underset{s\to\infty}{Lt}\, s[L f(t)] = \underset{t\to 0}{Lt}\, f(t)$$