CORE COURSE VII

VECTOR CALCULUS AND FOURIER SERIES

Objectives:

To provide the basic knowledge of vector differentiation & vector integration.
To solve vector differentiation & integration problems.

UNIT I

Vector differentiation –velocity & acceleration-Vector & scalar fields –Gradient of a vector- Directional derivative – divergence & curl of a vector solinoidal & irrotational vectors –Laplacian double operator –simple problems

UNIT II

Vector integration –Tangential line integral –Conservative force field –scalar potential- Work done by a force - Normal surface integral- Volume integral – simple problems.

UNIT III

Gauss Divergence Theorem – Stoke's Theorem- Green's Theorem – Simple problems & Verification of the theorems for simple problems.

UNIT IV

Fourier series- definition - Fourier Series expansion of periodic functions with Period 2π and period 2a – Use of odd & even functions in Fourier Series.

UNIT V

Half-range Fourier Series – definition- Development in Cosine series & in Sine series Change of interval – Combination of series

TEXT BOOK(S)

1. M.L. Khanna, Vector Calculus, Jai Prakash Nath and Co., 8th Edition, 1986.
2. S. Narayanan, T.K. Manicavachagam Pillai, Calculus, Vol. III, S. Viswanathan Pvt Limited, and Vijay Nicole Imprints Pvt Ltd, 2004.

UNIT – I - Chapter 1 Section 1 & Chapter 2 Sections 2.3 to 2.6 , 3 , 4 , 5 , 7 of [1]
UNIT – II - Chapter 3 Sections 1 , 2 , 4 of [1]
UNIT – III - Chapter 3 Sections 5 & 6 of [2]
UNIT – IV - Chapter 6 Section 1, 2, 3 of [2]
UNIT – V - Chapter 6 Section 4, 5.1, 5.2, 6, 7 of [2]

Reference:

1. P.Duraipandiyan and Lakshmi Duraipandian, Vector Analysis, Emarald publishers (1986).
2. Dr. S.Arumugam and prof. A.Thangapandi Issac, Fourier series, New Gamma publishing house (Nov 12)

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9/12/19

UNIT-I

Vector Differentiation

Velocity and acceleration:-

If $\vec{r}$ be the displacement at any time 't' of a moving point 'P' then,

velocity, $\vec{v} = \frac{d\vec{r}}{dt}$

acceleration

[Rate of change of displacement is called velocity]

acceleration, $\vec{a} = \frac{d^2\vec{r}}{dt^2}$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

[Rate of change of velocity is called acceleration].

Components of velocity $= \vec{v} \cdot \hat{b}$ $\qquad \therefore \hat{a} = \frac{\vec{a}}{|\vec{a}|}$

components of acceleration $= \vec{a} \cdot \hat{b}$

Magnitude of velocity $= |\vec{v}|$

Magnitude of acceleration $= |\vec{a}|$

The position vector, $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

where, (x, y, z) is the point 'P'.

Problem:

① A particle moves along the curve $x = e^{-t}$, $y = 2\cos 3t$, $z = 2\sin 3t$. Determine the velocity and acceleration at any time 't' and their magnitude ~~equal~~ at $t=0$.

Sol:

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{r} = e^{-t}\vec{i} + 2\cos 3t\,\vec{j} + 2\sin 3t\,\vec{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = -e^{-t}\,\vec{i} - 6\sin 3t\,\vec{j} + 6\cos 3t\,\vec{k} \rightarrow (1)$$

$$\vec{v}_{t=0} = -\vec{i} + 6\vec{k}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = e^{-t}\vec{i} - 18\cos 3t\,\vec{j} - 18\sin 3t\,\vec{k} \rightarrow (2)$$

$$\vec{a}_{t=0} = \vec{i} - 18\vec{j}$$

Magnitude of velocity, $|\vec{v}| = |-\vec{i} + 6\vec{k}|$

$$= \sqrt{(-1)^2 + (6)^2}$$

$$= \sqrt{37}$$

Magnitude of acceleration, $|\vec{a}| = |\vec{i} - 18\vec{j}|$

$$= \sqrt{(1)^2 + (18)^2}$$

$$= \sqrt{1 + 324}$$

$$= \sqrt{325}$$

② A particle moves along the curve

(i) $x = 4\cos t$, $y = 4\sin t$, $z = 6t$.

(ii) $x = a\cos t$, $y = a\sin t$, $z = at\tan\alpha$

Determine the velocity and acceleration any time 't' and their magnitude at t=0.

(2) A P

(i) $\vec{r} = 4\cos t\,\vec{i} + 4\sin t\,\vec{j} + 6t\,\vec{k}$

$$\vec{v} = \frac{d\vec{r}}{dt} = -4\sin t\,\vec{i} + 4\cos t\,\vec{j} + 6\vec{k}$$

$$\vec{v}_{t=0} = 4\vec{j} + 6\vec{k}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = -4\cos t\,\vec{i} - 4\sin t\,\vec{j} + 0\vec{k}$$

$$\vec{a}_{t=0} = -4\vec{i} + 0\vec{k}$$

Magnitude of, $|\vec{v}| = |4\vec{j} + 6\vec{k}|$

$$= \sqrt{(4)^2 + (6)^2}$$

$$= \sqrt{16 + 36}$$

$$= \sqrt{52}$$

Magnitude of, $|\vec{a}| = |-4\vec{i} + 0\vec{k}|$

$$= \sqrt{(-4)^2}$$

$$= \sqrt{16} = 4$$

(ii) $x = a\cos t$, $y = a\sin t$ $z = at\tan\alpha$

$$\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + at\tan\alpha\,\vec{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = -a\sin t\,\vec{i} + a\cos t\,\vec{j} + a\tan\alpha\,\vec{k}$$

$$\vec{v}_{t=0} = \cancel{a\cos t}\; a\vec{j} + a\tan\alpha\,\vec{k}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = -a\cos t\,\vec{i} - a\sin t\,\vec{j} + 0\,\vec{k}$$

$$\vec{a}_{t=0} = -a\vec{i}$$

Magnitude of velocity, $|\vec{v}| = |a\vec{j} + a\tan\alpha\,\vec{k}|$

$$= \sqrt{(a)^2 + (a\tan\alpha)^2}$$

$$= \sqrt{a^2(1+\tan^2\alpha)}$$

$$= a\sqrt{1+\tan^2\alpha} = a\sqrt{\sec^2\alpha}$$

$$= a\sec\alpha$$

Magitude of acceleration, $|\vec{a}| = |-a\vec{i}|$

$$= \sqrt{(-a)^2}$$

$$= a \quad //$$

(3) A particle moves along the curves, $x = 3t^2$, $y = t^2 - 2t$, $z = t^3$

Find $\vec{v}$, $\vec{a}$ at $t=1$ and their components.

Sol:-

$$\vec{r} = 3t^2\vec{i} + (t^2 - 2t)\vec{j} + t^3\vec{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$= 6t\,\vec{i} + (2t-2)\vec{j} + 3t^2\vec{k}$$

$$\vec{v}_{t=1} = 6\vec{i} + 0\vec{j} + 3\vec{k}$$

$$= 6\vec{i} + 3\vec{k}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$

$$\vec{a} = 6\vec{i} + 2\vec{j} + 6t\vec{k}$$

$$\vec{a}_{t=1} = 6\vec{i} + 2\vec{j} + 6\vec{k}$$

component of velocity $= \vec{v}\cdot\hat{b}$

where

Magnitude of Velocity, $|\vec{v}| = |6\vec{i} + 3\vec{k}|$

$$= \sqrt{6^2 + 3^2}$$

$$= \sqrt{36 + 9}$$

$$= \sqrt{45}$$

$$= \sqrt{9 \times 5}$$

$$= 3\sqrt{5}$$

Magnitude of acceleration, $|\vec{a}| = |6\vec{i} + 2\vec{j} + 6\vec{k}|$

$$= \sqrt{6^2 + 2^2 + 6^2}$$

$$= \sqrt{36 + 4 + 36}$$

$$= \sqrt{76}$$

$$= 2\sqrt{19}$$ //

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(5) A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$. Where t is the time. Find the components of its velocity and acceleration at time $t = 1$ in the direction $\vec{i} - 3\vec{j} + 2\vec{k}$.

Sol:

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{r} = 2t^2\,\vec{i} + (t^2 - 4t)\vec{j} + (3t - 5)\vec{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt} = 4t\,\vec{i} + (2t - 4)\vec{j} + 3\vec{k} \rightarrow (1)$$

$$\vec{v}_{t=1} = 4\vec{i} - 2\vec{j} + 3\vec{k}$$

components of velocity $= \vec{v} \cdot \hat{b}$

where $\vec{b} = \vec{i} - 3\vec{j} + 2\vec{k}$

$$\vec{v} \cdot \hat{b} = (4\vec{i} - 2\vec{j} + 3\vec{k}) \cdot \frac{(\vec{i} - 3\vec{j} + 2\vec{k})}{\sqrt{1^2 + (-3)^2 + (2)^2}}$$

$$= \frac{4 + 6 + 6}{\sqrt{1 + 9 + 4}} = \frac{16}{\sqrt{14}}$$

components of velocity $= \frac{16}{\sqrt{14}}$

Diff (1) w.r.to t

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2} = 4\vec{i} + 2\vec{j}$$

$\vec{a}_{t=1} = 4\vec{i} + 2\vec{j}$

components of acceleration $= \vec{a} \cdot \vec{b}$

$$= (4\vec{i} + 2\vec{j}) \frac{(\vec{i} - 3\vec{j} + 2\vec{k})}{\sqrt{1^2 + (-3)^2 + (2)^2}}$$

$$= \frac{4-6}{\sqrt{1+9+4}} = \frac{-2}{\sqrt{14}} \;//$$

(6) A particle moves along the curve $x = t^3+1$, $y = t^2$, $z = 2t+5$. where t is the time. Find components of its velocity and acceleration at $t=1$ in the direction $\vec{i} + \vec{j} + 3\vec{k}$.

$$\vec{r} = (t^3+1)\vec{i} + t^2\vec{j} + (2t+5)\vec{k}$$

$$\vec{v} = \frac{d\vec{r}}{dt}$$

$$= 3t^2\vec{i} + 2t\vec{j} + 2\vec{k}$$

$$\vec{v}_{t=1} = 3\vec{i} + 2\vec{j} + 2\vec{k}$$

component of velocity $= \vec{v} \cdot \vec{b}$

where $\vec{b} = \vec{i} + \vec{j} + 3\vec{k}$

$$\vec{v} \cdot \vec{b} = (3\vec{i} + 2\vec{j} + 2\vec{k}) \frac{(\vec{i} + \vec{j} + 3\vec{k})}{\sqrt{1^2 + 1^2 + 3^2}}$$

$$= \frac{3+2+6}{\sqrt{1+1+9}}$$

$$\vec{r}\cdot\vec{b} = \frac{11}{\sqrt{11}}$$
$$= \sqrt{11}$$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$$
$$= 6t\,\vec{i} + 2\vec{j}$$

$$\vec{a}_{t=1} = 6\vec{i} + 2\vec{j}$$

component of acceleration $= (6\vec{i}+2\vec{j})\frac{(\vec{i}+\vec{j}+3\vec{k})}{\sqrt{1^2+1^2+3^2}}$

$$= \frac{6+2}{\sqrt{11}}$$
$$= \frac{8}{\sqrt{11}}$$ //

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7. If $\vec{r} = a\cos nt + b\sin nt$ where $\vec{a}, \vec{b}$ and n are constant P.T $\frac{d^2\vec{r}}{dt^2} + n^2\vec{r} = 0$.

proof:

given $\vec{r} = a\cos nt + b\sin nt$

$$\frac{d\vec{r}}{dt} = a(-\sin nt)n + b\cos nt\, n$$
$$= -an\sin nt + bn\cos nt$$

$$\frac{d^2\vec{r}}{dt^2} = -an(\cos nt)n + bn(-\sin nt)n$$
$$= -an^2\cos nt - bn^2\sin nt$$
$$= -n^2[a\cos nt + b\sin nt]$$

$$\frac{d^2\vec{r}}{dt^2} = -n^2\vec{r}$$

$$\frac{d^2\vec{r}}{dt^2} + n^2\vec{r} = 0$$

Hence proved//

8. If $\vec{r} = \vec{a}e^{\omega t} + \vec{b}e^{-\omega t}$ where $\vec{a}, \vec{b}$ constant P.T $\frac{d^2\vec{r}}{dt^2} - \omega^2\vec{r} = 0$.

proof:-

$$\vec{r} = \vec{a}e^{\omega t} + \vec{b}e^{-\omega t}$$

$$\frac{d\vec{r}}{dt} = \omega\vec{a}e^{\omega t} - \vec{b}\omega e^{-\omega t}$$

$$\frac{d^2\vec{r}}{dt^2} = \vec{a}\omega^2 e^{\omega t} + \vec{b}\omega^2 e^{-\omega t}$$

$$= \omega^2(\vec{a}e^{\omega t} + \vec{b}e^{-\omega t})$$

$$= \omega^2\vec{r}$$

$$\frac{d^2\vec{r}}{dt^2} - \omega^2\vec{r} = 0$$

Hence proved//

9. If $\vec{r} = \cos nt\,\vec{i} + \sin nt\,\vec{j}$ where n is a constant 't' varies. S.T $r \times \frac{dr}{dt} = nk$.

proof:-

$$\vec{r} = \cos nt\,\vec{i} + \sin nt\,\vec{j}$$

Diff. w.r.to 't'

$$\frac{d\vec{r}}{dt} = -n\sin nt\,\vec{i} + n\cos nt\,\vec{j}$$

$$r \times \frac{dr}{dt} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos nt & \sin nt & 0 \\ -n\sin nt & n\cos nt & 0 \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(n\cos^2 nt + n\sin^2 nt)$$

$$= n\vec{k}(\cos^2 nt + \sin^2 nt)$$

$$r \times \frac{dr}{dt} = n\vec{k}$$

10. If $r = \sinh t\, a + (\cosh t)\, b$ where a and b are constant vectors then S.T $\frac{d^2r}{dt^2} = r$

Proof:-

$$\vec{r} = \sinh t\, a + \cosh t\, b \quad \cancel{\text{vectors}}$$

$$\frac{d\vec{r}}{dt} = \cosh t\, a + \sinh t\, b$$

$$\frac{d^2\vec{r}}{dt^2} = \sinh t\, a + \cosh t\, b$$

$$= \vec{r}$$

$$\therefore \frac{d^2\vec{r}}{dt^2} = \vec{r}$$

Hence proved

Scalar Point Function:-

To each point p of a region R there corresponds a scalar denoted by $\phi(P)$. Then ϕ is said to be a scalar point function for the region R.
If the co-ordinates of P be (x, y, z) then, $\phi(P) = \phi(x, y, z)$.

Vector point function:-

To each point P of a region R, there corresponds a vector defined by $f(\phi)$. Then f is called a vector point function for the region R.

If the co-ordinates of P be (x, y, z). Then $f(P) = f(x, y, z)$

Gradient of a scalar point function:-

Let $\phi(x, y, z)$ be a scalar point function, the expression $\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$ is called the gradient of the scalar point function ϕ.

It is denoted by the symbol $\nabla\phi$.

(i.e) $\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\phi$

$\text{Grad } \phi = \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$

Note:-

1. $\nabla = \vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ is a vector differential operator.

2. If ϕ defines a scalar field, then $\nabla\phi$ defines vector field.

10. If $\phi = 2x^2y^3 - 3y^2z^2$. Then find $\nabla\phi$ at the point $(1,-1,1)$.

Sol:-

Given, $\phi = 2x^2y^3 - 3y^2z^3$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\phi = 2x^2y^3 - 3y^2z^3$$

$$\frac{\partial\phi}{\partial x} = 4xy^3$$

$$\frac{\partial\phi}{\partial y} = 6x^2y^2 - 6yz^3$$

$$\frac{\partial\phi}{\partial z} = -9y^2z^2$$

$$\nabla\phi = \vec{i}\,(4xy^3) + \vec{j}\,(6x^2y^2 - 6yz^3) + \vec{k}\,(-9y^2z^2)$$

$$\nabla\phi_{(1,-1,1)} = [4(1)(-1)^3]\vec{i} + [6(1)^2(-1)^2 - 6(-1)(1)^3]\vec{j}$$

$$+ [-9(-1)^2(1)^2]\vec{k}$$

$$= -4\vec{i} + [6+6]\vec{j} - 9\vec{k}$$

$$\nabla\phi_{(1,-1,1)} = -4\vec{i} + 12\vec{j} - 9\vec{k}$$

11) If $\phi = x^2y + y^2x + z^2$. Find $\nabla\phi$ at the point $(1,1,1)$.

Sol:

$$\phi = x^2y + y^2x + z^2$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 2xy + y^2$$

$$\frac{\partial\phi}{\partial y} = x^2 + 2xy$$

$$\frac{\partial\phi}{\partial z} = 2z$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$= \vec{i}(2xy + y^2) + \vec{j}(x^2 + 2xy) + \vec{k}(2z)$$

$$= (2xy + y^2)\vec{i} + (x^2 + 2xy)\vec{j} + 2z\vec{k}$$

$$\nabla\phi_{(1,1,1)} = (2(1)(1)+(1)^2)\vec{i} + (1^2+2(1)(1))\vec{j} + (2(1))\vec{k}$$

$$\nabla\phi_{(1,1,1)} = 3\vec{i} + 3\vec{j} + 2\vec{k}$$

12. If $\phi = x^3 - y^3 + xz^2$. Find grad ϕ at $(1,-1,2)$.

$$\phi = x^3 - y^3 + xz^2$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 3x^2 + z^2$$

$$\frac{\partial\phi}{\partial y} = -3y^2$$

$$\frac{\partial\phi}{\partial z} = 2xz$$

$$\nabla\phi = \vec{i}(3x^2+z^2) + \vec{j}(-3y^2) + \vec{k}(2xz)$$

$$= (3x^2+z^2)\vec{i} + (-3y^2)\vec{j} + (2xz)\vec{k}$$

$$\nabla\phi_{(1,-1,2)} = (3(1)^2+(2)^2)\vec{i} + (-3(-1)^2)\vec{j} + 2(1)(2)\vec{k}$$

$$= 7\vec{i} - 3\vec{j} + 4\vec{k}$$

//

$\cos\theta = \hat{n}_1 \cdot \hat{n}_2$

where $\hat{n}_1 = \dfrac{\nabla\phi}{|\nabla\phi_1|}$, $\hat{n}_2 = \dfrac{\nabla\phi_2}{|\nabla\phi_2|}$

$$\cos\theta = \frac{\nabla\phi_1 \ \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

orthogonal vector $\Rightarrow \nabla\phi_1 . \nabla\phi_2 = 0$

14. Find the unit normal vector to the surface $x^2+3y^2+2z^2$ at the $P(2,0,1)$.

Sol:

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} \qquad \nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\phi = x^2+3y^2+2z^2 = 6$$

$$\frac{\partial\phi}{\partial x} = 2x$$

$$\frac{\partial\phi}{\partial y} = 6y$$

$$\frac{\partial\phi}{\partial z} = 4z$$

$$\nabla\phi = \vec{i}(2x) + \vec{j}(6y) + \vec{k}(4z)$$

$$\nabla\phi = 2x\vec{i} + 6y\vec{j} + 4z\vec{k}$$

$$\nabla\phi_{(2,0,1)} = 2(2)\vec{i} + 6(0)\vec{j} + 4(1)\vec{k}$$

$$= 4\vec{i} + 4\vec{k}$$

$$|\nabla\phi| = \sqrt{(4)^2+(4)^2} = \sqrt{16+16} = \sqrt{32} = 4\sqrt{2}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\hat{n} = \frac{4\vec{i}+4\vec{k}}{4\sqrt{2}}$$

$$= \frac{\cancel{4}(\vec{i}+\vec{k})}{\cancel{4}\sqrt{2}}$$

$$\hat{n} = \frac{\vec{i}+\vec{k}}{\sqrt{2}}$$ //

18. Find the unit normal vector to the surface $z = x^2+y^2$ at $P(-1,-2,-5)$.

Sol:

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} \qquad \phi = x^2+y^2-z$$

$$\frac{\partial\phi}{\partial x} = 2x \; ; \frac{\partial\phi}{\partial y} = 2y \; ; \frac{\partial\phi}{\partial z} = -1$$

$$\nabla\phi = \vec{i}\left(\frac{\partial\phi}{\partial x}\right)+\vec{j}\left(\frac{\partial\phi}{\partial y}\right)+\vec{k}\left(\frac{\partial\phi}{\partial z}\right)$$

$$= (2x)\vec{i}+(2y)\vec{j}+(-1)\vec{k}$$

$$= 2x\vec{i}+2y\vec{j}-\vec{k}$$

$$\nabla\phi_{(-1,-2,-5)} = -2\vec{i}-4\vec{j}-\vec{k}$$

$$|\nabla\phi| = \sqrt{(-2)^2+(-4)^2+(-1)^2} = \sqrt{21}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{-2\vec{i}-4\vec{j}-\vec{k}}{\sqrt{21}}$$ //

16. Find the unit Normal vector to the surface $x^2y + 2xz^2 = 8$ at $(1,0,2)$

Sol:-

$$\vec{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\phi = x^2y + 2xz^2 - 8$$

$$\frac{\partial\phi}{\partial x} = 2xy + 2z^2$$

$$\frac{\partial\phi}{\partial y} = x^2$$

$$\frac{\partial\phi}{\partial z} = 4xz$$

$$\nabla\phi = (2xy+2z^2)\vec{i} + x^2\vec{j} + 4xz\vec{k}$$

$$\nabla\phi_{(1,0,2)} = 8\vec{i} + \vec{j} + 8\vec{k}$$

$$|\nabla\phi| = \sqrt{8^2+1^2+8^2}$$
$$= \sqrt{129}$$

$$\vec{n} = \frac{8\vec{i}+\vec{j}+8\vec{k}}{\sqrt{129}}$$

17. Find the angle b/w the normal to the surface $xy - z^2 = 0$ at $(1,4,-2),(-3,-3,$

Sol:-

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\phi = xy - z^2$$

$$\frac{\partial \phi}{\partial x} = y \; ; \; \frac{\partial \phi}{\partial y} = x \; ; \; \frac{\partial \phi}{\partial z} = -2z$$

$$\nabla \phi = y\vec{i} + x\vec{j} - 2z\vec{k}$$

$$\nabla \phi_1 \; (1, 4, -2) = 4\vec{i} + \vec{j} - 2(-2)\vec{k}$$

$$\nabla \phi_1 = 4\vec{i} + \vec{j} + 4\vec{k}$$

$$|\nabla \phi_1| = \sqrt{(4)^2 + (1)^2 + (4)^2}$$

$$= \sqrt{16 + 1 + 16}$$

$$|\nabla \phi_1| = \sqrt{33}$$

$$\nabla \phi_{2\,(-3,-3,3)} = -3\vec{i} - 3\vec{j} - 2(3)\vec{k}$$

$$= -3\vec{i} - 3\vec{j} - 6\vec{k}$$

$$|\nabla \phi_2| = \sqrt{(-3)^2 + (-3)^2 + (-6)^2}$$

$$= \sqrt{9 + 9 + 36}$$

$$|\nabla \phi_2| = \sqrt{54}$$

$$\cos\theta = \frac{\nabla \phi_1 \cdot \nabla \phi_2}{|\nabla \phi_1| \cdot |\nabla \phi_2|}$$

$$= \frac{(4\vec{i} + \vec{j} + 4\vec{k}) / (-3\vec{i} - 3\vec{j} - 6\vec{k})}{\sqrt{33}\,\sqrt{54}}$$

$$= \frac{-12 - 3 - 24}{\sqrt{33}\,\sqrt{54}}$$

$$\cos\theta = \frac{-39}{\sqrt{33}\,\sqrt{54}}$$

$$\theta = \cos^{-1}\left(\frac{-39}{\sqrt{33}\,\sqrt{54}}\right)$$

10. Find the angle b/w the normal to the surface $x^2+y^2+z^2=9$ and $z=x^2+y^2-3$ at the $P(2,-1,2)$.

Sol:-

$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1||\nabla\phi_2|}$$

$\phi_1 = x^2+y^2+z^2-9$ $\qquad$ $\phi_2 = x^2+y^2-z-3$

$\frac{\partial\phi_1}{\partial x} = 2x$ $\qquad$ $\frac{\partial\phi_2}{\partial x} = 2x$

$\frac{\partial\phi_1}{\partial y} = 2y$ $\qquad$ $\frac{\partial\phi_2}{\partial y} = 3y^2$

$\frac{\partial\phi_1}{\partial z} = 2z$ $\qquad$ $\frac{\partial\phi_2}{\partial z} = -1$

$\nabla\phi_1 = 2x\vec{i}+2y\vec{j}+2z\vec{k}$ $\qquad$ $\nabla\phi_2 = 2x\vec{i}+3y\vec{j}-\vec{k}$

$\nabla\phi_{1\,(2,-1,2)} = 4\vec{i}-2\vec{j}+2\vec{k}$

$|\nabla\phi_1| = \sqrt{(4)^2+(-2)^2+(2)^2}$
$= \sqrt{16+4+4} = \sqrt{24} = 2\sqrt{6}$

$\nabla\phi_2(2,-1,2) = 4\vec{i}+3\vec{j}-2\vec{k}$

$|\nabla\phi_2| = \sqrt{(4)^2+(-2)^2+(-2)^2} = \sqrt{16+9+4}$
$= \cancel{2\sqrt{6}} = \sqrt{29}$

$$\cos\theta = \frac{(4\vec{i}-2\vec{j}+2\vec{k})(4\vec{i}+3\vec{j}-2\vec{k})}{2\sqrt{6}\cdot\cancel{2\sqrt{6}}\sqrt{29}}$$

$\cos\theta = \frac{16}{4\times6} = \frac{2}{3} \Rightarrow \theta = \cos^{-1}(2/3)$

$\cos\theta = \frac{6}{2\sqrt{16}\sqrt{29}} \Rightarrow \theta = \cos^{-1}\left(\frac{6}{2\sqrt{16}\sqrt{29}}\right)$

19. Find the angle b/w the normal to the surface $x^2+y^2+z^2=9$ and $x^2 3=0$ at $P(2,-1,2)$.

Sol:-
$$\cos\theta = \frac{\nabla\phi_1 \cdot \nabla\phi_2}{|\nabla\phi_1|\cdot|\nabla\phi_2|}$$

$$\phi_1 = x^2+y^2+z^2-9$$

$$\nabla\phi_1 = \vec{i}\frac{\partial\phi_1}{\partial x} + \vec{j}\frac{\partial\phi_1}{\partial y} + \vec{k}\frac{\partial\phi_1}{\partial z}$$

$$\frac{\partial\phi_1}{\partial x} = 2x \quad \frac{\partial\phi_1}{\partial y} = 2y \quad \frac{\partial\phi_1}{\partial z} = 2z$$

$$\nabla\phi_1 = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$\nabla\phi_{1\,(2,-1,2)} = 4\vec{i} - 2\vec{j} + 4\vec{k}$$

$$|\nabla\phi_1| = \sqrt{(4)^2+(-2)^2+(4)^2} = \sqrt{16+4+16} = \sqrt{34}$$

$$\phi_2 = x^2-3$$

$$\nabla\phi_2 = \vec{i}\frac{\partial\phi_2}{\partial x} + \vec{j}\frac{\partial\phi_2}{\partial y} + \vec{k}\frac{\partial\phi_2}{\partial z}$$

$$\frac{\partial\phi_2}{\partial x} = 2x \quad \frac{\partial\phi_2}{\partial y} = 0 \quad \frac{\partial\phi_2}{\partial z} = 0$$

$$\nabla\phi_2 = 2x\vec{i} + 0\vec{j} + 0\vec{k}$$

$$\nabla\phi_{2\,(2,-1,2)} = 4\vec{i}$$

$$|\nabla\phi_2| = \sqrt{4^2} = 2\,6$$

$$\cos\theta = \frac{(4\vec{i}-2\vec{j}+4\vec{k})(4\vec{i})}{\sqrt{34}\ \sqrt{4}}$$

$$= \frac{\cancel{16}}{\sqrt{34} \cdot \cancel{2}}$$

$$\cos\theta = \frac{8}{\sqrt{34}}$$

$$\theta = \cos^{-1}\left(\frac{8}{\sqrt{34}}\right)$$

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20. Show that the surface $5x^2 - 2yz - 9x = 0$ and $4x^2y + z^3 - 4 = 0$. are orthogonal at $P(1, -1, 2)$..

sol:-

orthogonal $\nabla\phi_1 \cdot \nabla\phi_2 = 0$

given $\phi_1 = 5x^2 - 2yz - 9x$

$\phi_2 = 4x^2y + z^3 - 4$

$$\nabla\phi_1 = \vec{i}\frac{\partial\phi_1}{\partial x} + \vec{j}\frac{\partial\phi_1}{\partial y} + \vec{k}\frac{\partial\phi_1}{\partial z}$$

$$\frac{\partial\phi_1}{\partial x} = 10x - 9 ; \quad \frac{\partial\phi_1}{\partial y} = -2z ; \quad \frac{\partial\phi_1}{\partial z} = -2y$$

$$\nabla\phi_1 = \vec{i}(10x - 9) + \vec{j}(-2z) + \vec{k}(-2y)$$

$$\nabla\phi_{1\,(1,-1,2)} = \vec{i}(10 - 9) + \vec{j}[-2(2)] + \vec{k}(-2(-1))$$

$$= \vec{i} - 4\vec{j} + 2\vec{k}$$

$$\nabla\phi_2 = \vec{i}\frac{\partial\phi_2}{\partial x} + \vec{j}\frac{\partial\phi_2}{\partial y} + \vec{k}\frac{\partial\phi_2}{\partial z}$$

$$\frac{\partial\phi_2}{\partial x} = 8xy ; \quad \frac{\partial\phi_2}{\partial y} = 4x^2 ; \quad \frac{\partial\phi_2}{\partial z} = 3z^2$$

$$\nabla\phi_2 = \vec{i}(8xy) + \vec{j}(4x^2) + \vec{k}(3z^2)$$

$$\nabla\phi_{2\,(1,-1,2)} = \vec{i}[8(1)(-1)] + \vec{j}[4(1)^2] + \vec{k}[3(2)^2]$$

$$= -8\vec{i} + 4\vec{j} + 12\vec{k}$$

$$\nabla\phi_1 \cdot \nabla\phi_2 = (\vec{i} - 4\vec{j} + 2\vec{k}) \cdot (-8\vec{i} + 4\vec{j} + 12\vec{k})$$

$$= -8 - 16 + 24$$

$$= 0.$$

Hence showed //.

Directional Derivetives:-

Directional derivative $= \nabla\phi \cdot \hat{n}$

(or)

$= \text{grad}\,\phi \cdot \hat{n}$

problems:-

① Find the directional derivative of xyz. at the point $(1,1,1)$. in the direc of vector $\vec{i} + \vec{j} + \vec{k}$.

sol:-

Directional derivative $= \nabla\phi \cdot \hat{n}$

$$\phi = xyz$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = yz \; ; \; \frac{\partial\phi}{\partial y} = xz \; ; \; \frac{\partial\phi}{\partial z} = xy$$

$$\nabla\phi = yz\vec{i} + xz\vec{j} + xy\vec{k}$$

$$\nabla\phi_{(1,1,1)} = (1)\vec{i} + (1)\vec{j} + (1)\vec{k}$$

$$\nabla\phi = \vec{i} + \vec{j} + \vec{k}$$

$$\vec{n} = \vec{i} + \vec{j} + \vec{k}$$

$$D.D = \nabla\phi \cdot \hat{n}$$

$$= (\vec{i}+\vec{j}+\vec{k}) \cdot \frac{(\vec{i}+\vec{j}+\vec{k})}{\sqrt{1^2+1^2+1^2}}$$

$$= \frac{3}{\sqrt{3}} = \sqrt{3}$$

(2) Find the directional derivative of $\phi = x^3+y^3+z^3$ at $(1,-1,2)$ in the direction of the vector $\vec{i}+2\vec{j}+\vec{k}$.

Sol:

$$\phi = x^3+y^3+z^2$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 3x^2 \; ; \; \frac{\partial\phi}{\partial y} = 3y^2 \; ; \; \frac{\partial\phi}{\partial z} = 3z^2$$

$$\nabla\phi = \vec{i}(3x^2) + \vec{j}(3y^2) + \vec{k}(3z^2)$$

$$\nabla\phi_{(1,-1,2)} = 3\vec{i} + 3\vec{j} + 12\vec{k}$$

$$\vec{n} = \vec{i} + 2\vec{j} + \vec{k}$$

$$D.D = \nabla\phi \cdot \hat{n}$$

$$= 3\vec{i}+3\vec{j}+12\vec{k} \cdot \frac{(\vec{i}+2\vec{j}+\vec{k})}{\sqrt{1^2+2^2+1^2}}$$

$$= \frac{3+6+12}{\sqrt{6}} = \frac{21}{\sqrt{6}}$$

(3) Find the D.D of $\phi = xy + yz + zx$ in the direction of the vector $\vec{i}+2\vec{j}+2\vec{k}$ at $P(1,2,0)$

Sol:

$$\phi = xy + yz + zx$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = y+z \; ; \; \frac{\partial\phi}{\partial y} = x+z \; ; \; \frac{\partial\phi}{\partial z} = x+y$$

$$\nabla\phi = \vec{i}(y+z) + \vec{j}(x+z) + \vec{k}(x+y)$$

$$\nabla\phi_{(1,2,0)} = 2\vec{i} + \vec{j} + 3\vec{k}$$

$$\vec{n} = \vec{i} + 2\vec{j} + 2\vec{k}$$

$$D.D = \nabla\phi \cdot \hat{n}$$

$$= (2\vec{i}+\vec{j}+3\vec{k}) \frac{(\vec{i}+2\vec{j}+2\vec{k})}{\sqrt{1^2+2^2+2^2}}$$

$$= \frac{2+2+6}{\sqrt{9}} = \frac{10}{3}$$

(4) Find the D.D of $\phi = 3x^2 - 2y - 3z$ at $P(1,1,1)$ in the direction specified by $2\vec{i} + 2\vec{j} - \vec{k}$.

Sol:

$$\phi = 3x^2 - 2y - 3z$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = 6x \; ; \; \frac{\partial\phi}{\partial y} = -2 \; ; \; \frac{\partial\phi}{\partial z} = -3$$

$$\nabla\phi = \vec{i}(6x) + \vec{j}(-2) + \vec{k}(-3)$$

$$= 6x\vec{i} - 2\vec{j} - 3\vec{k}$$

$$\nabla\phi_{(1,1,1)} = 6\vec{i} - 2\vec{j} - 3\vec{k}$$

$$\vec{n} = 2\vec{i} + 2\vec{j} - \vec{k}$$

$$D.D = \nabla\phi \cdot \hat{n}$$

$$= (6\vec{i} - 2\vec{j} - 3\vec{k}) \cdot \frac{(2\vec{i} + 2\vec{j} - \vec{k})}{\sqrt{2^2 + 2^2 + (-1)^2}}$$

$$= \frac{12 - 4 + 3}{\sqrt{9}} = 11/3$$

Generated by CamScanner

17/12/19

Equation of Tangent plane is

$$(\vec{r} - \vec{a}) \cdot \nabla\phi = 0$$

Equation of the Normal line is

$$\frac{x - x_1}{\frac{\partial\phi}{\partial x}} = \frac{y - y_1}{\frac{\partial\phi}{\partial y}} = \frac{z - z_1}{\partial\phi/\partial z}$$

1. Find the eqn of tangent and normal to the surface $xyz = 4$ at $(1, 2, 2)$.

Sol:

$$\phi = xyz - 4$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\frac{\partial\phi}{\partial x} = yz \quad \frac{\partial\phi}{\partial y} = xz \quad \frac{\partial\phi}{\partial z} = xy$$

$$\nabla\phi = yz\vec{i} + xz\vec{j} + xy\vec{k}$$

$$\nabla\phi_{(1,2,2)} = 4\vec{i} + 2\vec{j} + 2\vec{k}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$$

Eqn of tangent plane

$$(\vec{r} - \vec{a}) \cdot \nabla\phi = 0$$

$$((x\vec{i} + y\vec{j} + z\vec{k}) - \vec{i} - 2\vec{j} + 2\vec{k}) \cdot (4\vec{i} + 2\vec{j} + 2\vec{k}) = 0$$

$$[(x-1)\vec{i} + (y-2)\vec{j} + (z-2)\vec{k}]\cdot[4\vec{i} + 2\vec{j} + 2\vec{k}] = 0$$

$$4(x-1) + 2(y-2) + 2(z-2) = 0$$

$$4x - 4 + 2y - 4 + 2z - 4 = 0$$

$$4x + 2y + 2z - 12 = 0$$

$$2x + y + z - 6 = 0$$

which is the required eqn of tangent plane.

Eqn of normal line.

$$\frac{x - x_1}{\frac{\partial\phi}{\partial x}} = \frac{y - y_1}{\frac{\partial\phi}{\partial y}} = \frac{z - z_1}{\frac{\partial\phi}{\partial y}}$$

$$\phi = 2x + y + z - 6$$

$$\frac{\partial\phi}{\partial x} = 2 \ ; \ \frac{\partial\phi}{\partial y} = 1 \ ; \ \frac{\partial\phi}{\partial z} = 1$$

$$(x_1, y_1, z_1) = (1, 2, 2)$$

$$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-2}{1}$$

$$\frac{x-1}{2} = y - 2 = z - 2$$

which is the required equation of normal line.

③ Find the eqn of tangent and normal to the surface $yz - zx + xy + 5 = 0$ at $(1, -1, 2)$.

Sol:

$$\phi = yz - zx + xy + 5$$

$$\frac{\partial\phi}{\partial x} = -z+y \quad \frac{\partial\phi}{\partial y} = z+x \quad \frac{\partial\phi}{\partial z} = y-x$$

$$\nabla\phi = (-z+y)\vec{i} + (z+x)\vec{j} + (y-x)\vec{k}$$

$$\nabla\phi_{(1,-1,2)} = -3\vec{i} + 3\vec{j} - 2\vec{k}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{a} = \vec{i} - \vec{j} + 2\vec{k}$$

Eqn of tangent

$$(\vec{r} - \vec{a}) \cdot \nabla\phi = 0$$

$$(x\vec{i} + y\vec{j} + z\vec{k} - \vec{i} + \vec{j} - 2\vec{k}) \cdot (-3\vec{i} + 3\vec{j} - 2\vec{k}) = 0$$

$$((x-1)\vec{i} + (y+1)\vec{j} + (z-2)\vec{k}) \cdot (-3\vec{i} + 3\vec{j} - 2\vec{k}) = 0$$

$$-3(x-1) + 3(y+1) - 2(z-2) = 0$$

$$-3x + 3 + 3y + 3 - 2z + 4 = 0$$

$$-3x + 3y - 2z + 10 = 0$$

$$3x - 3y + 2z - 10 = 0$$

Eqn of normal line.

$$\frac{x - x_1}{\frac{\partial\phi}{\partial x}} = \frac{y - y_1}{\frac{\partial\phi}{\partial y}} = \frac{z - z_1}{\frac{\partial\phi}{\partial z}}$$

$$\phi = 3x - 3y + 2z - 10$$

$$\frac{\partial\phi}{\partial x} = 3 \quad \frac{\partial\phi}{\partial y} = -3 \quad \frac{\partial\phi}{\partial z} = 2$$

$$(x_1, y_1, z_1) = (1, -1, 2)$$

$$\frac{x-1}{3} = \frac{y+1}{-3} = \frac{z-2}{2}$$

③ Find the eqn of the tangent and normal to the surface $x^2 - 4y^2 + 3z^2 + 4 = 0$ at $(3, 2, 1)$.

Sol:-

$$\phi = x^2 - 4y^2 + 3z^2 + 4$$

$$\frac{\partial\phi}{\partial x} = 2x \quad \frac{\partial\phi}{\partial y} = -8y \quad \frac{\partial\phi}{\partial z} = 6z$$

$$\nabla\phi = 2x\vec{i} - 8y\vec{j} + 6z\vec{k}$$

$$\nabla\phi_{(3,2,1)} = 6\vec{i} - 16\cancel{y}\vec{j} + 6\vec{k}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \qquad \vec{a} = 3\vec{i} + 2\vec{j} + \vec{k}$$

Eqn of tangent

$$(\vec{r} - \vec{a}) \cdot \nabla\phi = 0$$

$$(x\vec{i} + y\vec{j} + z\vec{k} - 3\vec{i} - 2\vec{j} - \vec{k}) \cdot (6\vec{i} - 16\vec{j} + 6\vec{k}) = 0$$

$$((x-3)\vec{i} + (y-2)\vec{j} + (z-1)\vec{k}) \cdot (6\vec{i} - 16\vec{j} + 6\vec{k}) = 0$$

$$6(x-3) - 16(y-2) + 6(z-1) = 0$$

$$6x - 18 - 16y + 32 + 6z - 6 = 0$$

$$6x - 16y + 6z + 8 = 0$$

$$3x - 8y + 3z + 4 = 0$$

$$\phi = 3x - 8y + 3z + 4$$

$$\frac{\partial \phi}{\partial x} = 3 \quad \frac{\partial \phi}{\partial y} = -8 \quad \frac{\partial \phi}{\partial z} = 3$$

$$(x_1, y_1, z_1) = (3, 2, 1)$$

Eqn of normal line

$$\frac{x - x_1}{\frac{\partial \phi}{\partial x}} = \frac{y - y_1}{\frac{\partial \phi}{\partial y}} = \frac{z - z_1}{\frac{\partial \phi}{\partial z}}$$

$$\frac{x-3}{3} = \frac{y-2}{-8} = \frac{z-1}{3}$$

1. Find ϕ if $\nabla\phi = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)\vec{k}$.

Sol:

given $\nabla\phi = (6xy + z^3)\vec{i} + (3x^2 - z)\vec{j} + (3xz^2 - y)$

$$\nabla \phi = \vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z} \rightarrow (2)$$

Equating eqn (1) & (2)

$$\frac{\partial \phi}{\partial x} = 6xy + z^3 \rightarrow (3)$$

$$\frac{\partial \phi}{\partial y} = 3x^2 - z \rightarrow (4)$$

$$\frac{\partial \phi}{\partial z} = 3xz^2 - y \rightarrow (5)$$

Integrating eqn ③, ④ & ⑤ w.r.to x, y & z

$$\int \frac{\partial \phi}{\partial x} = \int (6xy + z^3)\, dx$$

$$\phi = \frac{6x^2 y}{2} + z^3 x + f_1(y,z)$$

$$\phi = 3x^2 y + z^3 x + f_1(y,z).$$

$$\int \frac{\partial \phi}{\partial y} = \int (3x^2 - z)\, dy$$

$$\phi = 3x^2 y - zy + f_2(x,z).$$

$$\int \frac{\partial \phi}{\partial z} = \int (3xz^2 - y)\, dz$$

$$\phi = \frac{3xz^3}{3} - yz + f_3(x,y).$$

$$\phi = xz^3 - yz + f_3(x,y).$$

$$\phi = 3x^2 y + xz^3 - yz + c$$

②. Find ϕ if $\nabla\phi = (y + \sin z)\vec{i} + x\vec{j} + x\cos z\,\vec{k}$

Sol:

given $\nabla\phi = (y + \sin z)\vec{i} + x\vec{j} + x\cos z\,\vec{k} \rightarrow ①$

$\nabla\phi = \vec{i}\,\frac{\partial \phi}{\partial x} + \vec{j}\,\frac{\partial \phi}{\partial y} + \vec{k}\,\frac{\partial \phi}{\partial z} \rightarrow ②$

Equating eqn ① & ②

$\frac{\partial \phi}{\partial x} = y + \sin z \rightarrow ③$

$\frac{\partial \phi}{\partial y} = x \rightarrow ④$

$$\frac{\partial \phi}{\partial z} = x\cos z \rightarrow (5)$$

Integrating eqn (3), (4) & (5) w.r.to x, y & z.

$$\int \frac{\partial \phi}{\partial x} = \int (y + \sin z)\,dx$$

$$\phi = xy + x\sin z + f_1(y,z)$$

$$\int \frac{\partial \phi}{\partial y} = \int x\,dy$$

$$\phi = xy + f_2(x,z)$$

$$\int \frac{\partial \phi}{\partial z} = \int x\cos z\,dz$$

$$\phi = x\sin z + f_3(x,y)$$

$$\phi = xy + x\sin z + c$$

(3) Find ϕ if $\nabla\phi = 2xyz\,\vec{i} + x^2z\,\vec{j} + x^2y\,\vec{k}$.

Sol:-

$$\nabla\phi = 2xyz\,\vec{i} + x^2z\,\vec{j} + x^2y\,\vec{k}$$

$$\int \frac{\partial \phi}{\partial x} = \int 2xyz\,dx$$

$$\phi = \frac{\cancel{2}x^2yz}{\cancel{2}} = x^2yz + f_1(y,z)$$

$$\int \frac{\partial \phi}{\partial y} = \int x^2z \cdot dy \Rightarrow \phi = x^2yz + f_2(x,z)$$

$$\int \frac{\partial \phi}{\partial z} = \int x^2 y \, dz$$

$$\phi = x^2 yz + f_1(x, y)$$

$$\phi = x^2 yz + c$$

(4) If $\nabla \phi = x(2yz+1)\vec{i} + x^2 z \vec{j} + x^2 y \vec{k}$

Find ϕ.

$$\frac{\partial \phi}{\partial x} = 2xyz + x \quad \frac{\partial \phi}{\partial y} = x^2 z \quad \frac{\partial \phi}{\partial z} = x^2 y$$

$$\int \frac{\partial \phi}{\partial x} = \int (2xyz + x) \, dx$$

$$\phi = \frac{\cancel{2} x^2 yz}{\cancel{2}} + \frac{x^2}{2} + f_1(y, z).$$

$$\phi = x^2 yz + \frac{x^2}{2} f_1(y, z)$$

$$\int \frac{\partial \phi}{\partial y} = \int x^2 z \, dy$$

$$\phi = x^2yz + f_2(x,z)$$

$$\int \frac{\partial \phi}{\partial z} = \int x^2 y \, dz$$

$$\phi = x^2yz + f_3(x,y)$$

$$\phi = x^2y + z + \frac{x^2}{2} + c$$

Divergent and curle of a vector point function

Divergents of a vector point function.

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

Then the divergence of vector function $\vec{F}$ can be written as

$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(F_1\vec{i} + F_2\vec{j} + F_3\vec{k})$$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

∴ The divergence of a vector function is denoted by $\nabla \cdot \vec{F}$ (or) div $\vec{F}$.

Solenoidal vector:-

If a vector $\vec{F}$ is said to be solenoidal then we satisfied the condition $\nabla \cdot \vec{F} = 0$ (or) div. $\vec{F} = 0$.

~~curl of a vec~~

Curl of a vector point function:-

Let $\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$

Then the curl of a vector function $\vec{F}$ can be written as

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ F_1 & F_2 & F_3 \end{vmatrix}$$

Irrotational vector:-

If a vector $\vec{F}$ is said to be irrotational then, we satisfies the condition.

$$\nabla \times \vec{F} = 0 \text{ (or)}$$

$$\text{curl } \vec{F} = 0$$

Note:-

1. diver $\vec{F}$ is a scalar quantity.
2. div $\vec{F}$ (or) $\nabla \vec{F} = \vec{i}\frac{\partial \vec{F}}{\partial x} + \vec{j}\frac{\partial \vec{F}}{\partial y} + \vec{k}\frac{\partial \vec{F}}{\partial z}$

3. curl $\vec{F}$ is a vector quantity.

4. curl $\vec{F}$ (or) $\nabla\times\vec{F} = \vec{i}\frac{\partial \vec{F}}{\partial x} + \vec{j}\frac{\partial \vec{F}}{\partial y} + \vec{k}\frac{\partial \vec{F}}{\partial z}$

1. If $\vec{f} = x^2z\vec{i} - 2y^3z^2\vec{j} + xy^2z\vec{k}$. Find div f, curl $\vec{f}$ at $(1,-1,1)$.

Sol:-

$$\text{div}\,\vec{f} = \nabla\cdot\vec{f} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\cdot\left(x^2z\vec{i} - 2y^3z^2\vec{j} + xy^2z\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2z) + \frac{\partial}{\partial y}(-2y^3z^2) + \frac{\partial}{\partial z}(xy^2z)$$

$$\nabla\cdot\vec{f} = 2xz - 6y^2z^2 + xy^2$$

$$\nabla\cdot\vec{f}_{(1,-1,1)} = 2(1)(1) - 6(-1)^2(1)^2 + (1)(-1)^2$$

$$= 2 - 6 + 1$$

$$\nabla\cdot\vec{f} = -3$$

$$\text{curl}\,\vec{f} = \nabla\times\vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & -2y^3z^2 & xy^2z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(xy^2z) - \frac{\partial}{\partial z}(2y^3z^2)\right] - \vec{j}\left[\frac{\partial}{\partial x}(xy^2z) - \frac{\partial}{\partial z}(x^2z)\right] + \vec{k}\left[\frac{\partial}{\partial x}(-2y^3z^2) - \frac{\partial}{\partial y}(x^2z)\right]$$

$= \vec{i}\ (2xyz) + 2(y^2z) - \vec{j}\ (y^2z - x^2) + \vec{k}\ (0-0)$

$\nabla \times \vec{f}_{(1,-1)} = \vec{i}\ (-2-4) - \vec{j}\ (1-1) + \vec{k}\ (0)$

$\nabla \times \vec{f} = -6\vec{i}$

(2) If $\vec{f} = x^2y\,\vec{i} + y^2z\,\vec{j} - z^2x\,\vec{k}$ Find $\nabla \times \vec{F}$ at $(1,2,3)$

$$\nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & y^2z & -z^2x \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(-z^2x) - \frac{\partial}{\partial z}(y^2z)\right) - \vec{j}\left(\frac{\partial}{\partial x}(-z^2x) - \frac{\partial}{\partial z}(x^2y)\right) + \vec{k}\left(\frac{\partial}{\partial x}(y^2z) - \frac{\partial}{\partial y}(x^2y)\right)$$

$= \vec{i}\,(0 - y^2) - \vec{j}\,(-z^2 - 0) + \vec{k}\,(0 - x^2)$

$= -y^2\vec{i} + z^2\vec{j} - x^2\vec{k}$

$\nabla \times \vec{f}_{(1,2,3)} = -4\vec{i} + 9\vec{j} - 1\vec{k}$ //

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3. If $\vec{f} = y(x+z)\vec{i} + z(x+y)\vec{j} + x(y+z)\vec{k}$. Find curl

$$\text{curl } \vec{f} = \nabla \times \vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y(x+z) & z(x+y) & x(y+z) \end{vmatrix}$$

$$= \vec{i}\left(\frac{\partial}{\partial y}(x(y+z)) - \frac{\partial}{\partial z}(z(x+y)\right) - \vec{j}\left[\frac{\partial}{\partial x}(x(y+z)) - \frac{\partial}{\partial z}(y(x+z)\right]$$

$$+ \vec{k}\left(\frac{\partial}{\partial x}(z(x+y)) - \frac{\partial}{\partial y}(y(x+z)\right]$$

$$= \vec{i}(x-(x+y) - \vec{j}(y+z-y) + \vec{k}(z-(x+z)]$$

$$= \vec{i}(-y) - \vec{j}(z) + \vec{k}(-x)$$

$$\nabla \times \vec{f} = -y\vec{i} - z\vec{j} - x\vec{k} \,//$$

$$\text{curl } \vec{f} = -y\vec{i} - z\vec{j} - x\vec{k} \,//$$

4. If $\vec{f} = xy^2\vec{i} - 2y^2z^3\vec{j} + xyz^3\vec{k}$. Find $\text{div } \vec{f}$ at $(1,-1,1)$

$$\text{div } \vec{f} = \nabla \cdot \vec{f} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(xy^2\vec{i} - 2y^2z^3\vec{j} + xyz^3\vec{k}$$

$$= \frac{\partial}{\partial x}(xy^2) + \frac{\partial}{\partial y}(-2y^2z^3) + \frac{\partial}{\partial z}(xyz^3)$$

$$= y^2 - 4yz^2 + 3xyz^2$$

$$\text{div } \vec{f}_{(1,-1,1)} = (-1)^2 - 4(-1)(1)^2 + 3(1)(-1)(1)^2$$

$$= 1 + 4 - 3$$

$$= 2 \,//$$

④ If $\vec{f} = (x^2y^2)\vec{i} + 2xy\,\vec{j} + (y^2 - 2xy)\,\vec{k}$.

Find $\operatorname{div}\vec{f}$ and $\operatorname{curl}\vec{f}$.

$$\operatorname{div}\vec{f} = \nabla\cdot\vec{f} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(x^2y^2\,\vec{i} + 2xy\,\vec{j} + (y^2-2xy)\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(x^2y^2) + \frac{\partial}{\partial y}(2xy) + \frac{\partial}{\partial z}(y^2-2xy).$$

$$= 2x\cancel{} + 2x + (0-0)$$

$$= \cancel{2xy^2} + 2x$$

$$\operatorname{curl}\vec{f} = \nabla\times\vec{f}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (x^2y^2) & 2xy & y^2-2xy \end{vmatrix}$$

$$= \vec{i}\cdot\left(\frac{\partial}{\partial y}(y^2-2xy) - \frac{\partial}{\partial z}(2xy)\right) - \vec{j}\left(\frac{\partial}{\partial x}(y^2-2xy) - \frac{\partial}{\partial z}(x^2y^2)\right) + \vec{k}\left(\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2y^2)\right)$$

$$= \vec{i}\,(2y-2x) - 0) - \vec{j}\,(-2y-0) + \vec{k}\,(2y+2y)$$

$$= (2y-2x)\vec{i} + (2y)\vec{j} + 4y\vec{k}$$

6. S.T the vector $\vec{F} = 3y^4z^2\vec{i} + 4x^2z^2\vec{j} - 3x^2y^2\vec{k}$ is solenoidal.

Proof:

To prove: Solenoidal Vector $\nabla \cdot \vec{F} = 0$.

$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(3y^4z^2\vec{i} + 4x^2z^2\vec{j} - 3x^2y^2\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(3y^4z^2) + \frac{\partial}{\partial y}(4x^2z^2) + \frac{\partial}{\partial z}(-3x^2y^2)$$

$$= 0 + 0 + 0$$

$$\nabla \cdot \vec{F} = 0.$$

$\therefore$ The given vector is solenoidal.

(7) P.T the vector $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ is solenoidal.

$$\nabla \cdot \vec{F} = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(z\vec{i} + x\vec{j} + y\vec{k}\right)$$

$$= \frac{\partial}{\partial x}(z) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(y)$$

$$= 0 + 0 + 0 = 0$$

The given vector is solenoidal.

(8) Find the value of 'a' if $\vec{F} = (x+3y)\vec{i} + (y-2z)\vec{j} + (x+az)\vec{k}$ is solenoidal.

solenoidal condition $\nabla \cdot \vec{F} = 0$.

$$\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left[(x+3y)\vec{i} + (y-2z)\vec{j} + (x+az)\vec{k}\right]$$

$$\frac{\partial}{\partial x}(x+3y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(x+az) = 0$$

$$1 + 1 + a = 0 \Rightarrow \boxed{a = -2}$$

⑨ Find the value of 'a' if

$\vec{F} = (z+2y)\vec{i} + (x-2z)\vec{j} + (x+az)\vec{k}$ is solenoidal

$\nabla \cdot \vec{F} = 0$

$$\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left((z+2y)\vec{i} + (x-2z)\vec{j} + (x+az)\vec{k}\right) = 0$$

$$\frac{\partial}{\partial x}(\cancel{z}+2y) + \frac{\partial}{\partial y}(x-2z) + \frac{\partial}{\partial z}(x+az) = 0$$

$$0 + 0 + a = 0$$

$$\boxed{a = 0}$$

⑩ $\vec{F} = 3x\vec{i} + (x+y)\vec{j} - az\vec{k}$ is solenoidal.

Find 'a'.

$\nabla \cdot \vec{F} = 0$

$$\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(3x\vec{i} + (x+y)\vec{j} - az\vec{k}\right) = 0$$

$$\frac{\partial}{\partial x}(3x) + \frac{\partial}{\partial y}(x+y) + \frac{\partial}{\partial z}(-az) = 0$$

$$3 + 1 - a = 0$$

$$\boxed{a = +4}$$

⑪. If $\vec{F} = (ax+3y+4z)\vec{i} + (x-2y+3z)\vec{j} + (3x+2y-z)\vec{k}$ is solenoidal. Find 'a'.

$\nabla \cdot \vec{F} = 0$

$$\left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left[(ax+3y+4z)\vec{i} + (x-2y+3z)\vec{j} + (3x+2y-z)\vec{k}\right] = 0$$

$$\frac{\partial}{\partial x}(ax+3y+4z) + \frac{\partial}{\partial y}(x-2y+3z) + \frac{\partial}{\partial z}(3x+2y-z) = 0$$

$$a - 2 - 1 = 0$$

$$\boxed{a = 3}$$

14. show that the vector $F = 2xy\vec{i} + (x^2+2yz)\vec{j} + (y^2+1)\vec{k}$ is irrotational.

proof:

Irrotational condition $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy & x^2+2yz & y^2+1 \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(y^2+1) - \frac{\partial}{\partial z}(x^2+2yz)\right] - \vec{j}\left[\frac{\partial}{\partial x}(y^2+1) - \frac{\partial}{\partial z}(2xy)\right] + \vec{k}\left[\frac{\partial}{\partial x}(x^2+2yz) - \frac{\partial}{\partial y}(2xy)\right]$$

$$= \vec{i}\,[2y - 2y] - \vec{j}\,[0-0] + \vec{k}\,(2x-2x)$$

$$\nabla \times \vec{F} = 0$$

15. $\vec{F} = (2xy - z^2)\vec{i} + (x^2+2yz)\vec{j} + (y^2-2xz)\vec{k}$ is irrotational.

proof:

Irrotational condition, $\nabla \times \vec{F} = 0$

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (2xy-z^2) & (x^2+2yz) & (y^2-2xz) \end{vmatrix}$$

$$\Rightarrow \vec{i}\,[2y-2y] - \vec{j}\,[-2z+2z) + \vec{k}\,[2x-2x$$

$$\Rightarrow -\vec{j}\,(-2z+2z) + \vec{k}\,(2x-2x) = 0$$

Equating the co-eff of $\vec{j}$ & $\vec{k}$ equal to zero.

$-(-az+2z)=0$ $\quad\quad$ $2x-ax=0$

$z(a-2)=0$ $\quad\quad$ $x(2-a)=0$

$a-2=0$ $\quad\quad$ $\boxed{a=2}$

$\boxed{a=2}$

③ Find the value of 'a' such that $\vec{F}=(axy-z^3)\vec{i}+(a-2)x^2\vec{j}+(1-a)xz^2\vec{k}$ is irrotational.

sol:-

Irrotational condition, $\nabla\times\vec{F}=0$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ axy-z^3 & (a-2)x^2 & (1-a)xz^2\end{vmatrix}$$

$\Rightarrow \vec{i}(0-0)-\vec{j}(z^2-az^2-0+3z^2)+\vec{k}(2xa-4x-ax)=0$

$-\vec{j}(4z^2-az^2)+\vec{k}(ax-4x)=0$

equating the co-eff of $\vec{j}$ and $\vec{k}$

$-(4z^2-az^2)=0$ $\quad\quad$ $ax-4x=0$

$-z^2(4-a)=0$ $\quad\quad$ $x(a-4)=0$

$\boxed{a=4}$ $\quad\quad$ $\boxed{a=4}$

⑮ S.T the vector $\vec{F}=(4xy-z^3)\vec{i}+2x^2\vec{j}-3xz^2\vec{k}$ is irrotational.

Irrotatational condition, $\nabla\times\vec{F}=0$

$$\nabla\times\vec{F}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 4xy-z^3 & 2x^2 & -3xz^2\end{vmatrix}$$

$=\vec{i}(0-$

$$= \vec{i}(0-0) - \vec{j}(-3z^2+3z^2) + \vec{k}(4x-4x)$$

$$= 0-0+0$$

$$= 0$$

The given vector is irrotational.

1. S.T (i) $\text{div}(\vec{r} \times \vec{a}) = 0$

(ii) $\text{curl}(\vec{r} \times \vec{a}) = -2\vec{a}$

(iii) $\text{grad}(\vec{r} \cdot \vec{a}) = \vec{a}$

where 'a' is constant vector.

proof:

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$$

$$\vec{r} \times \vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

$$= \vec{i}(a_3y - a_2z) - \vec{j}(a_3x - a_1z) + \vec{k}(a_2x - a_1y)$$

$$\text{div}(\vec{r} \times \vec{a}) = \nabla \cdot (\vec{r} \times \vec{a})$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left[\vec{i}(a_3y - a_2z) - \vec{j}(a_3x - a_1z) + \vec{k}(a_2x - a_1y)\right]$$

$$= \frac{\partial}{\partial x}(a_3y - a_2z) - \frac{\partial}{\partial y}(a_3x - a_1z) + \frac{\partial}{\partial z}(a_2x - a_1y)$$

$$= 0 - 0 + 0$$

$$= 0.$$

(ii) curl $(\vec{r}\times\vec{a}) = \nabla\times(\vec{r}\times\vec{a})$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (a_3y-a_2z) & -(a_3x-a_1z) & (a_2x-a_1y) \end{vmatrix}$$

$$= \vec{i}(-a_1-a_1)-\vec{j}(a_2+a_2)+\vec{k}(-a_3-a_3)$$

$$= -2a_1\vec{i}-2a_2\vec{j}-2a_3\vec{k}$$

$$= -2(a_1\vec{i}+a_2\vec{j}+a_3\vec{k})$$

$$= -2a$$

(iii) grad $(\vec{r}\cdot\vec{a}) = \vec{a}$

$$\vec{r}\cdot\vec{a} = (x\vec{i}+y\vec{j}+z\vec{k})(a_1\vec{i}+a_2\vec{j}+a_3\vec{k})$$

$$= a_1x+a_2y+a_3z$$

$$\nabla(\vec{r}\cdot\vec{a}) = \vec{i}\frac{\partial(\vec{r}\cdot\vec{a})}{\partial x}+\vec{j}\frac{\partial(\vec{r}\cdot\vec{a})}{\partial y}+\vec{k}\frac{\partial(\vec{r}\cdot\vec{a})}{\partial z}$$

$$= \vec{i}(a_1)+\vec{j}(a_2)+\vec{k}(a_3)$$

$$= \vec{a}.$$

2. P.T div $\vec{r}=3$ and curl $\vec{r}=0$ where $\vec{r}$ is the position vector of the point (x,y,z).

$$\vec{r} = x\vec{i}+y\vec{j}+z\vec{k}$$

(i) div $\vec{r} = \nabla\cdot\vec{r}$

$$= \left(\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}+\frac{\partial}{\partial z}\vec{k}\right)\cdot(x\vec{i}+y\vec{j}+z\vec{k})$$

$$= 1+1+1$$

$$= 3$$

(iii) curl $\vec{r} = \nabla \times \vec{r}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$

$$= 0$$

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2) P.T (i) $\nabla(r^n \vec{r}) = (n+3)\vec{r}$

(ii) $\nabla^2(r^n \vec{r}) = n(n+3)r^{n-2}\vec{r}$

proof:

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$r^n \vec{r} = r^n x\vec{i} + r^n y\vec{j} + r^n z\vec{k}$$

$$\nabla(r^n \vec{r}) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\left(\frac{\partial}{\partial z}\right)\right)(r^n x\vec{i} + r^n y\vec{j} + r^n z\vec{k})$$

$$= \sum \frac{\partial}{\partial x}(r^n x)$$

$$= \sum \left[r^n(1) + x n r^{n-1}\frac{\partial r}{\partial x}\right]$$

$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$r^2 = x^2 + y^2 + z^2$

$$= \sum \left[r^n + x n r^{n-1}\left(\frac{x}{r}\right)\right]$$

$2r\frac{\partial r}{\partial x} = 2x$

$$= \sum \left[r^n + x^2 n r^{n-2}\right]$$

$\frac{\partial r}{\partial x} = \frac{x}{r}$

$$= r^n + n r^{n-2} x^2$$

$$+ r^n + n r^{n-2} y^2$$

$\frac{\partial r}{\partial y} = \frac{y}{r}$

$$+ r^n + n r^{n-2} z^2$$

$\frac{\partial r}{\partial z} = \frac{z}{r}$

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$= 3r^n + nr^{n-2}[x^2+y^2+z^2]$

$= 3r^n + nr^{n-2} r^2$

$= 3r^n + nr^n$

$= r^n (3+n)$

$\nabla (r^n \vec{r}) = r^n (n+3)$,,

(ii) $\nabla^2 (r^n \vec{r}) = n(n+3) r^{n-2} \vec{r}$

$\nabla^2 (r^n \vec{r}) = \nabla [\nabla (r^n \vec{r})]$

$= \nabla [(n+3) r^n]$

$= \left[\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right][(n+3) r^n]$

$= \sum \vec{i} \frac{\partial}{\partial x} (n+3) r^n$

$= (n+3) \sum \vec{i} \frac{\partial}{\partial x} (r^n)$

$= (n+3) \sum \vec{i}\, n \cdot r^{n-1} \cdot \frac{\partial r}{\partial x}$

$= (n+3) \sum \vec{i}\, n \cdot r^{n-1} \left(\frac{x}{r}\right)$

$= n(n+3) \sum \vec{i}\, x\, r^{n-2}$

$= n(n+3) [\vec{i} x r^{n-2} + \vec{j} y r^{n-2} + \vec{k} z r^{n-2}]$

$= n(n+3) r^{n-2} [x\vec{i} + y\vec{j} + z\vec{k}]$

$\nabla^2 (r^n \vec{r}) = n(n+3) r^{n-2} \vec{r}$

(4) prove that grad $r^n = n r^{n-2}\vec{r}$.

$$\text{grad } r^n = \nabla r^n$$
$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) r^n$$
$$= \sum \vec{i}\frac{\partial}{\partial x} r^n$$
$$= \sum \vec{i}\, n r^{n-1}\frac{\partial r}{\partial x}$$
$$= \sum \vec{i}\, n r^{n-1}\left(\frac{x}{r}\right)$$
$$= n r^{n-2}\sum \vec{i} x$$
$$= n r^{n-2}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$$
$$= n r^{n-2}\left[\vec{r}\right]$$
$$\text{grad } r^n = n r^{n-2}\vec{r}$$

Hence proved

(5) P.T div $\frac{\vec{r}}{r} = \frac{2}{r}$

proof:
$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$
$$\frac{\vec{r}}{r} = \frac{x\vec{i} + y\vec{j} + z\vec{k}}{r}$$
$$= \frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}$$

$$\text{div}\left(\frac{\vec{r}}{r}\right) = \nabla\cdot\left(\frac{\vec{r}}{r}\right)$$
$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\frac{x}{r}\vec{i} + \frac{y}{r}\vec{j} + \frac{z}{r}\vec{k}\right)$$

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$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right) + \frac{\partial}{\partial y}\left(\frac{y}{r}\right) + \frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \sum \frac{\partial}{\partial x}\left(\frac{x}{r}\right)$$

$$= \sum \frac{r - x\frac{\partial r}{\partial x}}{r^2}$$

$$= \sum \frac{r - \frac{x^2}{r}}{r^2}$$

$$= \sum \frac{r^2 - x^2}{r^3}$$

$$= \frac{r^2 - x^2}{r^3} + \frac{r^2 - y^2}{r^3} + \frac{r^2 - z^2}{r^3}$$

$$= \frac{1}{r^3}\left[r^2 - x^2 + r^2 - y^2 + r^2 - z^2\right)$$

$$= \frac{1}{r^3}\left[3r^2 - (x^2 + y^2 + z^2)\right]$$

$$= \frac{1}{r^3}\left[3r^2 - r^2\right]$$

$$= \frac{1}{r^3}\left[2r^2\right]$$

$$\operatorname{div}\frac{\vec{r}}{r} = \frac{2}{r}$$

Hence proved.

6) P.T if $\vec{F} = \frac{\vec{r}}{r}$ Find grad div $\vec{f}$.

Sol:

$$\text{grad div}\vec{f} = \text{grad div}\frac{\vec{r}}{r}$$

$$= \text{grad}\left(\frac{2}{r}\right)$$

$$= \nabla\left(\frac{2}{r}\right)$$

$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\left(\frac{2}{r}\right)$$

$$= \sum \vec{i} \frac{\partial}{\partial x}\left(\frac{2}{r}\right)$$

$$= 2 \sum \vec{i} \; -r^{-2} \frac{\partial r}{\partial x}$$

$$= -2 \sum \vec{i} \; \frac{1}{r^2}\left(\frac{x}{r}\right)$$

$$= -2 \sum \vec{i} \; \frac{x}{r^3}$$

$$= \frac{-2}{r^3} \sum \vec{i} x$$

$$= \frac{-2}{r^3}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$$

$$\text{grad div } \vec{f} = \frac{-2}{r^3} \vec{r}.$$

7) If $\vec{V} = \vec{\omega} \times \vec{r}$ P.T $\vec{\omega} = \frac{1}{2} \text{curl } \vec{V}$, where ω is a constant vector and r is a position vector of a vector point function (x, y, z).

proof:

To prove: $\vec{\omega} = \frac{1}{2} \text{curl } \vec{V}$

R.H.S:- $\frac{1}{2} \text{curl } \vec{v} = \frac{1}{2} \text{curl } (\vec{\omega} \times \vec{r})$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{\omega} = \omega_1 \vec{i} + \omega_2 \vec{j} + \omega_3 \vec{k}$$

$$\vec{\omega} \times \vec{r} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix}$$

$$= \vec{i}(z\omega_2 - y\omega_3) - \vec{j}(\omega_1 z - x\omega_3) + \vec{k}(y\omega_1 - x\omega_2)$$

$$\text{curl}(\vec{\omega}\times\vec{r}) = \nabla \times (\vec{\omega}\times\vec{r})$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (z\omega_2 - y\omega_3) & -(\omega_1 z - x\omega_3) & (y\omega_1 - x\omega_2) \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(y\omega_1 - x\omega_2) - \frac{\partial}{\partial z}-(\omega_1 z - x\omega_3)\right]$$

$$-\vec{j}\left[\frac{\partial}{\partial x}(y\omega_1 - x\omega_2) - \frac{\partial}{\partial z}(z\omega_2 - y\omega_3)\right]$$

$$+\vec{k}\left[\frac{\partial}{\partial x}-(\omega_1 z - x\omega_3) - \frac{\partial}{\partial y}(z\omega_2 - y\omega_3)\right]$$

$$= \vec{i}\left[\omega_1 - 0) + \omega_1\right] - \vec{j}\left[-\omega_2 - \omega_2\right] + \vec{k}\left[\omega_3 + \omega_3\right]$$

$$= 2\omega_1\vec{i} + 2\omega_2\vec{j} + 2\omega_3\vec{k}$$

$$= 2(\omega_1\vec{i} + \omega_2\vec{j} + \omega_3\vec{k})$$

$$= 2\vec{\omega}$$

$$\frac{1}{2}\text{curl}(\vec{\omega}\times\vec{r}) = \frac{1}{\cancel{2}}(\cancel{2}\vec{\omega})$$

$$= \vec{\omega}$$

//

Hence proved//

8. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ S.T $\nabla \times (r^n \vec{r}) = 0$

$$r^n \vec{r} = r^n x\vec{i} + r^n y\vec{j} + r^n z\vec{k}$$

$$\nabla \times (r^n \vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ r^n x & r^n y & r^n z \end{vmatrix}$$

$$= \vec{i}(0-0) - \vec{j}(0-0) + \vec{k}(0-0)$$

$$= 0$$

$$\therefore \nabla \times (r^n \vec{r}) = 0$$

Hence proved.

4/1/2020

9. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ S.T $\nabla \times f(r)\vec{r} = 0$.

Sol:-

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$f(r)\vec{r} = f(r)x\vec{i} + f(r)y\vec{j} + f(r)z\vec{k}$$

$$\nabla \times (f(r)\vec{r}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f(r)x & f(r)y & f(r)z \end{vmatrix}$$

$$= \vec{i}\left[\frac{\partial}{\partial y}(f(r)z) - \frac{\partial}{\partial z}f(r)y\right] - \vec{j}\left[\frac{\partial}{\partial x}f(r)z - \frac{\partial}{\partial z} \ldots\right]$$
$$+ \vec{k}\left[\frac{\partial}{\partial x}(f(r)y) - \frac{\partial}{\partial y}(f(r)x)\right]$$

$$= \vec{i}\left[f(r)0 - zf'(r)\frac{\partial r}{\partial y} - f(r)0 - yf'(r)\frac{\partial r}{\partial z} \ldots\right.$$
$$- \vec{j}\left[f(r)0 - zf'(r)\frac{\partial r}{\partial x} - f(r)0 - xf'(r)\frac{\partial r}{\partial z} \ldots\right.$$
$$+ \vec{k}\left[f(r)(0) + yf'(r)\frac{\partial r}{\partial x} - f(r)(0) - xf'(r)\frac{\partial r}{\partial y} \ldots\right.$$

$$= \vec{i}\left[z f'(r)\left(\frac{y}{r}\right) - y f'(r)\left(\frac{z}{r}\right)\right] - \vec{j}\left[z f'(r)\left(\frac{x}{r}\right) - x f'(r)\left(\frac{z}{r}\right)\right]$$
$$+ \vec{k}\left[y f'(r)\left(\frac{x}{r}\right) - x f'(r)\frac{y}{r}\right]$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}(0)$$

$\nabla \times (f(r)\vec{r}) = 0$ //

(2m) 9. div $(r^n \vec{r})$ is irrotational. Find "n" when it is solenoidal.

Sol:-

w.k.T

$$\text{div}(r^n \vec{r}) = (n+3) r^n = 0$$
$$(n+3) r^n = 0$$
$$n+3 = 0$$
$$\boxed{n = -3}$$

- The Laplacian Operator ; ∇^2:-

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

If ϕ is any scalar function.

Then $\nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$

$$\nabla^2 \phi = \nabla(\nabla \phi) = (\nabla \cdot \nabla)\phi$$
$$= \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) \cdot \left(\vec{i}\frac{\partial \phi}{\partial x} + \vec{j}\frac{\partial \phi}{\partial y} + \vec{k}\frac{\partial \phi}{\partial z}\right)$$
$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \rightarrow (1)$$

$$(\nabla\cdot\nabla)\phi = \left[\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right]\cdot\left[\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right]$$

$$= \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\phi$$

$$(\nabla\cdot\nabla)\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \rightarrow (2)$$

From (1) & (2)

$$\nabla^2\phi = \nabla(\nabla\phi) = (\nabla\cdot\nabla)\phi.$$

Gradient, Divergence and curl of sums:-

1. grad $(\phi \pm \psi)$ = grad ϕ $\pm$ grad ψ

$$\nabla(\phi \pm \psi) = \nabla\phi + \nabla\psi$$

2. div $(f \pm g)$ = div f $\pm$ div g

$$\nabla(f \pm g) = \nabla f \pm \nabla g$$

3. curl $(f \pm g)$ = curl f $\pm$ curl g.

$$\nabla \times (f \pm g) = (\nabla \times f) \pm (\nabla \times g).$$

product:-

1. grad $(\phi\psi)$ = ϕ grad ψ + ψ grad ϕ

$$\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi$$

2. grad $(f\cdot g)$ = $f \times$ curl g + $g \times$ curl f

$$+ (f\cdot\nabla)g + (g\cdot\nabla)f$$

3. $\operatorname{div}(\phi f) = \phi \operatorname{div} f + f \operatorname{div}\phi$

4. $\operatorname{div}(f \times g) = \operatorname{curl} f \cdot g - \operatorname{curl} g \cdot f$

5. $\operatorname{curl}(\phi f) = \operatorname{grad}\phi \times f + \phi \operatorname{curl} f$

6. $\operatorname{curl}(f \times g) = f \operatorname{div} g - g \operatorname{div} f + (g\cdot\nabla)f - (f\cdot\nabla)g.$

06/01/2020

1. P.T $\operatorname{grad}(\phi\psi) = \phi \operatorname{grad}\psi + \psi \operatorname{grad}\phi$

Proof:-

$$\operatorname{grad}(\phi\psi) = \nabla(\phi\psi)$$
$$= \left[\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right]\phi\psi$$
$$= \vec{i}\frac{\partial}{\partial x}(\phi\psi) + \vec{j}\frac{\partial}{\partial y}(\phi\psi) + \vec{k}\frac{\partial}{\partial z}(\phi\psi)$$
$$= \vec{i}\left[\phi\frac{\partial\psi}{\partial x} + \psi\frac{\partial\phi}{\partial x}\right] + \vec{j}\left[\phi\frac{\partial\psi}{\partial y} + \psi\frac{\partial\phi}{\partial y}\right] + \vec{k}\left[\phi\frac{\partial\psi}{\partial z} + \psi\frac{\partial\phi}{\partial z}\right]$$

$$\operatorname{grad}(\phi\psi) = \phi\left[\vec{i}\frac{\partial\psi}{\partial x} + \vec{j}\frac{\partial\psi}{\partial y} + \vec{k}\frac{\partial\psi}{\partial z}\right] + \psi\left[\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right] \to (1)$$

$$\phi \operatorname{grad}\psi = \phi\left[\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right]\psi$$
$$= \phi\left[\vec{i}\frac{\partial\psi}{\partial x} + \vec{j}\frac{\partial\psi}{\partial y} + \vec{k}\frac{\partial\psi}{\partial z}\right]$$

$$\psi \operatorname{grad}\phi = \psi\left[\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right]$$

$$\phi \operatorname{grad}\psi + \psi \operatorname{grad}\phi = \phi\left[\vec{i}\frac{\partial\psi}{\partial x} + \vec{j}\frac{\partial\psi}{\partial y} + \vec{k}\frac{\partial\psi}{\partial z}\right] + \psi\left[\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}\right] \to (2)$$

From (1) and (2)

$$\operatorname{grad}(\phi\psi) = \phi \operatorname{grad}\psi + \psi \operatorname{grad}\phi$$

Q. If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $r = |\vec{r}|$. Then

P.T. ~~i.e. $\nabla r \cdot \frac{1}{r}$~~

(i) $\nabla r = \frac{1}{r}\vec{r}$ (ii) $\nabla\left(\frac{1}{r}\right) = \frac{-\vec{r}}{r^3}$

(iii) $\nabla r^n = n r^{n-2}\vec{r}$

(iv) $\nabla f(r) = f'(r)\frac{\vec{r}}{r} = f'(r)\nabla r$

(v) $\nabla \log r = \frac{\vec{r}}{r^2}$

(vi) $\nabla f(r) \times \vec{r} = 0$

proof:-

(i) $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

$2r\frac{\partial r}{\partial x} = 2x$

$\frac{\partial r}{\partial x} = \frac{x}{r}$

|||ly $\frac{\partial r}{\partial y} = \frac{y}{r}$

$\frac{\partial r}{\partial z} = \frac{z}{r}$

$\nabla r = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) r$

$= \vec{i}\frac{\partial r}{\partial x} + \vec{j}\frac{\partial r}{\partial x} + \vec{k}\frac{\partial r}{\partial z}$

$= \vec{i}\left(\frac{x}{r}\right) + \vec{j}\left(\frac{y}{r}\right) + \vec{k}\left(\frac{z}{r}\right)$

$= \frac{1}{r}\left[x\vec{i} + y\vec{j} + z\vec{k}\right]$

$\nabla r = \frac{\vec{r}}{r}$

(ii) $\nabla\left(\frac{1}{r}\right) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)\frac{1}{r}$

$= \vec{i}\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + \vec{j}\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \vec{k}\frac{\partial}{\partial z}\left(\frac{1}{r}\right).$

$$= -\frac{1}{r^2}\frac{\partial r}{\partial x}\vec{i} - \frac{1}{r^2}\frac{\partial r}{\partial y}\vec{j} + -\frac{1}{r^2}\frac{\partial r}{\partial z}\vec{k}$$

$$= -\frac{1}{r^3}x\vec{i} - \frac{1}{r^3}y\vec{j} - \frac{1}{r^3}z\vec{k}$$

$$= -\frac{1}{r^3}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$= -\frac{1}{r^3}\vec{r}$$

(iii) $\nabla r^n = n r^{n-2}\vec{r}$

$$\nabla r^n = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right) r^n$$

$$= \vec{i}\frac{\partial}{\partial x}(r^n) + \vec{j}\frac{\partial}{\partial y}(r^n) + \vec{k}\frac{\partial}{\partial z}(r^n)$$

$$= n r^{n-1}\frac{\partial r}{\partial x}\vec{i} + n r^{n-1}\frac{\partial r}{\partial y}\vec{j} + n r^{n-1}\frac{\partial r}{\partial z}$$

$$= n r^{n-1}\left(\frac{x}{r}\right)\vec{i} + n r^{n-1}\left(\frac{y}{r}\right)\vec{j} + n r^{n-1}\left(\frac{z}{r}\right)\vec{k}$$

$$= n r^{n-2}x\vec{i} + n r^{n-2}y\vec{j} + n r^{n-2}z\vec{k}$$

$$= n r^{n-2}(x\vec{i} + y\vec{j} + z\vec{k})$$

$$\nabla r^n = n r^{n-2}\vec{r}$$

(iv) $\nabla f(r) = f'(r)\frac{\vec{r}}{r} = f'(r)\nabla r$

$$\nabla f(r) = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(f(r))$$

$$= \frac{\partial}{\partial x}(f(r))\vec{i} + \frac{\partial}{\partial y}(f(r))\vec{j} + \frac{\partial}{\partial z}f(r)\vec{k}$$

$$= f'(r)\frac{\partial r}{\partial x}\vec{i} + f'(r)\frac{\partial r}{\partial y}\vec{j} + f'(r)\frac{\partial r}{\partial z}\vec{k}$$

$$= f'(r)\frac{x}{r}\vec{i} + f'(r)\frac{y}{r}\vec{j} + f'(r)\frac{z}{r}\vec{k}$$

$$= \frac{f'(r)}{r}(x\vec{i}+y\vec{j}+z\vec{k})$$

$$= \frac{f'(r)}{r}\vec{r}$$

$$= f'(r)\nabla r$$ //

(v) $\nabla \log r = \frac{\vec{r}}{r^2}$

$$\nabla \log r = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)\log r$$

$$= \frac{\partial}{\partial x}\log r\,\vec{i}+\frac{\partial}{\partial y}\log r\,\vec{j}+\frac{\partial}{\partial z}\log r\,\vec{k}$$

$$= \frac{1}{r}\frac{\partial r}{\partial x}\vec{i}+\frac{1}{r}\frac{\partial r}{\partial y}\vec{j}+\frac{1}{r}\frac{\partial r}{\partial z}\vec{k}$$

$$= \frac{1}{r}\left(\frac{x}{r}\right)\vec{i}+\frac{1}{r}\left(\frac{y}{r}\right)\vec{j}+\frac{1}{r}\left(\frac{z}{r}\right)\vec{k}$$

$$= \frac{1}{r^2}(x\vec{i}+y\vec{j}+z\vec{k})$$

$$= \frac{\vec{r}}{r^2}$$ //

(vi) $\nabla f(r)\times\vec{r} = 0$

$$\nabla f(r)\times\vec{r} = \frac{f'(r)}{r}\vec{r}\times\vec{r}$$ [By proof (iv)]

$$= \frac{f'(r)}{r}(\vec{r}\times\vec{r})$$

W.K.T $\vec{r}\times\vec{r} = 0$

$$\therefore \nabla f(r)\times\vec{r} = \frac{f'(r)}{r}(0)$$

$$\therefore \nabla f(r)\times\vec{r} = 0$$ //

Hence proved //

7.1.20

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2/1/2020 Unit - 2 1

Vector Integration

Line Integral:

Let $r = f(t)$, represent a continuously differentiable curve denoted by C and $f(r)$ be a continuous vector point function.

Then $\frac{dr}{ds}$ is unit vector functions along the tangent at any point P on the curve. The components of the vector function 'f' along the tangent is $F \cdot \frac{dr}{ds}$ which is a function of "s" for points on the curve. Then

$\int_c F \frac{dr}{ds} ds = \int_c F dr$ is called the line integral of tangent integral along c.

Work done by force:

If 'F' represent a force acting on a particle moving along the curve. Then the line integral '$F \cdot dr$' represent the work done the force. It is also called the circulation of $\vec{F}$ about c

When 'F' represent the velocity of time 't'.

problems:-

1. Evaluate: $\int_c \vec{F}\cdot d\vec{r}$ where $\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$ at the curve c is $\vec{r} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$ t varying from -1 to 1.

sol:

The paramatric eqn

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{r} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$$

where, $x = t$, $y = t^2$, $z = t^3$

$$\frac{d\vec{r}}{dt} = \vec{i} + 2t\vec{j} + 3t^2\vec{k}$$

$$d\vec{r} = (\vec{i} + 2t\vec{j} + 3t^2\vec{k})\,dt$$

$$\vec{F} = xy\vec{i} + yz\vec{j} + zx\vec{k}$$

$$= (t)(t^2)\vec{i} + (t^2)(t^3)\vec{j} + (t^3)(t)\vec{k}$$

$$\vec{F} = t^3\vec{i} + t^5\vec{j} + t^4\vec{k}$$

$$\vec{F}\cdot d\vec{r} = (t^3\vec{i} + t^5\vec{j} + t^4\vec{k})\cdot(\vec{i} + 2t\vec{j} + 3t^2\vec{k})$$

$$= (t^3 + 2t^6 + 3t^6)\,dt$$

$$\vec{F}\cdot d\vec{r} = (t^3 + 5t^6)\,dt$$

$$\int_c \vec{F}\cdot d\vec{r} = \int_{-1}^{1} (t^3 + 5t^6)\,dt$$

$$= \left[\frac{t^4}{4} + \frac{5t^7}{7}\right]_{-1}^{1}$$

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$$= \left[\frac{(1)^4}{4}+\frac{5(1)^7}{7}\right]-\left[\frac{(-1)^4}{4}+\frac{5(-1)^7}{7}\right]$$

$$= \left(\frac{1}{4}+\frac{5}{7}\right)-\left(\frac{1}{4}-\frac{5}{7}\right)$$

$$= \cancel{\frac{1}{4}}+\frac{5}{7}-\cancel{\frac{1}{4}}+\frac{5}{7}$$

$$\int \vec{F}\cdot d\vec{r} = \frac{10}{7}$$

H.W:

2) $\int_c \vec{F}\cdot d\vec{r}$ where $\vec{F} = x^2\vec{i}+y^2\vec{j}+z^2\vec{k}$ and the curve c is $\vec{r} = t\vec{i}+t^2\vec{j}+t^3\vec{k}$, t varying from 0 ~~to~~ 1. Ans : 1

3). Evaluate: $\int_c \vec{F}\,d\vec{r}$ where $\vec{F} = 3xy\vec{i}-5z\vec{j}+10x\vec{k}$ along the particle of the force is given by $x=t^2+1$, $y=2t^2$, $z=t^3$, $t = 1$ to 2. Ans 303

4) Evalute: $\int_c \vec{F}\,d\vec{r}$ where $F=(2x+y)\vec{i}+(3y-x)\vec{j}+yz\vec{k}$ and c is the curve $x=2t^2$, $y=t$, $z=t^3$ from $t=0$ to $t=1$. 277/42

5) If $F = (3x^2+6y)\vec{i}-14yz\vec{j}+20xz^2\vec{k}$. Evalute $\int_c \vec{F}\cdot d\vec{r}$ from $(0,0,0)$ to $(1,1,1)$ along the following paths $x=t$; $y=t^2$; $z=t^3$ 5

6) If $F = yz\vec{i}+zx\vec{j}-xy\vec{k}$ Find $\int_c \vec{F}d\vec{r}$ where c is given by $x=t$ $y=t^2$, $z=t^3$ from $P(0,0,0)$ to $Q(2,4,8)$. 0

2.

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{r} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$$

$$x = t \; ; \; y = t^2 \; ; \; z = t^3$$

$$\frac{d\vec{r}}{dt} = \vec{i} + 2t\vec{j} + 3t^2\vec{k}$$

$$d\vec{r} = (\vec{i} + 2t\vec{j} + 3t^2\vec{k})\,dt$$

$$\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$$

$$\vec{F} = t^2\vec{i} + t^4\vec{j} + t^6\vec{k}$$

$$\vec{F}\cdot d\vec{r} = (t^2\vec{i} + t^4\vec{j} + t^6\vec{k})\cdot(\vec{i} + 2t\vec{j} + 3t^2\vec{k})$$

$$= (t^2 + 2t^5 + 3t^8)\,dt$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (t^2 + 2t^5 + 3t^8)\,dt$$

$$= \left[\frac{t^3}{3} + \frac{2t^6}{6} + \frac{3t^9}{9}\right]_0^1$$

$$= \left[\frac{1}{3} + \frac{2}{6} + \frac{3}{9}\right] - [0]$$

$$= \left[\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right]$$

$$= \left[\frac{3}{3}\right]$$

$$\int_C \vec{F}\cdot d\vec{r} = 1$$

3.

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{r} = (t^2+1)\vec{i} + 2t^2\vec{j} + t^3\vec{k}$$

$$\frac{d\vec{r}}{dt} = 2t\vec{i} + 4t\vec{j} + 3t^2\vec{k}$$

$$d\vec{r} = (2t\vec{i} + 4t\vec{j} + 3t^2\vec{k})\,dt$$

$\vec{F} = 3xy\vec{i} - 5z\vec{j} + 10x\vec{k}$

$= 3(t^2+1)(2t^2)\vec{i} - 5(t^3)\vec{j} + 10(t^2+1)\vec{k}$

$= (3t^2+3)(2t^2)\vec{i} - 5(t^3)\vec{j} + (10t^2+10)\vec{k}$

$\vec{F} = (6t^4+6t^2)\vec{i} - 5t^3\vec{j} + (10t^2+10)\vec{k}$

$\vec{F}\cdot d\vec{r} = ((6t^4+6t^2)\vec{i} - 5t^3\vec{j} + (10t^2+10)\vec{k}).$

$(2t\vec{i} + 4t\vec{j} + 3t^2\vec{k})dt$

$= [(6t^4+6t^2)(2t) - (5t^3)(4t) + 3t^2(10t^2+10)]dt$

$= [12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2]\,dt$

$= [12t^5 + 10t^4 + 12t^3 + 30t^2]dt$

$\int_C \vec{F}\cdot d\vec{r} = \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)dt$

$= \left[\frac{12t^6}{6} + \frac{10t^5}{5} + \frac{12t^4}{4} + \frac{30t^3}{3}\right]_1^2$

$= \left[\frac{12(2)^6}{6} + \frac{10(2)^5}{5} + \frac{12(2)^4}{4} + \frac{30(2)^3}{3}\right]$

$- \left[\frac{12}{6} + \frac{10}{5} + \frac{12}{4} + \frac{30}{3}\right]$

$= [128 + 64 + 48 + 80] - [2 + 2 + 3 + 10]$

$= [320] - 17$

$= \cancel{263} \quad = 303$ //

4) $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$\vec{r} = 2t^2\vec{i} + t\vec{j} + t^3\vec{k}$

$\frac{d\vec{r}}{dt} = 4t\vec{i} + \vec{j} + 3t^2\vec{k}$

$d\vec{r} = (4t\vec{i} + \vec{j} + 3t^2\vec{k})dt.$

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$$\vec{F} = (2x+y)\vec{i} + (3y-x)\vec{j} + yz\vec{k}$$

$$= (4t^2+t)\vec{i} + (3t-2t^2)\vec{j} + t^4\vec{k}$$

$$\vec{F}\cdot d\vec{r} = ((4t^2+t)\vec{i} + (3t-2t^2)\vec{j} + t^4\vec{k}).$$
$$(4t\vec{i} + \vec{j} + 3t^2\vec{k})\,dt$$

$$= (4t(4t^2+t) + (3t-2t^2) + 3t^6)\,dt$$

$$= (16t^3 + 4t^2 + 3t - 2t^2 + 3t^6)\,dt$$

$$= (3t^6 + 16t^3 + 2t^2 + 3t)\,dt$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (3t^6 + 16t^3 + 2t^2 + 3t)\,dt$$

$$= \left[\frac{3t^7}{7} + \frac{16t^4}{4} + \frac{2t^3}{3} + \frac{3t^2}{2}\right]_0^1$$

$$= \left[\frac{3}{7} + \frac{16}{4} + \frac{2}{3} + \frac{3}{2}\right] - 0$$

$$= \left[\frac{27}{14} + \frac{14}{3}\right]$$

$$= \frac{81+196}{42}$$

$$\int_C \vec{F}\cdot d\vec{r} = \frac{277}{42}$$

5)
$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$
$$\vec{r} = t\vec{i} + t^2\vec{j} + t^3\vec{k}$$
$$\frac{d\vec{r}}{dt} = \vec{i} + 2t\vec{j} + 3t^2\vec{k}$$
$$d\vec{r} = (\vec{i} + 2t\vec{j} + 3t^2\vec{k})\,dt$$

$$\vec F = (3x^2+6y)\vec i - 14yz\vec j + 20xz^2\vec k$$

$$\vec F = (3t^2+6t^2)\vec i - 14t^5\vec j + 20t^7\vec k$$

$$= 9t^2\vec i - 14t^5\vec j + 20t^7\vec k$$

$$\vec F\cdot d\vec r = (9t^2\vec i - 14t^5\vec j + 20t^7\vec k)(\vec i + 2t\vec j + 3t^2\vec k)dt$$

$$= [9t^2 - 28t^6 + 60t^9]dt$$

$$\int_C \vec F\cdot d\vec r = \int_0^1 (9t^2 - 28t^6 + 60t^9)dt$$

$$= \left[\frac{9t^3}{3} - \frac{28t^7}{7} + \frac{60t^{10}}{10}\right]_0^1$$

$$= \left[\frac{9}{3} - \frac{28}{7} + \frac{60}{10}\right] - 0$$

$$= [3 - 4 + 6] - 0$$

$$= 5$$

6.

$$\vec r = x\vec i + y\vec j + z\vec k$$

$$\vec r = t\vec i + t^2\vec j + t^3\vec k$$

$$\frac{d\vec r}{dt} = \vec i + 2t\vec j + 3t^2\vec k$$

$$d\vec r = (\vec i + 2t\vec j + 3t^2\vec k)dt$$

$$\vec F = yz\vec i + zx\vec j - xy\vec k$$

$$\vec F = t^5\vec i + t^4\vec j - t^3\vec k$$

$$\vec F\cdot d\vec r = (t^5\vec i + t^4\vec j - t^3\vec k)\cdot(\vec i + 2t\vec j + 3t^2\vec k)dt$$

$$= (t^5 + 2t^5 - 3t^5)dt$$

$$= 0\,dt$$

$$\vec{F}\cdot d\vec{r} = 0$$

$$\therefore \int_C \vec{F}\, d\vec{r} = 0.$$

8/1/2020

7. Evaluate $\int_C F.dr$ where $\vec{F} = c[-3a\sin^2\theta\cos\theta\, \vec{i} + a(2\sin\theta - 3\sin^3\theta)\vec{j} + b\sin 2\theta\, \vec{k}]$ and is given by $\vec{r} = a\cos\theta\, \vec{i} + a\sin\theta\, \vec{j} + b\theta\, \vec{k}$ for $\theta = \pi/4$ to

$$\int_C F\cdot d\vec{r}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{r} = a\cos\theta\, \vec{i} + a\sin\theta\, \vec{j} + b\theta\, \vec{k}$$

$$\frac{d\vec{r}}{d\theta} = -a\sin\theta\, \vec{i} + a\cos\theta\, \vec{j} + b\vec{k}$$

$$d\vec{r} = [-a\sin\theta\, \vec{i} + a\cos\theta\, \vec{j} + b\vec{k}]\, d\theta$$

$$F\cdot d\vec{r} = c[3a^2\sin^3\theta\cos\theta + a^2[2\sin\theta\cos\theta - 3\sin^3\theta\cos\theta] + b^2\sin 2\theta]\, d\theta$$

$$= c[\cancel{3a^2\sin^3\theta\cos\theta} + 2a^2\sin\theta\cos\theta - 3\sin^3\theta\cos\theta + b^2\sin 2\theta]\, d\theta$$

$$= c[2a^2\sin\theta\cos\theta + b^2\sin 2\theta]\, d\theta$$

$$= c[a^2\sin 2\theta + b^2\sin 2\theta]\, d\theta$$

$$= c(a^2+b^2)\sin 2\theta\, d\theta$$

$$\int_C F\cdot d\vec{r} = \int_{\pi/4}^{\pi/2} c(a^2+b^2)\sin 2\theta\, d\theta$$

$$= c(a^2+b^2)\int_{\pi/4}^{\pi/2} \sin 2\theta \, d\theta$$

$$= c(a^2+b^2)\left[\frac{-\cos 2\theta}{2}\right]_{\pi/4}^{\pi/2}$$

$$= \frac{c(a^2+b^2)}{2}\left[-\cos 2(\pi/2) + \cos 2(\pi/4)\right]$$

$$= \frac{c(a^2+b^2)}{2}\left[-(-1) + 0\right]$$

$$\int_C F.d\vec{r} = \frac{c}{2}(a^2+b^2) \,//$$

8. Find the work done in moving the particle one around a circle c in xy plane. If the circle has centre at origin and radius 2. and if the force field is given by

$$\vec{F} = (2x - y + 2z)\vec{i} + (x + y - z^2)\vec{j} + (3x - 2y - 5z)\vec{k}$$

Sol:

xy plane means $z = 0$

$x = r\cos\theta$, $y = r\sin\theta$, $z = 0$

$x = 2\cos\theta$, $y = 2\sin\theta$ $z = 0$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{r} = 2\cos\theta\,\vec{i} + 2\sin\theta\,\vec{j}$$

$$\frac{d\vec{r}}{d\theta} = -2\sin\theta\,\vec{i} + 2\cos\theta\,\vec{j}$$

$$d\vec{r} = (-2\sin\theta\,\vec{i} + 2\cos\theta\,\vec{j})\,d\theta$$

$$\vec{F} = (2x - y + 2z)\vec{i} + (x + y - z^2)\vec{j} + (3\vec{i} - 2y - 5z)\vec{k}$$

$$= (4\cos\theta - 2\sin\theta)\vec{i} + (2\cos\theta + 2\sin\theta)\vec{j} + (6\cos\theta - 4\sin\theta)\vec{k}$$

$$\vec{F}\cdot d\vec{r} = -2\sin\theta(4\cos\theta - 2\sin\theta) + 2\cos\theta(2\cos\theta + 2\sin\theta)$$

$$= -8\sin\theta\cos\theta + 4\sin^2\theta + 4\cos^2\theta + 4\sin\theta\cos\theta$$

$$= -4\sin 2\theta + 4 + 2\sin 2\theta$$

$$= [4 - 2\sin 2\theta]\, d\theta$$

$$\int_C F\cdot dr = \int_0^{2\pi} (4 - 2\sin 2\theta)\, d\theta$$

$$= 4\int_0^{2\pi} d\theta - 2\int_0^{2\pi} \sin 2\theta\, d\theta$$

$$= 4[\theta]_0^{2\pi} - 2\left[-\frac{\cos 2\theta}{2}\right]_0^{2\pi}$$

$$= 4[2\pi - 0] - \frac{2}{2}[-\cos 2(2\pi) + \cos 2(0)]$$

$$= 8\pi - [-1+1] = 8\pi - 0$$

$$= 8\pi.$$

9. Evaluate $\int F\cdot d\vec{r}$ where $F = yz\vec{i} + 2x\vec{j} + xy\vec{k}$ and c is the portion of the curve $r = a\cos t\,\vec{i} + b\sin t\,\vec{j} + ct\,\vec{k}$ from $t = 0$ to $t = \frac{\pi}{2}$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\vec{r} = a\cos t\,\vec{i} + b\sin t\,\vec{j} + ct\,\vec{k}$$

$$\frac{d\vec{r}}{dt} = -a\sin t\,\vec{i} + b\cos t\,\vec{j} + c\,\vec{k}$$

$$d\vec{r} = (-a\sin t\,\vec{i} + b\cos t\,\vec{j} + c\vec{k})\, dt$$

$$\vec{F} = yz\vec{i} + zx\vec{j} + xy\vec{k}$$

$$= (b\sin t)(ct)\vec{i} + (ct)(a\cos t)\vec{i} + (a\cos t)(b\sin t)\vec{k}$$

$$\vec{F} = bct\sin t\,\vec{i} + act\cos t\,\vec{i} + ab\sin t\cos t\,\vec{k}$$

$$\vec{F}\cdot d\vec{r} = [(bct\sin t)(-a\sin t) + act\cos t\,(b\cos t) + c\,(ab\sin t\cos t)]\,dt$$

$$= [-abct\sin^2 t + abct\cos^2 t + abc\sin t\cos t]\,dt$$

$$= [abct[2\cos^2 t - 1] + abc\sin t\cos t]\,dt$$

$$= [abc[2t\cos^2 t - t + \sin t\cos t]]\,dt$$

$$\int_C \vec{F}\cdot d\vec{r} = \int abc\left[t\cos 2t + \frac{\sin 2t}{2}\right] dt$$

$$= \int_0^{\pi/2} abc\left[t\cos 2t + \frac{\sin 2t}{2}\right] dt$$

$$= abc\left[\int_0^{\pi/2} t\cos 2t\,dt + \int_0^{\pi/2} \frac{\sin 2t}{2}\,dt\right]$$

$$= abc\left[\int_0^{\pi/2} t\cos 2t\,dt + \frac{1}{2}\int_0^{\pi/2} \frac{\sin 2t}{2}\,dt\right.$$

$$\int u\,dv = uv - \int v\,du$$

$$u = t \quad u' = 1$$

$$\int dv = \int \cos 2t\,dt$$

$$v = \frac{\sin 2t}{2}$$

$$\int t\cos 2t\,dt = \frac{t\sin 2t}{2} - \int \frac{\sin 2t}{2}\,dt$$

$$\int_C \vec{F}\cdot d\vec{r} = abc\left[\left[\frac{t\sin 2t}{2}\right]_0^{\pi/2} - \int_0^{\pi/2} \frac{\sin 2t}{2}\,dt + \int_0^{\pi/2} \frac{\sin 2t}{2}\,dt\right]$$

$= \frac{a}{h} [0 \cdot 0]$

$= 0$

Hence proved.

20/01/2020

10. Evaluate $\int \vec{F} \cdot d\vec{r}$ where $\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$ and the curve c is the rectangle in the xy plane bounded by $y=0$, $x=a$, $y=b$, $x=0$.

Sol:

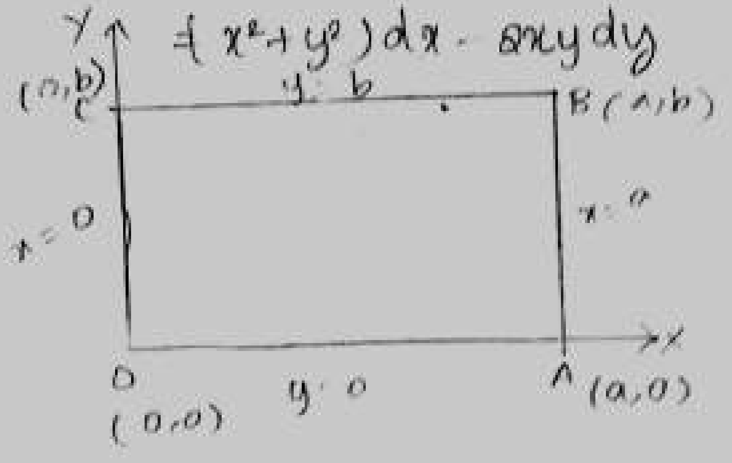
$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$

$\vec{F} = (x^2+y^2)\vec{i} - 2xy\vec{j}$

$F \cdot d\vec{r} = [(x^2+y^2)\vec{i} - 2xy\vec{j}] \cdot [dx\vec{i} + dy\vec{j} + dz\vec{k}]$

$= (x^2+y^2)dx - 2xy\,dy$



$$\int_C F \cdot dr = \int_{OA} F \cdot dr + \int_{AB} F \cdot dr + \int_{BC} F \cdot dr + \int_{CO} F \cdot dr \quad \rightarrow (1)$$

Along OA : $y=0$ $dy = 0$

$$\int_{OA} F \cdot dr = \int (x^2+y^2)\,dx - 2xy\,dy$$

$$= \int_0^a x^2 dy = \left[\frac{x^3}{3}\right]_0^a$$

$$= \frac{a^3}{3} \rightarrow (1)$$

Along AB: $x = a$, $dx = 0$

$$\int_{AB} F \cdot dr = \int_0^b 0 - 2(a)y \; dy$$

$$= -2a \int_0^b y \, dy$$

$$= -ab^2 \rightarrow (2)$$

Along BC: $y = b$, $dy = 0$

$$\int_{BC} F \cdot dr = \int_a^0 (x^2 + b^2) dx$$

$$= \left[\frac{x^3}{3} + b^2 x\right]_a^0$$

$$= 0 - \left[\frac{a^3}{3} + ab^2\right]$$

$$= \frac{-a^3}{3} - ab^2 \rightarrow (3)$$

Along CO: $x = 0$, $dx = 0$

$$\int_{CO} F \cdot dr = \int_b^0 0 - 2(0) y \, dy = 0$$

Sub (1), (2), (3), (4) in (*)

$$\int_C F \cdot dr = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 + 0$$

$$= -2ab^2.$$

H.W

11. Evaluate $\int_C F\cdot dr$.

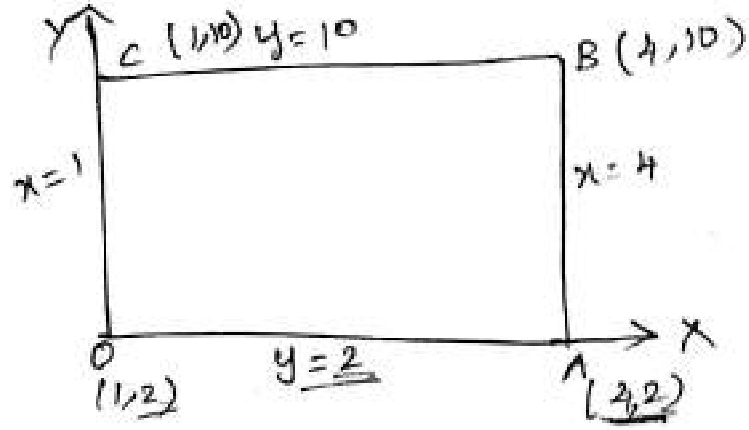
$\vec{F} = xy\vec{i} + (x^2+y^2)\vec{j}$ and c is the rectangle in the xy plane bounded by the curve $y=2$, $x=4$, $y=10$ and $x=1$.

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\vec{F} = xy\vec{i} + (x^2+y^2)\vec{j}$$

$$F\cdot d\vec{r} = xy\,dx + (x^2+y^2)\,dy.$$



$$\int_C \vec{F}\cdot dr = \int_{OA} \vec{F}\cdot d\vec{r} + \int_{AB} \vec{F}\cdot d\vec{r} + \int_{BC} \vec{F}\cdot d\vec{r} + \int_{CO} \vec{F}\cdot d\vec{r}.$$

Along OA, $y=2$ $dy=0$

$$\int_{OA} \vec{F}\cdot d\vec{r} = \int_2^4 2x\,dx = \left[\frac{2x^2}{2}\right]_2^4$$

$$= 16-1 = 15$$

Along AB, $x=4 \quad dx=0$

$$\int_{AB} F\cdot d\vec{r} = \int_{2}^{10} (16+y^2)dy$$

$$= \left[16y + \frac{y^3}{3}\right]_2^{10}$$

$$= \left[160 + \frac{1000}{3}\right] - \left[32 + \frac{8}{3}\right]$$

$$= \left[\frac{480+1000}{3} - \frac{96+8}{3}\right]$$

$$= \frac{1480}{3} - \frac{104}{3} = \frac{1376}{3}$$

$$\int_{AB} \vec{F} d\vec{r} = \frac{1376}{3}$$

Along BC, $y=10 \quad dy=0$

$$\int_{BC} F\cdot dr = \int xy\,dx + \int (x^2+y^2) dy$$

$$= \int_4^1 x(10)\,dx = 10\left[\frac{x^2}{2}\right]_4^1$$

$$= 10\left[\frac{1}{2} - \frac{16}{2}\right]$$

$$= -75 \rightarrow (3)$$

Along CO $\quad x=1 \quad dx=0$

$$\int_{CO} F\cdot d\vec{r} = \int xy\,dx + \int (x^2+y^2)dy$$

$$= \int_{10}^{2} (1^2+y^2) dy = y + \left[\frac{y^3}{3}\right]_{10}^{2}$$

$$= (2-10) + \left[\frac{2^3}{3} - \frac{10^3}{3}\right] = -8 + \left[\frac{8}{3} - \frac{1000}{3}\right]$$

$$= -8 - \frac{992}{3} = \frac{-1016}{3} \quad -(4)$$

Sub (1), (2), (3) & (4) in (*)

$$\int_C F\,dr = \int_{OA} F\cdot d\vec{r} + \int_{AB} F\cdot d\vec{r} + \int_{BC} F\cdot d\vec{r} + \int_{CO} F\cdot d\vec{r}$$

$$= 15 + \frac{1376}{3} - 75 - \frac{1016}{3}$$

$$= 120 - 60$$

$$\int F\cdot dr = 60.$$

12. If $\vec{F} = 3xy\vec{i} - y^2\vec{j}$. Evaluate $\int_c F \cdot dr$ where c is the curve in the xy plane $y = 2x^2$ from $(0,0)$ to $(1,2)$

Sol:

$$\vec{F} = 3xy\vec{i} - y^2\vec{j}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$F \cdot d\vec{r} = 3xy\,dx - y^2\,dy$$

$$\int_c F \cdot dr = \int 3xy\,dx - \int y^2\,dy$$

where $y = 2x^2$

$$= \int_0^1 3x(2x^2)\,dx - \int_0^2 y^2\,dy$$

$$= \int_0^1 6x^3\,dx - \int_0^2 y^2\,dy$$

$$= 6\left[\frac{x^4}{4}\right]_0^1 - \left[\frac{y^3}{3}\right]_0^2$$

$$= 6\left[\frac{1}{4}\right] - \frac{8}{3}$$

$$= \frac{3}{2} - \frac{8}{3}$$

$$\int F \cdot dr = -\frac{7}{6}$$

H.W:

13. Evalute $\int_c F \cdot dr$ where $F = xy\vec{i} + (x^2+y^2)\vec{j}$ where C is the arc of $y = x^2 - 4$ from $(2,0)$ to $(4,12)$ in the xy plane.

$$\vec{F} = xy\vec{i} + (x^2+y^2)\vec{j}$$
$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$
$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$
$$\vec{F}\cdot d\vec{r} = (xy)\,dx + (x^2+y^2)\,dy$$
$$\int_c F\cdot dr = \int xy\,dx + \int (x^2+y^2)\,dy$$
$$y = x^2-4 \qquad x^2 = y+4$$
$$= \int_2^4 (x^3-4x)\,dx + \int_0^{12} (y^2+y+4)\,dy$$
$$= \left[\frac{x^4}{4} - \frac{4x^2}{2}\right]_2^4 + \left[\frac{y^3}{3} + \frac{y^2}{2} + 4y\right]_0^{12}$$
$$= [64-32] - [4-8] + [576] + [72+48] - [0]$$
$$= 732$$

14) Evaluate $\int F\cdot dr$ where $F = y\vec{i} - x\vec{j}$ and c is the arc of the curve $y = x^2$ from $(0,0)$ to $(1,1)$.

$$\vec{F} = y\vec{i} - x\vec{j}$$
$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$
$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$y = x^2 \qquad x = \sqrt{y}$$

$$F \cdot d\vec{r} = y \cdot dx - x\,dy$$

$$= \int_0^1 x^2\,dx - \int_0^1 \sqrt{y}\,dy$$

$$= \left[\frac{x^3}{3}\right]_0^1 - \int_0^1 y^{1/2}\,dy$$

$$= \left[\frac{x^3}{3}\right]_0^1 - \left[\frac{y^{3/2}}{3/2}\right]_0^1$$

$$= \left[\frac{1}{3}\right] - \left[\frac{1}{3/2}\right] = \left[\frac{1}{3} - \frac{2}{3}\right]$$

$$F \cdot d\vec{r} = -\frac{1}{3}.$$

1. Find the workdone in the moving particle once around a circle $x^2+y^2=9$ $z=0$ and $\vec{F} = (2x-y-z)\vec{i} + (x+y-z)\vec{j} + (3x-2y-3z)\hat{k}$

$$x^2+y^2=9 \qquad \text{Here } r=3$$

The parameter.

$$x = r\cos t \qquad y = r\sin t$$

$$x = 3\cos t \qquad y = 3\sin t$$

$$r = x\vec{i} + y\vec{j}$$

$$r = 3\cos t\,\vec{i} + 3\sin t\,\vec{j}$$

$$\frac{dr}{dt} = -3\sin t\,\vec{i} + 3\cos t\,\vec{j}$$

$$\vec{F} = (2x-y-z)\vec{i} + (x+y-z)\vec{j} + (3x-2y-3z)\vec{k}$$

$$\vec{F} = (6\cos t - 3\sin t)\vec{i} + (3\cos t + 3\sin t)\vec{j} + (9\cos t - 6\sin t)\vec{k}$$

$$\vec{F}\cdot d\vec{r} = [(6\cos t - 3\sin t)\vec{i} + (3\cos t + 3\sin t)\vec{j} + (9\cos t - 6\sin t)\vec{k}]\cdot[-3\sin t\,\vec{i} + 3\cos t\,\vec{j}]dt$$

$$= [-3\sin t[6\cos t - 3\sin t] + [3\cos t + 3\sin t](3\cos t)]dt$$

$$= [-18\sin t\cos t + 9\sin^2 t + 9\cos^2 t + 9\sin t\cos t]dt$$

$$= [-9\sin t\cos t + 9(\sin^2 t + \cos^2 t)]dt$$

$$F\cdot dr = [-9\sin t\cos t + 9]dt$$

$$= 9[1 - \sin t\cos t]\,dt$$

$$= 9\left[1 - \frac{\sin 2t}{2}\right]dt \qquad \because [\cos 2\pi = 1]$$

$$\int F\cdot dr = 9\int_0^{2\pi}\left[1 - \frac{\sin 2t}{2}\right]dt$$

$$= 9\left[t + \frac{\cos 2t}{4}\right]_0^{2\pi}$$

$$= 9\left[2\pi + \frac{\cos 2(2\pi)}{4} - 0 - \frac{\cos 2(0)}{4}\right]$$

$$= 9\left[2\pi + \tfrac{1}{4} - 0 - \tfrac{1}{4}\right]$$

$$\int_C F\cdot dr = 18\pi.$$

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conservative Field:

vector function $\vec{F}$ is called conservative field. If $\int_{P_1}^{P_2} \vec{F}.d\vec{r}$ is independent if the path joining P_1 and P_2 and in this case $\vec{F} = \nabla\phi$.

Normal Surface Integral:

$$\int_C F.\hat{n}\,ds = \int_{S_1}\int F.\hat{n}\,\frac{dy\,dz}{\hat{n}.\vec{i}}$$

$$\int_S F.\hat{n}\,ds = \int_{S_2}\int F.\hat{n}.\,\frac{dz\,dx}{\hat{n}.\vec{j}}$$

$$\int_S F.\hat{n}\,ds = \int_{S_3}\int F.\hat{n}\,\frac{dx\,dy}{\hat{n}.\vec{k}}.$$

1. Evaluate $\int_S F.\hat{n}\,ds$ where $\vec{F} = yz\vec{i}+zx\vec{j}+xy\vec{k}$ and S is the part of the surface of the sphere $x^2+y^2+z^2=1$ which lies in the first octant xy plane.

soln:

In the xy plane

$$\int_S F.\hat{n}\,ds = \int_S\int F.\hat{n}\,.\frac{dx\,dy}{\hat{n}.\vec{k}}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2)$$

$$= \vec{i}(2x) + \vec{j}(2y) + \vec{k}(2z)$$

$$|\nabla\phi| = \sqrt{(2x)^2+(2y)^2+(2z)^2}$$

$$= \sqrt{4x^2+4y^2+4z^2}$$

$$= 2\sqrt{x^2+y^2+z^2}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2x\vec{i}+2y\vec{j}+2z\vec{k}}{2\sqrt{x^2+y^2+z^2}}$$

$$\hat{n} = \frac{x\vec{i}+y\vec{j}+z\vec{k}}{\sqrt{x^2+y^2+z^2}}$$

$$= \frac{x\vec{i}+y\vec{j}+z\vec{k}}{\sqrt{1}}$$

$$\hat{n} = x\vec{i}+y\vec{j}+z\vec{k}$$

$$\vec{F}\cdot\hat{n} = (yz\vec{i}+zx\vec{j}+xy\vec{k})(x\vec{i}+y\vec{j}+z\vec{k})$$

$$= xyz+xyz+xyz$$

$$= 3xyz$$

$$\hat{n}\cdot\vec{k} = (x\vec{i}+y\vec{j}+z\vec{k})\vec{k}$$

$$= z$$

$$x^2+y^2+z^2=1$$

x varies from $\Rightarrow$ $x=0$ to 1

y varies from $\Rightarrow$ $y^2=1-x^2$

$y=0,\ y=\sqrt{1-x^2}$

$$\int_S F\cdot\hat{n}\, ds = \int_0^1\int_0^{\sqrt{1-x^2}} 3xy\cancel{z}\ \frac{dx\,dy}{\cancel{z}}$$

$$= 3\int_0^1\int_0^{\sqrt{1-x^2}} 3xy\ dx\,dy$$

$$= 3\int_0^1 x\left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx$$

$$= 3\int_0^1 x\frac{(\sqrt{1-x^2})^2}{2} dx$$

$$= \frac{3}{2}\int_0^1 x(1-x^2)\,dx$$

$$= \frac{3}{2}\int_0^1 x - x^3\, dx$$

$$= \frac{3}{2}\left[\frac{x^2}{2} - \frac{x^4}{4}\right]_0^1$$

$$= \frac{3}{2}\left[\frac{1}{2} - \frac{1}{4}\right] = \frac{3}{2}\left[\frac{1}{4}\right]$$

$$\int_S F\hat{n}\,ds = 3/8$$

2. Evaluate $\int_S F\cdot\hat{n}\,ds$ where $\vec{F} = 2\vec{i} + x\vec{j} - 3y$ where S is the surface cylinder $x^2+y^2=16$ included in the first octan b/w $z=0$ to $z=5$.

Sol:

In the yz plane

$$\int_S F\cdot\hat{n}\,ds = \iint_S F\cdot\hat{n}\,\frac{dy\,dz}{|\hat{n}\cdot\vec{i}|}$$

$$\phi = x^2+y^2-16$$

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2+y^2-16)$$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j}$$

$$|\nabla\phi| = \sqrt{(2x)^2+(2y)^2}$$

$$= \sqrt{4x^2+4y^2}$$

$= 2\sqrt{x^2+y^2} = 2\sqrt{16}$

$= 2\times 4$

$|\nabla\phi| = 8$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i}+2y\vec{j}}{8}$$

$$= \frac{x\vec{i}+y\vec{j}}{4}$$

$$F\cdot\hat{n} = (z\vec{i}+x\vec{j}-3y^2z\vec{k})\left(\frac{x\vec{i}}{4}+\frac{y\vec{j}}{4}\right)$$

$$F\cdot\hat{n} = \frac{yz}{4}+\frac{xy}{4}$$

$$= \left(\frac{x\vec{i}}{4}+\frac{y\vec{j}}{4}\right)\vec{i}$$

$$\hat{n}\cdot\vec{i} = x/4$$

y varies from $\Rightarrow y^2 = 16-x^2$

$= \sqrt{16-x^2} = 4-x$

y varies from 0 to 4

z varies from 0 to 5

$$\int_S F\cdot\hat{n}\,ds = \iint_S F\cdot\hat{n}\frac{dy\,dz}{\hat{n}\cdot\vec{i}}$$

$$= \int_0^4\int_0^5\left(\frac{yz}{4}+\frac{xy}{4}\right)\frac{dy\,dz}{x/4}$$

$$= \int_0^4\int_0^5 \cancel{x/4}\,(z+y)\frac{dy\,dz}{\cancel{x/4}}$$

$$= \int_0^4\int_0^5 (z+y)\,dy\,dz$$

$$= \int_0^4\left[\frac{z^2}{2}+yz\right]_0^5 dy$$

$$= \int_0^4 \left(\frac{25}{2} + 5y\right) dy$$

$$= \left[\frac{25}{2} y + \frac{5y^2}{2}\right]_0^4$$

$$\left[\frac{25}{2}(4) + \frac{5(4)^2}{2}\right]$$

$$= \left(\frac{100}{2} + \frac{80}{2}\right) = \frac{180}{2} = 90$$

8. Evaluate $\int_S \vec{F}\cdot\hat{n}\, ds$ where $\vec{F} = z\vec{i} + x\vec{j} + y\vec{k}$ and S is the surface cylinder $x^2+y^2=1$ included in the first octant b/w $z=0$ to $z=2$.

Sol: In the yz plane

$$\int_S \vec{F}\cdot\hat{n}\, ds = \iint_S \vec{F}\cdot\hat{n} \frac{dy\,dz}{\hat{n}\cdot\vec{i}}$$

$$\phi = x^2+y^2-1$$

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2+y^2-1)$$

$$= 2x\vec{i} + 2y\vec{j}$$

$$|\nabla\phi| = \sqrt{(2x)^2+(2y)^2} = \sqrt{4x^2+4y^2}$$

$$= 2\sqrt{x^2+y^2} = 2\sqrt{1}$$

$$|\nabla\phi| = 2$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2x\vec{i} + 2y\vec{j}}{2}$$

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$\hat{n} = x\vec{i} + y\vec{j}$

$\vec{F} \cdot \hat{n} = (z\vec{i} + x\vec{j} + y\vec{k}) \cdot (x\vec{i} + y\vec{j})$

$= xz + xy$

$\hat{n} \cdot \vec{i} = (x\vec{i} + y\vec{j}) \cdot \vec{i}$

$= x$

y varies from $y^2 = 1$ ~~$-x^2$~~ $= \sqrt{\cancel{1-x^2}}$

$y = 1$ ~~$-x$~~

y varies from 0 to 1

z varies from 0 to 2

$$\int_S F \cdot \hat{n}\, ds = \iint_S F \cdot \hat{n} \frac{dy\, dz}{\hat{n} \cdot \vec{i}}$$

$$= \int_0^1 \int_0^2 xz + xy \frac{dy\, dz}{x}$$

$$= \int_0^1 \int_0^2 x(z+y) \frac{dy\, dz}{x}$$

$$= \int_0^1 \int_0^2 (z+y)\, dy\, dz$$

$$= \int_0^1 \left[\frac{z^2}{2} + yz\right]_0^2 dy$$

$$= \int_0^1 [2 + 2y]\, dy = \left[2y + \frac{2y^2}{2}\right]_0^1$$

$$= \left[2 + \frac{2}{2}\right] - 0 = 2 + 2/2 = 2 + 1$$

$$\int_S F \cdot \hat{n}\, ds = 3$$

4. Evaluate $\int_S F \cdot \hat{n}\, ds$ over the surface of the cylinder $x^2 + y^2 = 9$ is the first octant b/w $z = 0$ to 4. Where $F = z\vec{i} + x\vec{j} - yz\vec{k}$.

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Sol: In the yz plane,

$$\int_S F.\hat{n}\, ds = \int_S \int F.\hat{n} \frac{dy\, dz}{\hat{n}\, \vec{i}}$$

$$\phi = x^2+y^2-9$$

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2+y^2-9)$$

$$= 2x\vec{i} + 2y\vec{j}$$

$$|\nabla\phi| = \sqrt{(2x)^2+(2y)^2}$$

$$= \sqrt{4x^2+4y^2} = 2\sqrt{x^2+y^2}$$

$$= 2\sqrt{9} = 2\times 3$$

$$|\nabla\phi| = 6.$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2x\vec{i}+2y\vec{j}}{6}$$

$$\hat{n} = \frac{x\vec{i}+y\vec{j}}{3}$$

$$F.\hat{n} = (z\vec{i}+x\vec{j}-yz\vec{k})\left(\frac{x\vec{i}}{3}+\frac{y\vec{j}}{3}\right)$$

$$= \frac{xz}{3} + \frac{xy}{3}$$

$$\hat{n}.\vec{i} = \left(\frac{x\vec{i}}{3}+\frac{y\vec{j}}{3}\right).\vec{i}$$

$$= x/3$$

y varies from $y^2 = \sqrt{9-x^2}$

$y = 3$

y varies from 0 to 3

z varies from 0 to 4

$$\int_S F\hat{n}\, ds = \int\int_S F\hat{n}.\frac{dy\, dz}{\hat{n}.\vec{i}}$$

$$= \int_0 \int_0 \frac{\frac{xz}{3} + \frac{xy}{3}}{x/3} \, dydz$$

$$= \int_0^3 \int_0^4 \frac{\frac{x}{3}(z+y)}{x/3} \, dydz$$

$$= \int_0^3 \left[\frac{z^2}{2} + yz\right]_0^4 dy$$

$$= \int_0^3 (8+4y)\,dy$$

$$= \left[8y + \frac{4y^2}{2}\right]_0^3$$

$$= 8(3) + 2(3)^2 = 24 + 2(9)$$

$$\int_S F \cdot \vec{n} \cdot ds = 42$$

5. Evaluate $\int_S F \cdot \hat{n}\, ds$ where $\vec{F} = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ taken over the region bounded by $x^2+y^2=4$ $z=0$ to 3.

Sol:

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$\phi = x^2+y^2-4$$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j}$$

$$|\nabla\phi| = \sqrt{(2x)^2+(2y)^2} = \sqrt{4x^2+4y^2}$$

$$= 2\sqrt{4} = 2\times 2$$

$$|\nabla\phi| = 4$$

$$\hat{n} = \frac{2x\vec{i} + 2y\vec{j}}{4} = \frac{x\vec{i}}{2} + \frac{y\vec{j}}{2}$$

$$\vec{F}\cdot\hat{n} = (4x\vec{i} - 2y^2\vec{j} + z^2\vec{k})\left(\frac{x\vec{i}}{2} + \frac{y\vec{j}}{2}\right)$$

$$= \frac{4x^2}{2} - \frac{2y^3}{2} = 2x^2 - y^3$$

$$\hat{n}\cdot\vec{i} = \left(\frac{x_1}{2} + \frac{y_1}{2}\right)\vec{i}$$
$$= x/2$$

$x = 2\cos\theta \qquad y = 2\sin\theta$

$\qquad dy = 2\cos\theta\, d\theta$

y varies from 0 to 2π.

z varies from 0 to 2

$$\int_S F\cdot\hat{n}\, ds = \iint F\cdot\hat{n}\,\frac{dy\,dz}{\hat{n}\cdot\vec{i}}$$
$$= \int_0^{2\pi}\int_0^{2}(2x^2 - y^3)\,\frac{dy\,dz}{x/2}$$
$$= \int_0^{2\pi}\int_0^{2} 2(4\cos^2\theta) - (2\sin\theta)^3\,\frac{2\cos\theta}{\frac{2\cos\theta}{2}}$$
$$= 2\int_0^{2\pi}\int_0^{3}(8\cos^2\theta - 8\sin^3\theta)\,\frac{\cos\theta\, d\theta}{\cos\theta}\,dz$$
$$= 2\int_0^{2\pi} 8\left(\cos^2\theta\, z - \sin^3\theta\, z\right)_0^3 d\theta$$
$$= 16\int_0^{2\pi} 3(\cos^2\theta - \sin^3\theta)\, d\theta$$
$$= 48\int_0^{2\pi}\cos^2\theta\, d\theta - \int_0^{2\pi}\sin^3\theta\, d\theta$$
$$= 48\times 2\int_0^{\pi}\cos^2\theta\, d\theta - 0$$
$$= 96\int_0^{\pi}\frac{1+\cos 2\theta}{2}\, d\theta$$
$$= \frac{96}{2}\int_0^{\pi}(1+\cos 2\theta)\, d\theta = \frac{96}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi}$$
$$= 48\,[\pi + 0]$$
$$= 48\pi$$

Note 1:

$$F = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$dr = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$F \cdot dr = \int F_1 dx + \int F_2 dy + \int F_3 dz$$

Note 2:

$$\int_C F \cdot dr = \int F \times \frac{dr}{ds} \cdot ds$$

$$= \int_{S_1}^{S_2} F \times t \, ds$$

where t is unit tangent vector

$$F \times dr = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ F_1 & F_2 & F_3 \\ dx & dy & dz \end{vmatrix}$$

$$= \vec{i}[F_2 dz - F_3 dy] - \vec{j}[F_1 dz - F_3 dz] + \vec{k}[F_1 dy - F_2 dx]$$

$$= \vec{i}\int (F_2 dz - F_3 dy) - \vec{j}\int (F_1 dz - F_3 \cancel{dx}) + \vec{k}\int (F_1 dy - F_2 dx$$

$$\int_C F \cdot \frac{dr}{ds} ds = \int_C F \cdot dr$$

$$= \int_C (F_1\vec{i} + F_2\vec{j} + F_3\vec{k})(dx\vec{i} + dy\vec{j} + dz\vec{k})$$

$$= \int_C F_1 dx + F_2 dy + F_3 dz$$

$$= \int_{t_1}^{t_2} \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt}\right)$$

then $\vec{F}$ is a conservative field $\vec{F} = \nabla\phi$, where ϕ is a scalar potential of vector function $\vec{F}$.

Necessary Part:

Assume that $\vec{F}$ is conservative field we have, $\vec{F} = \nabla\phi$

$$\text{curl } \vec{F} = \nabla \times \vec{F}$$
$$= \nabla \times \nabla\phi$$
$$= 0$$

$\therefore \vec{F}$ is irrotational

Sufficient Part:

Conversely assume that curl $\vec{F} = 0$

(ie)
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \vec{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \vec{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \vec{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$
$$= 0$$

Hence, the coefficient of $\vec{i}, \vec{j}, \vec{k}$ are each zero separately

$$\frac{\partial F_3}{\partial y} = \frac{\partial F_2}{\partial z} \; ; \; \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z} \; ; \; \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$$

Let c be the path joining and two point $A(x,y,z)$ $P(x,y,z)$

Now,

$\int_C F\cdot dr$ is independent of the path joining A to P.

We choose it as consisting of 3 position AB, BC, CD respectively parallel to 3, co-ordinate axes.

Then the work done is

$$\phi(x,y,z) = \int_A^P \vec{F}\cdot dr$$

$$\int_A^P \vec{F}\cdot dr + \int_A^B \vec{F}\cdot dr + \int_B^C \vec{F}\cdot dr + \int_C^D \vec{F}\cdot dr$$

Now,

$$\vec{F}\cdot dr = f_1 dx + f_2 dy + f_3 dz$$

For 1st integral only x varies and for second intgral y varies & for 3rd integral only c varies

$$\phi(x,y,z) = \int_{x_1}^{x} f_1(x_1,y_1,z_1)\,dx + \int_{y_1}^{y} f_2(x_1,y_1,z_1)\,dy + \int_{z_1}^{z} f_3(x_1,y_1,z)\,dz$$

$$\therefore \frac{\partial \phi}{\partial z} = f_3(x,y,z)$$

Again, $\frac{\partial \phi}{\partial z} = f_2(x_1, y_1, z_1) + \int_{z_1}^{z} \frac{\partial}{\partial y} f_2(x_1, y_1, z_1) \partial z$

$$= f_2(x, y, z) + f_2(x_1, y_1 z_1)_{z_1}^{z}$$

$$= f_2(x, y, z)$$

Again, $\frac{\partial \phi}{\partial x} = f_1(x, y, z)$

$$= \int_{y_1}^{y} \frac{\partial}{\partial x} f_2(x, y, z) dy + \int_{z_1}^{z} \frac{\partial}{\partial x} f_3(x, y, z) dx$$

$\frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$ and $\frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}$ From (A)

$$\frac{\partial \phi}{\partial x} = f_1(x_1, y_1, z_1) + \int_{y_1}^{y} \frac{\partial f_1}{\partial y}(x, y, z) dy + \int_{z_1}^{z} \frac{\partial f_1}{\partial z}(x, y, z$$

$$= F_1(x, y, z) + [f_1(x, y, z)]_{y_1}^{y} + [f_1(x, y, z)]_{z_1}^{z}$$

$$= F_1(x, y, z) + f_1(x, y, z) - f_1(x, y, z) + f_1(x, y, z) - f_1(x, y, z)$$

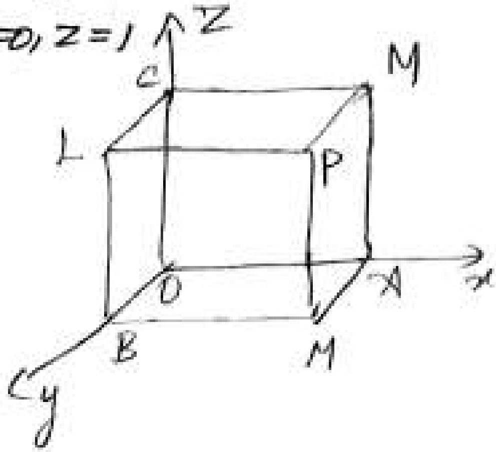
$$= f_1(x, y, z)$$

$\therefore \vec{F} = F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}$

$$= \frac{\partial \phi}{\partial x} \vec{i} + \frac{\partial \phi}{\partial y} \vec{j} + \frac{\partial \phi}{\partial z} \vec{k}$$

$\therefore \vec{F} = \nabla \phi$

1. Evaluate $\int_S F\cdot\hat{n}\,ds$ where $\vec{F} = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ where S is the surface of the cube bounded by $x=0,\ x=1,\ y=0,\ y=1,\ z=0,\ z=1$



Soln:

Face ANPM

$\hat{n} = \vec{i}\ ,\ x=1$

$F\cdot\hat{n} = 4xz$

$= 4z$

$\hat{n}\cdot\vec{i} = 1$

$$\int_{S_1} F\cdot\hat{n}\,ds = \iint_S F\cdot\hat{n}\,\frac{dy\,dz}{\hat{n}\cdot\vec{i}}$$

$$= \int_0^1\int_0^1 4z\,\frac{dy\,dz}{1}$$

$$= 4\int_0^1 \left[\frac{z^2}{2}\right]_0^1 dy$$

$$= 2\int_0^1 dy$$

$$= 2[y]_0^1$$

$$\int_{S_1} F\cdot\hat{n}\,ds = 2$$

OBLC Face,

$\hat{n} = -\vec{i}\ ,\ x=0$

$F\cdot\hat{n} = -4xz$

$F\cdot\hat{n} = 0$

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$$\int_{S_2} F\cdot\hat{n}\,ds = 0$$

Face, ORMA,

$$\hat{n} = \vec{j}, \quad y = 1$$

$$F\cdot\hat{n} = -y^2$$

$$= -1$$

$$\hat{n}\cdot\vec{j} = 1$$

$$\int_{S_3} F\cdot\hat{n}\,ds = \iint F\cdot\hat{n}\,\frac{dx\,dz}{\hat{n}\cdot\vec{j}}$$

$$= \int_0^1\int_0^1 (-1)\,dx\,dz$$

$$= -1\int_0^1 [z]_0^1\,dx$$

$$= -1\int_0^1 (1)\,dx$$

$$= -1\,[x]_0^1$$

$$= -1(1)$$

$$= -1$$

Face OCAN,

$$\hat{n} = \vec{k}, \quad z = 1$$

$$F\cdot\hat{n} = yz$$

$$= y$$

$$\hat{n}\cdot\vec{k} = 1$$

$$\int_{S_5} F\cdot\hat{n}\,ds = \iint F\cdot\hat{n}\,\frac{dy\,dx}{\vec{k}\cdot\hat{n}}$$

$$= \int_0^1\int_0^1 y\,dy\,dx$$

Face CLBN,

$$\hat{n} = -\vec{j}, \quad y = 0$$

$$F\cdot\hat{n} = y^2$$

$$= 0$$

$$\int_{S_4} F\cdot\hat{n}\,ds = 0$$

$$= \int_0^1 \left[\frac{y^2}{2}\right]_0^1 dx$$

$$= \int_0^1 \frac{1}{2} dx$$

$$= \frac{1}{2}(x)_0^1$$

$$\int_{S5} F\cdot\hat{n}\,ds = \frac{1}{2}$$

Face BMPL,

$\hat{n} = -\vec{k}$, $z=0$

$F\cdot\hat{n} = -yz$

$= 0$

$$\int_{S_6} F\cdot\hat{n}\,ds = 0$$

$$\int_S F\cdot\hat{n}\,ds = 2+0+0-1+\frac{1}{2}+0$$

$$= 2-1+\frac{1}{2}$$

$$= 1+\frac{1}{2}$$

$$\int_S F\cdot\hat{n}\,ds = \frac{3}{2}$$

2. Evaluate $\int_S F\cdot\hat{n}\,ds$ where $F = 2xy\vec{i} - zy\vec{j} - x^2\vec{k}$ where S is the surface of the cube bounded by the co-ordination plane and the plane $x=a$, $y=a$, $z=a$

soln:

Face ANPM,

$$\hat{n} = \vec{i} \quad , x = a$$

$$F \cdot \hat{n} = (2xy\vec{i} - zy\vec{j} + x^2\vec{k}) \cdot \vec{i}$$

$$= 2xy$$

$$= 2ay$$

$$\hat{n} \cdot \vec{i} = 1$$

$$\int_{S_1} F \cdot \hat{n}\, ds = \iint F \cdot \hat{n} \frac{dy\,dz}{\hat{n} \cdot \vec{i}}$$

$$= \int_0^a \int_0^a 2ay \; dy\,dz$$

$$= 2a \int_0^a [yz]_0^a dy$$

$$= 2a \int_0^a ay\, dy$$

$$= 2a^2 \int_0^a y\, dy$$

$$= 2a^2 \left[\frac{y^2}{2}\right]_0^a$$

$$= a^2 (a^2)$$

$$\int_{S_1} F \cdot \hat{n}\, ds = a^4$$

Face OBLC,

$$\int_{S_2} F \cdot \hat{n}\, ds = \iint F \cdot \hat{n} \cdot \frac{dy\,dz}{\hat{n} \cdot \vec{i}}$$

$$x = 0 , \quad \hat{n} = -\vec{i}$$

$$F \cdot \hat{n} = -2xy$$

$$= 0$$

$$\int_{S_2} F \cdot \hat{n}\, ds = 0$$

Face OBMA,

$$y=0, \quad \hat{n}=\vec{j}$$

$$F\cdot\hat{n} = -zy$$

$$= -az$$

$$\hat{n}\cdot\vec{j} = 1$$

$$\int_{S_3} F\cdot\hat{n}\, ds = \int_0^a\int_0^a F\cdot\hat{n}\,\frac{dx\,dz}{\hat{n}\cdot\vec{j}}$$

$$= \int_0^a\int_0^a -az\, dx\,dz$$

$$\cancel{= -a\int_0^a \left[\frac{z^2}{2}\right] dx}$$

$$= -a\int_0^a \left[\frac{z^2}{2}\right]_0^a dx$$

$$= -a\int_0^a \left(\frac{a^2}{2}\right) dx$$

$$= \frac{-a^3}{2}\left[x\right]_0^a$$

$$= \frac{-a^3}{2}(a)$$

$$\int_{S_3} F\cdot\hat{n}\, ds = \frac{-a^4}{2}$$

Face CLPN,

$$y=0, \quad \hat{n}=-\vec{j}$$

$$F\cdot\hat{n} = zy$$

$$=0$$

$$\hat{n}\cdot\vec{j} = 1$$

$$\int_{S_4} F\cdot\hat{n}\,ds = 0$$

Face OANC,

$\hat{n} = \vec{k}$, $z = a$

$F\cdot\hat{n} = -x^2$

$= -x^2$

$\hat{n}\cdot\vec{k} = 1$

$$\int_{S_5} F\cdot\hat{n}\,ds = \int_0^a\int_0^a -x^2\,dx\,dy$$

$$= \int_0^a\int_0^a -x^2\,dx\,dy$$

$$= \int_0^a \left[\frac{-x^3}{3}\right]_0^a dy$$

$$= \frac{-a^3}{3}\int_0^a dy$$

$$= \frac{-a^3}{3}(y)_0^a$$

$$= \frac{-a^4}{3}$$

Face PMBL,

$\hat{n} = -\vec{k}$, $z = 0$

$F\cdot\hat{n} = x^2$

$\hat{n}\cdot\vec{k} = -\vec{k}\cdot\vec{k}$

$= -1$

$$\int_{S_6} F\cdot\hat{n}\,ds = \iint F\cdot\hat{n}\,\frac{dx\,dy}{\hat{n}\cdot\vec{k}}$$

$$= \int_0^a \int_0^a x^2 \frac{dx\,dy}{-1}$$

$$= -\int_0^a \left[\frac{x^3}{3}\right]_0^a dy = \frac{-a^3}{3}\int_0^a dy$$

$$= \frac{-a^3}{3}(y)_0^a$$

$$= \frac{-a^4}{3}$$

$$\int_{S_6} F.\hat{n}\,ds = \frac{-a^4}{3}$$

$$\int_S F.\hat{n}\,ds = a^4 + 0 - \frac{a^4}{2} + 0$$

$$= a^4 - \frac{a^4}{2}$$

$$= \frac{2a^4 - a^4}{2}$$

$$= \frac{a^4}{2}$$

3. Evaluate $\int_S \frac{r}{r^3}\,da$ where S denotes the sphere of radius 'a' with centre at the origin

Soln:

Let the equation of sphere be,

$$x^2+y^2+z^2 = a^2$$

$$\phi = x^2+y^2+z^2-a^2$$

$$\nabla\phi = \vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}$$

$$\nabla\phi = \vec{i}\,2x + \vec{j}\,2y + \vec{k}\,2z$$

$$= 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$|\nabla\phi| = \sqrt{(2x)^2+(2y)^2+(2z)^2}$$

$$= \sqrt{4x^2+4y^2+4z^2}$$

$$= 2\sqrt{x^2+y^2+z^2}$$

$$= 2a$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= \frac{2x\vec{i}+2y\vec{j}+2z\vec{k}}{2a}$$

$$\hat{n} = \frac{x\vec{i}+y\vec{j}+z\vec{k}}{a}$$

$$F = \frac{r}{r^3} = \frac{x\vec{i}+y\vec{j}+z\vec{k}}{(x^2+y^2+z^2)^{3/2}}$$

$$= \frac{x\vec{i}+y\vec{j}+z\vec{k}}{(a^2)^{3/2}}$$

$$= \frac{x\vec{i}+y\vec{j}+z\vec{k}}{a^3}$$

$$F\cdot\hat{n} = \frac{x^2+y^2+z^2}{a^4}$$

$$= \frac{a^2}{a^4}$$

$$= 1/a^2$$

$$\int_s \frac{r}{r^3}\,da = \int_s \frac{r}{r^3}\hat{n}\,ds$$

Sphere formula = $4\pi r^2$

$$= \int_s \frac{1}{a^2}\cdot ds$$

$$= 1/a^2 \int_s ds$$

$$= \frac{1}{a^2}\,4\pi a^2$$

$$\int_s \frac{r}{r^2}\,da = 4\pi$$

A. Evaluate $\int_S F \cdot \hat{n}\, ds$ where $\vec{F} = 18z\vec{i} - 12\vec{j} + 3y\vec{k}$ and S is the part of the plane $2x+3y+6z=12$ which is located in the first octant.

Soln:

$$\phi = 2x+3y+6z-12$$

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(2x+3y+6z-12)$$

$$= 2\vec{i} + 3\vec{j} + 6\vec{k}$$

$$|\nabla\phi| = \sqrt{(2)^2+(3)^2+(6)^2}$$

$$= \sqrt{4+9+36}$$

$$= \sqrt{49}$$

$$= 7$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i}+3\vec{j}+6\vec{k}}{7}$$

$$F \cdot \hat{n} = (18z\vec{i} - 12\vec{j} + 3y\vec{k}) \cdot \frac{(2\vec{i}+3\vec{j}+6\vec{k})}{7}$$

$$= \frac{36z}{7} - \frac{36}{7} + \frac{18y}{7}$$

$$= \frac{36z - 36 + 18y}{7}$$

$$\hat{n} \cdot \vec{i} = \frac{2\vec{i}+3\vec{j}+6\vec{k}}{7} \cdot \vec{i}$$

$$= 2/7$$

$$\int_S F \cdot \hat{n}\, ds = \iint F \cdot \hat{n} \frac{dy\,dz}{\hat{n} \cdot \vec{i}}$$

$$2x+3y+6z=12$$

y varies from $y=0$ to $3y=12$, $y=4$

z varies from $z=0$ to $6z=12-3y$

$$z=\frac{12-3y}{6}$$

$$=\frac{4-y}{2}$$

$$\int_S F\cdot\hat{n}\,ds=\int_0^4\int_0^{\frac{4-y}{2}}\frac{36z-36+18y}{7}\,\frac{dy\,dz}{2/7}$$

$$=\int_0^4\int_0^{\frac{4-y}{2}}\frac{36z-36+18y}{2}\,dy\,dz$$

$$=\int_0^4\int_0^{\frac{4-y}{2}}(18z-18+9y)\,dy\,dz$$

$$=\int_0^4\int_0^{\frac{4-y}{2}}9(2z-2+y)\,dy\,dz$$

$$=9\int_0^4\int_0^{\frac{4-y}{2}}(2z-2+y)\,dy\,dz$$

$$=9\int_0^4\int_0^{\frac{4-y}{2}}(2z-2+y)\,dy\,dz$$

$$=9\int_0^4\left[\frac{2z^2}{2}-2z+yz\right]_0^{\frac{4-y}{2}}dy$$

$$=9\int_0^4\left[\left(\frac{4-y}{2}\right)^2-2\left(\frac{4-y}{2}\right)+y\left(\frac{4-y}{2}\right)\right]dy$$

$$=9\int_0^4\left[\left(\frac{16+y^2-8y}{2}\right)-4+y+\left(\frac{4y-y^2}{2}\right)\right]dy$$

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$$= \frac{9}{4}\int_0^4 (16+y^2-8y-16+4y+8y-2y^2)\,dy$$

$$= \frac{9}{4}\int_0^4 (4y-y^2)\,dy$$

$$= \frac{9}{4}\left[\frac{4y^2}{2}-\frac{y^3}{3}\right]_0^4$$

$$= \frac{9}{4}\left[2y^2-\frac{y^3}{3}\right]_0^4$$

16
4
64

$$= \frac{9}{4}\left[2(16)-\frac{64}{3}\right]$$

$$= \frac{9}{4}\left[32-\frac{64}{3}\right]$$

$$= \frac{9}{4}\left[\frac{96-64}{3}\right]$$

$$= \frac{9}{4}\left[\frac{32}{3}\right]$$

$$= 3(8)$$

$$\int_S F\cdot\hat{n}\,ds = 24$$

5. Evaluate $\int_S F\cdot\hat{n}\,ds$, where $F = y\vec{i}+2x\vec{j}+z\vec{k}$ and s is the surface of the plane $2x+y=6$ in the first octant cut off by the plane $z=4$

Soln:-

Given, $\phi = 2x+y-6$

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x}+\vec{j}\frac{\partial}{\partial y}+\vec{k}\frac{\partial}{\partial z}\right)(2x+y-6)$$

Evaluate $\int_S F\cdot\hat{n}\,ds$ where $F = y\vec{i} + 2x\vec{j} + z\vec{k}$ and S is the surface of the plane $2x+y=6$ in the first octant cut off by the plane $z=4$

Sol:

given $\phi = 2x+y-6$

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(2x+y-6)$$

$$= 2\vec{i} + \vec{j}$$

$$|\nabla\phi| = \sqrt{(2)^2+(1)^2} = \sqrt{4+1} = \sqrt{5}$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2\vec{i}+\vec{j}}{\sqrt{5}}$$

$$\bar{F}\cdot\hat{n} = (y\vec{i}+2x\vec{j}+z\vec{k})\left(\frac{2\vec{i}+\vec{j}}{\sqrt{5}}\right) = \frac{2x+2y}{\sqrt{5}}$$

$$\int_S F\cdot\hat{n}\,ds = \iint F\cdot\hat{n}\,\frac{dx\,dy}{\hat{n}\cdot\vec{j}}$$

$$\hat{n}\cdot\vec{j} = \frac{2\vec{i}+\vec{j}}{\sqrt{5}}\cdot\vec{j} = 1/\sqrt{5}$$

z varies from $z=0$ to $z=4$

x varies from $x=0$ to $2x=6$
$x=3$

$$\int_S F\cdot\hat{n}\,ds = \int_0^4\int_0^3 2/\sqrt{5}\,(x+y)\,\frac{dx\,dy}{\sqrt{5}}$$

$$= 2\int_0^4\int_0^3 (x+y)\,dx\,dz$$

$$= 2\int_0^4\int_0^3 (6-x)\,dx\,dz$$

$$= 2\int_0^3 \left[(6-x)z\right]_0^4\,dx$$

$$= 2\int_0^3 4(6-x)dx = 8\int_0^3 (6-x)dx$$

$$= 8\left[6x - \frac{x^2}{2}\right]_0^3 dx$$

$$= 8\left[6(3) - (3)^2/2\right]$$

$$= 8\left[18 - 9/2\right] = 8\left[27/2\right] = 108$$

$$\int_S F\cdot\hat{n}\,ds = 108$$

6. If $f = y\vec{i} + (x-2xz)\vec{j} - xy\vec{k}$ Evaluate $\int(\nabla\times f)\hat{n}ds$ where S is the surface of the sphere $x^2+y^2+z^2 = a^2$ above the xy plane.

Sol:

given $\phi = x^2+y^2+z^2-a^2$

$$\nabla\phi = \left(\vec{i}\frac{\partial}{\partial x} + \vec{j}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}\right)(x^2+y^2+z^2-a^2)$$

$$\nabla\phi = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}$$

$$|\nabla\phi| = \sqrt{(2x)^2+(2y)^2+(2z)^2}$$

$$= 2\sqrt{x^2+y^2+z^2} = 2\sqrt{a^2} = 2a.$$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\vec{i} + 2y\vec{j} + 2z\vec{k}}{2a}$$

$$= \frac{x\vec{i} + y\vec{j} + z\vec{k}}{a}$$

$$F\cdot\hat{n} = (y\vec{i} + (x-2xz)\vec{j} - xy\vec{k})\frac{(x\vec{i} + y\vec{j} + z\vec{k})}{a}$$

$$= \frac{xy + xy - 2xyz - xyz}{a}$$

$$= \frac{2xy - 3xyz}{a}$$

$$\int_S F\cdot\hat{n}\,ds = \iint F\cdot\hat{n}\,\frac{dx\,dy}{\vec{n}\cdot\vec{k}}$$

$$\vec{n}\cdot\vec{k} = \frac{x\vec{i}+y\vec{j}+z\vec{k}}{a}\cdot\vec{k}$$

$$= z/a$$

$$= \frac{\sqrt{a^2-x^2-y^2}}{a}$$

~~F·n =~~

$$\nabla\times\vec{f} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x-2xz & -xy \end{vmatrix}$$

$$= \vec{i}[-x+2x] - \vec{j}[-y-0] + \vec{k}[1-2z-1]$$

$$= x\vec{i} + y\vec{j} - 2z\vec{k}$$

$$\vec{F}\cdot\hat{n} = (x\vec{i}+y\vec{j}-2z\vec{k})\left(\frac{x\vec{i}+y\vec{j}+z\vec{k}}{a}\right)$$

$$= \frac{x^2+y^2-2z^2}{a}$$

$$= \frac{x^2+y^2-2(a^2-x^2-y^2)}{a}$$

$$= \frac{x^2+y^2-2a^2+2x^2+2y^2}{a}$$

$$= \frac{3x^2+3y^2-2a^2}{a}$$

$$= \frac{3(x^2+y^2)-2a^2}{a}$$

$$\int_S F\cdot\hat{n}\,ds = \iint \frac{3(x^2+y^2)-2a^2}{\cancel{a}}\frac{dx\,dy}{\frac{\sqrt{a^2-x^2-y^2}}{\cancel{a}}}$$

$$= \iint \frac{3(x^2+y^2)-2a^2}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$$

$x = r\cos\theta \qquad y = r\sin\theta$

$x^2 = r^2\cos^2\theta \quad y^2 = r^2\sin^2\theta$

$x^2+y^2 = r^2(\sin^2\theta+\cos^2\theta) = r^2$

$dx\,dy = r\,dr\,d\theta$

θ varies from 0 to 2π

and r varies from 0 to a.

$$\int_S F\cdot\hat{n}\,ds = \int_0^{2\pi}\int_a^{0}\frac{3r^2-2a^2}{\sqrt{a^2-r^2}}\,r\,dr\,d\theta$$

$$= \int_a^{0}\frac{3r^2-2a^2}{\sqrt{a^2-r^2}}\,r\,dr\,[\theta]_0^{2\pi}$$

$$= 2\pi\int_a^{0}\frac{3r^2-2a^2}{\sqrt{a^2-r^2}}\,r\,dr$$

$a^2-r^2 = t^2$

$-2r\,dr = 2t\,dt$

$r\,dr = -t\,dt$

$$\int_S F\cdot\hat{n}\,ds = 2\pi\int_a^{0}\frac{3(a^2-t^2)-2a^2}{\sqrt{t^2}}(-t\,dt)$$

$$= 2\pi\int_a^{0}\frac{3a^2-3t^2-2a^2}{t}(-t\,dt)$$

$$= 2\pi \int_{a}^{0} \frac{-3t^2+a^2}{t}(-t\,dt)$$

$$= 2\pi \int_{a}^{0} (a^2-3t^2)(-dt)$$

$$= 2\pi \int_{0}^{a} (a^2-3t^2)\,dt$$

$$= 2\pi \left[a^3 - \frac{3t^3}{3} \right]_0^a$$

$$= 2\pi \left[a^3 - \frac{3a^3}{3} \right]$$

$$= 2\pi (0)$$

$$\int_S F.\hat{n}\,ds = 0$$

5|2|2020 **Volume Integral**

The limit of the sum $\int \sum f_p \Delta V_p$ when the number of volume elements tends to infinity and each element tends to zero defined as the volume integral and is written as $\int_V \vec{F}.dv$.

1. If $\vec{F} = (2x^2-3z)\vec{i} - 2xy\vec{j} - 4x\vec{k}$. Evaluate $\int$ and $\iiint \nabla \times \vec{F}\,dv$, where v is closed region bounded by the planes $x=0$ $y=0$, and $z=0$ & $2x+2y+z=4$.

$$= 2\int_0^2 4x(2-x) - x(2-x)^2 - 2x^2(2-x)\, dx$$

$$= 2\int_0^2 \left[8x - \cancel{4x^2} - x^3 - 4x + \cancel{4x^2} - 4x^2 + 2x^3\right] dx$$

$$= 2\int_0^2 \left[x^3 - 4x^2 + 4x\right] dx$$

$$= 2\left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{4x^2}{2}\right]_0^2$$

$$= 2\left[\frac{2^4}{4} - \frac{4(2)^3}{3} + 2(2)^2 - 0\right]$$

$$= 2\left[\cancel{64} - \cancel{256} + \cancel{32}\right]$$

$$= 2\left[4 - \frac{32}{3} + 8\right]$$

$$= 2\left[12 - \frac{32}{3}\right]$$

$$= 2\left[\frac{36 - 32}{3}\right]$$

$$= 2\left[\frac{4}{3}\right]$$

$$= 8/3$$

$$\int_V \nabla\cdot\vec{F}\, dv = 8/3$$

$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \end{vmatrix}$$

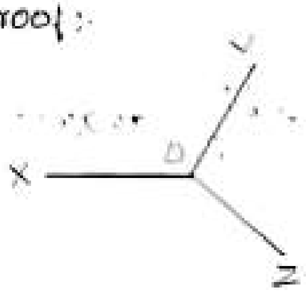
Gauss Divergent theorem:

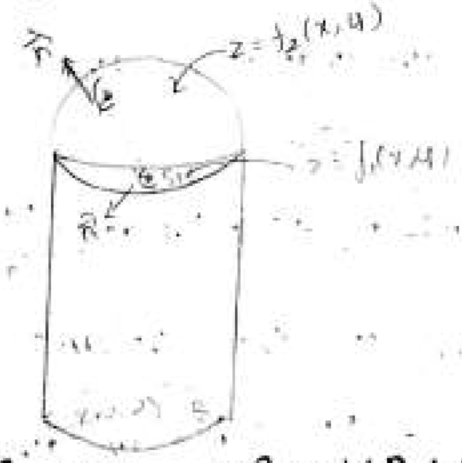
Reduction of Surface Integral to volume integral.

Statement:

The normal surface integral of a vector point function F which is continuously differentiable over the boundary of a closed the region is equal to the volume integral of div F taken throughout the region.

proof:





If F be a continuously differentiable vector points function and S is closed surface enclosing a region V. Then,

$$\int_S \vec{F} \cdot n\,ds = \int_V \operatorname{div} F \cdot dv \longrightarrow (1)$$

where n is the unit outward drawn normal vector. Let us suppose that the region V is such that it is possible for us to choose co-ordinate axes so that each line parallel to any co-ordinate axis which has integral points in common with the region cuts the boundary S in two points. Let S_1 and S_2 be the lower & upper boundaries.

Again let R denote the projection of the regin V on the xy plane. Any line through $(x, y, 0)$ a point in R meets the boundary S in two points. The lower boundary S_1 is given by $z = f_1(x, y)$ and the upper boundary S_2 is given by $z = f_2(x, y)$. where,

$$f_2(x, y) > f_1(x, y)$$

Now if $F = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$. Also,

$dv = dx\,dy\,dz$, Hence from ① we have to prove that,

$$\int_S (F_1\vec{i} + F_2\vec{j} + F_3\vec{k}) \cdot n\,ds = \iiint_v \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz \rightarrow ②$$

consider the volume integral,

$$= \iiint_v \left[\frac{\partial F_3}{\partial z} dx\right] dy\,dz = \iint_R \left[\frac{F_2\,\partial F_3}{f_1\,dz}\right] dx\,dy$$

$$= \iint_R \left[F_3(x, y, f_2) \cdot F_3(x, y, f_1)\right] dx\,dy \rightarrow ③$$

Now take n to be the unit outward drawn normal at any point to the surface S then,

$$ds = \frac{dx\,dy}{n \cdot k} \quad \text{and} \quad ds = \frac{-dx}{n} \cdot \frac{dy}{k}$$

The normal n makes an acute angle with +ve direction of z-axis for any point of S_2.

Where as it makes an obtuse angle with z-axis for any point on S_1.

$$\therefore \iint_R F_3(x, y, f_2)\, dx\, dy = \int_S F_3 (n \cdot k)\, ds$$

$$\iint_R F_3(x, y, f_1)\, dx\, dy = \int_{S_1} F_2 (n, k)\, ds$$

Subracting we get.

$$\iiint_V \frac{\partial F_3}{dz}\, dx\, dy\, dz = \int_{S_2} F_3 (n \cdot k)\, ds + \int_{S_1} F_3 (n \cdot k)\, ds$$

$$= \int_S F_3\, n \cdot k\, ds \rightarrow (4)$$

In a similar manner it can be shown that

$$\iiint_V \frac{\partial F_2}{\partial y}\, dx\, dy\, dz = \int_S F_2 (n \cdot j)\, ds \rightarrow (5)$$

$$\iiint_V \frac{\partial F_1}{\partial y}\, dx\, dy\, dz = \int_S F_3\, n \cdot i\, ds \rightarrow (6)$$

Adding: (4), (5) & (6) we get.

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\, dy\, dz$$

$$= \int_S (F_1 \vec{i} + F_2 \vec{j} + F_3 \vec{k}) \cdot \vec{n}\, ds$$

$$\Rightarrow \int_V \operatorname{div} F\, dv = \int_S F \cdot n\, ds$$

Hence the proof.

Green's Theorem:

If ϕ and ψ are scalar point functions together with their derivatives in any direction are uniform and continuous within the region V bo.

bounded by a closed surface 'S'. Then

$$\int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, dv = \int_S (\phi \nabla \psi - \psi \nabla \phi) n\, ds$$

proof:-

By Gauss's divergence theorem, we have

$$\int_S F \cdot \vec{n}\, ds = \int_V \text{div } F\, dv \longrightarrow (1)$$

choosing $F = \phi \nabla \psi$

$$\text{div } F = \nabla(\phi \nabla \psi) = \nabla \phi \cdot \nabla \psi + \phi \nabla^2 \psi$$

$$\int \phi \nabla \psi \cdot n ds = \int \nabla \phi \cdot \nabla \psi\, dv + \int \psi \nabla^2 \phi\, dv \longrightarrow (2)$$

Interchanging ϕ and ψ in (2), we get

$$\int \psi \nabla \phi\, n ds = \int_V \nabla \psi \nabla \phi\, dv + \int \psi \nabla^2 \phi\, dv \longrightarrow (3)$$

subtracting (2) & (3) we get,

$$\int_S (\phi \nabla \psi - \psi \nabla \phi)\, n ds = \int_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dv$$

Green's Theorem in the plane:

Statement:

If S is the plane surface in the xy plane bounded by a simple closed curve 'c' and 'F_1 & F_2' are continuous functions of x & y having continuous derivaties in the region 'S'.

Then,

$$\int (F_1 dx + F_2 dy) = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy.$$

Note:

M and N are sometimes written in place of F_1 and F_2 respectively.

i.e, $\int_C (M dx + N dy) = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial x}{\partial y}\right) dx\, dy.$

Green's theorem in vector notation:

$$\int \vec{F} \cdot d\vec{r} = \int_S \text{curl } \vec{F} \cdot \vec{n}\, ds$$

where $\vec{n} = \vec{k}$ for xy plane.

$ds = dx\,dy$ and curl $\vec{F} \cdot \vec{n} = \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}$

Proof of Green's Theorem in plane:

Assume that the lines drawn parallel to either axes meets the boundary curve in at the most two points.

Now let us consider the case when a closed curve c, be such that in which lines drawn parallel to axes may meet c in more than two points in the adjoining figure.

Draw a line AC dividing the whole region into two regions S_1 & S_2 which are now such that any line $\parallel^{el}$ to axis meets them in at the most two points and hence by Green's theorem we have

$$\int_{ABCA} (F_1 dx + F_2 dy) = \iint_{S_1} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\, dy \rightarrow (1)$$

$$\int_{ACDA} (F_1 dx + F_2 dy) = \iint_{S_2} \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy \rightarrow ②$$

$$\int_{ABCA} = \int_{ABC} + \int_{CA} \quad \int_{ACDA} = \int_{CDA} + \int_{AC}$$

Adding these we get,

$$\int_{ABCA} + \int_{ACDA} = \int_{ABC} + \int_{CDA} + \left[\int_{CA} + \int_{AC}\right]$$

$$= \int_{ABCDA} + 0$$

$$\int_{CA} = -\int_{AC}$$

$$= \int_{C}$$

Also adding R.H.S. of ① & ②
we get,

$$\iint_{S_1} + \iint_{S_2} = \iint_{S}$$

Hence $\int_C (F_1 dx + F_2 dy) = \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy$

Hence the proof//

(*) **Problems:-**

① Evaluate $\int F \cdot \hat{n}\, ds$ where $F = 4xz\vec{i} - y^2\vec{j} + yz\vec{k}$ and S is the surface of the cube bounded by $x=0$; $x=1$, $y=0$, $y=1$, $z=0$, $z=1$ and veri by divergent theorem.

proof:- given,

$$F = 4xz\,\vec{i} - y^2\vec{j} + yz\,\vec{k}$$

$$\vec{F} = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)$$

$$\operatorname{div} F = 4z - 2y + y$$

$$\operatorname{div} F = 4z - y$$

$$\int \operatorname{div} F\, dv = \int_0^1\int_0^1\int_0^1 (4z - y)\, dx\, dy\, dx$$

$$= \int_0^1\int_0^1 \left[\frac{4z^2}{2} - yz\right]_0^1 dy\, dx$$

$$= \int_0^1\int_0^1 (2 - y)\, dy\, dx$$

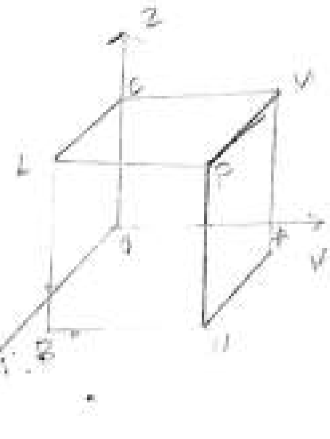
$$= \int_0^1 \left[2y - \frac{y^2}{2}\right]_0^1 dx$$

$$= \int_0^1 (2 - 1/2)\, dx$$

$$= \int_0^1 3/2\, dx$$

$$= 3/2\, [x]_0^1$$

$$\int_V \operatorname{div} F\, dv = 3/2 \longrightarrow (1)$$



To Find:

(i) Face ANPM

$$\hat{n} = \vec{i}$$

$$\vec{F}\cdot\hat{n} = (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\,\vec{i}$$

$$F\cdot\hat{n} = 4xz$$

$$\hat{n}\cdot\vec{i} = |\vec{i}\cdot\vec{i}| = 1$$

$$\int_S F\cdot\hat{n}\, ds = \int_0^1\int_0^1 4xz\, \frac{dy\, dz}{1} \quad (\because x = 1)$$

$$= \int_0^1 \left[\frac{4z^2}{2}\right]_0^1 dy$$

$$\int_S F\cdot\vec{n}\,ds = \int_0^1 2\,dy = 2[y]_0^1 = 2 \rightarrow (1)$$

(ii) Face OBLC $(x=0)$

$\vec{n} = -\vec{i}$

$$F\cdot\vec{n} = (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot(-\vec{i})$$

$$= -4xz = 0 \quad \because (x=0)$$

$$\int_S F\cdot\vec{n}\,ds = \int_0^1\int_0^1 0\,\frac{dy\,dz}{1} = 0 \rightarrow (2)$$

(iii) Face PLBN

$y=1$, $\vec{n} = \vec{j}$

$$|\hat{n}\cdot\vec{j}| = |\vec{j}\cdot\vec{j}| = 1$$

$$F\cdot\vec{n} = (4xz\vec{i} - y^2\vec{j} + yz\vec{k})\cdot\vec{j}$$

$$= -y^2$$

$$\int_S F\cdot\vec{n}\,ds = \int_0^1\int_0^1 -y^2\,dx\,dz \quad [\because y=1]$$

$$= -\int_0^1\int_0^1 dx\,dz$$

$$= -\int_0^1 [z]_0^1\,dx = -\int_0^1 dx$$

$$= -[x]_0^1$$

$$\int_S F\cdot\vec{n}\,ds = -1 \rightarrow (3)$$

(iv) Face MCOA, $y=0$

$\vec{n} = -\vec{j}$

$$\vec{n}\cdot\vec{j} = |-\vec{j}\cdot\vec{j}| = 1$$

$$\vec{F}\cdot\vec{n} = (4xy\vec{i} - y^2\vec{j} + yz\vec{k})(-\vec{j}) = -y^2$$

$$\int_S F\cdot\vec{n}\,ds = \int_0^1\int_0^1 y^2\,dx\,dz \qquad [\because y=0]$$

$$\int_S F\cdot\vec{n}\,ds = 0 \qquad \rightarrow (4)$$

(v) <u>Face</u> <u>CLPM</u>, $z=1$

$$\vec{n} = \vec{k}$$

$$\vec{n}\cdot\vec{k} = |\vec{k}\cdot\vec{k}| = 1$$

$$F\cdot\vec{n} = (4xy\vec{i} - y^2\vec{j} + yz\vec{k})\,\vec{k}$$

$$= yz$$

$$\int_S F\cdot\vec{n}\,ds = \int_0^1\int_0^1 yz\,dx\,dy$$

$$= \int_0^1\int_0^1 y\,dx\,dy$$

$$= \int_0^1 \left(\frac{y^2}{2}\right)_0^1 dx$$

$$= \frac{1}{2}\int_0^1 dx = \frac{1}{2}(x)_0^1$$

$$\int_S F\cdot\vec{n}\,ds = \frac{1}{2} \longrightarrow (5)$$

(vi) <u>Face</u> <u>OBNA</u>, $z=0$

$$\vec{n} = -\vec{k}$$

$$\vec{n}\cdot\vec{k} = |-\vec{k}\cdot\vec{k}| = 1$$

$$F\cdot\vec{n} = (4xz\vec{i} + y^2\vec{j} + yz\vec{k})(-\vec{k})$$

$$= -yz$$

$$\int_S F\cdot\vec{n}\,ds = \int_0^1\int_0^1 -yz\,dx\,dy \quad [z=0]$$

$$= 0 \longrightarrow (6)$$

Adding eqn ① to ⑥, we get;

$$\int_S F\cdot\hat{n}\,ds = 2+0-1+0+1/2$$
$$= 3/2 \rightarrow \text{(II)}$$

From eqn I & II

$$\int_V \text{div}\,F\cdot dv = \iint F\cdot\hat{n}\,ds$$

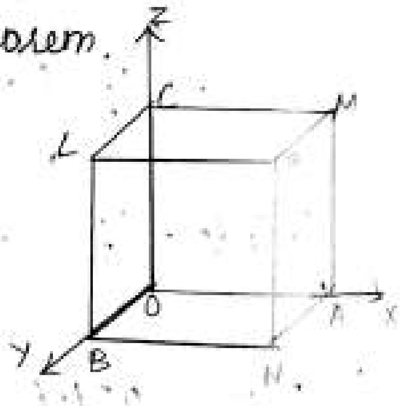
$$3/2 = 3/2$$

∴ The ~~g~~ Gauss divergence

Theorem is verified.

2. Evaluate $\int_S F\cdot\hat{n}\,ds$ when $F = 4xy\vec{i} + yz\vec{j} - xz\vec{k}$ and S is the surface of the cube bounded by the planes $x=0$, $x=2$, $y=0$, $y=2$ and $z=0, z=2$ and verify divergence theorem.

Proof:



$$\int_V \text{div}\,F\cdot dv = \iint_S F\cdot\hat{n}\,ds$$

$$F = F_1\vec{i} + F_2\vec{j} + F_3\vec{k}$$

$$\text{div}\,F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$F = \frac{\partial}{\partial x}(4xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(-xz)$$

$$\text{div}\,F = 4y + z - x$$

$$\int_V \text{div}\,F\,dv = \int_0^2\int_0^2\int_0^2 (4y+z-x)\,dx\,dy\,dz$$

$$= \int_0^2\int_0^2 \left[4yz + \frac{z^2}{2} - xz\right]_0^2 dx\,dy$$

$$= \int_0^2 \int_0^2 \left[8y + \frac{4}{2} - 2x\right] dx\, dy$$

$$= \int_0^2 \left[\frac{8y^2}{2} + 2y - 2xy\right]_0^2 dx$$

$$= \int_0^2 \left[4(2)^2 + 2(2) - 2(x)(2)\right] dx$$

$$= \int_0^2 [16 + 4 - 4x]\, dx$$

$$= \left[20x - \frac{4x^2}{2}\right]_0^2$$

$$= 40 - 8$$

$$\int \text{div}\, F \cdot dv = 32 \rightarrow \text{(I)}$$

To Find : $\int_S F \cdot \vec{n}\, ds$

(i) Face ANPM

$$F \cdot \hat{n} = (4xy\,\vec{i} + yz\,\vec{j} - xz\,\vec{k})\ \vec{i}$$

$$= 4xy$$

$$\vec{n} \cdot \vec{i} = |\vec{i} \cdot \vec{i}| = 1$$

$$\int_S F \cdot \vec{n}\, ds = \int_0^2 \int_0^2 4xy \frac{dy\, dz}{1} \quad (\because x = 2)$$

$$= \int_0^2 (8yz)_0^2\, dy$$

$$= \int_0^2 (8y(2))\, dy = \left[16y^2/2\right]_0^2$$

$$= [8(2)^2]$$

$$\int_S F \cdot \vec{n}\, ds = 32 \rightarrow (1)$$

(ii) Face OBLC : $x = 0$

$$\vec{n} = -\vec{i}$$

$$F\cdot\vec{n} = (4xy\vec{i} + yz\vec{j} - xz\vec{k})\cdot(-\vec{i})$$

$$= -4xy$$

$$\int_S F\cdot\vec{n}\,ds = 0 \longrightarrow (2)$$

(iii) Face BLPM:

$$\vec{n} = \vec{j}$$

$$F\cdot\vec{n} = (4xy\vec{i} + yz\vec{j} - xz\vec{k})\cdot\vec{j}$$

$$= yz$$

$$\int F\cdot\vec{n}\,ds = \int_0^2\int_0^2 F\cdot\vec{n}\,\frac{dx\,dz}{\vec{n}\cdot\vec{j}} \qquad (y=2)$$

$$= \int_0^2\int_0^2 2z\,dx\,dz$$

$$= \int_0^2 \left[\frac{2z^2}{2}\right]_0^2 dx$$

$$= \int_0^2 (2)^2\,dx$$

$$= [4x]_0^2$$

$$\int F\cdot\vec{n}\,ds = 8 \longrightarrow (3)$$

(iv) Face OCMA:

$$\vec{n} = -\vec{j}$$

$$F\cdot\vec{n} = (4xy\vec{i} + yz\vec{j} - xz\vec{k})\cdot(-\vec{j}) \qquad (y=0)$$

$$= -yz$$

$$\int F\cdot\vec{n}\,ds = 0 \rightarrow (4)$$

(v) Face CLPM:-

$$\vec{n} = \vec{k}$$

$$F\cdot\vec{n} = (4xy\vec{i} + yz\vec{j} - xz\vec{k})\cdot\vec{k} \qquad (z=0)$$

$$= -xz$$

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$\vec{n}\cdot\vec{k} = |\vec{k}\cdot\vec{k}|$
$= 1$

$\int F\cdot\vec{n}\,ds = \int_0^2\int_0^2 -2x\,dx\,dy$

$= \int_0^2 [-2xy]_0^2\,dx$

$= \int_0^2 (-4x)\,dx = \left[-\frac{4x^2}{2}\right]_0^2$

$= [-2(2)^2]$

$\int F\cdot\vec{n}\,ds = -8 \rightarrow (5)$

(vi) Face OBNA:

$\vec{n} = -\vec{k}$

$F\cdot\vec{n} = (4xy\,\vec{i} + y^2\,\vec{j} - xz\,\vec{k})\cdot(-\vec{k}) \quad (z=0)$

$= xz$

$\int F\cdot\vec{n}\,ds = 0 \rightarrow (6)$

Adding (1) to (6)

$\int F\cdot\vec{n}\,ds = 32 + 0 + 8 + 0 - 8 + 0$

$\int F\cdot\vec{n}\,ds = 32 \rightarrow (II)$

From (I) & (II), we get

$\int \operatorname{div} F\,dv = \iint_S F\cdot\vec{n}\,ds$

$32 = 32$

Hence the proof //

(3) Verify divergence theorem for $F = x\vec{i} - y\vec{j} + (z^2-1)\vec{k}$ taken over the region bounded by $x^2+y^2=4$, $z=0$, $z=1$

Sol:-

Divergence theorem

$$\int_S F\cdot\vec{n}\,ds = \int_V \operatorname{div}\vec{F}\,dv$$

$$F = x\vec{i} - y\vec{j} + (z^2-1)\vec{k}$$

$z=0, \quad z=1$

$x^2=4 \qquad x^2+y^2=4$

$x=\pm 2 \qquad y^2 = 4-x^2$

$y = \pm\sqrt{4-x^2}$

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial (x)}{\partial x} + \frac{\partial (-y)}{\partial y} + \frac{\partial (z^2-1)}{\partial z}$$

$$\operatorname{div} F = 1-1+2z = 2z$$

$$\int_V \operatorname{div} F\cdot dv = \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^1 2z\,dx\,dy\,dz$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 2\left[\frac{z^2}{2}\right]_0^1 dx\,dy$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx\,dy$$

$$= 2\int_{-2}^{2} [y]_0^{\sqrt{4-x^2}}$$

$$= 2\int_{-2}^{2} \sqrt{4-x^2}\,dx$$

$$= 4\int_0^2 \sqrt{2^2-x^2}\,dx$$

$$= 4\left[\frac{x}{2}\sqrt{4-x} + \frac{4}{2}\sin^{-1}\left(\frac{x}{2}\right)\right]_0^2$$

$= 4\left[0 + 2\sin^{-1}(1)\right]$

$= 4\left[2\left(\pi/2\right)\right]$

$\int \operatorname{div} F \, dv = 4\pi \rightarrow (I)$

$$\iint_S F\cdot\hat{n}\,ds = \iint_{S_1} F\cdot\hat{n}\,ds_1 + \iint_{S_2} F\cdot\hat{n}\,ds_2 + \iint_{S_3} F\cdot\hat{n}\,ds_3$$

(i) <u>On S_1</u>: $\hat{n} = -\vec{k}$, $z = 0$

$\vec{F} = x\vec{i} - y\vec{j} + (z^2-1)\vec{k}$

$\vec{F}\cdot\hat{n} = (x\vec{i} - y\vec{j} - \vec{k})\cdot(-\vec{k})$

$\vec{F}\cdot\hat{n} = 1$

$$\iint_{S_1} F\cdot\hat{n}\,ds = \iint_{S_1} ds_1$$

$= S_1$

$=$ Area of $S_1 = \pi(2^2)$

$= 4\pi \rightarrow (1)$ [$\because S_1$ is the circle of radius 2]

(ii) <u>On S_2</u>: $\hat{n} = \vec{k}$, $z = 1$

$\vec{F} = x\vec{i} - y\vec{j} + (z^2-1)\vec{k}$

$\vec{F}\cdot\hat{n} = (x\vec{i} - y\vec{j})\cdot\vec{k}$

$= 0$

$$\iint_{S_2} F\cdot\hat{n}\,ds = 0 \rightarrow (2)$$

(iii) <u>On S_3</u> $x^2+y^2 = 4$

The unit normal outward vector,

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}}{|\nabla\phi|}$$

$$= \frac{2x\vec{i} + 2y\vec{j}}{\sqrt{(2x)^2+(2y)^2}} = \frac{2(x\vec{i}+y\vec{j})}{\sqrt{4x^2+4y^2}}$$

$$= \frac{2(x\vec{i}+y\vec{j})}{2\sqrt{x^2+y^2}} = \frac{x\vec{i}+y\vec{j}}{\sqrt{4}}$$

$$\vec{n} = \frac{x\vec{i}+y\vec{j}}{2} \qquad [\because x^2+y^2=4]$$

$$F\cdot\vec{n} = (x\vec{i}-y\vec{j}+(z^2-1)\vec{k})\cdot\left(\frac{x\vec{i}}{2}+\frac{y\vec{j}}{2}\right)$$

$$= \frac{x^2}{2} - \frac{y^2}{2}$$

$$\iint_{S_3} F\cdot\vec{n}\,ds_3 = \tfrac{1}{2}\iint_{S_3}(x^2-y^2)\,ds_3$$

Here S_3 is a curved surface, Hence to find elemental area ds_3, we consider polar co-ordinates. $\because S_3$ is the circle $x^2+y^2=4$. Its polar co-ordinates are

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$x = 2\cos\theta, \quad y = 2\sin\theta, \quad ds_3 = 2\,d\theta\,dz$$

$$\iint_{S_3} F\cdot\vec{n}\,ds = \tfrac{1}{2}\int_0^{2\pi}\int_0^1 (x^2-y^2)\,2\,d\theta\,dz$$

$$= \frac{2}{2}\int_0^{2\pi}\int_0^1 (2\cos\theta)^2-(2\sin\theta)^2\,d\theta\,dz$$

$$= \int_0^{2\pi}\int_0^1 (4\cos^2\theta-4\sin^2\theta)\,d\theta\,dz$$

$$= 4\int_0^{2\pi}\int_0^1 (\cos^2\theta-\sin^2\theta)\,d\theta\,dz$$

$$= 4\int_0^{2\pi}\int_0^1 (2\cos^2\theta-1)\,d\theta\,dz$$

$$= 4\int_0^{2\pi}[2z\cos^2\theta - z]_0^1\,d\theta$$

$$= 4\int_0^{2\pi}[2\cos^2\theta-1]\,d\theta$$

$$= 4\int_0^{2\pi}\cancel{2}\left[\frac{1+\cos 2\theta}{\cancel{2}}\right]-1\,]\,d\theta$$

$$= 4\int_0 [1+\cos 2\theta - 1]\, d\theta$$

$$= 4\int_0^{2\pi} \cos 2\theta \, d\theta$$

$$= 0 \longrightarrow (3)$$

$$\iint_S F\cdot\hat{n}\, ds = \iint_{S_1} F\cdot\hat{n}\, ds + \iint_{S_2} F\cdot\hat{n}\, ds + \iint_{S_3} F\cdot\hat{n}\, ds$$

Adding (1), (2) & (3)

$$\iint_S F\cdot\hat{n}\, ds = 4\pi + 0 + 0$$

$$\iint_S F\cdot\hat{n}\, ds = 4\pi \longrightarrow (II)$$

$$(I) = (II)$$

$$\iint_S F\cdot\hat{n}\, ds = \int_V \operatorname{div} F \cdot dv$$

$\therefore$ Gauss divergence theorem is verified//.

4. Verify Divergence theorem for $F = 4x\vec{i} - 2y^2\vec{j} + z^2\vec{k}$ taken over the region bounded by $x^2+y^2=4$. $z=0$ and $z=3$.

Sol: By Gauss divergence theorem,

$$\int_S \vec{F}\cdot\hat{n}\, ds = \int_V \operatorname{div} \vec{F}\, dv$$

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)$$

$$\operatorname{div} \vec{F} = 4 - 4y + 2z$$

$$\int_V \operatorname{div} \vec{F}\, dv = \int_V (4 - 4y + 2z)\, dv$$

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$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{0}^{3} (4 - 4y + 2z)\, dx\, dy\, dz$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (12 - 12y + 9)\, dx\, dy$$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (21 - 12y)\, dx\, dy$$

$$= \int_{-2}^{2} \left[21y - 12\frac{y^2}{2} \right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx$$

$$= \int_{-2}^{2} \left[21\sqrt{4-x^2} - 6(4-x^2)\right] - \left[-21\sqrt{4-x^2} - 6(4-x^2)\right] dx$$

$$= 2\int_{0}^{2} 42\sqrt{4-x^2}\, dx.$$

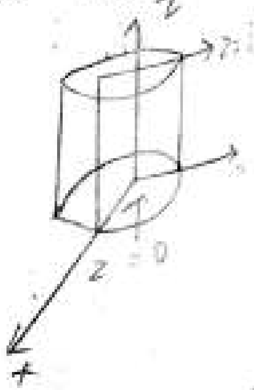
$$= 84 \int_{0}^{2} \sqrt{4-x^2}\, dx$$

$$= 84 \left[\frac{x}{2}\sqrt{4-x^2} + \frac{4}{2}\sin^{-1}(x/2) \right]_0^2$$

$$= 84\left[0 + 2\pi/2\right]$$

$$\int_V \operatorname{div} \vec{F}\, dv = 84\pi \longrightarrow \text{(II)}$$

The surface S will consist of the base in $z=0$, the top S_3 in $z=3$ and the convex portion.



(i) For S_1. $z=0$ $\hat{n} = -\vec{k}$

$$\vec{F}\cdot\hat{n} = (4x\vec{i} - 2y^2\vec{j} + z^2\vec{k})(-\vec{k})$$

$$= -z^2$$

$$\vec{F}\cdot\hat{n} = 0 \longrightarrow (1) \quad (\because z=0)$$

(ii) For S_3 : $z = 3$, $\hat{n} = \vec{k}$

$\vec{F}\cdot\hat{n} = (4x\vec{i} - 2y^2\vec{j} + z^2\vec{k})\cdot\vec{k}$

$\vec{F}\cdot\hat{n} = z^2$

$\vec{F}\cdot\hat{n} = 9 \quad (z = 3)$

$$\int_{S_3} \vec{F}\cdot\hat{n}\, ds_3 = \int_{S_3} 9\, ds_3$$

$$= 9\,(S_3)$$

$$= 9\,(4\pi)$$

$$\int_{S_3} \vec{F}\cdot\hat{n}\, ds = 36\pi \rightarrow (2)$$

Because for S_3, the area of surface is $2\pi r$

$2\pi . 2 = 4\pi$

For convex portion, $x^2+y^2=4$

$\phi = x^2+y^2$, $\nabla\phi = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)(x^2+y^2)$

$$\hat{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{\vec{i}\frac{\partial\phi}{\partial x} + \vec{j}\frac{\partial\phi}{\partial y} + \vec{k}\frac{\partial\phi}{\partial z}}{|\nabla\phi|}$$

$$= \frac{2x\vec{i} + 2y\vec{j}}{\sqrt{4x^2+4y^2}} = \frac{\cancel{2}(x\vec{i}+y\vec{j})}{\cancel{2}\sqrt{x^2+y^2}}$$

$$= \frac{x\vec{i}+y\vec{j}}{\sqrt{4}}$$

$$\hat{n} = \frac{x\vec{i}+y\vec{j}}{2}$$

$$\vec{F}\cdot\hat{n} = (4x\vec{i} - 2y^2\vec{j} + z^2\vec{k})\cdot\left(\frac{x\vec{i}+y\vec{j}}{2}\right)$$

$$\vec{F}\cdot\hat{n} = 2x^2 - y^2$$

$$\int_{S_2} \vec{F}\cdot\hat{n}\,ds_2 = \int_{S_2} (2x^2 - y^3)\,ds_2$$

Here S_2 is a curved surface; Hence to find elementary area ds_2 we consider polar co-ordinates. Since S_2 is the circle $x^2+y^2=4$, it polar co-ordinates are,

$x = r\cos\theta \quad , \quad y = r\sin\theta$

$x = 2\cos\theta \quad , \quad y = 2\sin\theta$

$$\int \vec{F}\cdot\hat{n}\,ds = \iint \vec{F}\cdot\hat{n}\,\frac{dy\,dz}{\hat{n}\cdot\vec{i}}$$

$$\hat{n}\cdot\vec{i} = \frac{x\vec{i}+y\vec{j}}{2} \quad , \quad \vec{i} = x/2$$

$y = 2\sin\theta, \quad dy = 2\cos\theta\,d\theta$

$$\int \vec{F}\cdot\hat{n}\,ds = \iint (2x^2 - y^3)\,\frac{2\cos\theta\,d\theta\,dz}{x/2}$$

$$= \iint \frac{2(4\cos^2\theta) - 8\sin^3\theta}{2\cos\theta}\,2(2\cos\theta)\,d\theta\,dz$$

$$= 2\int_0^{2\pi}\int_0^{3} (8\cos^2\theta - 8\sin^3\theta)\,dz\,d\theta$$

$$= 16\int_0^{2\pi} (\cos^2\theta - \sin^3\theta)\,[z]_0^3\,d\theta$$

$$= 16\int_0^{2\pi} 3(\cos^2\theta - \sin^3\theta)\,d\theta$$

$$= 48\int_0^{2\pi} \cos^2\theta\,d\theta - 48\int_0^{2\pi} \sin^3\theta\,d\theta$$

$$= 48\int_0^{2\pi} \cos^2\theta\,d\theta \qquad \left[\because \int_0^{2\pi} \sin^3\theta\,d\theta = 0\right]$$

$$= \int_0^{2\pi} 48\left(\frac{\cos 2\theta + 1}{2}\right) d\theta$$

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$$= 48\left[\frac{2\sin 2\theta}{4} + \frac{\theta}{2}\right]_0^{2\pi}$$

$$= 48\left[\cancel{\frac{2\pi}{2}} + 0\right]$$

$$\int_{S_2} \vec{F}\cdot\hat{n}\, ds_2 = 48\pi \longrightarrow (3).$$

Adding (1), (2) & (3)

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3}$$

$$= 0 + 48\pi + 36\pi$$

$$\therefore \int_S \vec{F}\cdot\hat{n}\, ds = 84\pi \longrightarrow \text{II}$$

$$\text{I} = \text{II}$$

$$\iint_S \vec{F}\cdot\hat{n}\, ds = \int_V \operatorname{div} F\, dv$$

GDT is verified//

5. verify Divergence theorem for the functions $F = y\vec{i} + x\vec{j} + z^2\vec{k}$ over the cylindrical region S bounded by $x^2+y^2=a^2$, $z=0$ and $z=h$.

Sol:- By Gauss divergence theorem.

$$\int_S \vec{F}\cdot\hat{n}\, ds = \int_V \operatorname{div}\vec{F}\, dv$$

$$\operatorname{div}\vec{F} = \frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z^2)$$

$$\operatorname{div}\vec{F} = 2z$$

$$z=0,\quad z=h$$

$$x^2 = a^2 \qquad\qquad x^2+y^2=a^2$$

$$x = \pm a \qquad\qquad y^2 = a^2 - x^2$$

$$y = \pm\sqrt{a^2-x^2}$$

$$\int_V \operatorname{div}\vec{F}\,dv = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\int_0^h 2z\,dx\,dy\,dz$$

$$= \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}}\left[\frac{2z^2}{2}\right]_0^h dx\,dy$$

$$= h^2\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx\,dy$$

$$= 2h^2\int_{-a}^{a}\int_0^{\sqrt{a^2-x^2}} dx\,dy$$

$$= 4h^2\int_0^a \sqrt{a^2-x^2}\,dx$$

$$= 4h^2\left[\frac{x}{2}\sqrt{a^2-x^2}+\frac{1}{2}a^2\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 4h^2\left[0+\frac{1}{2}a^2\frac{\pi}{2}\right]$$

$$\int_V \operatorname{div}\vec{F}\,dv = \pi a^2 h^2 \rightarrow \text{I}$$

$$\int_S \vec{F}\cdot\hat{n}\,ds = \int_{S_1} + \int_{S_2} + \int_{S_3}$$

(i) For S_1, $z=0$, $\hat{n} = -\vec{k}$

$$\vec{F}\cdot\hat{n} = (y\vec{i}+x\vec{j}+z^2\vec{k})\cdot(-\vec{k})$$

$$\vec{F}\cdot\hat{n} = -z^2$$

$$\vec{F}\cdot\hat{n} = 0 \qquad (\because z=0)$$

For S_3, $z=h$, $\hat{n} = \vec{k}$

$$\vec{F}\cdot\hat{n} = z^2 \Rightarrow \vec{F}\cdot\hat{n} = h^2 \qquad [\because z=h]$$

$$\int_{S_3} \vec{F}\cdot\hat{n}\,ds_3 = \int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} h^2\,dx\,dy$$

$$= 2h^2\int_{-a}^{a}[y]_0^{\sqrt{a^2-x^2}}\,dx$$

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$$= 2h^2 \int_{-a} \sqrt{a^2 - x^2} \, dx$$

$$= 2h^2 \left[\frac{x}{a} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right]_{-a}^{a}$$

$$= 2h^2 \left[\frac{a^2}{2} \sin^{-1}(1) - \frac{a^2}{2} \sin^{-1}(-1) \right]$$

$$= 2h^2 \left[\frac{a^2}{2} (\pi/2) + \frac{a^2}{2} (\pi/2) \right]$$

$$= 2h^2 \left[\frac{2\pi a^2}{2} \right]$$

$$\int_{S_3} \vec{F} \cdot \hat{n} \, dS_3 = \pi h^2 a^2 \longrightarrow (2)$$

For convex portion, $x^2 + y^2 = a^2$

$$\nabla \phi = 2x\vec{i} + 2y\vec{j}$$

$$\hat{n} = \frac{\nabla \phi}{|\nabla \phi|} = \frac{2x\vec{i} + 2y\vec{j}}{\sqrt{(2x)^2 + (2y)^2}} = \frac{2x\vec{i} + 2y\vec{j}}{\sqrt{4x^2 + 4y^2}}$$

$$= \frac{\cancel{2}(x\vec{i} + y\vec{j})}{\cancel{2}\sqrt{x^2 + y^2}}$$

$$= \frac{x\vec{i} + y\vec{j}}{\sqrt{x^2 + y^2}}$$

$$= \frac{x\vec{i} + y\vec{j}}{\sqrt{a^2}}$$

$$= \frac{x\vec{i} + y\vec{j}}{a}$$

$$\vec{F} \cdot \hat{n} = (y\vec{i} + x\vec{j} + z^2\vec{k}) \cdot \left(\frac{x\vec{i} + y\vec{j}}{a} \right)$$

$$= \frac{xy + yx}{a}$$

$$\vec{F} \cdot \hat{n} = \frac{2xy}{a}$$

$$\hat{n} \cdot \vec{i} = \frac{x\vec{i} + y\vec{j}}{a} \cdot \vec{i}, \quad \vec{i} = x/a$$

$$\int_{S_2} \vec{F} \cdot \hat{n} \, dS_2 = \iint \frac{2xy}{a} (a/x) \, dy \, dz$$

$$\int_{S_2} \vec{F}\cdot\vec{n}\,ds_2 = \iint 2y\,dy\,dz$$

Since S_2 is a curved surface. Hence to find elemental surface area ds_2 we consider polar co-ordinates. Since S_2 is the circle $x^2+y^2=a^2$ it polar co-ordinates are,

$x = r\cos\theta$; $y = r\sin\theta$

$x = a\cos\theta$; $y = a\sin\theta$, $dy = a\cos\theta\,d\theta$

$$\int_{S_2} \vec{F}\cdot\vec{n}\,ds_2 = 2\int_0^{2\pi}\int_0^{h} a\sin\theta \cdot a\cos\theta\,d\theta\,dz$$

$$= 2\int_0^{2\pi}\int_0^{h} a^2\sin\theta\cos\theta\,d\theta\,dz$$

$$= 2a^2\int_0^{h}\int_0^{2\pi}\frac{\sin2\theta}{2}\,d\theta\,dz$$

$$= a^2\int_0^{h}\left[\frac{-\cos2\theta}{2}\right]_0^{2\pi}dz$$

$$= a^2\int_0^{h}\left(-\tfrac{1}{2}+\tfrac{1}{2}\right)dz$$

$$\int_{S_2}\vec{F}\cdot\vec{n}\,ds_2 = 0 \rightarrow (3)$$

Adding (1), (2) & (3)

$$\int_S \vec{F}\cdot\vec{n}\,ds = 0 + \pi h^2a^2 + 0$$

$$\int_S \vec{F}\cdot\vec{n}\,ds = \pi h^2a^2 \longrightarrow (\text{II})$$

$$\text{I} = \text{II}$$

$$\iint_S \vec{F}\cdot\vec{n}\,ds = \int \operatorname{div}\vec{F}\,dv$$

∴ Hence verified of GDT.

6. Evaluate $\int_S (x\vec{i}+y\vec{j}+z\vec{k}) \cdot \vec{n}\, ds$ where s denotes the surface of the cube bounded by the planes $x=0$, $x=a$, $y=a$, $y=0$, $z=0$, $z=a$ by the application of Gauss theorem and verify the result by direct multiplication.

Sol:

$$\int_S F\cdot\vec{n}\, ds = \iiint_V \text{div}\, F \cdot dv$$

$$\text{div}\, \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial (x)}{\partial x} + \frac{\partial (y)}{\partial y} + \frac{\partial (z)}{\partial z}$$

$$\text{div}\, \vec{F} = 1+1+1 = 3$$

$$\int_V \text{div}\, F\cdot dv = \int_0^a\int_0^a\int_0^a 3\, dx\, dy\, dz$$

$$= 3\int_0^a\int_0^a [z]_0^a\, dx\, dy$$

$$= 3\int_0^a\int_0^a a\, dx\, dy$$

$$= 3a\int_0^a [y]_0^a\, dx$$

$$= 3a\int_0^a a\, dx = 3a^2\int_0^a dx = 3a^2[x]_0^a$$

$$\int_V \text{div}\, F\, dv = 3a^3 \longrightarrow \text{I}$$

$$\vec{F} = x\vec{i}+y\vec{j}+z\vec{k}$$

(i) Face ANPM, $x=a$

$$\vec{n} = \vec{i}$$

$$\vec{F}\cdot\vec{n} = (x\vec{i}+y\vec{j}+z\vec{k})\cdot\vec{i}$$

$$= x \quad [\because x=a]$$

$$\vec{F}\cdot\vec{n} = a$$

$$\int_S \vec{F}\cdot\vec{n}\, ds = \int_0^a\int_0^a a\,\frac{dy\, dz}{1} = a\int_0^a [z]_0^a\, dy$$

$$= a\int_0^a a\,dy$$

$$= a^2 \int_0^a dy$$

$$= a^2 [y]_0^a = a^2(a)$$

$$\int_S F\cdot\hat{n}\,ds = a^3 \rightarrow (1)$$

(ii) Face OBLC: $x = 0$

$\hat{n} = \vec{i}$, $\vec{F}\cdot\hat{n} = (x\vec{i}+y\vec{j}+z\vec{k})\cdot(-\vec{i}) = -x$

$$|\hat{n}\cdot\vec{i}| = |-\vec{i}\cdot\vec{i}| = 1$$

$$\int_S F\cdot\hat{n}\,ds = \int_0^a\int_0^a -x\,dy\,dz \qquad \because x = 0$$

$$= 0 \rightarrow (2)$$

(iii) Face PLBN: $y = a$

$$\hat{n} = \vec{j}$$

$$\vec{F} = (x\vec{i}+y\vec{j}+z\vec{k})\cdot\vec{j} = y$$

$$|\hat{n}\cdot\vec{j}| = |\vec{j}\cdot\vec{j}| = 1$$

$$\int_S F\cdot\hat{n}\,ds = \int_0^a\int_0^a y\,dx\,dz = \int_0^a\int_0^a a\,dx\,dz \quad [\because y=$$

$$= \int_0^a\int_0^a a\,dx\,dz$$

$$= a\int_0^a [z]_0^a\,dx = a\int_0^a a\,dx = a^2\int_0^a dx$$

$$= a^2[x]_0^a = a^2(a) = a^3 \rightarrow (3)$$

(iv) Face MCOA, $y = 0$

$$\hat{n} = -\vec{j}$$

$$\vec{F}\cdot\hat{n} = (x\vec{i}+y\vec{j}+z\vec{k})\cdot(-\vec{j}) = -y$$

$$\vec{F}\cdot\hat{n} = 0 \qquad [\because y = 0]$$

$$\int_S F\cdot\hat{n}\,ds = 0 \rightarrow (4)$$

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(v) Face CLPM, $z = a$

$\hat{n} = \vec{k}$

$\vec{F} \cdot \hat{n} = (x\vec{i} + y\vec{j} + z\vec{k}) \vec{k}$

$\vec{F} \cdot \hat{n} = z$

$|\hat{n} \cdot \vec{k}| = |\vec{k} \cdot \vec{k}| = 1$

$$\int_S F \cdot \hat{n}\, ds = \int_0^a \int_0^a z\, dx\, dy \quad [\because z = a]$$

$$= \int_0^a \int_0^a a\, dx\, dy = a \int_0^a [y]_0^a\, dx$$

$$= a \int_0^a a\, dx = a^2 \int_0^a dx = a^2 [x]_0^a = a^2 (a)$$

$$\int_S F \cdot \hat{n}\, ds = a^3 \rightarrow ⑤$$

(vi) Face OBNA; $z = 0$

$\hat{n} = -\vec{k}$

$\vec{F} \cdot \hat{n} = (x\vec{i} + y\vec{j} + z\vec{k}) \cdot (-\vec{k})$

$= -z$

$\vec{F} \cdot \hat{n} = 0 \quad [\because z = 0]$

$|\hat{n} \cdot \vec{k}| = |-\vec{k} \cdot \vec{k}| = 1$

$$\int_S \vec{F} \cdot \hat{n}\, ds = 0 \rightarrow ⑥$$

Adding eqn ① to ⑥ we get

$$\int_S F \cdot \hat{n}\, ds = a^3 + 0 + a^3 + 0 + a^3 + 0 = 3a^3 \rightarrow (II)$$

By eqn I & II we get

$$\int_S F \cdot \hat{n}\, ds = \int_V \text{div } F\, dv$$

Hence the Gauss divergence theorem.

7. If v is the volume enclosed by a closed surface and $F = x\vec{i} + 2y\vec{j} + 3z\vec{k}$. Show that $\int_S F \cdot \hat{n}\, ds = 6v$.

Sol:

Given,

$$\vec{F} = x\vec{i} + 2y\vec{j} + 3z\vec{k}$$

$$\int_S F \cdot \hat{n}\, ds = \int_V \text{div}\, \vec{F}\, dv \rightarrow (1)$$

$$\text{div}\, \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial (x)}{\partial x} + \frac{\partial (2y)}{\partial y} + \frac{\partial (3z)}{\partial z}$$

$$= 1 + 2 + 3$$

$$\text{div}\, \vec{F} = 6$$

$$\int_S F \cdot \hat{n}\, ds = \int_V 6\, dv$$

$$= 6 \int_V dv$$

$$\int_S F \cdot \hat{n}\, ds = 6v$$

Hence proved //.

8. S.T $\frac{1}{3} \int \vec{r} \cdot \hat{n}\, ds = v$ where v is the volume (a) volume enclosed by surface s.

Sol:

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

$$\text{div}\, \vec{r} = \partial/\partial x\, (x) + \partial/\partial y\, (y) + \partial/\partial z\, (z)$$

$$\text{div}\, \vec{r} = 3$$

$$\frac{1}{3} \int \vec{r} \cdot \hat{n}\, ds = \frac{1}{3} \int_V \text{div}\, \vec{r}\, dv$$

$$= \frac{1}{3} \int_V 3\, dv$$

$$= \frac{3}{3} v$$

$$\frac{1}{3} \int \vec{r} \cdot \hat{n}\, ds = v \,//$$

(b) If $\vec{n}$ is a unit outward normal vector at any closed surface S. P.T $\int_V \operatorname{div} \vec{n}\, dv = S$.

proof:

By Gauss divergence theorem.

$$\int_S \vec{F}\cdot\hat{n}\, ds = \int_V \operatorname{div} \vec{F}\, dv$$

Here, $\vec{F} = \hat{n}$ and $\hat{n}\cdot\hat{n} = 1$

$$\int_S \hat{n}\cdot\hat{n}\, ds = \int_V \operatorname{div} \vec{n}\, dv$$

$$\int_V \operatorname{div} \hat{n}\, dv = \int_S ds = S$$

$$\int_V \operatorname{div} \vec{n}\, dv = S$$

(C) Prove that $\int \vec{n}\,(\nabla\times\vec{f})\, ds = 0$ where $\vec{F}$ is a vector point function and S is a closed surface.

Sol:-

let $\vec{f} = \nabla\times\vec{F} = \operatorname{curl} \vec{F}$

given integral $= \int \vec{n}\,\vec{f}\, ds = \int_S \vec{F}\cdot\hat{n}\, ds$

$= \int_V \operatorname{div} \vec{F}\, dv$ (By Gauss divergence theorem

$= \int_V \operatorname{div}.(\operatorname{curl} \vec{F}).dv$ $[\because \vec{F} = \operatorname{curl} \vec{F})$

[Formula $[\operatorname{div} \operatorname{curl} \vec{F}] = 0$ (or) $\nabla\cdot(\nabla\times\vec{F}) = 0]$

$\therefore \int \vec{n}\,(\nabla\times\vec{F})\, ds = 0$

(d) Prove that $\int_S \vec{F}\cdot\hat{n}\, ds = 4\pi \int \vec{F}\, dv$ where $\vec{F} = \nabla\phi$ and $\nabla^2\phi = -4\pi\vec{F}$.

Sol:-

By Gauss Divergence theorem

$$\int_S \vec{F}\cdot\hat{n}\, ds = \int_V \operatorname{div} \vec{F}\, dv$$

$$\int_V \operatorname{div} \vec{F}\, dv = \int_V \nabla\cdot\vec{F}\, dv = \int \nabla\cdot(\nabla\phi)\, dv$$

$$= \int \nabla^2\phi \, dv$$

$$= \int -4\pi \vec{\rho}$$

$$\int_v \mathrm{div}\, \vec{F}\, dv = -4\pi \int_v \rho \, dv$$

$$\therefore \int_s \vec{F}\cdot\hat{n}\, ds = \int_v \mathrm{div}\, \vec{F}\, dv$$

$$= -4\pi \int \vec{\rho}\, dv$$

(e) show that $\int \vec{F}\cdot\hat{n}\, ds = \int_v A^2\, dv$ where $\vec{F} = \phi A$

$A = \nabla\phi$ & $\nabla^2\phi = 0$

Sol:-

Formula: $\mathrm{div}\,(\phi\vec{F}) = \phi\, \mathrm{div}\,\vec{f} + f\,\nabla\phi$

$$\mathrm{div}\,\vec{F} = \mathrm{div}\,(\vec{A}\phi) = \phi\, \mathrm{div}\,\vec{A} + \vec{A}\cdot\nabla\phi$$

$$= \phi\nabla\cdot(\nabla\phi) + \vec{A}\cdot\vec{A}$$

$$= \phi(\nabla^2\phi) + \vec{A}^2$$

$$= 0 + \vec{A}^2$$

$$\mathrm{div}\,\vec{F} = A^2 \rightarrow (1)$$

By Gauss Divergence Theorem,

$$\int_s \vec{F}\cdot\hat{n}\, ds = \int_v \mathrm{div}\, \vec{F}\cdot dv \qquad \text{(By eqn (1))}$$

$$\int_s \vec{F}\cdot\hat{n}\, ds = \int_v A^2\, dv$$

9. Evaluate $\int_S (y^2z^2\vec{i} + z^2x^2\vec{j} + z^2y^2\vec{k})\cdot\vec{n}\,ds$. where S is the part of the sphere $x^2+y^2+z^2=1$ above the xy plane.

Sol:

x varies -1 to 1
y varies -1 to 1
z varies 0 to 1

By Gauss Divergence theorem,

$$\int_S \vec{F}\cdot\vec{n}\,ds = \int_V \text{div}\,\vec{F}\,dv$$

$$\text{div}\,\vec{F} = \partial/\partial x\,(y^2z^2) + \partial/\partial y\,(z^2x^2) + \partial/\partial z\,(z^2y^2)$$

$$\text{div}\,\vec{F} = 2zy^2$$

$$\int_S \vec{F}\cdot\vec{n}\,ds = \int_V \text{div}\,\vec{F}\,dv$$

$$= \int_{-1}^{1}\int_{-1}^{1}\int_{0}^{1} 2zy^2\,dx\,dy\,dz$$

$$= \int_{-1}^{1}\int_{-1}^{1} y^2\left[\frac{2z^2}{2}\right]_0^1 dy\,dx$$

$$= \int_{-1}^{1}\int_{-1}^{1} y^2\,dx\,dy = 2\int_{-1}^{1}\left(y^3/3\right)_0^1 dx$$

$$= 2\int_{-1}^{1} 1/3\,dx = 2/3 \quad 2[x]_0^1$$

$$\int_S \vec{F}\cdot\vec{n}\,ds = 4/3 \;//$$

10. S.T $\int_S (x^2\vec{i} + y^2\vec{j} + z^2\vec{k})\cdot\vec{n}\,ds = 0$ where S denotes the surface of the ellipsed $x^2/a + y^2/b + z^2/c = 1$.

Sol:

By Gauss Divergence theorem,

$$\int_S \vec{F}\cdot\vec{n}\,ds = \int_V \text{div}\,\vec{F}\,dv$$

$x = -a$ to a ; $y = -b$ to b ; $z = -c$ to c

$\text{div } \vec{F} = \partial/\partial x (x^2) + \partial/\partial y (y^2) + \partial/\partial z (z^2)$

$\text{div } \vec{F} = 2(x+y+z)$

$$\int_S \vec{F}\cdot\vec{n}\,ds = \int_{-a}^{a}\int_{-b}^{b}\int_{-c}^{c} 2(x+y+z)\,dx\,dy\,dz$$

$$= 2\int_{-a}^{a}\int_{-b}^{b}\left[(x+y)z + \frac{z^2}{2}\right]_{-c}^{c} dx\,dy$$

$$= 2\int_{-a}^{a}\int_{-b}^{b}\left[(x+y)c + \frac{c^2}{2} + (x+y)c - \frac{c^2}{2}\right] dx\,dy$$

$$= 2\int_{-a}^{a}\int_{-b}^{b}\left[2(x+y)c\right] dx\,dy$$

$$= 4c\int_{-a}^{a}\left[xy + \frac{y^2}{2}\right]_{-b}^{b} dx$$

$$= 4c\int_{-a}^{a}\left[xb + b^2/2 + xb - b^2/2\right] dx$$

$$= 4c\int_{-a}^{a} 2bx\,dx = 8bc\left[\frac{x^2}{2}\right]_{-a}^{a}$$

$$= 8bc\left[\frac{a^2}{2} - \frac{a^2}{2}\right]$$

$$\int_S \vec{F}\cdot\vec{n}\,ds = 0 \;//$$

(b) Evaluate $\int_S \vec{F}\cdot\vec{n}\,ds$ where $\vec{F} = x^2\vec{i} + y^2\vec{j} + z^2\vec{k}$ and S is the surface of the solid cut off by the plane $x+y+z = a$ from the first octant.

Sol:

$x = 0$ to a ; $y = 0$ to $a-x$, $z = 0$ to $a-x-y$

$$\int_S \vec{F}\cdot\vec{n}\,ds = \int_V \text{div } \vec{F}\,dv$$

$\text{div } \vec{F} = \partial/\partial x (x^2) + \partial/\partial y (y^2) + \partial/\partial z (z^2)$

$$\text{div}\ \vec{F} = 2(x+y+z)$$

$$\int_S \vec{F}\cdot\hat{n}\,ds = \int_V \text{div}\ \vec{F}\,dv$$

$$= \int_0^a \int_0^{a-x} \int_0^{a-x-y} 2(x+y+z)\,dx\,dy\,dz$$

$$= 2\int_0^a \int_0^{a-x} \left[xy + yz + \frac{z^2}{2}\right]_0^{a-x-y} dx\,dy$$

$$= 2\int_0^a \int_0^{a-x} \left[(x+y)(a-x-y) + \frac{(a-x-y)^2}{2}\right] dx\,dy$$

$$= \frac{2}{2}\int_0^a \int_0^{a-x} (a-x-y)(2x+2y+a-x-y)\,dx\,dy$$

$$= \int_0^a \int_0^{a-x} (a-(x+y))(a+(x+y))\,dx\,dy$$

$$= \int_0^a \int_0^{a-x} \left[a^2 - (x+y)^2\right] dx\,dy$$

$$= \int_0^a \int_0^{a-x} \left[a^2 - x^2 - y^2 - 2xy\right] dx\,dy$$

$$= \int_0^a \left[a^2y - x^2y - y^3/3 - 2x\,y^2/2\right]_0^{a-x} dx$$

$$= \int_0^a \left[a^2(a-x) - x^2(a-x) - \frac{(a-x)^3}{3} - x(a-x)^2\right] dx$$

$$= \int_0^a \left[(a-x)(a^2-x^2) - (a-x)^2\left[\frac{a-x+3x}{3}\right]\right] dx$$

$$= \int_0^a \left[(a-x)^2\left(a+x - \frac{a+2x}{3}\right)\right] dx$$

$$= \frac{1}{3}\int_0^a \left[(a^2 - 2ax + x^2)(2a+x)\right] dx$$

$$= \frac{1}{3}\int_0^a \left[(2a^3 + xa^2 - 4a^2x - 2ax^2 + 2ax^2 + x^3)\right] dx$$

$$= \frac{1}{3}\int_0^a (2a^3 - 3a^2x + x^3)\,dx$$

$$= \frac{1}{3}\left[2a^3x - \frac{3a^2x^2}{2} + \frac{x^4}{4}\right]_0^a$$

$$= \frac{1}{3}\left[2a^4 - \frac{3a^4}{2} + \frac{a^4}{4}\right]$$

$$= \frac{1}{3}a^4\left[2 - \frac{3}{2} + \frac{1}{4}\right]$$

$$= \frac{1}{3}a^4\left[\frac{8-6+1}{4}\right]$$

$$= \frac{1}{12}a^4(3)$$

$$\int_S \vec{F}\cdot\hat{n}\,ds = \frac{a^4}{4} \;//$$

11. Evalaate $\int_S (x\vec{i} + y\vec{j} + z^2\vec{k})\cdot\hat{n}\,ds$ where S is the classed surface bounded by the cone $x^2+y^2=z^2$ and the plane $z=1$.

Sol:-

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z^2)$$

$$\text{div}\,\vec{F} = 2+2z$$

$$x^2+y^2 = z^2 \Rightarrow x^2+y^2=1 \quad [\because z=1]$$

$$x = -1 \text{ to } 1 \;;\; y = -\sqrt{1-x^2} \text{ to } \sqrt{1-x^2} \;;\; z = 0 \text{ to } 1$$

$$\int_S \vec{F}\cdot\hat{n}\,ds = \int_V \text{div}\,\vec{F}\,dv$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^1 (2+2z)\,dx\,dy\,dz$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left(2z + \frac{2z^2}{2}\right)_0^1 dx\,dy$$

$$= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 3\,dx\,dy = 6\int_{-1}^{1}[y]_0^{\sqrt{1-x^2}}\,dx$$

$$= 6 \int_{-1} \sqrt{1-x^2}\, dx$$

$$= 12 \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2} \sin^{-1}(x) \right]_0^1$$

$$= 12 \left[\frac{1}{2}\sqrt{1-1} + \frac{1}{2} \sin^{-1}(1) \right]$$

$$= 12 \left(\frac{1}{2} \times \frac{\pi}{2} \right)$$

$$\int_S \vec{F} \cdot \hat{n}\, ds = 3\pi$$

12. If $\vec{OA} = a\vec{i}$, $OB = a\vec{j}$, $OC = a\vec{k}$, from the three coterminous edge of a cube and S denotes the surface of the cube evaluate $\int_S \{(x^3 - yz)\vec{i} - 2x^2y\vec{j} + 2\vec{k}\} \cdot \hat{n}\, ds$ by expressing it as a volume integral. Also verify the result by direct evaluation of the surface integral.

Sol: By Divergence theorem

$$\int_S \vec{F} \cdot \hat{n}\, ds = \int_V \operatorname{div} \vec{F}\, dv$$

$$\operatorname{div} \vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

$$= \frac{\partial}{\partial x}(x^3 - yz) + \frac{\partial}{\partial y}(-2x^2y) + \frac{\partial}{\partial z}(2)$$

$$= 3x^2 - 2x^2$$

$$\operatorname{div} \vec{F} = x^2$$

$$\int_V \operatorname{div} \vec{F}\, dv = \int_0^a \int_0^a \int_0^a x^2\, dx\, dy\, dz = \int_0^a \int_0^a \left[\frac{x^3}{3} \right]_0^a dy\, dz$$

$$= \int_0^a \int_0^a \frac{a^3}{3}\, dy\, dz = \int_0^a \frac{a^3}{3} [y]_0^a\, dz$$

$$= \left[\frac{a^4}{3} z \right]_0^a$$

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$$\int_V \operatorname{div} \vec{F}\, dv = \frac{a^5}{5} \rightarrow (1)$$

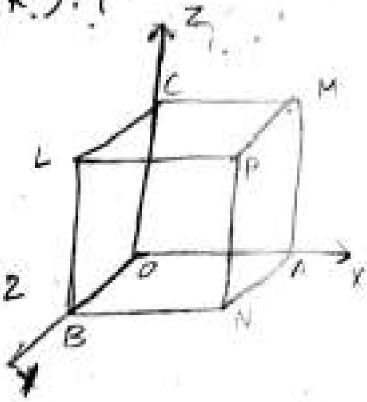
Now $\iint_S \vec{F}\cdot\hat{n}\, ds = \iint_{S_1} + \iint_{S_2} + \iint_{S_3} + \iint_{S_4} + \iint_{S_5} + \iint_{S_6}$

(i) Face ANPM :-

A unit normal $\hat{n} = \vec{i}$ and $x = a$

$\vec{F}\cdot\hat{n} = \left((x^3\,yz)\vec{i} - 2x^2y\vec{j} - 2xy\vec{k}\right)\cdot\vec{i}$

$= x^3 - yz$



$\vec{F}\cdot\hat{n} = a^3 - yz \quad (\because x = a)$

$ds = \dfrac{dy\,dz}{\hat{n}\cdot\vec{i}} = \dfrac{dy\,dz}{\vec{i}\cdot(\vec{i})} = dy\,dz$

$\int_{S_1} \vec{F}\cdot\hat{n}\, ds_1 = \int_0^a\int_0^a (a^3 - yz)\, dy\, dz$

$= \int_0^a \left[a^3y - \frac{y^2}{2}z\right]_0^a dz$

$= \int_0^a \left[a^4 - \frac{a^2}{2}z\right] dz$

$= \left[a^4z - \frac{a^2}{2}\left(\frac{z^2}{2}\right)\right]_0^a$

$\int_{S_1} \vec{F}\cdot\hat{n}\, ds_1 = a^5 - \frac{a^4}{4} \longrightarrow (1)$

(ii) Face OBLC :

A unit normal $\hat{n} = -\vec{i}$ and $x = 0$

$\vec{F}\cdot\hat{n} = x^3 + yz = yz \quad (\because x = 0)$

$\int_{S_2} \vec{F}\cdot\hat{n}\, ds_2 = \int_0^a\int_0^a yz\, dy\, dz$

$= \int_0^a \left[\frac{y^2}{2}z\right]_0^a dz$

$$= \int_0^a \frac{a^2}{2} z\,dz$$

$$\int_{S} \vec{F}\cdot\hat{n}\,ds_2 = \frac{a^2}{2}\left[\frac{z^2}{2}\right]_0^a = \frac{a^4}{4} \rightarrow (2)$$

(iii) Face PLBN:

A unit normal $\hat{n} = \vec{j}$ and $y = a$

$$\int_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a \{(x^3-yz)\vec{i} - 2x^2y\vec{j} + 2\vec{k}\}\,\vec{j}\,ds$$

$$\vec{F}\cdot\hat{n} = \{(x^3-yz)\vec{i} - 2x^2y\vec{j} + 2\vec{k}\}\,\vec{j}$$

$$\vec{F}\cdot\hat{n} = -2x^2y$$

$$\vec{F}\cdot\hat{n} = -2x^2a \quad (\because y = a)$$

$$ds = \frac{dx\,dz}{\hat{n}\cdot\vec{j}} = \frac{dx\,dy}{\vec{j}\cdot\vec{j}} = dx\,dy$$

$$\int_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a -2x^2a\,dx\,dy$$

$$= -\int_0^a \left[\frac{2ax^3}{3}\right]_0^a dz$$

$$= -\int_0^a \frac{2a^4}{3}\,dz$$

$$= -\left[\frac{2a^4}{2}z\right]_0^a = \frac{-2a^5}{3}$$

$$\int_{S_3} \vec{F}\cdot\hat{n}\,ds_3 = -\frac{2a^5}{3} \rightarrow (3)$$

(iv) Face AOCM:

A unit normal $\hat{n} = -\vec{j}$ and $y = 0$

$$\vec{F}\cdot\hat{n} = [(x^3-yz)\vec{i} - 2x^2y\vec{j} + 2\vec{k}]\,(-\vec{j})$$

$\vec{F}\cdot\vec{n} = 2x^2y = 0 \quad [\because y=0]$

$$\int_{S4} \vec{F}\cdot\vec{n}\, ds_4 = \int_0^a\int_0^a 0 \cdot dx\, dz$$

$$\int_{S4} \vec{F}\cdot\vec{n}\, ds_4 = 0 \longrightarrow (4)$$

(v) Face MPLC:

A unit normal $\vec{n} = \vec{k}$ and $z=a$

$\vec{F}\cdot\vec{n} = \{(x^2 - yz)\vec{i} - 2x^2y\vec{j} + 2\vec{k}\}\cdot\vec{k}$

$\vec{F}\cdot\vec{n} = 2$

$$\int_{S5} \vec{F}\cdot\vec{n}\, ds_5 = \int_0^a\int_0^a 2\, dx\, dy$$

$$ds_5 = \frac{dx\, dy}{\vec{n}\cdot\vec{k}} = \frac{dx\, dy}{\vec{k}\cdot\vec{k}} = dx\, dy$$

$$\int_{S5} \vec{F}\cdot\vec{n}\, ds_5 = \int_0^a 2a\, dy = 2a^2$$

$$\int_{S5} \vec{F}\cdot\vec{n}\, ds_5 = 2a^2 \longrightarrow (5)$$

(vi) Face OANB

A unit normal $\vec{n} = -\vec{k}$ and $z=0$

$\vec{F}\cdot\vec{n} = ((x^2 - yz)\vec{i} - 2x^2y\vec{j} + 2\vec{k})\cdot(-\vec{k})$

$\vec{F}\cdot\vec{n} = -2$

$$\int_{S6} \vec{F}\cdot\vec{n}\, ds_6 = -\int_0^a\int_0^a 2\, dx\, dy$$

$$= -\int_0^a 2a\, dy$$

$$\int_{S6} \vec{F}\cdot\vec{n}\, ds_6 = -2a^2 \longrightarrow (6)$$

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Adding 1 to 6

$$\int_S \vec{F}\cdot\vec{n}\,ds = a^5 - \cancel{\frac{a^4}{4}} + \cancel{\frac{a^4}{4}} - \frac{2a^5}{3} + 0 + \cancel{2a^5} - \cancel{2a^5}$$

$$= a^5\left(1-\frac{2}{3}\right) = \frac{a^5}{3} \rightarrow \text{II}$$

From I & II

$$\text{I} = \text{II}$$

$$\int_V \operatorname{div} F\,dv = \int_S \vec{F}\cdot\vec{n}\,ds$$

$\therefore$ verified the GDT //

13. S.T $\int_S \vec{n}\,ds = 0$ over a closed surface.

(a) choose any arbitary vector a.

Sol:-

$$\therefore \vec{a}\cdot\int_S \vec{n}\,ds = \int_S \vec{a}\cdot\vec{n}\,ds = \int_V \operatorname{div}\vec{a}\,dv$$

But $\operatorname{div}\vec{a} = \frac{\partial}{\partial x}(a_1) + \frac{\partial}{\partial y}(a_2) + \frac{\partial}{\partial z}(a_3) = 0$

Since $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$

$$\vec{a}\int_S \vec{n}\,ds = \int_V 0\,dv$$

$\vec{a}\cdot\int_S \vec{n}\,ds = 0$, But $\vec{a}$ is arbitary

$$\therefore \int_S \vec{n}\,ds = 0 \;//$$

(b) P.T $\int_S (r\times\vec{n})\,ds = 0$ for any closed surface S.

Sol:- choose any arbitary constant vector $\vec{a}$

$$\therefore \vec{a}\int_S \vec{r}\times\vec{n}\,ds = \int \vec{a}\cdot(\vec{r}\times\vec{n})\,ds$$

$$= \int (\vec{a}\cdot\vec{r})\,\vec{n}\,ds$$

$= \int \vec{F} \cdot \vec{n}\, ds$

$\vec{a} \cdot \int_S \vec{r} \times \vec{n}\, ds = \int \text{div}\, \vec{F}\, dv$ (by GDT)

$\hookrightarrow (1)$

Now $\text{div}\, \vec{F} = \text{div}\, (\vec{a} \times \vec{r})$

$= \text{curl}\, \vec{a} \cdot \vec{r} - \vec{a} \cdot \text{curl}\, \vec{r}$

$= 0$ $(\because \text{curl}\, \vec{r} = 0)$

and $\text{curl}\, \vec{a} = 0$ as $\vec{a}$ is constant.

eqn (1) $\Rightarrow \vec{a} \int \vec{r} \times \vec{n}\, ds = \int 0\, dv$

$\vec{a} \int \vec{r} \times \vec{n}\, ds = 0$

$\int \vec{r} \times \vec{n}\, ds = 0$

as $\vec{a}$ is a arbitary constant vector.

(c) P.T $\int \frac{dv}{r^2} = \int \frac{\vec{r} \cdot n}{r^2}\, ds.$

Sol:

By Gauss Divergence theorem.

$\int_S \vec{F} \cdot \vec{n}\, ds = \int_V \text{div}\, \vec{F}\, dv$

$\iint_S \frac{\vec{r}}{r^2} \vec{n}\, ds = \iiint_V \nabla \cdot \frac{\vec{r}}{r^2}\, dv \rightarrow (1)$

$\nabla \frac{\vec{r}}{r^2} = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right) \cdot \left(\frac{x\vec{i} + y\vec{j} + z\vec{k}}{x^2+y^2+z^2}\right)$

$= \sum \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2+z^2}\right)$

$= \sum \frac{(x^2+y^2+z^2)(1) - x(2x)}{(x^2+y^2+z^2)^2}$

$= \sum \frac{x^2+y^2+z^2-2x^2}{(x^2+y^2+z^2)^2}$

$$= \sum \frac{r^2 - 2x^2}{r^4}$$

$$= \frac{3r^2 - 2(x^2+y^2+z^2)}{r^4}$$

$$= \frac{3r^2 - 2r^2}{r^4} = \frac{r^2}{r^4} = \frac{1}{r^2}$$

$$\nabla \cdot \frac{\vec{r}}{r^2} = \frac{1}{r^2}$$

$$① \Rightarrow \iint_S \frac{\vec{r}}{r^2} \cdot \vec{n}\, ds = \iiint_V \frac{1}{r^2}\, dv$$

$$\iint_S \frac{\vec{r}}{r^2} \cdot \vec{n}\, ds = \iiint_V \frac{dv}{r^2} \;//$$

14. S.T $\int_S (ax\vec{i} + by\vec{j} + cz\vec{k}) \cdot \vec{n}\, ds = \frac{4}{3}\pi(a+b+c)$ where S is the surface of the sphere $x^2+y^2+z^2=1$.

Sol:

$$\int_S \vec{F} \cdot \vec{n}\, ds = \int_V \operatorname{div} \vec{F}\, dv$$

$$\operatorname{div} \vec{F} = \frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)$$

$$\operatorname{div} \vec{F} = a+b+c$$

$$\int_S \vec{F} \cdot \vec{n}\, ds = \int_V \operatorname{div} \vec{F}\, dv = \int (a+b+c)\, dv$$

$$= (a+b+c)\, V$$

Now, V = volume of a sphere of unit radius

$$= 4/3\pi r^3$$

$$= 4/3\, \pi \quad (\because r=1)$$

$$\int_S \vec{F} \cdot \vec{n}\, ds = (a+b+c)\frac{4}{3\pi} \;//$$

15. If $\vec{F} = 2yx\,\vec{i} - 2y\,\vec{j} + x^2\vec{k}$. Evaluate $\int_s \vec{F}\cdot\vec{n}\,ds$ where s denote the entire surface of the cube bounded by the co-ordinate planes and the planes $x=a$, $y=a$, $z=a$ by the application of Gauss theorem and verify it by direct evaluation of surface integral.

Sol:

Gauss divergence theorem $\int_s \vec{F}\cdot\vec{n}\,ds = \int_v \text{div}\,\vec{F}\,dv$

$$\text{div}\,\vec{F} = \frac{\partial}{\partial x}(2yx) + \frac{\partial}{\partial y}(-2y) + \frac{\partial}{\partial z}(x^2)$$

$$\text{div}\,\vec{F} = 2y - 2$$

$$\int_v \text{div}\,\vec{F}\,dv = \int_0^a\int_0^a\int_0^a (2y-z)\,dx\,dy\,dz$$

$$= \int_0^a\int_0^a \left[2yz - \frac{z^2}{2}\right]_0^a dx\,dy$$

$$= \int_0^a\int_0^a \left(2ya - \frac{a^2}{2}\right) dx\,dy$$

$$= \int_0^a \left[2a\frac{y^2}{2} - \frac{a^2}{2}y\right]_0^a dx$$

$$= \int_0^a \left[a^3 - \frac{a^3}{2}\right] dx = \int_0^a \frac{a^3}{2}\,dx$$

$$= \frac{a^3}{2}[x]_0^a = \tfrac{1}{2}a^4$$

$$\int_v \text{div}\,\vec{F}\,dv = \frac{a^4}{2} \rightarrow (1)$$

$$\int_s \vec{F}\cdot\vec{n}\,ds = \int_{s_1} + \int_{s_2} + \int_{s_3} + \int_{s_4} + \int_{s_5} + \int_{s_6}$$

(i) Face ANPM:

$\vec{n} = \vec{i}$ & $x = a$

$$\vec{F}\cdot\vec{n} = (2yx\,\vec{i} - 2y\,\vec{j} + x^2\vec{k})\cdot\vec{i}$$

$= 2yx$

$\vec{F}\cdot\hat{n} = 2ya \quad (\because x=a)$

$$ds = \frac{dy\,dz}{\hat{n}\cdot\vec{i}} = \frac{dy\,dz}{\vec{i}\cdot\vec{i}} = dy\,dz$$

$$\int_{S_1} \vec{F}\cdot\hat{n}\,ds = \int_0^a\int_0^a 2ya\,dy\,dz$$

$$= \int_0^a 2a\left[\frac{y^2}{2}\right]_0^a dz$$

$$= 2a\int_0^a \frac{a^2}{2}\,dz$$

$$= a^3[z]_0^a$$

$$\int_{S_1} \vec{F}\cdot\hat{n}\,ds = a^4 \rightarrow (1)$$

(ii) <u>Face</u> <u>OBLC:</u>

$\hat{n} = -\vec{i}$ & $x=0$

$\vec{F}\cdot\hat{n} = -2yx$

$\vec{F}\cdot\hat{n} = 0$

$$\int_{S_2} \vec{F}\cdot\hat{n}\,ds_2 = 0 \rightarrow (2)$$

(iii) <u>Face</u> <u>PLBN:</u>

$\hat{n} = \vec{j}$ & $y=a$

$\vec{F}\cdot\hat{n} = (2yx\vec{i} - 2y\vec{j} + x^2\vec{k})\cdot\vec{j}$

$= -2y$

$\vec{F}\cdot\hat{n} = -2a \quad (\because y=a)$

$$\int_{S_3} \vec{F}\cdot\hat{n}\,ds_3 = \int_0^a\int_0^a -a z\,dx\,dz$$

$$ds = \frac{dx\,dz}{\hat{n}\cdot\vec{j}} = \frac{dx\,dy}{\vec{j}\cdot\vec{j}} = dx\,dz$$

$$\int_{S_3} \vec{F}\cdot\vec{n}\,ds_3 = -\int_0^a a\left[\frac{z^2}{2}\right]_0^a dx$$

$$= -\int_0^a \frac{a^3}{2}\,dx$$

$$\int_{S_3} \vec{F}\cdot\vec{n}\,ds_3 = -\frac{a^4}{2} \rightarrow (3)$$

(iv) <u>Face</u> <u>AOCM:</u>

$\vec{n} = -\vec{j}$ & $y = 0$

$$\vec{F}\cdot\vec{n} = (2yx\vec{i} - zy\vec{j} + x^2\vec{k})\cdot\vec{j}$$

$$= -zy$$

$$\vec{F}\cdot\vec{n} = 0 \qquad (\because y = 0)$$

$$\int_{S_4} \vec{F}\cdot\vec{n}\,ds_4 = 0 \rightarrow (4)$$

(v) <u>Face</u> <u>MPLC:</u>

$\vec{n} = \vec{k}$ and $z = a$

$$\vec{F}\cdot\vec{n} = (2yx\vec{i} - zy\vec{j} + x^2\vec{k})\cdot\vec{k}$$

$$\vec{F}\cdot\vec{n} = x^2$$

$$\int_{S_5} \vec{F}\cdot\vec{n}\,ds_5 = \int_0^a\int_0^a x^2\,dx\,dy$$

$$ds = \frac{dx\,dy}{\vec{n}\cdot\vec{k}} = \frac{dx\,dy}{\vec{k}\cdot\vec{k}} = dx\,dy$$

$$\int_{S_5} \vec{F}\cdot\vec{n}\,ds = \int_0^a \left[\frac{x^3}{3}\right]_0^a dy$$

$$= \frac{a^3}{3}\int_0^a dy$$

$$\int_{S_5} \vec{F}\cdot\vec{n}\,ds = \frac{a^4}{3} \rightarrow (5)$$

(vi) Face OANB:-

$$\vec{n} = -\vec{k} \quad \text{and} \quad z = 0$$

$$\vec{F} \cdot \vec{n} = -x^2$$

$$\int_{S_6} \vec{F} \cdot \vec{n}\, ds_6 = -\int_0^a \int_0^a x^2 dx\, dy$$

$$= -\int_0^a \left[\frac{x^3}{3}\right]_0^a dy$$

$$= \frac{-a^3}{3} \int_0^a dy$$

$$\int_{S_6} \vec{F} \cdot \vec{n}\, ds_6 = -\frac{a^4}{3} \rightarrow (6)$$

Adding (1) to (6)

$$\int_S \vec{F} \cdot \vec{n}\, ds = a^4 + 0 - \frac{a^4}{2} + 0 + \cancel{\frac{a^4}{3}} - \cancel{\frac{a^4}{3}}$$

$$= \frac{2a^4 - a^4}{2}$$

$$\int_S \vec{F} \cdot \vec{n}\, ds = \frac{a^4}{2} \rightarrow (II)$$

$$(I) = (II)$$

Hence the verification of Gauss Divergence theorem.//

10/02/2020

UNIT - IV

Definition:-

Periodic function:-

A function $f(x)$ is said to have a period T for all x.

$$f(x+T) = f(x)$$

where T is a +ve constant. The least value of $T>0$ is called the period of $f(x)$.

Ex: $\sin x, \cos x$ are periodic function with period 2π.

Fourier series:-

If $f(x)$ is a periodic function and satisfies Dirichlet condition then it can be represented by an infinite series is known as fourier series.

which can be written as

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos x + \dots + a_n \cos nx + \dots \infty + b_1 \sin x + b_2 \sin 2x + \dots + b_n \sin nx + \dots + \infty$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos x + b_n \sin x).$$

where a_0, a_n, b_n are called fourier co-efficients in the fourier series for the function $f(x)$ in the interval $\lambda < x < \lambda + 2\pi$ is given by $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$

where a_0, a_n, b_n are fourier coefficients

$$a_0 = \frac{1}{\pi} \int_{\lambda}^{\lambda+2\pi} f(x)\, dx.$$

$$a_n = \frac{1}{\pi} \int_{\lambda}^{\lambda+2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{\lambda}^{\lambda+2\pi} f(x) \sin nx\, dx$$

corollary ①:

putting $\lambda = 0$ in the interval $\lambda < x < \lambda + 2\pi$ we get $0 < x < 2\pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi} \int_{0}^{2\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \cos nx\, dx$$

$$b_n = \frac{1}{\pi} \int_{0}^{2\pi} f(x) \sin nx\, dx$$

corollory ②:

putting $\lambda = -\pi$ in the interval $\lambda < x < \lambda + 2\pi$ we get $-\pi < x < \pi$.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi} f(x)\cos nx \, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx.$$

Note:

$\cos n\pi = (-1)^n$ | $\cos n(0) = 1$

$\cos(2\pi) = 1$ | $\sin n(\pi) = 0$

where n is an integer.

problems:

① example. Express $f(x) = \frac{1}{2}(\pi - x)$ as a fourier series with period 2π to be valid in the interval 0 to 2π

sol:

fourier series expansion.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{2}{2\pi - 0}\int_0^{2\pi} \frac{1}{2}(\pi - x)\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(\pi - x)\,dx$$

$$= \frac{1}{2\pi}\left[\pi x - \frac{x^2}{2}\right]_0^{2\pi}$$

$= \frac{1}{2\pi}\left[2\pi^2 - \frac{4\pi^2}{2}\right] - 0$

$= \frac{1}{2\pi}[0]$

$\boxed{a_0 = 0}$

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos nx\,dx$$

$$= \frac{2}{2\pi - 0}\int_0^{2\pi} \frac{1}{2}(\pi - x)\cos nx\,dx$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi}(\pi - x)\cos nx\,dx$$

$$a_n = \frac{1}{2\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$\int u\,dv = uv - u'v_1 + u''v_2 - u'''v_3 \ldots$

$$= \frac{1}{2\pi}\left[\cancel{(\pi - x)}\left(-\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{\cos n(2\pi)}{n^2} + \frac{\cos n(0)}{n^2}\right]$$

$$= \frac{1}{2\pi}\left[-\frac{1}{n^2} + \frac{1}{n^2}\right] = 0$$

$a_n = 0$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}\frac{1}{2}(\pi - x)\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(\pi - x)\sin nx\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi}(\pi - x)\sin nx\,dx$$

$$= \frac{1}{2\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[(\pi-2\pi)\left(\frac{-\cos n\pi}{n}\right)-(\pi-0)\left(\frac{-\cos n(0)}{n}\right)\right]$$

$$= \frac{1}{2\pi}\left[(-\pi)\left(\frac{-1}{n}\right)-(\pi)\left(\frac{-1}{n}\right)\right]$$

$$= \frac{1}{2\pi}\left[\frac{\pi}{n}+\frac{\pi}{n}\right]$$

$$= \frac{1}{2\pi}\left[\frac{2\pi}{n}\right]$$

$$b_n = \frac{1}{n}$$

Fourier series,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{0}{2} + \sum_{n=1}^{\infty} \left((0)\cos nx + \frac{1}{n}\sin nx\right)$$

$$f(x) = \sum_{n=1}^{\infty} \left[\frac{1}{n}\sin nx\right]$$

put, $x = \pi/2$

$$f(x) = \frac{1}{1}\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \dots$$

$$= \sin \pi/2 + \frac{1}{2}\sin 2(\pi/2) + \frac{1}{3}\sin 3(\pi/2) + \dots$$

$$= 1 - 1/3 + 1/5 - 1/7 \dots$$

2. Find the fourier series for the function $f(x) = x$ in $(0, 2\pi)$.

Sol:

The fourier series equation

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int^{2\pi} x\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\,\frac{\overset{2}{\cancel{4}}\pi^2}{\cancel{2}}$$

$$= 2\pi$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\cos nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - (1)\left[\frac{-\cos nx}{n^2}\right]\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x\left(\frac{\sin nx}{n}\right) + \left(\frac{\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{\cos n(2\pi)}{n^2} - \frac{\cos n(0)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{1}{n^2} - \frac{1}{n^2}\right]$$

$$a_n = 0$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\frac{x(-\cos nx)}{n} - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) + \frac{\sin nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right)\right]_0^{2\pi} \qquad \boxed{\therefore \sin\pi = 0}$$

$$= \frac{1}{\pi}\left[(2\pi)\left(\frac{-\cos n(2\pi)}{n}\right) - (0)\left(\frac{-\cos\ldots}{n}\right)\right]$$

$$= \frac{1}{\pi}\left[(2\pi)\left[\frac{-1}{n}\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{-2\pi}{n}\right]$$

$$b_n = \frac{-2}{n}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \frac{2\pi}{2} + \sum_{n=1}^{\infty} \left[0 + \left(\frac{-2}{n}\right)\sin nx\right]$$

$$= \frac{2\pi}{2} + \sum_{n=1}^{\infty} \left[\frac{-2}{n}\sin nx\right]$$

$$f(x) = \pi + \sum_{n=1}^{\infty} \frac{-2}{n}\sin nx$$

7. A function $f(x)$ is defined within the range $(0, 2\pi)$ by the relations.

$$f(x) = \begin{cases} x, & \text{in the range } (0,\pi) \\ 2\pi - x, & \text{in the range } (\pi, 2\pi) \end{cases}$$

Express $f(x)$ as a fourier series in the range $(0, \pi)$.

Soln.

The fourier series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_0^\pi x\,dx + \int_\pi^{2\pi}(2\pi - x)\,dx\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^\pi + \left[2\pi x - \frac{x^2}{2}\right]_\pi^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} - 0\right) + \left[2\pi(2\pi) - \frac{(2\pi)^2}{2}\right] - \left[\frac{2\pi(\pi)}{1} - \frac{\pi^2}{2}\right]\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \left[4\pi^2 - \frac{4\pi^2}{2} - 2\pi^2 + \frac{\pi^2}{2}\right]\right]$$

$$= \frac{1}{\pi}\left[2\pi^2 + \frac{2\pi^2}{2}\right]$$

$$= \frac{\pi}{2} + \left[4\pi^2 - 2\pi^2 - \frac{4\pi^2}{2} + \frac{\pi^2}{2}\right]$$

$$\frac{\pi}{2} + \left(2\pi^2 - \frac{3\pi^2}{2}\right)$$

$$= \frac{\pi}{2} + \left(\frac{4\pi^2 - 3\pi^2}{2}\right)$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + 4\pi^2 - \frac{4\pi^2}{2} - 2\pi^2 + \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi^2}{2} + 2\pi^2\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi^2}{2}\right]$$

$$a_0 = \pi$$

$$a_n = \frac{1}{\pi}\left[\int_0^\pi f(x)\cos nx\,dx + \int_\pi^{2\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^\pi x\cos nx\,dx + \int_\pi^{2\pi}(2\pi - x)\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left[x\frac{\sin nx}{n} - (1) - \frac{\cos nx}{n^2}\right]_0^\pi + \left[(2\pi - x)\frac{\sin nx}{n} - (-1)\frac{-\cos nx}{n^2}\right]_\pi^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^\pi + \left[(2\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right]_\pi$$

$$= \frac{1}{\pi}\left[\frac{\cos nx}{n^2}\right]_0^{\pi} + \left[\frac{-\cos nx}{n^2}\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{\cos n(\pi)}{n^2} - \frac{\cos n(0)}{n^2}\right] + \left[-\frac{\cos n(2\pi)}{n^2} + \frac{\cos n(\pi)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right] + \left[\frac{-1}{n^2} + \frac{(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{-2}{n^2} + \frac{2(-1)^n}{n^2}\right]$$

$$a_n = \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$b_n = \frac{1}{\pi}\left[\int_0^{\pi} f(x)\sin nx\,dx + \int_{\pi}^{2\pi} f(x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - (1)\frac{-\sin nx}{n^2}\right]_0^{\pi} + \left[(2\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left[x\left(-\frac{\cos nx}{n}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(\frac{-\cos nx}{n}\right)\right]_{\pi}^{2\pi}$$

$$= \frac{1}{\pi}\left\{\left[\pi\left(\frac{-\cos n(\pi)}{n}\right) - 0\right] + \left[(2\pi - 2\pi)\left(\frac{-\cos n(2\pi)}{n}\right) - (2\pi - \pi)\left(-\frac{\cos n(\pi)}{n}\right)\right]\right.$$

$$= \frac{1}{\pi}\left[\pi \frac{(-1)^n}{n} + 0 - \left[\pi - \frac{(-1)^n}{n}\right]\right.$$

$$= \frac{1}{\pi}\left[-\frac{\pi(-1)^n}{n} + \pi\frac{(-1)^n}{n}\right]$$

$$= \frac{1}{\pi}(0)$$

$b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]\cos nx + 0\right]$$

$$= \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]\cos nx$$

$$= \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n^2} - \frac{(-1)^n}{n^2}\right]\cos nx$$

When, n is even,

$1-(-1)^n = 0$

when n is add,

$1-(-1)^n = 2$

$$f(x) = \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1-(-1)^n}{n^2}\cos nx$$

$$= \frac{\pi}{2} - \frac{2}{\pi}\left[2\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right)\right]$$

$$f(x) = \frac{\pi}{2} - \frac{2}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right]$$

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14/2/20

Even and odd functions:

Even functions:-

If $f(x) = f(-x)$

$\therefore$ Then $f(x)$ is said to be even function.

Ex:- $x^2, \cos x, x^4+3x^2+2\cos x$.

odd functions:

If $f(-x) = -f(x)$

(or)

$f(x) = -f(-x)$

Then $f(x)$ is said to be odd function

Ex:-

$x^3, \sin x, \sin^3 x, 2x^3+3x$.

Properties of odd and even functions:-

(i) If $f(x)$ is odd then, PT $\int_{-a}^{a} f(x)\,dx = 0$.

Proof:

Let $f(x)$ is odd

$f(x) = -f(-x)$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= \int_{-a}^{0} -f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

$$= -\int_{-a}^{0} f(-x)\,dx + \int_{0}^{a} f(x)\,dx$$

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put $-x = y$

$-dx = dy$

$dx = -dy$

$x = -a, \quad y = -(-a) = a$

$x = 0, \quad y = -(0) = 0$

$$\int_{-a}^{a} f(x)dx = -\int_{a}^{0} f(y)(-dy) + \int_{0}^{a} f(x)dx$$

$$= -\int_{0}^{a} f(y)\,dy + \int_{0}^{a} f(x)dx$$

$$= -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)dx$$

$$= 0$$

2. If $f(x)$ is an even function. Then,

P.T $\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx.$

Proof :

If $f(x)$ is an even

$f(x) = f(-x)$

$$\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_{0}^{a} f(x)dx$$

$$= \int_{-a}^{0} f(-x)dx + \int_{0}^{a} f(x)dx$$

put, $-x = y$

$-dx = dy$

$dx = -dy$

$x = -a, \quad y = -(-a) = a$

$x = 0, \quad y = -(0) = 0$

$$\int_{-a}^{a} f(x)dx = \int_{-a}^{a} f(y)(-dy) + \int_{0}^{a} f(x)dx.$$

$$= \int_{a} f(y)dy + \int_{0}^{a} f(x)dx.$$

$$= \int_0 f(x)\,dx + \int_0 f(x)\,dx$$

$$= 2\int_0^a f(x)\,dx.$$

Even function :- $(-\pi < x < \pi)$

If $f(x)$ is an even function, th
can be expanded as a series of the

$$b_n = 0$$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where,

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx.$$

odd function :- $(-\pi < x < \pi)$

If $f(x)$ is add function. Then $f(x)$ can be expanded as a series of the form. $a_0 = 0$, $a_n = 0$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Where, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx.$

1. Express $f(x) = x$ $(-\pi < x < \pi)$ as a fourier series with period 2π.

Sol:-

Given $f(x) = x$

The given function is odd

$a_0 = 0$ and $a_n = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin nx\,dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{-\cos n\pi}{n}\right) - (-\pi)\left(\frac{-\cos n(-\pi)}{n}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} - \frac{\pi(-1)^n}{n}\right]$$

$$= \frac{1}{\pi}\left[\frac{-2\pi(-1)^n}{n}\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{2\pi(-1)^{n+1}}{n}\right]$$

$$f(x) = \sum_{n=1}^{\infty} b_n\sin nx$$

$$= \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n}\sin nx$$

$$= 2\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin nx$$

$$= 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right]$$

$$f(x) = x$$

$$x = 2\left[\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots\right]$$

Q S.T $x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}$ in the interval $(-\pi < x < \pi)$. Deduce that,

(i) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$

(ii) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

(iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

Sol:-

Given $f(x) = x^2$ in $(-\pi < x < \pi)$

$f(x)$ is even,

$b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{\pi^3}{3} - \frac{(-\pi)^3}{3}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^3}{3} + \frac{\pi^3}{3}\right] \qquad \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\frac{2\pi^3}{3}\right]$$

$$= \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right.$$

$$= \frac{2}{\pi}\left[(2x)\left(\frac{\cos n\pi}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[(2\pi)\left(\frac{\cos n\pi}{n^2}\right)\right]$$

$$= 4\left[\frac{(-1)^n}{n^2}\right)$$

$$= \frac{4}{n^2}(-1)^n$$

$\because$ Since terms will be zero while applying all values.

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$= \frac{2\pi^2}{3\times 2} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} (-1)^n \frac{\cos nx}{n^2}$$

$$x^2 = \frac{\pi^2}{3} + 4\left[\frac{-1}{1^2}\cos x + \frac{1}{2^2}\cos 2x - \frac{1}{3^2}\cos 3x + \dots\right]$$

(i) put, $x = 0$ in ①

$$0 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots\right]$$

$$\frac{\pi^2}{3} = 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \dots\right]$$

$$\frac{\pi^2}{3\times 4} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

$$\boxed{\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} + \dots}$$

(ii) put, $x = \pi$ in ①.

$$\pi^2 = \frac{\pi^2}{3} + 4\left[\frac{-1}{1^2}\cos\pi + \frac{1}{2^2}\cos 2\pi - \frac{1}{3^2}\cos 3\pi + \dots\right]$$

$$\pi^2 - \frac{\pi^2}{3} = 4\left[\frac{-1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots\right]$$

$$\frac{2\pi^2}{3\times 4} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

$$\boxed{\frac{\pi^2}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots}$$

From ① & (ii)

Adding (i) & (ii)

$$\frac{\pi^2}{12}+\frac{\pi^2}{6} = \left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\cdots\right)+\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\right)$$

$$\frac{6\pi^2+12\pi^2}{72} = \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}-\cdots\right)+\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots$$

$$\frac{18\pi^2}{72} = \frac{2}{1^2}+\frac{2}{3^2}+\frac{2}{5^2}+\cdots$$

$$\frac{\pi^2}{4} = 2\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

$$\boxed{\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots}$$

Q. Find the fourier series expansion of $x+x^2$ in the interval $(-\pi,\pi)$ and deduce that $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \frac{\pi^2}{6}$ and prove that

$$f(x) = \frac{\pi^2}{3} - 4\left(\frac{\cos x}{1^2}-\frac{\cos 2x}{2^2}+\cdots\right) + 2\left(\frac{\sin x}{1}-\frac{\sin 2x}{2}+\frac{\sin 3x}{3}-\cdots\right)$$

Sol:-

Fourier series Expansion,

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

given,

$f(x) = x+x^2$ under the interval $(-\pi,\pi)$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\,dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2}+\frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2}+\frac{\pi^3}{3}\right)-\left(\frac{(-\pi)^2}{2}+\frac{(-\pi)^3}{3}\right)\right\}$$

$$= \frac{1}{\pi}\left\{\left[\frac{\pi^2}{2}+\frac{\pi^3}{3}-\left[\frac{\pi^2}{2}-\frac{\pi^3}{3}\right]\right\}\right.$$

$$= \frac{1}{\pi}\left\{\frac{\pi^2}{2}+\frac{\pi^3}{3}-\frac{\pi^2}{2}+\frac{\pi^3}{3}\right]$$

$$= \frac{1}{\pi}\left(\frac{2\pi^3}{3}\right)$$

$$= \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(\frac{\sin nx}{n}\right)-(1+2x)\left(\frac{-\cos nx}{n^2}\right)+(2)\left(\frac{-\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(1+2x)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1+2x)\left(\frac{\cos nx}{n^2}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[(1+2\pi)\left(\frac{\cos n\pi}{n^2}\right)-(1+2(-\pi))\left[\frac{\cos n(-\pi)}{n^2}\right]\right\}\right.$$

$$= \frac{1}{\pi}\left\{(1+2\pi)\frac{(-1)^n}{n^2}-\left[(1-2\pi)\frac{(-1)^n}{n^2}\right]\right\}$$

$$= \frac{1}{\pi}\left\{(1+2\pi)\frac{(-1)^n}{n^2}-(1-2\pi)\frac{(-1)^n}{n^2}\right]$$

$$= \frac{1}{\pi}\frac{(-1)^n}{n^2}\left[1+2\pi-(1-2\pi)\right]$$

$$= \frac{1}{\pi}\frac{(-1)^n}{n^2}(1+2\pi-1+2\pi)$$

$$= \frac{1}{\pi}\frac{(-1)^n}{n^2}4\pi = \frac{4(-1)^n}{n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi}(x+x^2)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(\frac{-\cos nx}{n}\right) - (1+2x)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[(x+x^2)\left(\frac{-\cos nx}{n}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left[(\pi+\pi^2)\left(\frac{-\cos n\pi}{n}\right) + 2\frac{\cos n\pi}{n^3}\right] - \left[\ldots + 2\frac{\cos n\pi}{n^3}\right]\right\}$$

$$= \frac{1}{\pi}\left\{\cancel{(\pi+\pi^2)\left(\frac{-\cos n\pi}{n}\right) + \frac{2\cos n\pi}{n^3}}\right]$$

$$= \frac{1}{\pi}\left[(\pi+\pi^2)\left(\frac{-(-1)^n}{n}\right) + 2\frac{(-1)^n}{n^3}\right] - \left[(-\pi+\pi^2)\ldots + 2\frac{\ldots}{n^3}\right]$$

$$= \frac{1}{\pi}\left\{\left[-(\pi+\pi^2)\frac{(-1)^n}{n} + \frac{2(-1)^n}{n^3}\right] + \left[(-\pi+\pi^2)\frac{(-1)^n}{n} - \frac{2(-1)^n}{n^3}\right]\right.$$

$$= \frac{1}{\pi}\left\{(-\pi-\pi^2)\frac{(-1)^n}{n} + \frac{2(-1)^n}{n^3} - (-\pi+\pi^2)\frac{-(-1)^n}{n} - \frac{2(-1)^n}{n^3}\right.$$

$$= \frac{1}{\pi}\left\{(-\pi-\pi^2)\frac{(-1)^n}{n} + \frac{2(-1)^n}{n^3} + (-\pi+\pi^2)\frac{(-1)^n}{n} - \frac{2(-1)^n}{n^3}\right.$$

$$= \frac{1}{\pi}\left\{(-\pi-\pi^2)\frac{(-1)^n}{n} + (-\pi+\pi^2)\frac{(-1)^n}{n}\right\}$$

$$= \frac{1}{\pi}\frac{(-1)^n}{n}\left[-\pi-\cancel{\pi^2}-\pi+\cancel{\pi^2}\right]$$

$$b_n = \frac{1}{\pi}\frac{(-1)^n}{n}(-2\pi) = \frac{-2(-1)^n}{n}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx)$$

$$= \frac{2\pi^2}{3\times 2} + \sum_{n=1}^{\infty}\left[\frac{4(-1)^n}{n^2}\cos nx + \frac{-2(-1)^n}{n}\sin nx\right]$$

$$= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\left[\frac{4(-1)^n}{n^2}\cos nx - \frac{2(-1)^n}{n}\sin nx\right]$$

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n^2}\cos nx\right] - 2\sum_{n=1}^{\infty}\left[\frac{(-1)^n}{n}\sin nx\right]$$

$$= \frac{\pi^2}{3} + 4\left[\frac{-1}{1^2}\cos x + \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x + \cdots\right]$$

$$-2\left[\frac{(-1)}{1^2}\sin x + \frac{1}{2^2}\sin 2x - \frac{1}{3^2}\sin 3x + \cdots\right]$$

$$f(x) = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2}\cos x - \frac{1}{2^2}\cos 2x + \frac{1}{3^2}\cos 3x - \cdots\right]$$

$$+2\left[\frac{1}{1^2}\sin x - \frac{1}{2^2}\sin 2x + \frac{1}{3^2}\sin 3x - \cdots\right]$$

Hence proved//.

$$x + x^2 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2}\cos x - \frac{1}{2^2}\cos 2x + \cdots\right]$$

$$+2\left[\frac{1}{1^2}\sin x - \frac{1}{2^2}\sin 2x + \cdots\right]$$

put, $x = \pi$,

$$\pi + \pi^2 = \frac{\pi^2}{3} - 4\left[\frac{1}{1^2}\cos\pi - \frac{1}{2^2}\cos 2\pi + \cdots\right]$$

$$+2\left[\frac{1}{1^2}\sin\pi - \frac{1}{2^2}\sin 2\pi + \cdots\right]$$

$$\pi + \pi^2 = \frac{\pi^2}{3} - 4\left[\frac{-1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} + \cdots\right] + 2(0)$$

$$\pi + \pi^2 - \frac{\pi^2}{3} = 4\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\pi + \frac{3\pi^2 - \pi^2}{3} = 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{3\pi + 2\pi^2}{3} = 4\left(1/1^2 + 1/2^2 + \cdots\right]$$

$$\frac{3\pi + 2\pi^2}{12} = \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{3\pi}{12} + \frac{2\pi^2}{12} = 1/1^2 + 1/2^2 + \cdots$$

$$\pi^2/6 + \pi/4 = 1/1^2 + 1/2^2 + \cdots$$

$$\pi^2/6 = 1/1^2 + 1/2^2 + 1/3^2 + \cdots //$$

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4. Find that range $-\pi$ to π, a fourier series from,

$$f(x) = y = \begin{cases} 1+x \ ; & 0 < x < \pi \\ -1+x \ ; & -\pi < x < 0 \end{cases}$$

<u>soln:</u>

The fourier series expansion,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} (1+x)\, dx + \int_{-\pi}^{0} (-1+x)\, dx \right]$$

$$= \frac{1}{\pi} \left[\left(x + \frac{x^2}{2} \right)_0^{\pi} + \left(-x + \frac{x^2}{2} \right)_{-\pi}^{0} \right]$$

$$= \frac{1}{\pi} \left[\left(\pi + \frac{\pi^2}{2} \right) + \left(0 - \left(\pi + \frac{\pi^2}{2} \right) \right) \right]$$

$$= \frac{1}{\pi} \left[\pi + \frac{\pi^2}{2} - \pi - \frac{\pi^2}{2} \right]$$

$$= 0$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi} \left[\int_0^{\pi} (1+x) \cos nx\, dx + \int_{-\pi}^{0} (-1+x) \cos nx\, dx \right]$$

$$= \frac{1}{\pi} \left[(1+x) \left(\frac{\sin nx}{n} \right) - (1) \left(\frac{-\cos nx}{n^2} \right) \right]_0^{\pi} + \left[(-1+x) \frac{\sin nx}{n} - (1) \left(\frac{-\cos nx}{n^2} \right) \right]_{-\pi}^{0} \Bigg\}$$

$$= \frac{1}{\pi} \left\{ \left[\frac{\cos nx}{n^2} \right]_0^{\pi} + \left[\frac{\cos nx}{n^2} \right]_{-\pi}^{0} \right\}$$

$$= \frac{1}{\pi} \left\{ \frac{\cos n(\pi)}{n^2} - \frac{\cos n(0)}{n^2} + \frac{\cos n(0)}{n^2} - \frac{\cos n(-\pi)}{n^2} \right.$$

$$= \frac{1}{\pi} \left\{ \frac{(-1)^2}{n^2} - \frac{1}{n^2} + \frac{1}{n^2} - \frac{(-1)^n}{n^2} \right\}$$

$$\boxed{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi}(1+x)\sin nx\,dx + \int_{-\pi}^{0}(-1+x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left\{\left[(1+x)\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} + \left[(-1+x)\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{0}\right\}$$

$$= \frac{1}{\pi}\left\{\left[(1+x)\left(\frac{-\cos nx}{n}\right)\right]_0^{\pi} + \left[(-1+x)\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{0}\right\}$$

$$= \frac{1}{\pi}\left\{(1+\pi)\left(\frac{-\cos n(\pi)}{n}\right) - (1)\left(\frac{-\cos n(0)}{n}\right) + (-1)\left(\frac{-\cos n(0)}{n}\right) - (-1-\pi)\left(\frac{-\cos n(-\pi)}{n}\right)\right\}$$

$$= \frac{1}{\pi}\left\{-(1+\pi)\frac{(-1)^n}{n} + \frac{1}{n} + \frac{1}{n} - (1+\pi)\frac{(-1)^n}{n}\right\}$$

$$= \frac{1}{\pi}\left[-2(1+\pi)\frac{(-1)^n}{n} + \frac{2}{n}\right]$$

$$b_n = \frac{1}{\pi}\left[-2(1+\pi)\frac{(-1)^n}{n} + \frac{2}{n}\right]$$

If n is even,

$$b_n = \frac{1}{\pi}\left[\frac{-2(1+\pi)}{n} + \frac{2}{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-2}{n} - \frac{2\pi}{n} + \frac{2}{n}\right]$$

$$= \frac{1}{\pi}\left[\frac{-2\pi}{n}\right]$$

$$b_n = \frac{-2}{n}$$

If n is odd,

$$b_n = \frac{1}{\pi}\left[\frac{-2(1+\pi)(-1)}{n} + \frac{2}{n}\right]$$

$$= \frac{1}{\pi}\left[\frac{2}{n} + \frac{2\pi}{n} + \frac{2}{n}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi}{n} + \frac{4}{n}\right]$$

$$= \frac{2}{n} + \frac{4}{n\pi}$$

$$b_n = \frac{2(\pi+2)}{n\pi}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \sum_{n=1}^{\infty} \frac{2(\pi+2)}{n\pi} \sin nx$$

$$= \frac{2(\pi+2)}{\pi} \sin x + \frac{2(\pi+2)}{3\pi} \sin 3x + \frac{2(\pi+2)}{5\pi} \sin 5x + \ldots$$

$$f(x) = \frac{2(\pi+2)}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$$

4. If $f(x) = \begin{cases} -x & \text{in } -\pi < x < 0 \\ x & \text{in } 0 < x \le \pi \end{cases}$, expand $f(x)$ as fourier series in interval $-\pi$ to π. Deduce that $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

Sol:

The fourier series expansion,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\,dx + \int_{0}^{\pi} x\,dx\right]$$

$$= \frac{1}{\pi}\left\{\left[\frac{-x^2}{2}\right]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[0 - \left(\frac{-\pi^2}{2}\right) + \left(\frac{\pi^2}{2} - 0\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi^2}{2}\right]$$

$$a_0 = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\cos nx\, dx + \int_{0}^{\pi} x\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\cos nx\, dx + \int_{0}^{\pi} x\cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left\{\left[(-x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{0} + \left[(x)\left(\frac{\sin nx}{n}\right) - (1)\left(-\frac{\cos nx}{n^2}\right)\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[-\frac{\cos nx}{n^2}\right]_{-\pi}^{0} + \left[\frac{\cos nx}{n^2}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-\cos n(0)}{n^2} - \left(\frac{-\cos n(-\pi)}{n^2}\right) + \frac{\cos n(\pi)}{n^2} - \frac{\cos n(0)}{n^2}\right\}$$

$$= \frac{1}{\pi}\left\{\frac{-1}{n^2} - \frac{-(-1)^n}{n^2} + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{-2}{n^2} + \frac{2(-1)^n}{n^2}\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{-2}{n^2} + \frac{2(-1)^n}{n^2}\right]$$

If n is even,

$$a_n = \frac{1}{\pi}\left[\frac{-2}{n^2} + \frac{2}{n^2}\right]$$

$$a_n = 0$$

If n is odd,

$$a_n = \frac{1}{\pi}\left[\frac{-2}{n^2} - \frac{2}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{-4}{n^2}\right]$$

$$a_n = \frac{-4}{\pi n^2}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} -x\sin nx\, dx + \int_{0}^{\pi} x\sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left\{\left[(-x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{0} + \left[(x)\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{\left[(-x)\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{0} + \left[(x)\left(\frac{-\cos nx}{n}\right)\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi}\left\{(\pi)\left(\frac{-\cos n(-\pi)}{n}\right) + \pi\left(\frac{-\cos n\pi}{n}\right) - 0\right]$$

$$= \frac{1}{\pi}\left[\pi\left(\frac{-(-1)^n}{n}\right) + \pi\,\frac{-(-1)^n}{n}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} - \frac{\pi(-1)^n}{n}\right]$$

$$b_n = \frac{1}{\pi}\left[-2\pi\frac{(-1)^n}{n}\right] = -2\frac{(-1)^n}{n}$$

If n is even,

$$b_n = \frac{-2}{n}$$

If n is odd,

$$b_n = \frac{2}{n}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n\cos nx + b_n\sin nx) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\left[\frac{2}{n}\sin nx\right]$$

$$= \frac{\pi}{2} + \left[2\sin x + \sin 2x + \frac{2}{3}\sin 3x + \cdots\right]$$

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8/1/2020

Unit - 5

25/2/2020

Half Range Fourier Series

Developement in cosine series:- let f(x) can be expressed as a series containing cosines only. And let

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

where,

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

Developement in sine series:

let f(x) can be expressed as a series containing sines only. And let

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

Where, $b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$.

Problem:-

X ① Find the sine series for $f(x) = c$ in the range 0 to π.

Sol:- $f(x) = c$

Half range sine series

$a_0 = 0 \quad a_n = 0$

The fourier series expansion

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} c\sin nx\,dx$$

$$= \frac{2c}{\pi}\int_0^{\pi} \sin nx\,dx$$

$$= \frac{2c}{\pi}\left[\frac{-\cos nx}{n}\right]_0^{\pi}$$

$$= \frac{2c}{\pi}\left[\frac{-\cos n\pi}{n} + \frac{\cos n(0)}{n}\right]$$

$$= \frac{2c}{\pi}\left[-\frac{(-1)^n}{n} + \frac{1}{n}\right]$$

If n is add $b_n = \frac{2c}{\pi}\left(\frac{1}{n} + \frac{1}{n}\right) = \frac{2c}{\pi}\frac{(2)}{n}$

$$= \frac{4c}{n\pi}$$

If n is even $b_n = \frac{2c}{\pi}\left(\frac{-1}{n} + \frac{1}{n}\right) = 0$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= \sum_{n=1}^{\infty} \frac{4c}{n\pi} \frac{\sin nx}{}$$

$$f(x) = \frac{4c}{\pi}\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

$$f(x) = \frac{4c}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \cdots\right]$$

put $x = \pi/2$

$$c = \frac{4c}{\pi}\left[\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$$

$$1 = \frac{4}{\pi}\left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$$

$$\frac{\pi}{4} = \left[1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right]$$

② If $f(x) = \begin{cases} x & \text{when } 0 < x < \pi/2 \\ \pi - x & \text{when } x > \pi/2 \end{cases}$

expand $f(x)$ as a sine series in the interval $(0, \pi)$.

<u>sol:</u>

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$a_0 = 0 \;;\; a_n = 0$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx \right]$$

$$= \frac{2}{\pi} \left[\left[x\left(-\frac{\cos nx}{n}\right) - \left(-\frac{\sin nx}{n^2}\right) \right]_0^{\pi/2} + \left[(\pi - x)\left(-\frac{\cos nx}{n}\right) - (-1)\left(-\frac{\sin nx}{n^2}\right) \right. \right.$$

$$= \frac{2}{\pi} \left[\frac{\sin n(\pi/2)}{n^2} - \frac{(0)\cos n(0)}{n} \right]$$

$$+ \left[(+1)\left(-\frac{\sin n(\pi)}{n^2}\right) - \left(\frac{\pi}{2}\right)(0) + \frac{\sin n}{n^2} \right.$$

$$= \frac{2}{\pi} \left[\frac{1}{n^2} + 0 - 0 + \frac{1}{n^2} \right]$$

$$= \frac{2}{\pi} \left[\frac{2}{n^2} \right]$$

$$= \frac{4}{\pi n^2}$$

Fourier Series expansion

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi n^2} \sin nx$$

$$f(x) = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$$

$$f(x) = \frac{4}{\pi} \left[\frac{\sin x}{1^2} + \frac{\sin 3x}{3^2} + \frac{\sin 5x}{5^2} + \cdots \right]$$

put $x = \pi/2$

$$\pi/2 = 4/\pi \left[\frac{1}{1^2} - \frac{1}{3^2} + 1/5^2 - 1/7^2 + \cdots \right]$$

$$\pi/2 \times \frac{\pi}{4} = \frac{1}{1^2} - 1/3^2 + 1/5^2 - 1/7^2 + \cdots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots \quad //$$

———— x ————

③ Expand a cosine series in the range $(0, \pi)$ $f(x) = \begin{cases} x, & 0 < x < \pi/2 \\ \pi - x, & \pi/2 < x < \pi \end{cases}$

Sol:

The cosine series expansion

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx \right]$$

$$= \frac{2}{\pi} \left\{ \left[\frac{x^2}{2} \right]_0^{\pi/2} + \left[\pi x - \frac{x^2}{2} \right]_{\pi/2}^{\pi} \right\}$$

$$= \frac{2}{\pi} \left\{ \frac{(\pi/2)^2}{2} + \pi(\pi) - \frac{\pi^2}{2} - \pi(\pi/2) + \frac{(\pi/2)^2}{2} \right\}$$

$$= \frac{2}{\pi} \left\{ \frac{\pi^2}{8} + \pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8} \right\}$$

$$= \frac{2}{\pi} \left\{ \frac{2\pi^2}{8} - \frac{\cancel{2}\pi^2}{\cancel{2}} + \pi^2 \right\}$$

$$= \frac{2}{\pi} \left\{ \frac{\pi^2}{4} \right\}$$

$$a_0 = \pi/2$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\,dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} x \cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x) \cos nx\,dx \right]$$

$$= \frac{2}{\pi} \left[\left(x \frac{\sin nx}{n} \right) - (1)\left(-\frac{\cos nx}{n^2} \right) \right]_0^{\pi/2} + (\pi - x)\left(\frac{\sin nx}{n} \right) - (1)\left(-\frac{\cos nx}{n^2} \right) \Big]_{\pi/2}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2}\left(\frac{[illegible]}{n}\right) - \left(\frac{-\cos n(0)}{n^2}\right) + \left(\frac{-\cos n(\pi)}{n^2}\right) - \left(\pi \cdot \frac{\pi}{2}\right)\left(\frac{\sin n(\pi/2)}{n}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2}\left(1/n\right) - \left(1/n^2\right) - \left(\frac{(-1)^n}{n^2}\right) - \left(\frac{\pi}{2}(1/n)\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2n} - \frac{1}{n^2} - \frac{(-1)^n}{n^2} - \frac{\pi}{2n}\right]$$

$$a_n = \frac{-2}{\pi}\left[\frac{1}{n^2} + \frac{(-1)^n}{n^2}\right]$$

If n is odd

$$a_n = \frac{-2}{\pi}\left[1/n^2 - 1/n^2\right] = 0$$

If n is even.

$$a_n = \frac{-2}{\pi}\left[\frac{1}{n^2} + \frac{1}{n^2}\right] = \frac{-2}{\pi}\left[\frac{2}{n^2}\right] = \frac{-4}{\pi n^2}$$

$$f(x) = \frac{\pi}{4} - \sum_{n=1}^{\infty} \frac{4}{\pi n^2} \cos nx$$

$$= \frac{\pi}{4} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$

$$f(x) = \frac{\pi}{4} - \frac{4}{\pi}\left(\frac{\cos 2x}{2^2} + \frac{\cos 4x}{4^2} + \cdots\right)$$

change of intervals

To find a fourier series for an interval which is not of length π (or) 2π in many problems. The period of the function 'n' is to be expanded is not 2π but some other interval by $2l$.

Suppose we have to expand $f(x)$ in the interval $-l$ to l as a fourier series.

let $X = \frac{\pi x}{l}$ (i.e) $x = \frac{Xl}{\pi}$

when $x = -l$, $X = -\pi$ and

when $x = l$ $X = \pi$

Hence the function becomes

$f\left(\frac{Xl}{\pi}\right)$ when $-\pi < X < \pi$.

$f\left(\frac{Xl}{\pi}\right)$ can be expanded as a fourier series of the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

where, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lX}{\pi}\right) \cos nX \, dx$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f\left(\frac{lX}{\pi}\right) \sin nX \, dx$.

$(n = 0, 1, 2, 3, \ldots)$

Reveling back to the original variable

$x = \frac{lX}{\pi}$, $dx = \frac{l dx}{\pi}$

where, $X = \pi$, $x = l$, when $X = -\pi$, $x = -l$

$a_n = \frac{1}{\pi}\int_{-l}^{l} f(x) \cos n\pi \frac{x}{l}, \frac{\pi dx}{l}$

$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos n\pi \frac{x}{l} \, dx$

III^ly $b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l}\right)$$

(i) If $f(x)$ is an even function $f(x)$ can be expanded as a fourier series consisting of cosine terms only in the interval of length $2l$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx.$$

(ii) If $f(x)$ is an odd function $f(x)$ can be expanded as a fourier series consisting of sine terms only in the interval of length $2l$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$b_n = 2/l\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$

(iii) If $f(x)$ can be expanded as a sine series in half range $(0-l)$ with period l of the form.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}.$$

where, $b_n = 2/l\int_0^l f(x)\sin\frac{n\pi}{l}\,dx$

(iv) $f(x)$ can be expanded as a cosine series in a half range $(0,-l)$ with period l of the form.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}$$

where, $a_0 = 2/l\int_0^l f(x)\,dx$

$$a_n = 2/l\int_0^l f(x)\cos\frac{n\pi}{l}\,dx$$

① In the range $(0, 2l)$ $f(x)$ is defined by the relations $f(x) = \begin{cases} a & \text{when } 0<x<l \\ a & \text{when } 0<x<2l \end{cases}$

Expand $f(x)$ as a fourier series of period $2l$.

Sol:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$a_0 = 1/l \int_{l}^{2l} f(x).dx$$

$$= 1/l \int_{l}^{2l} a\,dx$$

$$= a/l \left[x\right]_{l}^{2l}$$

$$= \frac{a}{l}\left[2l - l\right]$$

$$\boxed{a_0 = a}$$

$$a_n = 1/l \int_{l}^{2l} f(x)$$

$$= 1/l \int_{l}^{2l} a \cos\frac{n\pi x}{l}\,dx$$

$$= a/l \left[\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi x}{l}}\right]_{l}^{2l}$$

$$= 0$$

[∴ since term will be zero applying all the limit value except $\pi/2$]

$$b_n = 1/l \int_{l}^{2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= 1/l \int_{l}^{2l} a \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= a/l \left[\frac{-\cos \pi (x/l)}{n\pi/l} \right]_{l}^{2l}$$

$$= -a/l \left[\frac{\cos n\pi \frac{(2l)}{l}}{n\pi/l} + \frac{\cos n\pi (l/l)}{n\pi/l} \right]$$

$$= -a/l \left[\frac{\cos 2n\pi}{n\pi/l} + \frac{\cos n\pi}{n\pi/l} \right]$$

$$= -a/l \times l/n\pi \cdot [1 + (-1)^n]$$

$$b_n = \frac{-a}{n\pi} [1 + (-1)^n]$$

When n is even

$$b_n = 0$$

When n is odd

$$b_n = \frac{-2a}{n\pi}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= a/2 + \sum_{n=1}^{\infty} \frac{-2a}{n\pi} \sin \frac{n\pi x}{l}$$

$$= \frac{a}{2} - \frac{2a}{\pi} \left[\sum_{n=1}^{\infty} 1/n \sin \frac{\pi x}{l} \right]$$

$$= \frac{a}{2} - \frac{2a}{\pi} \left[\sin \frac{\pi x}{l} + 1/3 \sin \frac{3\pi x}{l} + \frac{1}{5} \frac{\sin 5\pi x}{l} \right]$$

//

2. Express $f(x) = c-x$, where $0<x<c$ as a half range cosine series with period $2c$.

Soln:

Half range cosine series

$b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c}$$

$$a_0 = \frac{2}{c}\int_0^c f(x)\,dx$$

$$= \frac{2}{c}\int_0^c (c-x)\,dx$$

$$= \frac{2}{c}\left[cx - \frac{x^2}{2}\right]_0^c$$

$$= \frac{2}{c}\left[c(c) - \frac{c^2}{2}\right]$$

$$= \frac{2}{c}\left[c^2 - \frac{c^2}{2}\right]$$

$$= \frac{2}{c}\left(\frac{c^2}{2}\right)$$

$$a_0 = c$$

$$a_n = \frac{2}{c}\int_0^c f(x)\cos\frac{nx\pi}{c}\,dx$$

$$= \frac{2}{c}\int_0^c (c-x)\cos\frac{nx\pi}{c}\,dx$$

$$= \frac{2}{c}\left[(c-x)\left(\frac{\sin\frac{n\pi x}{c}}{\frac{n\pi}{c}}\right) - (-1)\left(\frac{-\cos\frac{n\pi x}{c}}{\frac{n^2\pi^2}{c^2}}\right)\right]_0^c$$

$$= -\frac{2}{c}\left[\frac{\cos\frac{n\pi x}{c}}{\frac{n^2\pi^2}{c^2}}\right]_0^c$$

$$= \frac{-2}{c}\times\frac{c^2}{n^2\pi^2}\left[\cos\frac{n\pi x}{c}\right]_0^c$$

$$= \frac{-2c}{n^2\pi^2}\left(\cos\frac{n\pi c}{c} - \frac{\cos n\pi(0)}{c}\right)$$

$$= \frac{-2c}{n^2\pi^2}\left[(-1)^n - 1\right]$$

If n is odd, $a_n = \frac{-2c}{n^2\pi^2}\left[(-1)^n - 1\right]$

$$= \frac{4c}{n^2\pi^2}$$

If n is even, $a_n = \frac{-2c}{n^2\pi^2}\left[-1+1\right]$

$$= 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{c}$$

$$= \frac{c}{2} + \sum_{n=1}^{\infty} \frac{4c}{n^2\pi^2}$$

2) Find a fourier series with period 3 to represent $f(x) = 2x - x^2$ in the range $(0,3)$

Soln: Given, $2l = 3$

$l = 3/2$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\frac{2n\pi x}{3} + b_n \sin\frac{2n\pi x}{3}\right]$$

$$a_0 = \frac{2}{3}\int_0^3 f(x)\,dx$$

$$= \frac{2}{3}\int_0^3 (2x - x^2)\,dx$$

$$= \frac{2}{3}\left[\frac{2x^2}{2} - \frac{x^3}{3}\right]_0^3$$

$$= \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3$$

$$= \frac{2}{3}\left[9 - \frac{27}{3}\right]$$

$$= \frac{2}{3}\left[9 - 9\right]$$

$$a_n = \frac{2}{3}\int_0^3 f(x)\cos\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\int_0^3 (2x-x^2)\cos\frac{2n\pi x}{3}\,dx$$

$$= \frac{2}{3}\left\{\left[(2x-x^2)\frac{\sin\frac{2n\pi x}{3}}{\frac{2n\pi}{3}}\right] - \left[(2-2x)\left(\frac{-\cos\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^2}\right)\right] + \left[(-2)\frac{-\sin\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^3}\right]\right\}_0^3$$

$$= \frac{2}{3}\left\{-(2-2x)\left(\frac{-\cos\frac{2n\pi x}{3}}{\left(\frac{2n\pi}{3}\right)^2}\right)\right\}_0^3$$

$$= \frac{2}{3}\left\{(2-2x)\left(\frac{\cos\frac{2n\pi x}{3}}{\frac{4n^2\pi^2}{9}}\right)\right]_0^3$$

$$= \frac{\cancel{2}}{3}\times\frac{9}{\cancel{4}n^2\pi^2}\left[[2-2(3)]\left(\cos\frac{2n\pi(3)}{3}\right) - [(2-2(0))]\left[\cos\frac{2n\pi(0)}{3}\right]\right]$$

$$= \frac{3}{2n^2\pi^2}\left[(2-6)(\cos 2n\pi) - (2)(1)\right]$$

$$= \frac{3}{2n^2\pi^2}\left[(-4)(+1)^{2n} - 2\right]$$

$$= \frac{3}{2n^2\pi^2}\left[(+1)^{n}(-4) - 2\right]$$

$$= \frac{3}{2n^2\pi^2}\left[-4-2\right]$$

$$= \frac{-3(-6)^3}{\cancel{2}n^2\pi^2} = \frac{-9}{n^2\pi^2}$$

$$b_n = \frac{2}{3}\int_0^3 f(x)\, \sin\frac{2n\pi x}{3}\, dx$$

$$= \frac{2}{3}\int_0^3 (2x - x^2)\sin\frac{2n\pi x}{3}\, dx$$

$$= \frac{2}{3}\left[(2x-x^2)\left(\frac{-\cos\frac{2n\pi x}{3}}{2n\pi/3}\right) - (2-2x)\left(\frac{-\sin\frac{2\pi x}{3}}{\frac{4n^2\pi^2}{9}}\right) + (-2)\left[\frac{\cos\frac{2n\pi x}{3}}{8n^3\pi^3/27}\right]\right]_0^3$$

$$= \frac{2}{3}\left[(2x-x^2)\left[\frac{-\cos\frac{2n\pi x}{3}}{2n\pi/3}\right] - 2\left(\frac{\cos\frac{2n\pi x}{3}}{8n^3\pi^3/27}\right)\right]_0^3$$

$$= \frac{2}{3}\left[(2(3)-3^2)\left(\frac{-\cos\frac{2n\pi(3)}{3}}{2n\pi/3}\right) - 2\left(\frac{\cos\frac{2n\pi(3)}{3}}{8n^3\pi^3/27}\right) + 2\frac{\cos 2n\pi(0)}{3}\right]$$

$$= \frac{2}{3}\left[(-3)\left(\frac{-3}{2n\pi}\right) - 2\left(\frac{27}{8n^3\pi^3}\right) + 2\left(\frac{27}{8n^3\pi^3}\right)\right]$$

$$= \frac{2}{3}\left[\frac{9}{2\pi n}\right]$$

$$= 3/n\pi$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$2x - x^2 = \sum_{n=1}^{\infty}\left(\frac{-9}{n^2\pi^2}\cdot\cos\frac{2n\pi x}{l} + b_n\ 3/n\pi \sin\frac{2n\pi x}{3}\right)$$