Unit - I

Vector space.

A non empty set V said to be a vector space over a field F if

(i) V is an abelian group under an operation called addition which we denote by +

(ii) For every $\alpha \in F$ and $v \in V$ there is defined an element αv in V subject to the following conditions.

(a) $\alpha(u+v) = \alpha u + \alpha v$; $\forall$ $u, v \in V$ and $\alpha \in F$

(b) $(\alpha+\beta)u = \alpha u + \beta u$; $\forall$ $u \in V$ and $\alpha, \beta \in F$

(c) $\alpha(\beta u) = (\alpha\beta)u$; $\forall$ $u \in V$ and $\alpha, \beta \in F$

(d) $u = u$; $\forall$ $u \in V$

Remark:

The Elements of F are called scalar and the elements of V are called vectors

The Rule which associated with each scalar $\alpha \in F$ and a vector $u \in V$, a vector αv is called the scalar multiplication.

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Thus a scalar multiplication gives rise to a function from $F \times V \to V$ defined by $(\alpha, u) \to \alpha v$.

problem :-

1. $R \times R$ is a vector spaces over R under addition and scalar multiplication defined by, $(x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$ and

$$\alpha (x_1, x_2) = \alpha x_1 + \alpha x_2 .$$

proof :-

Clearly the binary operation $+$ is cummulative and associative and $(0,0)$ is the zero element.

The Inverse of (x_1, x_2) is $(-x_1, -x_2)$

Hence, $(R \times R, +)$ is an abelian group.

Now, let $u = (x_1, x_2)$ and $v = (y_1, y_2)$ and $\alpha, \beta \in R$.

Then,

$$\alpha (u+v) = \alpha [(x_1, x_2) + (y_1, y_2)]$$
$$= \alpha [x_1 + y_1, x_2 + y_2)]$$
$$= [\alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2]$$
$$= \alpha (x_1, x_2) + \alpha (y_1, y_2)$$
$$\boxed{\alpha (u+v) = \alpha u + \alpha v}$$

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Now, $(\alpha+\beta) u = (\alpha+\beta)(x_1, x_2)$

$$= (\alpha+\beta) x_1, (\alpha+\beta) x_2$$

$$= \alpha x_1 + \beta x_1, \alpha x_2 + \beta x_2$$

$$= \alpha x_1, \alpha x_2 + \beta x_1, \beta x_2$$

$$= \alpha(x_1, x_2) + \beta(x_1, x_2)$$

$$\therefore \boxed{(\alpha+\beta)u = \alpha u + \beta u}$$

Also, $\alpha(\beta u) = \alpha[\beta(x_1, x_2)]$

$$= \alpha[\beta x_1, \beta x_2)$$

$$= \alpha\beta x_1, \alpha\beta x_2$$

$$= \alpha\beta(x_1, x_2)$$

$$\therefore \boxed{\alpha(\beta u) = \alpha\beta(u)}$$

obviously, $\boxed{u = u}$

$R \times R$ is a vector over R.

Q. $R^n = \{(x_1, x_2, \ldots, x_n) / x_i \in R, 1 \leq i \leq n\}$ then R^n is a vector space over R under addition and scalar Multiplication defind by $(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n) =$

$$(x_1+y_1, x_2+y_2, \ldots, x_n+y_n)$$

and $\alpha(x_1, x_2, \ldots, x_n) = (\alpha x_1, \alpha x_2, \ldots, \alpha x_n)$.

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proof:

clearly. the binary operation + is commutative and Associative

$(0, 0, \ldots, 0)$ is the zero element

The inverse of $(x_1, x_2, \ldots, x_n)$ is $(-x_1, -x_2, \ldots, -x_n)$

Hence, $(R^n, +)$ is an abelian group

Now,

$u = (x_1, x_2, \ldots, x_n)$ and

$v = (y_1, y_2, \ldots, y_n)$

$\alpha, x \in R$

Then, $\alpha(u+v) = \alpha[(x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n)]$

$= \alpha[(x_1+y_1), (x_2+y_2), \ldots, (x_n+y_n)]$

$= \alpha x_1 + \alpha y_1, \alpha x_2 + \alpha y_2, \ldots, \alpha x_n + \alpha y_n$

$= \alpha(x_1, x_2, \ldots, x_n) + \alpha(y_1, y_2, \ldots, y_n)$

$$\boxed{\alpha(u+v) = \alpha u + \alpha v}$$

Similarly,

$(\alpha + \beta) u = (\alpha + \beta)(x_1, x_2, \ldots, x_n)$

$= \alpha(x_1, x_2, \ldots, x_n) + \beta(x_1, x_2, \ldots, x_n)$

$$\boxed{(\alpha + \beta) u = \alpha u + \beta u}$$

$\alpha(\beta u) = \alpha[\beta(x_1, x_2, \ldots, x_n)]$

$= \alpha[\beta x_1, \beta x_2, \ldots, \beta x_n]$

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$\alpha(\beta u) = \alpha\beta(x_1, x_2, \ldots, x_n)$

$$\boxed{\alpha(\beta u) = \alpha\beta(u)}$$

obviously

$$\boxed{u = u}$$

R^n is vector space over R.

Note:

we denote this vector space by $V_n(R)$.

2. Let V denote the set of all solutions of differential equation $2\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 3y = 0$. Then V is a vector space over R.

Proof:

Let $f, g \in V$ and $\alpha \in R$

Then,
$$2\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 3y = 0$$

$$\Rightarrow 2\frac{d^2f}{dx^2} - 7\frac{df}{dx} + 3f = 0$$

$$\Rightarrow 2\frac{d^2g}{dx^2} - 7\frac{dg}{dx} + 3g = 0$$

$$\therefore \left[2\frac{d^2f}{dx^2} - 7\frac{df}{dx} + 3f\right] + \left[2\frac{d^2g}{dx^2} - 7\frac{dg}{dx} + 3g\right] = 0$$

$$2\left[\frac{d^2f}{dx^2} + \frac{d^2g}{dx^2}\right] - 7\left[\frac{df}{dx} + \frac{dg}{dx}\right] + 3\left[f + g\right] = 0$$

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$$\alpha \frac{d^2}{dx^2}[f+g] - 7\frac{d}{dx}[f+g] + 3[f+g] = 0$$

Hence, $f+g \in V$

Also,

$$\alpha \frac{d^2}{dx^2}(\alpha f) - 7\frac{d}{dx}(\alpha f) + 3(\alpha f) = 0$$

Hence, $\alpha f \in V$

Since the operation are usual addition and scalar multiplication. The Axioms of vector space are true.

Hence, V is a vector space over R.

4. R is not a vector space over C.

Proof:

clearly $(R, +)$ is an abelian group.

But the scalar multiplication is not defined, for if $\alpha = a+ib \in C$ and $u \in R$, then $\alpha u = au + ibu \notin R$

$\therefore$ R is not a vector space over C.

5. consider $R \times R$ with usual addition, we define scalar multiplication by $\alpha(x, y) = (\alpha x, \alpha^2 y)$. Then $R \times R$ is not a vector space over R.

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Proof:-

clearly $R \times R$ with addition is an abelian group.

$$(\alpha+\beta)(x,y) = ((\alpha+\beta)x, (\alpha+\beta)^2 y)$$
$$= (\alpha x+\beta x, \alpha^2 y+\beta^2 y+2\alpha\beta y)$$

Also,

$$\alpha(x,y)+\beta(x,y) = (\alpha x, \alpha^2 y) + (\beta x, \beta^2 y)$$
$$= (\alpha x+\beta x, \alpha^2 y+\beta^2 y)$$

Hence $(\alpha+\beta)(x,y) \neq \alpha(x,y)+\beta(x,y)$.

$R \times R$ is not a vector space over R.

6. Consider $R \times R$ with usual addition, Define the scalar multiplication as $\alpha(a,b) = (0,0)$.

Proof:-

clearly $R \times R$ is an abelian group also,

$$\alpha(u+v) = 0$$
$$\alpha u+\alpha v = 0+0 = 0$$
$$\alpha(u+v) = \alpha u+\alpha v$$

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III'ly $(\alpha+\beta)u = \alpha u + \beta u = 0$

$\alpha(\beta u) = (\alpha\beta)u = 0$

$1(a,b) = 0,0$

Hence it is not a vector space.

7. Let V be the set of all ordered pairs of real numbers. Addition and multiplication are defined by $(x+y)+(x_1+y_1) = (x+x_1, y+y_1)$ and $\alpha(x,y) = (x, \alpha y)$ where x, y, x_1, y_1, and α are real numbers. Then V is not a vector space over R.

proof:

Clearly V is an abelian group under the operation + defined above.

Let $\alpha, \beta \in R$ and $(x,y) \in V$

Now, $(\alpha+\beta)(x,y) = (x, (\alpha+\beta)y)$

$= (x, \alpha y + \beta y)$

$\therefore (\alpha+\beta)(x,y) \neq \alpha(x,y) + \beta(x,y)$

Hence V is not a vector space over R.

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8. Let R^+ be the set of all positive real numbers. Define addition and scalar multiplication as follows $u+v = uv \ \forall u,v \in R^+$

$\alpha u = u^\alpha \ \forall \ u \in R^+$ and $\alpha \in R$. Then R^+ is a real vector.

proof:

clearly $(R^+, +)$ is an abelian group with identity 1.

Now, $\alpha(u+v) = \alpha(uv)$

$= (uv)^\alpha$

$= u^\alpha v^\alpha = u^\alpha + v^\alpha$

$= \alpha u + \alpha v$

$(\alpha+\beta)u = u^{\alpha+\beta} = u^\alpha u^\beta$

$= u^\alpha + u^\beta \Rightarrow \alpha u + \beta u$

$\alpha(\beta u) = \alpha u^\beta$

$= (u^\beta)^\alpha = (u)^{\alpha\beta}$

$= (\alpha\beta)u$

Also. $1u = u^1 = u$

$\therefore R^+$ is a vector space over R.

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Remark :-

commutativity of addition in a vector space can be derived from the other axioms of the vector space (i.e), the axiom of ~~addition~~ commutativity of addition in a vector space is redundant for

$$(1+1)(u+v) = 1(u+v) + 1(u+v)$$
$$= 1u + 1v + 1u + 1v$$
$$= u + v + u + v \rightarrow (1)$$

Also
$$(1+1)(u+v) = (1+1)u + (1+1)v$$
$$= u + u + v + v \rightarrow (2)$$
$$u + v + u + v = u + u + v + v$$
$$v + u = u + v$$

Theorem 5.1 :-

Let V be a vector space over a field F. Then,

(i) $\alpha 0 = 0 \ \forall \alpha \in F$.

(ii) $0v = 0 \ \forall v \in V$

(iii) $(-\alpha)v = \alpha(-v) = -(\alpha v) \ \forall \alpha \in F$ and $v \in V$

(iv) $\alpha v = 0 \Rightarrow \alpha = 0$ or $v = 0$

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without Basicss of Dimension / generality we can

assume that
Linear Independent:- $s_1, s_2, \ldots, s_m \in S$

and Let V be a vector space over a
field F. A finite set of vectors $v_1, v_2, \ldots, v_n$ ($s_{m+1}, \ldots, s_n \in T$)

in V is said to be Linear Independent. Hence $\alpha_1 s_1 + \alpha_2 s_2 + \cdots + \alpha_m s_m$

$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$; $\alpha_{m+1} s_{m+1} + \cdots + \alpha_n s_n \in L(T)$.

$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$. ; $s = (\alpha_1 s_1 + \cdots + \alpha_m s_m)$

$+ (\alpha_{m+1} s_{m+1} + \cdots + \alpha_n s_n) \in L(S) + L(T)$

If $v_1, v_2, \ldots, v_n$ are not linearly

Hence $L(S \cup T) \subseteq L(S) + L(T)$

independent then they are said to be

Also by (a) $L(S) \subseteq L(S \cup T)$ and

linearly dependent. $L(T) \subseteq L(S \cup T)$.

Note:- Hence $L(S) + L(T) \subseteq L(S \cup T)$

If $v_1, v_2, \ldots, v_n$ are linear

Hence $L(S \cup T) = L(S) + L(T)$

dependent then there exist scalars $\alpha_1, \alpha_2, \ldots, \alpha_n$

(c) Let $L(S) = S$. By them 5.9 $L(S) = S$ is a

not all zero, such that $\alpha_1 v_1 + \cdots + \alpha_n v_n = 0$.

subspace of V.

Example:-

① In $V_n(F)$, $\{e_1, e_2, \ldots, e_n\}$ is linear

Conversely, let S be a subspace of V. Then the

independent set of vectors

smallest subspace containing S is S itself.

$\alpha_1 e_1 + \alpha_2 e_2 + \cdots + \alpha_n e_n = 0$

Hence $L(S) = S$

$\Rightarrow \alpha_1 (1, 0, \ldots, 0) + \cdots$; Corollary:- $L(L(S)) = L(S)$

$+ \alpha_n (0, 0, \ldots, 1) = (0, 0, \ldots, 0)$

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$\Rightarrow (\alpha_1 \alpha_2, \ldots, \alpha_n) = (0, 0, \ldots, 0)$

$\Rightarrow \alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$

② In $V_3(R)$ the vectors $(1,2,1)$; $(2,1,0)$ & $(1,-1,2)$ are linearly independent. For,

let $\alpha_1(1,2,1) + \alpha_2(2,1,0) + \alpha_3(1,-1,2) = (0,0,0)$

$\therefore (\alpha_1 + 2\alpha_2 + \alpha_3,\ 2\alpha_1 + \alpha_2 - \alpha_3,\ \alpha_1 + 2\alpha_3) = (0,0,0)$.

$\therefore \alpha_1 + 2\alpha_2 + \alpha_3 = 0 \longrightarrow (1)$

$2\alpha_1 + \alpha_2 - \alpha_3 = 0 \longrightarrow (2)$

$\alpha_1 + 2\alpha_3 = 0 \longrightarrow (3)$

Solving equation (1), (2), & (3) we get

$\alpha_1 = \alpha_2 = \alpha_3 = 0$.

$\therefore$ The given vectors are linearly independent.

③ In $V_3(R)$ the vectors $(1,4,-2)$, $(-2,1,3)$ and $(-4,11,5)$ are linearly dependent.

For, let

$\alpha_1(1,4,-2) + \alpha_2(-2,1,3) + \alpha_3(-4,11,5) = (0,0,0)$

$\therefore \alpha_1 - 2\alpha_2 - 4\alpha_3 = 0 \longrightarrow (1)$

$\therefore 4\alpha_1 + \alpha_2 + 11\alpha_3 = 0 \longrightarrow (2)$

$\therefore -2\alpha_1 + 3\alpha_2 + 5\alpha_3 = 0 \longrightarrow (3)$

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From (1) & (2)

$$\frac{\alpha_1}{-18} = \frac{\alpha_2}{-27} = \frac{\alpha_3}{9} = k \text{ (say)}.$$

$\therefore \alpha_1 = -18k, \alpha_2 = -27k, \alpha_3 = 9k.$

These values of α_1, α_2, and α_3 for any k satisfy (3) also,

Taking $k = 1$, we get

$\alpha_1 = -18, \alpha_2 = -27, \alpha_3 = 9$ as a non-trivial solution.

Hence the three vectors are linearly dependent.

(4) Let V be a vector space over a field F. Then any subset S of V containing the zero vector is linearly dependent.

Proof: Let $S = \{0, v_1, \ldots\ldots, v_n\}$

Clearly $\alpha 0 + 0v_1 + 0v_2 + \cdots + 0v_n = 0$, where α is any element of F. Hence for any $\alpha \neq 0$, we get non-trivial linear combination of vectors in S giving the zero vector. Hence S is linearly dependent.

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Theorem :- 5.11

Any subset of a linearly independent set is linearly independent.

Proof :-

Let V be a vector space over a field F.

Let $S = \{v_1, v_2, \ldots\ldots, v_n\}$ be a linearly independent set.

Let S' be a subset of S. Without loss of generality we take $S' = \{v_1, v_2, \ldots, v_k\}$ where $k \leq n$.

Suppose S' is a linearly dependent set. Then there exist $\alpha_1, \alpha_2, \ldots\ldots, \alpha_k$ in F not all zero, such that.

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_k v_k = 0.$$

Hence $\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_k v_k = 0$

Hence $\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_k v_k + 0 v_{k+1} + \ldots + 0 v_n =$ is a non-trivial linear combination giving the zero vector.

Here S is a linearly dependent set which is a contradiction.

Hence S' is linearly independent.

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Theorem 5.12 :-

Any set containing a linearly dependent set is also linearly dependent.

proof :-

Let V be a vector space.

Let S be a linearly dependent set.

$$S' \supset S$$

If S' is linearly independent S is also linearly independent (by thm 5.11) which is a contradiction.

Hence S' is linearly dependent.

Theorem 5.13 :-

Let $S = \{v_1, v_2, \ldots, v_n\}$ be a linearly independent set of vectors in a vector space V over a field F. Then every element of $L(S)$ can be uniquely written in the form.

$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$ where $\alpha_i \in F$.

Proof :-

By definition every elements of $L(S)$ is of the form

$$\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_n v_n$$

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Now, Let $\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = \beta_1 v_1 + \beta_2 v_2 + \cdots + \alpha_n v_n$

Hence, $(\alpha_1 - \beta_1)v_1 + (\alpha_2 - \beta_2)v_2 + \cdots + (\alpha_n - \beta_n)v_n = 0$

Since S is a linearly independent set

$$\alpha_i - \beta_i = 0 \quad \forall i$$

$$\therefore \alpha_i = \beta_i \quad \forall i$$

Hence the theorem.

Theorem 5.14:-

$S = \{v_1, v_2, \ldots, v_n\}$ is a linearly dependent set of vectors in V iff there exists a vector $v_k \in S$ such that v_k is a linear combination of the preceding vector $v_1, v_2, \ldots, v_{k-1}$.

proof:-

Suppose $v_1, v_2, \ldots, v_n$ are linearly depent.

Then there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in F$ not all zeros, such that

$$\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n = 0$$

Let k be the largest integer for which $\alpha_k \neq 0$

Then $\alpha_1 v_1 + \cdots + \alpha_k v_k = 0$

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$\therefore \alpha_k v_k = -\alpha_1 v_1 - \alpha_2 v_2 \cdots\cdots - \alpha_{k-1} v_{k-1}$

$$v_k = (-\alpha_k^{-1}\alpha_1) v_1 + \cdots\cdots + (-\alpha_k^{-1} \alpha_{k-1}) v_{k-1}$$

$\therefore$ v_k is a linear combination of the preceding vectors.

Conversely, suppose there exists a vector v_k such that $v_k = \alpha_1 v_1 + \cdots\cdots + \alpha_{k-1} v_{k-1}$

Hence $-\alpha_1 v_1 - \cdots\cdots - \alpha_{k-1} v_{k-1} + v_k + 0v_k + \cdots + 0v_n = 0$

Since the coefficient of $v_k = 1$, we have $S = \{v_1, \cdots\cdots, v_n\}$ is linearly dependent.

Example:

In $V_3(R)$, let $S = \{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$

Here $(1,1,1) = (1,0,0) + (0,1,0) + (0,0,1)$.

Thus $(1,1,1)$ is a linear combination of the preceding vectors. Hence S is a linearly dependent set.

Theorem 5.15:

Let V be a vector space over F. Let $S = \{v_1, v_2, \cdots\cdots, v_n\}$ and $L(S) = W$. Then there exists a linearly independent subset S' of S such that $L(S') = W$.

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proof: Let $S = \{v_1, v_2, \ldots, v_n\}$.

If S is linearly independent there is nothing to pro

If not, let v_k be the first vector in S which is a linear combination of the preceding vectors.

Let $S_1 = \{v_1, v_2, \ldots, v_{k-1}, v_{k+1}, \ldots, v_n\}$.

(i.e) S_1 is obtained by deleting the vector v_k from S.

we claim that $L(S_1) = L(S) = W$.

Since $S_1 \subseteq S$, $L(S_1) \subseteq L(S)$.

Not, let $v \in L(S)$.

Then $v = \alpha_1 v_1 + \cdots + \alpha_k v_k + \cdots + \alpha_n v_n$

Now, v_k is a linear combination of the preceding vectors.

Let $v_k = \beta_1 v_1 + \cdots + \beta_{k-1} v_{k-1}$

Hence $v = \alpha_1 v_1 + \cdots + \alpha_{k-1} v_{k-1} + \alpha_k(\beta_1 v_1 + \cdots + \beta_{k-1} v_{k-1}) + \alpha_{k+1} v_{k+1} \cdots + \alpha_n v_n$

$\therefore v$ can be expressed as a linear combination of the vectors of S_1 so that $v \in L(S_1)$. Hence $L(S) \subseteq L(S_1)$.

Thus $L(S) = L(S_1) = W$

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Now, if S_1 is linearly independent, the proof is compelete.

If not, we continue the above process of removing a vector from S_1, which is a linear combination of the preceeding vectors until we arrive at a linearly independent subset S' of S such that $L(S') = W$.

Basis and Dimension:-

A linearly independent subset S of a vector space V which spans the whole space V is called a basis of the vector space.

Theorem 5.16:-

Any finite-dimensional vector space V contains a finite number of linearly independent vectors which span V. (i.e) A finite dimensional vector spacer has a basis consisting of a finite number of vectors.

proof:- Since V is finite dimensional there exists a finite subset S of V such that $L(S) = V$. By theorem this set S contains a linearly independent subset $S' = \{v_1, v_2, \ldots, v_n\}$ such that $L(S') = L(S) = V$.

Hence S' is a basis for V.

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Theorem 5.17:

Let V be a vector space over a field F. Then $S = \{v_1, v_2, \dots, v_n\}$ is a basis for V iff every element of V can be uniquely expressed as a linear combination of element of S.

Proof:

Let S be a basis for V

Then by definition S is linearly independent and $L(S) = V$. Hence by theorem, every element of V can be uniquely expressed as a linear combination of elements of S.

conversely, suppose every element of V can be uniquely expressed as a linear combination of elements of S.

clearly $L(S) = V$.

Now, let $\alpha_1 v_1 + \alpha_2 v_2 + \dots + \alpha_n v_n = 0$

Also $0v_1 + 0v_2 + \dots + 0v_n = 0$

Thus we have expressed 0 as a linear combination of vectors of S in two ways.

$\therefore$ By hypothesis $\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

Hence S is linearly independent. Hence S is a basis.

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Theorem:

Any two bases of a finite dimensional vector space V have the same number of elements.

proof:

Since V is finite dimensional, it has a basis say $S = \{v_1, v_2, \ldots, v_n\}$, $S' = \{w_1, w_2, \ldots, w_m\}$ be any other basis for V.

Now, $L(S) = V$ and S' is a set of m linearly independent vectors. Hence by Theorem, $m \leq n$.

Also, since $L(S') = V$ and S is a set of n linearly independent vectors, $n \leq m$ Hence $m = n$.

Definition:

Let V be a finite dimensional vector space over a field F. The number of elements in any basis of V is called the dimension of V and is denoted by dim V.

Theorem:

Let V be a vector space of dimension n. Then, (i) any set of m vectors where $m > n$ is linearly dependent.

(ii) any set of m vectors where $m < n$ cannot span V.

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proof:

(i) Let $S = \{v_1, v_2, \dots, v_n\}$ be a basis of V. Hence $L(S) = V$.

Let S' be any set consisting of m vectors where $m > n$. Suppose S' is linearly independent since S span V by Theom, $m \leq n$ which is a contradiction.

Hence S' is linearly dependent.

(ii) Let S' be a set consisting of m vectors where $m < n$. Suppose $L(S') = V$

Now, $S = \{v_1, v_2, \dots, v_n\}$ is a basis for V and hence linearly independent. Hence by Them, $n \leq m$ which is a contradiction.

Hence S' cannot span V.

Theorem:

Let V be a finite dimensional vector space over a field F. Any linearly independent set of vectors in V is part of a basis.

proof:

Let $S = \{v_1, v_2, \dots, v_r\}$ be a linearly independent set of vectors.

If $L(S) = V$ then S itself is a basis.

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If $L(S) \neq V$, choose an element $v_{r+1} \in V - L(S)$

Now, consider $S_1 = \{v_1, v_2, \ldots, v_r, v_{r+1}\}$

we shall prove that S_1 is linearly independent by showing that no vector in S_1 is a linear combination of the preceeding vectors.

Since $\{v_1, v_2, \ldots, v_r\}$ is linearly independent, v_i where $1 \leq i \leq r$ is not a linear combination of the preceeding vectors.

Also $v_{r+1} \notin L(S)$ and hence v_{r+1} is not a linear combination of $v_1, v_2, \ldots, v_r$.

Hence S_1 is linearly independent.

If $L(S_1) = V$, then S_1 is a basis for V. If not we take an element $v_{r+2} \in V - L(S_1)$ and proceed as before. Since the dimension of V is finite, this process must stop at a certain stage giving the required basis containing S.

Theorem:

Let V be a finite dimensional vector space over a field F. Let A be a subspace of V. Then there exists a subspace B of V such that $V = A \oplus B$

proof:

let $S = \{v_1, v_2, \ldots, v_r\}$ be a basis of A.

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By theorem, we can find $w_1, w_2, \ldots, w_s \in V$ such that $S' = \{v_1, v_2, \ldots, v_r, w_1, w_2, \ldots, w_s\}$ is a basis of V.

Now, let $B = L\{w_1, w_2, \ldots, w_s\}$

We claim that $A \cap B = \{0\}$ and $V = A + B$

Now, let $v \in A \cap B$. Then $v \in A$ and $v \in B$.

Hence $v = \alpha_1 v_1 + \ldots + \alpha_r v_r$

$= \beta_1 w_1 + \ldots + \beta_s w_s$

$\therefore \alpha_1 v_1 + \ldots + \alpha_r v_r - \beta_1 w_1 - \ldots - \beta_s w_s = 0$

Now, since S' is linearly independent,

$\alpha_i = 0 = \beta_j \quad \forall \; i$ and j

Hence $v = 0$. Thus $A \cap B = \{0\}$

Now, let $v \in V$.

Then $v = (\alpha_1 v_1 + \ldots + \alpha_r v_r) + (\beta_1 w_1 + \ldots + \beta_s w_s)$

$\in A + B$

Hence $A + B = V$ so that $V = A \oplus B$.

Definition:

Let V be a vector space and $S = \{v_1, v_2, \ldots, v_n\}$ be a set of independent vectors in V. Then S is called a maximal linearly independent set if for every $v \in V - S$, the set $\{v, v_1, v_2, \ldots, v_n\}$ is linearly dependent

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20/11/2020

Rank & Nullity: UNIT-3

Let $T: V \to W$ be a linear transformation.

Matrix and Inner product space:

Then the dimension of $T(V)$ is rank and dimension

Matrix of a Linear Transformation:

of $\ker T$ is called Nullity.

Let V and W be finite dimensional

Theorem:

vector space over a field F. Let $\dim V = m$

Let $T: V \to W$ be a linear transformation.

and $\dim W = n$. Fix an ordered basis

Then $\dim V = \text{rank } T + \text{nullity } T$.

Proof: $\{v_1, v_2, \ldots, v_m\}$ for V and an ordered basis

We know that $V/\ker T \cong T(V)$

$\{w_1, w_2, \ldots, w_n\}$ for W.

$\dim V - \dim(\ker T) = \dim(T(V))$

Note: $\dim V - \text{nullity } T = \text{rank } T$

The $m \times n$ matrix which we have associated

$\dim V = \text{nullity } T + \text{rank } T$.

with a linear transformation $T: V \to W$

Definition:

depends on the choice of basis for V and W

A linear transformation $T: V \to W$ is

Solved Problems:

called non-singular if T is 1-1; otherwise T is

1. Obtain the matrix representing the linear

called singular.

transformation $T: V_3(R) \to V_3(R)$ given by

$T(a,b,c) = [3a, a-b, 2a+b+c]$, w.r.t the

standard basis $\{e_1, e_2, e_3\}$

Soln.

$$T(e_1) = T(1,0,0) = (3,1,2) = 3e_1 + e_2 + 2e_3$$

$$T(e_2) = T(0,1,0) = (0,-1,1) = -e_2 + e_3$$

$$T(e_3) = T(0,0,1) = (0,0,1) = e_3$$

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Thus the matrix representing T is $\begin{pmatrix} 3 & 1 & 2 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Definition:

Let $A(a_{ij})$ be an arbitary matrix over a field F. Let $\alpha \in F$ we defined $\alpha A = (\alpha a_{ij})$

Theorem 5.21:

Let V and W be a two finite dimensional vector space over a field F. Then $L(V, W)$ in a vector space of dimensional mn over F.

Proof:

By theorem 5.8 $L(V, W)$ is a vector space over F. Fix a basis $\{v_1, v_2, \ldots, v_m)$ for V and a basis $\{w_1, w_2, \ldots, w_n\}$ for W.

Let T be represented by $M(T)$. This function $M : L(V, W) \to M_{m\times n}$ is clearly 1-1 and onto.

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$$M(T_1) = (a_{ij}) \Rightarrow T_1(v_i)$$

$$= \sum_{j=1}^{n} a_{ij} w_j$$

$$M(T_2) = (b_{ij}) \Rightarrow T_2(v_i)$$

$$= \sum_{j=1}^{n} b_{ij} w_j$$

$$\therefore (T_1+T_2)(v_i) = \sum_{j=1}^{\infty} (a_{ij}+b_{ij})$$

$$M(T_1+T_2) = (a_{ij}+b_{ij})$$

$$= (a_{ij}) + (b_{ij})$$

$$= M(T_1)+M(T_2)$$

Similarly, $M(\alpha T_1) = 2M(T_1)$

Hence M is the required isomorphism from $L(V,W)$ to $M_{m \times n}(F)$.

(2) Find the linear transformation $T: V_3(R) \to V_3(R)$. determined by the matrix $\begin{pmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ -1 & 3 & 4 \end{pmatrix}$ w.r.t the standard basis $\{e_1, e_2, e_3\}$.

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Sol.

$T(e_1) = e_1 + 2e_2 + e_3 = (1, 2, 1)$

$T(e_2) = 0e_1 + e_2 + e_3 = (0, 1, 1)$

$T(e = -e_1 + 3e_2 + 4e_3 = (-1, 3, 4)$

Now $(a, b, c) = a(1,0,0) + b(0,1,0) + c(0,0,1)$

$= ae_1 + be_2 + ce_3$

$T(a, b, c) = T(ae_1 + be_2 + ce_3)$

$= aT(e_1) + bT(e_2) + cT(e_3)$

$= a(1,2,1) + b(0,1,1) + c(-1,3,4)$

$T(a, b, c) = [a-c, 2a+b+3c, a+b+4c]$

This is the required linear transformation.

Q. $T: V_3(R) \to V_2(R)$ given by $T(a,b,c) = (3a+c, -2a+b, a+2b+4c)$ w.r.t standard basis.

Sol:

$T(e_1) = (1,0,0) = (3,-2,1) = 3e_1 - 2e_2 + e_3$

$T(e_2) = (0,1,0) = (0,1,2) = e_2 + 2e_3$

$T(e_3) = (0,0,1) = (1,0,4) = e_1 + 4e_3$

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This is the matrix representing T is $\begin{pmatrix} 3 & -2 & 1 \\ 0 & 1 & 2 \\ 1 & 0 & 4 \end{pmatrix}$. //

4. obtain the linear transformation determin by the following matrices.

a). $V: V_2(R) \to V_2(R)$ given by $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ w.r.t the standard basis.

Sol:

$$T(e_1) = \cos\theta e_1 - \sin\theta e_2 = (\cos\theta, -\sin\theta)$$

$$T(e_2) = \sin\theta e_1 + \cos\theta e_2 = (\sin\theta, \cos\theta)$$

Now, $T(a,b,c) = a(1,0,0) + b(0,1,0)$

$$= ae_1 + be_2$$

$$T(a,b) = T(ae_1 + be_2)$$

$$= aT(e_1) + b(Te_2)$$

$$= a(\cos\theta, -\sin\theta) + b(\sin\theta, \cos\theta)$$

$$T(a,b) = [a\cos\theta + b\sin\theta, -a\sin\theta + b\cos\theta]$$

This is the required linear transformation

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b) $T: V_3(R) \to V_3(R)$ given by $\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ w.r.t the standard basis.

Sol:-

$$T(e_1) = ae_1 + be_2 + ce_3 = (a, b, c)$$

$$T(e_2) = be_1 + ce_2 + ae_3 = (b, c, a)$$

$$T(e_3) = ce_1 + ae_2 + be_3 = (c, a, b).$$

$$T(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$$

$$= xe_1 + ye_2 + ze_3$$

$$T(x, y, z) = T(xe_1 + ye_2 + ze_3)$$

$$= x\,T(e_1) + y\,T(e_2) + z\,T(e_3)$$

$$= x(a, b, c) + y(b, c, a) + z(c, a, b)$$

$$T(x, y, z) = (xa + yb + zc), xb + yc + za, xc + ya + zb)$$

$$= ax + by + cz, bx + cy + az, cx + ay + bz.$$

(c) $T: V_2(R) \to V_3(R)$ given by $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$ w.r.t the standard basis.

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Sol:-

$$T(e_1) = (2e_1 + e_2 - e_3) = (2, 1, -1)$$

$$T(e_2) = (e_1 + e_2 - e_3) = (1, 1, -1)$$

Now,

$$T(a, b) = a(1, 0, 0) + b(0, 1, 0)$$
$$= ae_1 + be_2$$

$$T(a, b) = T(ae_1 + be_2)$$
$$= aT(e_1) + bT(e_2)$$
$$= a(2, 1, -1) + b(1, 1, -1)$$
$$= (2a + b, a + b, -a - b).$$

Inner product space:-

Definition:-

Let V be a vector space over F. An inner product on V is a function which assings to each ordered pair of vector u, v in V a scalar in F denoted by $\langle u, v \rangle$. satisfing the following conditions,

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(i) $\langle u+v, w\rangle = \langle u, w\rangle + \langle v, w\rangle$

(ii) $\langle \alpha u, v\rangle = \alpha \langle u, v\rangle$

(iii) $\langle u, v\rangle = \overline{\langle v, u\rangle}$ where $\overline{\langle v, u\rangle}$ is the complex conjugate of $\langle u, v\rangle$.

(iv) $\langle u, v\rangle \geq 0$ and $\langle u, v\rangle = 0$ iff $u = 0$.

Solved problems:-

1. Let V be the vector space of polynomials with inner product given by $\langle f, g\rangle = \int_0^1 f(t)\, g(t)\, dt$, Let $f(t) = t+2$ and $g(t) = t^2 - 2t - 3$.

Find (i) $\langle f, g\rangle$. (ii) $\|f\|^2$

Sol:- (i) $\langle f, g\rangle = \int_0^1 f(t)\, g(t)\, dt$

$$= \int_0^1 (t+2)(t^2 - 2t - 3)\, dt$$

$$= \int_0^1 (t^3 - 7t - 6)\, dt$$

$$= \left[\frac{t^4}{4} - \frac{7t^2}{2} - 6t\right]_0^1$$

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$$= 1/4 - 7/2 - 6$$

$$= \frac{-37}{4}$$

(iii) $\|f\|^2 = \langle f, f \rangle$

$$= \int_0^1 [f(t)]^2 \, dt$$

$$= \int_0^1 (t+2)^2 \, dt$$

$$= \int_0^1 (t^2 + 4t + 4) \, dt$$

$$= \left[\frac{t^3}{3} + \frac{4t^2}{2} + 4t\right]_0^1$$

$$= 1/3 + 2 + 4 = \frac{19}{3}$$

$$\|f\| = \sqrt{19}/\sqrt{3}$$

Note : 1

$$\langle u, \alpha v \rangle = \bar{\alpha} \langle u, v \rangle$$

For $\langle u, \alpha v \rangle = \langle \overline{\alpha v}, \bar{u} \rangle$

$$= \overline{\alpha \langle v, u \rangle}$$

$$= \bar{\alpha} \langle u, v \rangle.$$

Note : 2

$$\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$$

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Types of Matrices:-

(i) Row Matrix:-

Any matrix which has only one row is called as row matrix.

Ex: $\begin{pmatrix} 1 & 5 & 3 & 2 & 6 \end{pmatrix}_{1\times 5}$

(ii) column Matrix:

Any matrix which has only one column is called as column matrix.

Ex: $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}_{3\times 1}$

(iii) square Matrix:

Any matrix which has equal number of rows and columns is called a square matrix.

Ex: $\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}_{2\times 2}$

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(iv) scalar Matrix:

A diagonal matrix whose diagonal elements are all equal is called as scalar matrix.

$$Ex: \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3 \times 3}$$

(v) Unit Matrix (or) Identity Matrix:

A scalar matrix in which each diagonal elements is 1 is called a unit matrix (or) Identity matrix. It is denoted by I_n, the matrix order of n.

$$Ex: \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3}$$

(vi) zero matrix (or) Null matrix:

It the elements of the matrix are all zero, it is called a null matrix (or) zero matrix.

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It is denoted by O.

Ex: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times3}$

(vii) Triangular Matrix:

A square in which all the ~~element~~ entries above the main diagonal are zero is called as lower triangular matrix.

Ex: $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{bmatrix}$

vii If all the entries are below the main diagonal are zero, It is called as upper triangular matrix.

Ex: $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 5 \end{bmatrix}$

(viii) Equality of Matrix:

Two matrices of $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal if and only if

(i) Both the matrices A and B are of the same order.

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(ii) The corresponding entries in both the matrices A and B are equal. (i.e.), $a_{ij} = b_{ij}$ for all i and j.

(ix) Transpose of Matrix:

The matrix obtained from the given matrix A by interchanging its rows into column and its column by rows is called transpose matrix of A.

Denoted by A' (or) A^T

Eg: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}_{3\times2}$ $A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}_{2\times3}$

(x) Negative of a Matrix:

Let A be any matrix the negative of a matrix A is (-A) and is obtained by changing the sign of the entries of A.

Ex: $A = \begin{bmatrix} 4 & -5 \\ 6 & 7 \end{bmatrix}$ $-A = \begin{bmatrix} -4 & 5 \\ -6 & -7 \end{bmatrix}$

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20/12 Symmetric matrix:

A square matric $A=[a_{ij}]$ is said to be symmetric if $(i,j)^{th}$ elements is same as its $(j,i)^{th}$ element.

(i.e) $A=A^T$

$a_{ij}=a_{ji} \ \forall$ i and j

Skew symmetric matrix:

A square matrix $A=[a_{ij}]$ is said to be skew-symmetric if $(i,j)^{th}$ element is negative of its $(j,i)^{th}$ element.

(i.e) $A^T=-A$

$a_{ij}=-a_{ji} \ \forall$ i and j.

problems:-

1. $A=\begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ and $B=\begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 1 \\ 2 & 1 & 2 \end{bmatrix}$

Evaluate (i) A+B and (ii) AB.

Sol:-

$$A+B=\begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}+\begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

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$$= \begin{bmatrix} 3 & 3 & 5 \\ 7 & 6 & 2 \\ 3 & 3 & 5 \end{bmatrix}$$

(ii) $$AB = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2+8+4 & 1+10+2 & 3+2+4 \\ 6+4+2 & 3+5+1 & 9+1+2 \\ 2+8+6 & 1+10+3 & 3+2+6 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 13 & 9 \\ 12 & 9 & 12 \\ 16 & 14 & 11 \end{bmatrix}$$

(2) $A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$ $B = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

(i) $3A + 2B + C$ (ii) $3A - AB + 2B$ (iii) $A - B - C$

(iv) $AB + BC$.

(i) $$3A + 2B + C = \begin{bmatrix} 3 & 6 & 6 \\ 9 & 3 & 3 \\ 3 & 6 & 9 \end{bmatrix} + \begin{bmatrix} 4 & 2 & 6 \\ 8 & 10 & 2 \\ 4 & 2 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 10 & 13 \\ 18 & 14 & 6 \\ 9 & 9 & 14 \end{bmatrix}$$

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(ii) $3A - AB + 2B = \begin{bmatrix} 3 & 6 & 6 \\ 9 & 3 & 3 \\ 3 & 6 & 9 \end{bmatrix} - \begin{bmatrix} 2+8+4 & 1+10+2 & 3+2+4 \\ 6+4+2 & 3+5+1 & 9+1+2 \\ 2+8+6 & 1+10+3 & 3+2+6 \end{bmatrix}$

$$+ \begin{bmatrix} 4 & 2 & 6 \\ 8 & 10 & 2 \\ 4 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 6 & 6 \\ 9 & 3 & 3 \\ 3 & 6 & 9 \end{bmatrix} - \begin{bmatrix} 14 & 13 & 9 \\ 12 & 9 & 12 \\ 16 & 14 & 11 \end{bmatrix} + \begin{bmatrix} 4 & 2 & 6 \\ 8 & 10 & 2 \\ 4 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -7 & -5 & 3 \\ 5 & 4 & -7 \\ -9 & -6 & 2 \end{bmatrix}$$

(iii) $A - B - C = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 1 & 3 \\ 4 & 5 & 1 \\ 2 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$

$$= \begin{bmatrix} -2 & -1 & -2 \\ -2 & -5 & -1 \\ -3 & 0 & 0 \end{bmatrix}$$

(iv) $AB + BC = \begin{bmatrix} 14 & 13 & 9 \\ 12 & 9 & 12 \\ 16 & 14 & 11 \end{bmatrix} + \begin{bmatrix} 2+1+6 & 4+1+3 & 2+1+3 \\ 4+5+2 & 8+5+1 & 4+5+1 \\ 2+1+4 & 4+1+2 & 2+1+2 \end{bmatrix}$

$$= \begin{bmatrix} 14 & 13 & 9 \\ 12 & 9 & 12 \\ 16 & 14 & 11 \end{bmatrix} + \begin{bmatrix} 9 & 8 & 6 \\ 11 & 14 & 10 \\ 7 & 7 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 23 & 21 & 15 \\ 23 & 23 & 22 \\ 23 & 21 & 16 \end{bmatrix}$$ //

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3. If $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$. P.T $(A-I)(A-4I) = 0$.

Sol:-

$$(A-I)\cancel{(A-4I)} = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$A - 4I = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$(A-I).(A-4I) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2+1+1 & 1-2+1 & 1+1-2 \\ -2+1+1 & 1-2+1 & 1+1-2 \\ -2+1+1 & 1-2+1 & 1+1-2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 \,//$$

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4. $A = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ and the $A^3 - 3A^2 - A + 9I = 0$

Sol:- $A^2 = \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 1+0+3 & 0+0+-3 & 3+0+3 \\ 2+2-1 & 0+1+1 & 6-1-1 \\ 1-2+1 & 0+-1 & 3+1+1 \end{bmatrix}$

$= \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix}$

$A^3 = \begin{bmatrix} 4 & -3 & 6 \\ 3 & 2 & 4 \\ 0 & -2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 2 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 4-6+6 & 0-3-6 & 12+3+6 \\ 3+4+4 & 0+2-4 & 9-2+4 \\ 0-4+5 & 0-2-5 & 0+2+5 \end{bmatrix}$

$= \begin{bmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{bmatrix}$

$-|A| =$

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$$-A = \begin{bmatrix} -1 & 0 & -3 \\ -2 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

$$9\underline{I} = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$A^3 - 3A^2 - A + 9I = \begin{bmatrix} 4 & -9 & 21 \\ 11 & -2 & 11 \\ 1 & -7 & 7 \end{bmatrix} - \begin{bmatrix} 12 & -9 & 18 \\ 9 & 6 & 12 \\ 0 & -6 & 15 \end{bmatrix}$$

$$+ \begin{bmatrix} -1 & 0 & -3 \\ -2 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= 0$$

Hence proved //

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3/1/20

Inverse of the matrix:

1. compute the inverse of the matrix

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

Sol:

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$$

$$|A| = \begin{vmatrix} 2 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{vmatrix}$$

$$= 2(12-10) + 1(-30+25) + 1(30-30)$$

$$= 2(2) + \cancel{(-60)} \cancel{+10} - 5 + 0$$

$$= 4 - \cancel{60} - 5$$

$$\cancel{=} -\cancel{56} = -1$$

$$\operatorname{adj} A = [a_{ij}]^T$$

$$= \begin{bmatrix} +\begin{vmatrix} 6 & -5 \\ -2 & 2 \end{vmatrix} & -\begin{vmatrix} -15 & -5 \\ 5 & 2 \end{vmatrix} & +\begin{vmatrix} -15 & 6 \\ 5 & -2 \end{vmatrix} \\ -\begin{vmatrix} -1 & 1 \\ -2 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 5 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 5 & 2 \end{vmatrix} \\ +\begin{vmatrix} -1 & 1 \\ 6 & -5 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ -15 & -5 \end{vmatrix} & +\begin{vmatrix} 2 & -1 \\ -15 & +6 \end{vmatrix} \end{bmatrix}^T$$

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$$= \begin{bmatrix} 2 & +5 & 0 \\ 0 & -1 & -9 \\ -1 & -5 & -24 \end{bmatrix}^T$$

$$= \begin{bmatrix} 2 & 0 & -1 \\ 5 & -1 & -5 \\ 0 & -9 & -24 \end{bmatrix}.$$

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} 2 & 0 & -1 \\ 5 & -1 & -5 \\ 0 & -9 & -24 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 0 & 1 \\ -5 & 1 & 5 \\ 0 & 9 & 24 \end{bmatrix}$$

//

2. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 4 \\ -2 & 2 & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj}_A$$

$$|A| = 1(-1-8) - 2(0+8) + 3(0-2)$$

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$$= 1(-9) - 2(8) + 3(-2)$$

$$= -9 - 16 - 6 = -31$$

$$\text{adj}\,A = \begin{bmatrix} +\begin{vmatrix} -1 & 4 \\ 2 & 1 \end{vmatrix} & -\begin{vmatrix} 0 & 4 \\ -2 & 1 \end{vmatrix} & +\begin{vmatrix} 0 & -1 \\ -2 & 2 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ -2 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ -2 & 2 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ -1 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 0 & 4 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} -9 & -8 & -2 \\ 4 & 7 & -6 \\ 11 & -4 & -1 \end{bmatrix}^T$$

$$= \begin{bmatrix} -9 & 4 & 11 \\ -8 & 7 & -4 \\ -2 & -6 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-31} \begin{bmatrix} -9 & 4 & 11 \\ -8 & 7 & -4 \\ -2 & -6 & -1 \end{bmatrix}$$

$$= \frac{1}{31} \begin{bmatrix} +9 & -4 & -11 \\ 8 & -7 & 4 \\ 2 & 6 & 1 \end{bmatrix} //$$

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3. $A = \begin{bmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$

$$|A| = 3(-3+4) - 3(2) + 4(-2)$$

$$= 3(1) - 6 - 8$$

$$= -11$$

$$\text{Adj } A = \begin{bmatrix} +\begin{vmatrix} -3 & 4 \\ -1 & 1 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 0 & 1 \end{vmatrix} & +\begin{vmatrix} 2 & -3 \\ 0 & -1 \end{vmatrix} \\ -\begin{vmatrix} 3 & 4 \\ -1 & 1 \end{vmatrix} & +\begin{vmatrix} 3 & 4 \\ 0 & 1 \end{vmatrix} & -\begin{vmatrix} 3 & 3 \\ 0 & -1 \end{vmatrix} \\ +\begin{vmatrix} 3 & 4 \\ -3 & 4 \end{vmatrix} & -\begin{vmatrix} 3 & 4 \\ 2 & 4 \end{vmatrix} & +\begin{vmatrix} 3 & 3 \\ 2 & -3 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -2 & -2 \\ -7 & 3 & 3 \\ 24 & -4 & -15 \end{bmatrix}^T$$

$$= \begin{bmatrix} 1 & -7 & 24 \\ -2 & 3 & -4 \\ -2 & 3 & -15 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-11} \begin{bmatrix} 1 & -7 & 24 \\ -2 & 3 & -4 \\ -2 & 3 & -15 \end{bmatrix}$$

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4. $\omega = e^{2\pi i/3}$ find the inverse of the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}$

Sol:-

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$$

$$= 1(\omega^2 - \omega) - 1(\omega - \omega^2) + 1(\omega^2 - \omega)$$

$$= \omega^2 - \omega - \omega + \omega^2 + \omega^2 - \omega.$$

$$= 3\omega^2 - 3\omega$$

$$|A| = 3\omega(\omega - 1).$$

$$\operatorname{adj} A = [a_{ij}]^T$$

$$= \begin{bmatrix} +\begin{vmatrix} \omega & \omega^2 \\ \omega^2 & \omega \end{vmatrix} & -\begin{vmatrix} 1 & \omega^2 \\ 1 & \omega \end{vmatrix} & +\begin{vmatrix} 1 & \omega \\ 1 & \omega^2 \end{vmatrix} \\ -\begin{vmatrix} 1 & 1 \\ \omega^2 & \omega \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 1 & \omega \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & \omega^2 \end{vmatrix} \\ +\begin{vmatrix} 1 & 1 \\ \omega & \omega^2 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 1 & \omega^2 \end{vmatrix} & +\begin{vmatrix} 1 & 1 \\ 1 & \omega \end{vmatrix} \end{bmatrix}^T$$

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$$= \begin{bmatrix} \omega^2-\omega & -(\omega-\omega^2) & \omega^2-\omega \\ -(\omega-\omega^2) & \omega-1 & -(\omega^2-1) \\ \omega^2-\omega & -(\omega^2-1) & \omega-1 \end{bmatrix}^T$$

$$= \begin{bmatrix} \omega^2-\omega & \omega^2-\omega & \omega^2-\omega \\ \omega^2-\omega & \omega-1 & -(\omega^2-1) \\ \omega^2-\omega & -(\omega^2-1) & \cancel{\omega}\,1 \end{bmatrix}^T$$

$$= \begin{bmatrix} \omega(\omega-1) & \omega(\omega-1) & \omega(\omega-1) \\ \omega(\omega-1) & \omega-1 & -(\omega+1)(\omega-1) \\ \omega(\omega-1) & -(\omega+1)(\omega-1) & \cancel{\omega}-1 \end{bmatrix}^T$$

$$= (\omega-1) \begin{bmatrix} \omega & \omega & \omega \\ \omega & \cancel{\omega}\,1 & -(\omega+1) \\ \omega & -(\omega+1) & \cancel{\omega}\,1 \end{bmatrix}^T$$

$$\operatorname{adj} A = (\omega-1) \begin{bmatrix} \omega & \omega & \omega \\ \omega & 1 & -(\omega+1) \\ \omega & -(\omega+1) & \cancel{\omega}\,1 \end{bmatrix}$$

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$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$$

$$= \frac{\cancel{(\omega-1)}}{3\omega\cancel{(\omega-1)}} \begin{bmatrix} \omega & \omega & \omega \\ \omega & 1 & -(\omega+1) \\ \omega & -(\omega+1) & 1 \end{bmatrix}$$

$$= \frac{1}{3\omega} \begin{bmatrix} \omega & \omega & \omega \\ \omega & 1 & -(\omega+1) \\ \omega & -(\omega+1) & 1 \end{bmatrix}$$

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1. $x+2y+3z=2$; $2x+4y+5z=3$; $3x+5y+6z=$

Sol:.

The given equation can be written as matrix,

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

$$AX = B$$

$$X = A^{-1}B$$

$$A^{-1} = \frac{1}{|A|} \text{adj } A$$

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{vmatrix}$$

$$= 1(24-25) - 2(12-15) + 3(10-12)$$

$$= -1 + 6 - 6$$

$$= -1$$

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$$\text{Adj } A = \begin{bmatrix} +\begin{vmatrix} 4 & 5 \\ 5 & 6 \end{vmatrix} & -\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} & +\begin{vmatrix} 2 & 4 \\ 3 & 5 \end{vmatrix} \\ -\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} & +\begin{vmatrix} 1 & 3 \\ 3 & 6 \end{vmatrix} & -\begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} \\ +\begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 3 \\ 2 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 3 & -2 \\ +3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}^T$$

$$= \begin{bmatrix} -1 & 3 & -2 \\ 3 & -3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & -3 & 2 \\ -3 & 3 & -1 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$$

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$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2-9+8 \\ -6+9-4 \\ 4-3+0 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\therefore$ The solution is $x=1;\ y=-1;\ z=1$

H.W:-

1. $\begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Let $A = \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$$

$$|A| = \begin{vmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \cos\alpha(\cos\alpha - 0) + \sin\alpha(\sin\alpha - 0)$$

$$= \cos^2\alpha + \sin^2\alpha = 1$$

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$$\operatorname{Adj} A = \begin{bmatrix} \begin{vmatrix} \cos\alpha & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} \sin\alpha & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} \sin\alpha & \cos\alpha \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} -\sin\alpha & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} \cos\alpha & 0 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} \cos\alpha & -\sin\alpha \\ 0 & 0 \end{vmatrix} \\ \begin{vmatrix} -\sin\alpha & 0 \\ \cos\alpha & 0 \end{vmatrix} & \begin{vmatrix} \cos\alpha & 0 \\ \sin\alpha & 0 \end{vmatrix} & \begin{vmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & \cos^2\alpha + \sin^2\alpha \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}^T$$

$$= \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$A^{-1} = \begin{bmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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2) $A = \begin{bmatrix} 2 & 2 & -3 \\ -3 & 2 & 2 \\ 2 & -3 & 2 \end{bmatrix}$

Sol:-

$$A^{-1} = \frac{1}{|A|} \text{adj} A$$

$$|A| = \begin{vmatrix} 2 & 2 & -3 \\ -3 & 2 & 2 \\ 2 & -3 & 2 \end{vmatrix}$$

$$= 2(4+6) - 2(-6-4) - 3(9-4)$$

$$= \cancel{4 + 20}$$

$$= 20 + 20 - 15$$

$$= 25$$

$$\text{adj} A = \begin{bmatrix} +\begin{vmatrix} 2 & 2 \\ -3 & 2 \end{vmatrix} & -\begin{vmatrix} -3 & 2 \\ 2 & 2 \end{vmatrix} & +\begin{vmatrix} -3 & 2 \\ 2 & -3 \end{vmatrix} \\ -\begin{vmatrix} 2 & -3 \\ -3 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & -3 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 2 \\ 2 & -3 \end{vmatrix} \\ +\begin{vmatrix} 2 & -3 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & -3 \\ -3 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 2 \\ -3 & 2 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} 10 & 10 & 5 \\ 5 & 10 & 10 \\ 10 & 5 & 10 \end{bmatrix}^T$$

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$$= \begin{bmatrix} 10 & 5 & 10 \\ 10 & 10 & 5 \\ 5 & 10 & 10 \end{bmatrix}$$

$$A^{-1} = \frac{1}{25} \begin{bmatrix} 10 & 5 & 10 \\ 10 & 10 & 5 \\ 5 & 10 & 10 \end{bmatrix}.$$

$$= \frac{1}{5} \begin{bmatrix} 2 & 1 & 2 \\ 2 & 2 & 1 \\ 1 & 2 & 2 \end{bmatrix}.$$

3. $6x+2y-2z=6$; $-2x+2y+2z=2$; $2x+2y+2z=6$.

Sol:-

The given equ can be written in matrix as, $AX = B$

$$\begin{bmatrix} 6 & 2 & -2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 6 \end{bmatrix}$$

$$X = A^{-1} B$$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$$

$$|A| = \begin{vmatrix} 6 & 2 & -2 \\ -2 & 2 & 2 \\ 2 & 2 & 2 \end{vmatrix}$$

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UNIT-V

Characteristic Equation & Cayley Hamilton Theorem:

Definition:

An expression of the form $A_0 + A_1x + A_2x^2 + \cdots + A_nx^n$ where $A_0, A_1, \ldots, A_n$ are square matrices of same order and $A_n \neq 0$ is called a matrix polynomial of degree n.

For example, $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} + \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}x + \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}x^2$ is a matrix polynomial of degree 2 and its simply the matrix $\begin{pmatrix} 1+x+2x^2 & 2+x \\ 2x+3x^2 & 3+x+x^2 \end{pmatrix}$

Definition:

Let A be any square matrix of order n and let 'I' be the identity matrix of order 'n'. Then the matrix polynomial given by $A - xI$ is called the characteristic matrix of A.

The determinant $|A - xI|$ which is an ordinary polynomial in x of degree 'n' is called the characteristic polynomial of A.

The equation $|A - xI| = 0$ is called the characteristic equation of A.

Example 1:- Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, Then the characteristics matrix of A is $A - xI$ given by

$$A - xI = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - x\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1-x & 2 \\ 3 & 4-x \end{pmatrix}$$

$$= (1-x)(4-x) - 6$$

$$= 4 - x - 4x + x^2 - 6$$

$$= x^2 - 5x - 2$$

$\therefore$ The characteristic equation of A is

$|A - xI| = 0$

$\therefore x^2 - 5x - 2 = 0$ is the characteristics equation of A.

② Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$

The characteristic matrix of A is $|A - xI|$ given by

$$|A - xI| = \begin{pmatrix} 1-x & 0 & 2 \\ 0 & 1-x & 2 \\ 1 & 2 & -x \end{pmatrix}$$

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The characteristic polynomial of A is

$$|A - xI| = \begin{vmatrix} 1-x & 0 & 2 \\ 0 & 1-x & 2 \\ 1 & 2 & x \end{vmatrix}$$

$$= (1-x)[(1-x)(-x)-4] - 0[0-2] + 2[0-(1-x)]$$

$$= (1-x)[-x+x^2-4] - 0 + 2[-1+x]$$

$$= (1-x)(x^2-x-4) + 2x - 2$$

$$= x^2 - x - 4 - x^3 + x^2 + 4x + 2x - 2$$

$$= -x^3 + 2x^2 + 5x - 6$$

$\therefore$ The characteristic equation A is $x^3 - 2x^2 - 5x + 6 = 0$ //

3. $A = \begin{bmatrix} 1 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

The characterictic matrix of A is

$$|A - xI| = \begin{vmatrix} \cancel{7}-x & 2 & -2 \\ -6 & -1-x & 2 \\ 6 & 2 & -1-x \end{vmatrix}$$

polynomial of $A = (7-x)[(1+x)^2 - 4]$
$-2(+6(1+x)-12) - 2[12 + (6(1+x)]$

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$= (7-x)[(1+2x+x^2-4) - 2[+6+6x+12]$

$-72 \quad -12 - 12x$

$= 7 + 14x + 7x^2 - 28 - x - 2x^2 - x^3 + 4x$

$+ \cancel{12} + 12x + 24 - 72 - \cancel{12} - 12x$

$= -x^3 + 5x^2 - 17x + 3$

$\therefore$ C.E is

$x^3 - 5x^2 - 17x + 3 = 0$

4. Find the C.E of the matrix $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Sol:-

The C.E $|A - \lambda I| = 0$

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$|A - \lambda I| = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}$

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$$= (8-x)\left[(7-x)(3-x)-16\right]+6\left[-6(-6(3-x)+8\right]$$
$$+2\left[24-2(7-x)\right]$$

$$= (8-x)\left[21-7x-3x+x^2\right]-16\left]+6\right[-18+6x+8$$
$$+2\left[24-14+2x\right]$$

$$= (8-x)\left[x^2-10x+5\right]+6\left[6x-10\right]+2\left[2x+10\right]$$

$$= 8x^2-80x+40-x^3+10x^2-5x$$
$$+36x-60+4x+20$$

$$= -x^3+18x^2-45x=0$$

C.E is $x^3-18x^2+45x=0$ //

Definition :-

Cayley Hamilton's theorem :-

Any square matrix A satisfies it characteristic equation that is $a_0+a_1x+a_2x^2+\dots+a_nx^n$ is the characteristic polynomial of degree 'n'

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at 'A' then $a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n = 0$

(5) verify cayley Hamilton theorem to the matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

sol:.

The characteristic equation $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix} = (1-\lambda)(3-\lambda) - 8$$

$$= 3 - \lambda - 3\lambda + \lambda^2 - 8$$

$$= \lambda^2 - 4\lambda - 5$$

$$\lambda^2 - 4\lambda - 5 = 0$$

The cayley's Hamilton theorem

$$A^2 - 4A - 5I = 0$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+8 & 2+6 \\ 4+12 & 8+9 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 9 & 8 \\ 16 & 17 \end{bmatrix}$$

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$$4A = \begin{bmatrix} 4 & 8 \\ 16 & 12 \end{bmatrix}$$

$$5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^2 - 4A - 5I = 0$$

$$\begin{bmatrix} 9 & 8 \\ 16 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 8 \\ 16 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = 0$$

$$\begin{bmatrix} 9-4-5 & 8-8-0 \\ 16-16-0 & 17-12-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

6

$$A = \begin{bmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

The C.E $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{bmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 3 & 4 \\ 2 & -3-\lambda & 4 \\ 0 & -1 & 1-\lambda \end{bmatrix}$$

$$= (3-\lambda)\left[-(3+\lambda)(1-\lambda) + 4\right] - 3\left[2 - 2\lambda\right]$$

$$+ 4\left[-2\right]$$

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$$= (3-\lambda)\left[-3+3\lambda-\lambda+\lambda^2\right] - 6 + 6\lambda - 8$$

$$= -9+9\lambda-\cancel{3\lambda}+\cancel{3\lambda^2}+\cancel{3\lambda}-\cancel{3\lambda^2}+\lambda^2-\lambda^3$$
$$-14+6\lambda+12-4\lambda$$

$$= -\lambda^3+\lambda^2+11\lambda-11 = 0$$

$$\lambda^3-\lambda^2-11\lambda+11 = 0$$

The cayley's Hamilton theorem,

$$A^3-A^2-11A+11I = 0$$

$$A^2 = \begin{bmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9+6+0 & 9-9-4 & 12+12+4 \\ 6-6+0 & 6+9-4 & 8-12+4 \\ 0-2+0 & 0+3-1 & 0-4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & -4 & 28 \\ 0 & 11 & 0 \\ -2 & 2 & -3. \end{bmatrix}$$

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$$A^3 = A^2 \times A = \begin{bmatrix} 15 & -4 & 28 \\ 0 & 11 & 0 \\ -2 & 2 & -3 \end{bmatrix} \begin{bmatrix} 3 & 3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 45-8+0 & 45+12-28 & 60-16+28 \\ 0+22+0 & 0-33+0 & 0+44+0 \\ -6+4+0 & -6-6+3 & -8+8-3 \end{bmatrix}$$

$$= \begin{bmatrix} 37 & 29 & +72 \\ 22 & -33 & 44 \\ -2 & -9 & -3 \end{bmatrix}$$

$$-11A = \begin{bmatrix} -33 & -33 & -44 \\ -22 & +33 & -44 \\ 0 & 11 & -11 \end{bmatrix}$$

$$11I = \begin{bmatrix} +11 & 0 & 0 \\ 0 & +11 & 0 \\ 0 & 0 & +11 \end{bmatrix}$$

$$A^3 - A^2 - 11A - 11I = 0$$

$$\begin{bmatrix} 37 & 29 & 72 \\ 22 & -33 & 44 \\ -2 & -9 & -3 \end{bmatrix} - \begin{bmatrix} 15 & -4 & 28 \\ 0 & 11 & 0 \\ -2 & 2 & -3 \end{bmatrix} + \begin{bmatrix} -33 & -33 & -44 \\ -22 & 33 & -44 \\ 0 & 11 & -11 \end{bmatrix}$$

$$+ \begin{bmatrix} +11 & 0 & 0 \\ 0 & +11 & 0 \\ 0 & 0 & +11 \end{bmatrix} = 0$$

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$$\begin{bmatrix} 37-15-33+11 & 29+4-33 & 72-28-44 \\ 22-0-22-0 & -33-11+33+11 & 44-44-0 \\ -2+2+0+0 & -9-2+11+0 & -3+3-11+11 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$A^3 - A^2 - 11A + 11I = 0 \;//$$

(4) $A = \cancel{\begin{bmatrix} 8 & -6 & 2 \\ & & \end{bmatrix}}$ $\quad A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

$|A - \lambda I| = 0$

$$|A-\lambda I| = \begin{bmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{bmatrix}$$

$$= (7-\lambda)\left(+(1+\lambda)^2 - 4\right) - 2(6+6\lambda-12$$

$$-2(-12+6+6\lambda)$$

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$$= (7-\lambda)(1+\lambda^2+2\lambda-4) - 12 - 12\lambda + 24$$
$$+ 24 - 12 - 12\lambda$$

$$= 7 + 7\lambda^2 + \cancel{14\lambda} - 28 - \lambda - \lambda^3 - 2\lambda^2 + 4\lambda$$
$$+ 12 - \cancel{12\lambda} + 12 - 12\lambda$$

$$= -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = 0$$

$$\lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By cayley's Hamilton theorem,

$$A^3 - 5A^2 + 7A - 3 = 0$$

$$A^2 = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 49-12-12 & 14-2-4 & -14+4+2 \\ -42+6+12 & -12+1+4 & 12-2-2 \\ 42-12-6 & 12-2-2 & -12+4+1 \end{bmatrix}$$

$$= \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}$$

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$$A^3 = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$$

$$\frac{25 \times 7}{175}$$

$$= \begin{bmatrix} 175-48-48 & 50-8-16 & -50+16+8 \\ -168+42+48 & -48+7+16 & 48-14-8 \\ 168-48-42 & 48-8-14 & -48+16+7 \end{bmatrix}$$

$$= \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$$

$$7A = 7 \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 49 & 14 & -14 \\ -42 & -7 & 14 \\ 42 & 14 & -7 \end{bmatrix}$$

$$3I = 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

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$$A^3 - 5A^2 + 7A - 3I = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix} - \begin{bmatrix} 125 & 40 & -40 \\ -120 & -35 & 40 \\ 120 & 40 & -35 \end{bmatrix}$$

$$+ \begin{bmatrix} 49 & 14 & -14 \\ -42 & -7 & 14 \\ 42 & 14 & -7 \end{bmatrix} - \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 79-125+49-3 & 26-40+14 & -26+40-14 \\ -78+120-42 & -25+35-7-3 & 26-40+14 \\ 78-120+42 & 26-40+14+0 & -25+35-7-3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

5m

8) Show that the matrix, $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$ satisfy the eqn $A(A-I)(A+2I) = 0$.

Sol:-

$$A - I = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -3 & 1 \\ 3 & 0 & 3 \\ -5 & 2 & -5 \end{bmatrix}$$

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$$A+2I = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -3 & 1 \\ 3 & 3 & 3 \\ -5 & 2 & -2 \end{bmatrix}$$

$$(A-I)(A+2I) = \begin{bmatrix} 1 & -3 & 1 \\ 3 & 0 & 3 \\ -5 & 2 & -5 \end{bmatrix} \begin{bmatrix} 4 & -3 & 1 \\ 3 & 3 & 3 \\ -5 & 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 4-9-5 & -3-9+2 & 1-9-2 \\ 12+0-15 & -9+0+6 & 3+0-6 \\ -20+6+25 & 15+6-10 & -5+6+10 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & -10 & -10 \\ -3 & -3 & -3 \\ 11 & 11 & 11 \end{bmatrix}$$

$$A(A-I)(A+2I) = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix} \begin{bmatrix} -10 & -10 & -10 \\ -3 & -3 & -3 \\ 11 & 11 & 11 \end{bmatrix}$$

$$= \begin{bmatrix} -20+9+11 & -20+9+11 & -20+9+11 \\ -30-3+33 & -30-3+33 & -30-3+33 \\ 50-6-44 & 50-6-44 & 50-6-44 \end{bmatrix}$$

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$$A(A-I)(A+2I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

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$$\begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \end{bmatrix}$$

1. S.T the non-singular matrix $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ satisified the equation $A^2 - 2A - 5I = 0$. Henc evaluate A^{-1}.

Sol:

The characteristic equation $|A - \lambda I| = 0$

$$|A - \lambda I| = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 \\ 3 & 1-\lambda \end{bmatrix}$$

$$= [1-\lambda][1-\lambda] - 6] = 1 - \lambda - \lambda + \lambda^2 - 6$$

$$= \lambda^2 - 2\lambda - 5 = 0$$

The Cayley's Hamilton's theorem

$$A^2 - 2A - 5I = 0$$

$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

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$$= \begin{bmatrix} 1+6 & 2+2 \\ 3+3 & 6+1 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 6 & 7 \end{bmatrix}$$

$$2A = \begin{bmatrix} 2 & 4 \\ 6 & 2 \end{bmatrix}$$

$$5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^2 - 2A - 5I = 0$$

$$\begin{bmatrix} 7 & 4 \\ 6 & 7 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 6 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$A^2 - 2A - 5I = 0$$

Multiply A^{-1},

$$A^2(A^{-1}) - 2AA^{-1} - 5A^{-1} = 0$$

$$A - 2I - 5A^{-1} = 0$$

$$-5A^{-1} = -A + 2I$$

$$A^{-1} = \frac{1}{5}[A - 2I]$$

$$A^{-1} = \frac{1}{5}\left[\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right]$$

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$$A^{-1} = \frac{1}{5}\begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix}$$

2. Using cayley Hamilton's theorem for the matrix $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$ Find (i) A^{-1} (ii) A^4

The C.E, $|A - \lambda I| = 0$

$$[A - \lambda I] = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 0 & -2 \\ 2 & 2-\lambda & 4 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

$$= (1-\lambda)(2-\lambda)^2 - 2(0)$$

$$= (1-\lambda)(4 + \lambda^2 - 4\lambda)$$

$$= 4 + \lambda^2 - 4\lambda - 4\lambda - \lambda^3 + 4\lambda^2$$

$$= -\lambda^3 + 5\lambda^2 - 8\lambda + 4 = 0$$

$$\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$$

By the cayley Hamilton's theorem,

$$A^3 - 5A^2 + 8A - 4I = 0$$

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$$A^2 = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & -2+0-4 \\ 2+4+0 & 0+4+0 & -4+8+8 \\ 0+0+0 & 0+0+0 & 0+0+4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 & 0+0+0 & -2+0-12 \\ 6+8+0 & 0+8+0 & -12+16+24 \\ 0+0+0 & 0+0+0 & 0+0+8 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & -14 \\ 14 & 8 & 28 \\ 0 & 0 & 8 \end{bmatrix}$$

$$8A = \begin{bmatrix} 8 & 0 & -16 \\ 16 & 16 & 32 \\ 0 & 0 & 16 \end{bmatrix}$$

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$$A^3 - 5A^2 + 8A - 4I = 0$$

$$\begin{bmatrix} 1 & 0 & -14 \\ 14 & 8 & 28 \\ 0 & 0 & 8 \end{bmatrix} - \begin{bmatrix} 5 & 0 & -30 \\ 30 & 20 & 60 \\ 0 & 0 & 20 \end{bmatrix} + \begin{bmatrix} 8 & 0 & -16 \\ 16 & 16 & 32 \\ 0 & 0 & 16 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 0 & 4 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1-5+8-4 & 0+0+0+0 & -14+30-16-0 \\ 14-30+16-0 & 8-20+16-4 & 28-60+32-0 \\ 0+0+0+0 & 0+0+0+0 & 8-20+16-4 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

(ii) A^{-1}

$$A^3 - 5A^2 + 8A - 4I = 0$$

$$A^3(A^{-1}) - 5A^2(A^{-1}) + 8A(A^{-1}) - 4A^{-1}I = 0$$

$$A^2 - 5A + 8I - 4A^{-1} = 0$$

$$-4A^{-1} = -A^2 + 5A - 8I$$

$$A^{-1} = \frac{1}{4}\left[A^2 - 5A + 8I\right]$$

$$= \frac{1}{4}\left[\begin{bmatrix} 1 & 0 & -6 \\ 6 & 4 & 12 \\ 0 & 0 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & -10 \\ 10 & 10 & 20 \\ 0 & 0 & 10 \end{bmatrix} + \begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 8 \end{bmatrix}\right]$$

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$$= \frac{1}{4}\begin{bmatrix} 1-5+8 & 0-0-0 & -6+10+0 \\ 6-10+0 & 4-10+8 & 12-20+0 \\ 0+0+0 & 0+0+0 & 4-10+8 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 4 & 0 & 4 \\ -4 & 2 & -8 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1/2 & -2 \\ 0 & 0 & 1/2 \end{bmatrix}$$

(ii) A^4

$$A^3 - 5A^2 + 8A - 4I = 0$$

$$A^4 - 5A^3 + 8A^2 - 4A = 0$$

$$A^4 = 5A^3 - 8A^2 + 4A$$

$$= \begin{bmatrix} 5 & 0 & -70 \\ 70 & 40 & 140 \\ 0 & 0 & 40 \end{bmatrix} - \begin{bmatrix} 8 & 0 & -48 \\ 48 & 32 & 96 \\ 0 & 0 & 32 \end{bmatrix} + \begin{bmatrix} 4 & 0 & -8 \\ 8 & 8 & 16 \\ 0 & 0 & 8 \end{bmatrix}$$

$$= \begin{bmatrix} 5-8+4 & 0+0+0 & -70+48-8 \\ 70-48+8 & 40-32+8 & 140-96+16 \\ 0+0+0 & 0+0+0 & 40-32+8 \end{bmatrix}$$

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$$A^{4} = \begin{bmatrix} -1 & 0 & -30 \\ 30 & 16 & 60 \\ 0 & 0 & 16 \end{bmatrix}$$ //

1. Show that the non singular matrix $A=\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ satisfied the equation $A^2 - 2A - 5I = 0$. Hence evaluate A^{-1}!

Sol:-

The characteristic equation $|A-\lambda I|=0$

$$|A-\lambda I| = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$= \begin{bmatrix} 1-\lambda & 2 \\ 3 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda)(1-\lambda) - 6$$

$$= 1 - \lambda - \lambda + \lambda^2 - 6$$

$$= \lambda^2 - 2\lambda - 5$$

By the cayley Hamilton's theorem,

$$A^2 - 2A - 5 = 0$$

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$$A^2 = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6 & 2+2 \\ 3+3 & 6+1 \end{bmatrix} = \begin{bmatrix} 7 & 4 \\ 6 & 7 \end{bmatrix}$$

$$2A = \begin{bmatrix} 2 & 4 \\ 6 & 2 \end{bmatrix}$$

$$5I = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$A^2 - 2A - 5I = \begin{bmatrix} 7 & 4 \\ 6 & 7 \end{bmatrix} - \begin{bmatrix} 2 & 4 \\ 6 & 2 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 7-2-5 & 4-4-0 \\ 6-6-0 & 7-2-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

$$A^2 - 2A - 5I = 0$$

$$A^2(A^{-1}) - 2A(A^{-1}) - 5(A^{-1}).I = 0$$

$$A - 2I - 5A^{-1} = 0$$

$$-5A^{-1} = -A + 2I$$

$$A^{-1} = \frac{1}{5}[A - 2I]$$

$$A^{-1} = 1/5 \begin{bmatrix} -1 & 2 \\ 3 & -1 \end{bmatrix} //$$

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2. If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$ P.T $A^3\ 2A^2 - 5A + 6I = 0$

Sol:-

$$A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+2 & 0+0+4 & 2+0+0 \\ 0+0+2 & 0+1+4 & 0+2+0 \\ 1+0+0 & 0+2+0 & 2+4+0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 2 \\ 1 & 2 & 6 \end{bmatrix} \Rightarrow 2A^2 = \begin{bmatrix} 6 & 8 & 4 \\ 4 & 10 & 4 \\ 2 & 4 & 12 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 3 & 4 & 2 \\ 2 & 5 & 2 \\ 1 & 2 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3+0+2 & 0+4+4 & 6+8+0 \\ 2+0+2 & 0+5+4 & 4+10+0 \\ 1+0+6 & 0+2+12 & 2+4+0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 8 & 14 \\ 4 & 9 & 14 \\ 7 & 14 & 6 \end{bmatrix}$$

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