CHAPTER III

PROJECTILES

Introduction: In this chapter we shall consider the motion of a particle projected from a point on the earth making the following assumptions. (*a*) The resistance offered by air to the moving particle is negligibly small. (*b*) The acceleration due to gravity remains constant at all points in the paths.

3·1. Vertical motion under gravity: When a particle is projected vertically upwards from a point on the earth, we regard the upward direction as the positive direction. The force of gravity produces an acceleration g downwards on the particle. g is therefore taken as negative.

The equation of motion of the particle projected vertically upwards from the earth are obtained by substituting $-g$ instead of a in the equations of motion of a particle moving with uniform acceleration along a straight line.

The velocity of the particle t seconds after projection is given by

$$v = u - gt \quad \text{......(1)}$$

The displacement of the particle in time t is given by

$$s = ut - \tfrac{1}{2}gt^2 \quad \text{......(2)}$$

and the relation between the velocity of projection and the velocity after t seconds is

$$v^2 = u^2 - 2gs \quad \text{......(3)}$$

3·2. Motion of a particle projected horizontally from a point above the earth: Let a particle be projected horizontally with velocity u from a point P at a height y above the earth. In this case, the force due to gravity which acts vertically downwards has no effect on the motion of the particle in the horizontal direction. Hence the horizontal velocity remains constant throughout the motion of the particle. But due to the force of gravity, the initial velocity vertically downwards is zero. The

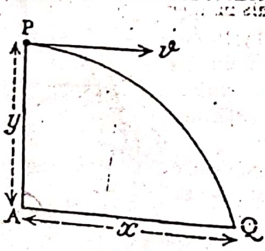

Fig 15

particle will have an acceleration g vertically downwards. The velocity with which the particle hits the ground vertically after t seconds is given by

$$v = gt \quad \text{......(4)}$$

The vertical distance described by the particle is

$$y = \tfrac{1}{2}gt^2 \quad \text{......(5)}$$

also

$$v^2 = 2gy \quad \text{......(6)}$$

The horizontal displacement of the projectile in t sec is

$$x = ut \quad \text{......(7)}$$

Therefore $t = \dfrac{x}{u}$

Substituting this value of t in equation (5), we have

$$y = \tfrac{1}{2}g \times \frac{x^2}{u^2} \quad \text{......(8)}$$

In equation (8) since g and u are constant, y is quadratic function of x. The graph showing the relation between y and x^2 is a parabola fig. 15.

3·3. Particle projected in any direction: When a particle is projected in any direction from a point on the earth, the angle which the direction of projection makes with the horizontal plane through the point of projection is called the *angle of projection*. The path described by the particle is called its *trajectory*. The distance measured from the point of projection to the point, where the particle reaches the horizontal plane through the point of projection is called the *range* on the horizontal plane. The interval of time from the instant of projection to the instant the particle reaches the horizontal plane through the point of projection is called the *time of flight*.

Let a particle be projected from a point P on the ground with velocity u in a direction making an angle α with the horizontal through the point of projection, fig. 16. Resolving the velocity of projection into components along the horizontal and vertical through the point of projection, the horizontal and vertical components are $u \cos\alpha$ and $u \sin\alpha$ respectively. By the principle of physical independence of vectors, we can consider the horizontal and vertical motions separately. Since the force of gravity acts vertically downwards it has no effect on the horizontal velocity. Hence the horizontal component of the velocity of projection remains constant throughout the motion. The displacement of the particle in the horizontal direction in time t seconds after projection is given by

$$x = u \cos\alpha \times t \qquad ......(9)$$

The component velocity in the vertical direction is retarded by the gravitational acceleration. Hence the vertical displacement of the projectile in time t seconds after projection is given by

$$y = u \sin\alpha \,.\, t - \tfrac{1}{2}gt^2 \qquad ......(10)$$

The velocity of the projectile t seconds after projection is given by

$$v = u \sin\alpha - gt \qquad ......(11)$$

Also

$$v^2 = u^2 \sin^2\alpha - 2gy \qquad ......(12)$$

Substituting the value of t from equation (9) in equation (10)

$$y = u \sin\alpha \cdot \frac{x}{u\cos\alpha} - \tfrac{1}{2}g \cdot \frac{x^2}{u^2\cos^2\alpha}$$

$$y = x \tan\alpha - \frac{gx^2}{2u^2\cos^2\alpha} \qquad ......(13)$$

From equation (13) it is easily seen that, y is a quadratic function of x. Hence the path of the projected particle is a parabola.

(1) *Velocity t seconds after projection.*

Let the particle projected from P reach the point Q on the parabola t seconds after projection. The horizontal component at Q is $u \cos\alpha$, while the vertical component is reduced to $u \sin\alpha - gt$. Therefore, the resultant velocity of the particle to Q is given by

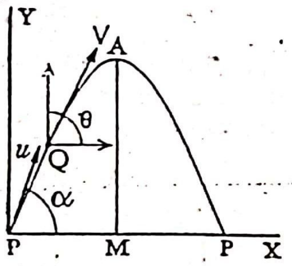

Fig. 16

$$V^2 = u^2\cos^2\alpha + (u \sin\alpha - gt)^2$$

$$= u^2 - 2ugt \sin\alpha + g^2t^2$$

or

$$V = \sqrt{u^2 - 2ugt \sin\alpha + g^2t^2} \qquad ......(14)$$

If θ be the inclination of the resultant velocity to the horizontal at Q

$$\tan\theta = \frac{u \sin\alpha - gt}{u\cos\alpha} \qquad ......(15)$$

(2) *Time of flight.*

The interval of time from the instant of projection to the instant the particle reaches the horizontal plane through the point of projection is called time of flight.

Let T be the time of flight. Then, in this time the vertical distance travelled $y = 0$.

Therefore substituting this condition in equation. (10)

$$0 = u \sin\alpha \,.\, T - \tfrac{1}{2}gT^2$$

Therefore

$$T = \frac{2u \sin\alpha}{g}$$

(3) *Greatest height attained by the projectile.*

At the highest point of the trajectory, the vertical component of velocity is reduced to zero.

At the highest point, we have

$$0^2 = u^2 \sin^2\alpha - 2gh$$

or

$$h = \frac{u^2 \sin^2\alpha}{2g}$$

(4) *Range on the horizontal plane.*

$$R = u \cos\alpha \,.\, T$$

$$= u\cos\alpha \,.\, \frac{2u\sin\alpha}{g} = \frac{2u^2 \sin\alpha\cos\alpha}{g} = \frac{u^2 \sin 2\alpha}{g}$$

The range on the horizontal plane is maximum for a given value of u, when $\sin 2\alpha = 1$, *i.e.*, when $2\alpha = 90°$ or $\alpha = 45°$. Maximum range is u^2/g.

(3·1) **Path of a Projectile is a Parabola:** Let a particle be projected from a point P with velocity u. Let α be the angle of projection, A the highest point and PP_1 the range on the horizontal plane and let AM be drawn perpendicular to PP_1. Let Q be a point on the path of the particle after a time t from the instant of projection (Fig. 17).

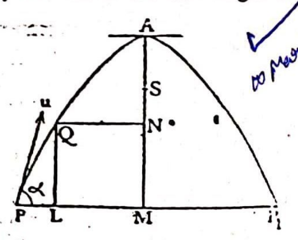

Fig. 17

Draw QL and QN perpendicular to PP_1 and AM respectively. It has been already shown that

$$AM = \frac{u^2 \sin^2\alpha}{2g} \qquad ......(1)$$

and

$$PP_1 = \frac{2u^2}{g} \sin\alpha\cos\alpha \qquad ......(2)$$

Hence, $PM = \frac{u^2}{g}\sin\alpha\cos\alpha$,

Also $QL = u\sin\alpha\cdot t - \frac{1}{2}gt^2$ ......(3)

and $PL = u\cos\alpha\cdot t$ ......(4)

$AN = AM - NM = AM - QL$ ......(5)

Substituting the values of AM and QL in (5a), ...(5a)

$$AN = \frac{u^2\sin^2\alpha}{2g} - (u\sin\alpha\cdot t - \tfrac{1}{2}gt^2),$$

$$= \tfrac{1}{2}g\left[\frac{u\sin\alpha}{g} - t\right]^2 \quad ......(6)$$

Also $QN = PM - PL$ ......(6a)

$$= \frac{u^2\sin\alpha\cos\alpha}{g} - u\cos\alpha\cdot t$$

$$= u\cos\alpha\left(\frac{u\sin\alpha}{g} - t\right) \quad ......(7)$$

Squaring equation (7),

$$QN^2 = u^2\cos^2\alpha\left[\frac{u\sin\alpha}{g} - t\right]^2 \quad ......(8)$$

Substituting for $\left[\frac{u\sin\alpha}{g} - t\right]^2$ the value $\frac{2AN}{g}$ as given by equation (6), we have

$$QN^2 = u^2\cos^2\alpha \times \frac{2AN}{g} = \frac{2u^2\cos^2\alpha}{g}AN \quad ......(9)$$

If S be a point on AM such that

$$AS = \frac{u^2\cos^2\alpha}{2g}$$

equation (9) reduces to $QN^2 = 4AS \times AN$ ......(10)

Equation (10) represents a parabola, having S as its focus with its axis vertical, with the vertex at A and having a latus rectum which is $4AS$ equal to $\frac{2u^2\cos^2\alpha}{g}$

Aliter. Consider the position P of the projectile at any instant t when its horizontal displacement $x = u_x\cdot t = u\cos\alpha\cdot t$

or $t = \frac{x}{u\cos\alpha}$ ......(1)

The vertical displacement y at this instant is such that

$$y = u_y\cdot t - \tfrac{1}{2}gt^2$$

$$= u\sin\alpha\cdot\frac{x}{u\cos\alpha} - \tfrac{1}{2}g\frac{x^2}{u^2\cos^2\alpha} \quad ......(2)$$

If this relation between x and y is to satisfy the equation of a parabola it should be of the forms

$$(x-h)^2 = -4a(y-k)$$

In order to make the coefficient of x^2 to be one multiply equation (2) throughout by $\frac{-2u^2\cos^2\alpha}{g}$ we get

$$-\frac{2u^2\cos^2\alpha}{g}y = \frac{2u^2\sin\alpha\cos\alpha}{g}x + x^2$$

i.e., $$x^2 - \frac{2u^2\sin\alpha\cos\alpha}{g}x + \frac{u^4\sin^2\alpha\cos^2\alpha}{g^2}$$

$$= -\frac{2u^2\cos^2\alpha}{g}y + \frac{u^4\sin^2\alpha\cos^2\alpha}{g^2}$$

or $$\left[x - \frac{u^2\sin\alpha\cos\alpha}{g}\right]^2 = -\frac{2u^2\cos^2\alpha}{g}\left[y - \frac{1}{2}\frac{u^2\sin^2\alpha}{g}\right]$$

This equation is of the form

$$(x-h)^2 = -4a[y-k]$$

This is the equation of an inverted parabola with the point (h, k) as its vertex and $\frac{2u^2\cos^2\alpha}{g}$ as its latus rectum.

Therefore the path of a projectile is a parabola.

(10m) **3·5. Range of a projectile on a plane inclined to the horizontal:** Let a particle be projected from a point A with velocity u in a direction making an angle α with the horizontal plane through A. It is required to find the range AB on a plane inclined at an angle β with the horizontal. The direction of projection lies in a vertical plane through AB. Let BC be the perpendicular from B to the horizontal through A.

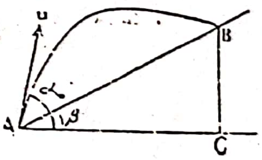

Fig. 18

The initial velocity of projection u can be resolved into a component $u\cos(\alpha-\beta)$ along the plane and a component $u\sin(\alpha-\beta)$ perpendicular to the plane. The acceleration due to gravity g, which acts vertically down can be resolved into a component $-g\sin\beta$ up the plane and $-g\cos\beta$ perpendicular to the plane. Let T be the time which the particle takes to go from A to B.

Then in this time the distance traversed by the projectile perpendicular to the plane is zero.

So $0 = u \sin(\alpha-\beta) \,.\, T - \frac{1}{2} g \cos\beta \,.\, T^2$.

Hence, $$T = \frac{2u \sin(\alpha-\beta)}{g \cos\beta} \quad \text{......(1)}$$

During this time T, the horizontal velocity of the projectile ($u \cos\alpha$) remains constant. Hence the horizontal distance described is given by

$$AC = u \cos\alpha \,.\, T = \frac{2u^2 \sin(\alpha-\beta) \cos\alpha}{g \cos\beta} \quad \text{......(2)}$$

The range on the inclined plane

$$AB = \frac{AC}{\cos\beta} = \frac{2u^2 \sin(\alpha-\beta) \cos\alpha}{g \cos^2\beta}$$

Range on the inclined plane

$$= \frac{2u^2 \sin(\alpha-\beta) \cos\alpha}{g \cos^2\beta} \quad \text{......(3)}$$

Aliter. During this time T, consider the motion parallel to the plane AB.

$$AB = u \cos(\alpha-\beta)\, T - \frac{1}{2} g \sin\beta \,.\, T^2$$

$$= \frac{u \cos(\alpha-\beta)\, 2u \sin(\alpha-\beta)}{g \cos\beta} - \frac{1}{2} g \sin\beta \,.\, \frac{4u^2 \sin^2(\alpha-\beta)}{g^2 \cos^2\beta}$$

$$= \frac{2u^2 \sin(\alpha-\beta) \cos(\alpha-\beta)}{g \cos\beta} - \frac{2u^2 \sin^2(\alpha-\beta) \sin\beta}{g \cos^2\beta}$$

$$= \frac{2u^2 \sin(\alpha-\beta)}{g \cos\beta} \left[\cos(\alpha-\beta) - \frac{\sin(\alpha-\beta) \sin\beta}{\cos\beta} \right]$$

$$= \frac{2u^2 \sin(\alpha-\beta)}{g \cos\beta} \left[\frac{\cos(\alpha-\beta) \cos\beta - \sin(\alpha-\beta) \sin\beta}{\cos\beta} \right]$$

$$= \frac{2u^2 \sin(\alpha-\beta) \cos\alpha}{g \cos^2\beta}$$

3·6. Maximum Range on the Inclined Plane:

$$R = \frac{2u^2 \sin(\alpha-\beta) \cos\alpha}{g \cos^2\beta}$$

$$R = \frac{u^2}{g \cos^2\beta} [\sin(2\alpha-\beta) - \sin\beta]$$

For given values of u and β, R is maximum, when $\sin(2\alpha-\beta) = 1$, *i.e.*, when $(2\alpha-\beta) = 90°$ or $\alpha = (45° + \frac{1}{2}\beta)$

If R_m represents the maximum range on the inclined plane

$$R_m = \frac{u^2}{g \cos^2\beta} (1 - \sin\beta) = \frac{u^2}{g(1+\sin\beta)}$$

3·7. For a given velocity of projection there are two directions of projection, in order to obtain a given range on the inclined plane and these two directions of projection are equally inclined to the direction giving the maximum range.

Now, $$R = \frac{u^2}{g \cos^2\beta} [\sin(2\alpha-\beta) - \sin\beta] \quad \text{......(1)}$$

For given values of u, β and u and R, $\sin\beta$ is constant. There are two values of $(2\alpha-\beta)$ each less than 180° that can satisfy the above equation. Let $(2\alpha_1-\beta)$ and $(2\alpha_2-\beta)$ be the two values.

Then $$2\alpha_1 - \beta = 180° - (2\alpha_2 - \beta) \quad \text{......(2)}$$

Hence. $$\alpha_1 - \frac{\beta}{2} = 90° - \left(\alpha_2 - \frac{\beta}{2}\right)$$

$$\alpha_1 - \left(45° + \frac{\beta}{2}\right) = \left(45° + \frac{\beta}{2}\right) - \alpha_2. \quad \text{......(3)}$$

Since $\{45° + \beta/2\}$ is the angle of projection giving the maximum range, it follows that the direction giving maximum range bisects the angle between the two angles of projection that can give a particular range.

3·8. The velocity at any point in the path of a projectile is equal in magnitude to that acquired by it in falling freely from the directrix to that point.

Let PAP_1 be the path of a particle projected from P with velocity u at angle α with the horizontal through P. Let XTX_1 be the directrix and S the focus of the parabola:

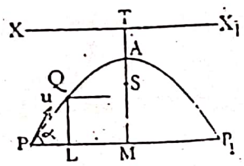

Fig. 19

Then $$AT = AS = \frac{u^2 \cos^2\alpha}{2g}$$

The height of the directrix above $PP_1 = MT$

$$= AM + AT$$

$$= \frac{u^2 \sin^2\alpha}{2g} + \frac{u^2 \cos^2\alpha}{2g}$$

$$= \frac{u^2}{2g}$$

2m. **5·1. Impulse of a force :** If a constant force acts on a body for a given interval of time, the product of the force and the time during which it acts, measures the impulse of the force. If F be the constant force and t the time during which it acts, the impulse of the force is given by $$I = F \times t$$

By Newton's second law of motion,

$$F = ma$$

where m is the mass of the body and a the acceleration produced.

$$\text{Therefore} \quad I = mat$$

If u be the initial velocity of the body and v the velocity after time t, $$a = \frac{v-u}{t}$$

$$\text{Therefore} \quad I = m(v-u)$$

The impulse of a force acting on a body for an interval of time is measured by the change of momentum it produces.

When the force is variable, the impulse of the force is calculated as follows. Let f be the force at any instant of time t and let this force act for a short time dt. The impulse during the time dt is fdt it being assumed that the force remains constant for the short interval of time. The impulse of the force during a definite interval of time t is given by

$$I = \int_0^t f dt.$$

By Newton's second law of motion

$$f = m\frac{dv}{dt}$$

$$\text{Therefore} \quad I = \int_0^t m \frac{dv}{dt} \cdot dt$$

$$= m(v-u)$$

Hence the impulse of a force is measured by the change of momentum produced, whether the force is constant or variable.

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5·2. Impulsive force: The effect of a finite force acting on a body for a finite time is measured by (1) the displacement of the body during the time and (2) the change of momentum produced. If the magnitude of the force becomes indefinitely large and the time during which the force acts is infinitely small, the displacement produced in the body is negligible and the entire effect of the force is measured by the change of momentum produced in the body. Such an enormous force acting for a very short time producing a finite effect is called an impulsive force and the entire effect of such a force is measured by the change of momentum produced. Some examples of impulsive force are (i) the blow of a hammer on a pile (ii) the force exerted by the bat on a cricket ball.

5·3. Impact between two smooth bodies: It is a matter of common observation that, if smooth balls of different materials like glass, ivory and steel are dropped from the same height above a marble floor, they rise to different heights after rebounding. This shows that the velocities with which the different balls rebound from the floor are different, even though they strike the floor with the same velocity. Again it will be observed that the velocity of rebound also depends on the nature of the material of the floor. This property of bodies by virtue of which they rebound from the floor with different velocities is attributed to their elasticity.

When two bodies like two smooth spheres impinge, the only force acting at their point of contact is directed along the common normal at the point of contact. The forces between the spheres by Newton's third law of motion are equal in magnitude but opposite in direction. Consequently the gain of momentum along the common normal for one smooth sphere must be equal to the loss of momentum for the other in the same direction. Hence the total momentum of the two spheres along the common normal before the impact must be equal to the total momentum of the system after impact in the same direction. This is in accordance with principle of conservation of momentum.

Impact between two smooth spheres is said to be direct, if the direction of motion of each smooth sphere before impact, is along the common normal at their point of contact. It is oblique if the direction of motion of one or both the smooth spheres is inclined to the common normal at the point of contact. In all cases of impact between two smooth bodies, the following principles must always hold good:

1. The total momentum of the two bodies after impact measured along the common normal must be equal to their total momentum before impact measured along the same direction.

2. The relative velocity of the spheres after impact along the common normal bears a constant ratio to their relative velocity before impact along the same direction and is of opposite sign. This constant ratio is known as the coefficient of restitution or coefficient of elasticity and is denoted by the letter e.

3. There is no tangential action between the two spheres at the point of contact. From this it follows that due to the impact, there is no change in the velocity of each sphere in a direction perpendicular to the common normal at their point of contact.

5·4. Direct impact between two smooth spheres: Let a smooth sphere of mass m_1 moving with a velocity u_1, impinge directly on another smooth sphere of mass m_2 moving with velocity u_2 in the same direction. Let e be the coefficient of restitution between them. Since the impact is direct, there is no force along the common tangent between the two spheres at the point of contact. Hence, the velocities of two spheres after the impact will be along the common normal at the point of contact. Let these velocities be v_1 and v_2. By the principle of conservation of momentum the total momentum after the impact along the common normal at the point of contact is equal to the total momentum before impact in the same direction.

$$m_1v_1 + m_2v_2 = m_1u_1 + m_2u_2 \qquad \text{......(1)}$$

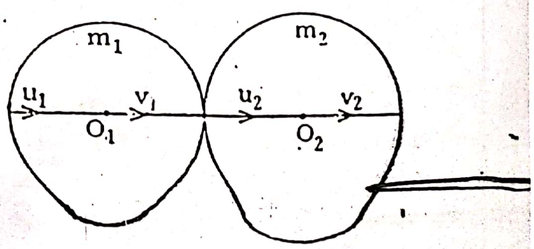

Fig. 26

By Newton's experimental law, the relative velocity between the spheres along the common normal after impact is equal to e times the relative velocity between them along the same direction but is opposite in sign.

$$v_1 - v_2 = -e(u_1 - u_2) \qquad \text{......(2)}$$

Multiplying equation (2) by m_2 and (1) adding to

$$(m_1 + m_2)v_1 = m_2u_2(1+e) + u_1(m_1 - em_2)$$

$$v_1 = \frac{m_2u_2(1+e) + u_1(m_1 - em_2)}{(m_1+m_2)} \qquad \text{...(3)}$$

Multiplying equation (2) by m_1 and subtracting from (1);

$$(m_1 + m_2)v_2 = m_1u_1(1+e) + u_2(m_2 - em_1)$$

$$v_2 = \frac{m_1u_1(1+e) + u_2(m_2 - em_1)}{m_1+m_2} \qquad \text{......(4)}$$

Equations (3) and (4) gives the velocities of two spheres after impact along the common normal.

COROLLARY 1. If the two spheres are of equal mass and are perfectly elastic, $m_1 = m_2$ and $e = 1$, therefore $v_1 = u_2$ and $v_2 = u_1$. The two spheres interchange their velocities after impact.

COROLLARY 2. The impulse of the blow on the sphere of mass m_1 is equal to the change of momentum produced in it.

$$I = m_1(v_1 - u_1)$$

$$= \frac{m_1m_2(1+e)(u_2-u_1)}{m_1+m_2}$$

The impulse of the blow on the sphere of mass m_2 is equal and opposite to that on m_1.

COROLLARY 3. If the two spheres are inelastic, $e = 0$ and therefore $v_1 = v_2$.

5·5. Oblique impact between two smooth spheres; Let a smooth sphere of mass m, moving with velocity u, impinge obliquely on a smooth sphere of mass m_2 moving with velocity u_2. Let the directions of motion of the spheres before impact make angles α and β with the common normal at their point of contact and the velocities of the spheres be v_1 and v_2 making angle θ and ϕ with the common normal after impact. By the principle of conservation of momentum, the total momentum after the impact along the

common normal is equal to the total momentum before the impact in the same direction.

$$m_1v_1\cos\theta + m_2v_2\cos\phi = m_1u_1\cos\alpha + m_1u_2\cos\beta \qquad \text{......(1)}$$

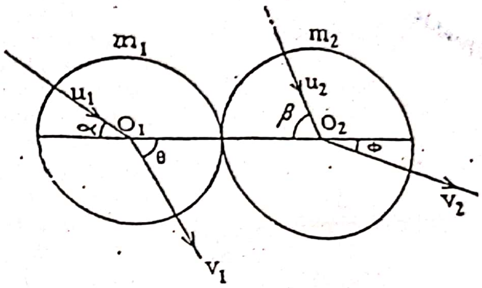

Fig. 27

By Newton's experimental law

$$v_1\cos\theta - v_2\cos\phi = -e(u_1\cos\alpha - u_2\cos\beta) \qquad \text{......(2)}$$

Since there is no tangential action, there is no change in the velocity of either sphere perpendicular to the common normal.

Therefore $v_1\sin\theta = u_2\sin\alpha$ ......(3)

$v_2\sin\phi = u_2\sin\beta$ ......(4)

Multiplying equation (2) by m_2 and adding to (1)

$$(m_1+m_2)v_1\cos\theta = m_2u_2\cos\beta(1+e) + u_1\cos\alpha(m_1 - em_2) \qquad \text{......(5)}$$

Multiplying equation (2) by m_1 and subtracting

$$v_2\cos\phi = m_1u_1\cos\alpha(1+e) + u_2\cos\beta(m_2 - em_1) \qquad \text{......(6)}$$

v_1 can be obtained by squaring (3) and (5) and adding

v_2 can be obtained by squaring (4) and (6) and adding

θ is obtained by dividing (3) by (5) and ϕ by diving (4) by (6)

COROLLARY 1. If $e = 1$ and $m_1 = m_2$

$v_1\cos\theta = v_2\cos\beta$ and $v_2\cos\theta = u_1\cos\alpha$

COROLLARY 2. The impulse of the blow on m_1

$$I = m_1(v_1\cos\theta - u_1\cos\alpha)$$

$$= \frac{m_1m_2(1+e)(v_1\cos\theta - u_1\cos\alpha)}{m_1+m_2}$$

The impulse of the blow on m_2 is equal and opposite to the impulse of the blow on m_1.

§5·6. Impact of a smooth sphere on a smooth fixed horizontal plane: Let a smooth sphere of mass m and whose coefficient of restitution is e, impinge obliquely on a smooth fixed horizontal plane PQ. Let A be the point of contact and AO the common normal at the point of contact.

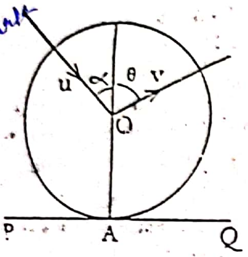

Fig. 28

Let u be the velocity of the sphere before impact in a direction making an angle α with the common normal. Let the velocity of the sphere after the impact be v inclined at an angle θ with the common normal.

By Newton's experimental law

$$v \cos \theta - 0 = -e\,[-u \cos \alpha - 0]$$

$$\text{or} \quad v \cos \theta = eu \cos \alpha \qquad \ldots\ldots (1)$$

Since both the sphere and the plane are smooth, there is no change in the velocity of the sphere in a direction perpendicular to the common normal.

$$\text{Therefore} \quad v \sin \theta = u \sin \alpha \qquad \ldots\ldots(2)$$

Squaring equation (1) and (2) and adding

$$v^2 = u^2 \sin^2\alpha + e^2 u^2 \cos^2\alpha$$

$$\text{or} \quad v = u\sqrt{(\sin^2\alpha + e^2 \cos^2\alpha)} \qquad \ldots\ldots(3)$$

Dividing equation (2) by equation (1)

$$\tan \theta = \frac{\tan \alpha}{e} \qquad \ldots\ldots(4)$$

COROLLARY 1. If $e = 1$, *i.e.*, if the sphere is perfectly elastic $v = u$ and $\theta = \alpha$. Thus, if a perfectly elastic sphere impinges obliquely on a fixed smooth plane, the velocity is unaltered in magnitude but the direction of motion before and after impact make equal angles with the common normal.

COROLLARY 2. If $\alpha = 0$, $\theta = 0$ and $v = eu$. If a smooth sphere impinges directly on a smooth fixed plane, it rebounds along the

common normal with its velocity reduced to e times its velocity before impact.

COROLLARY 3. If $e = 0$, *i.e.*, if the sphere is inelastic $\theta = 0$ and $v = u \sin \alpha$.

An elastic smooth sphere after oblique impact with a smooth fixed plane slides along the plane with velocity $u \sin \alpha$.

COROLLARY 4. The impulse of the pressure on the sphere is measured by the change of momentum produced in the sphere.

$$I = m\,[v \cos \theta - (-u \cos \alpha)]$$

$$= m\,(v \cos \theta + u \cos \alpha) = (en \cos \alpha + u \cos \alpha)$$

$$= mu \cos \alpha\,(1 + e).$$

The impulse of the force on the plane is equal and opposite to the impulse of the pressure on the sphere.

COROLLARY 5. The change in K.E. of the sphere due to impact on the plane is given by $\frac{1}{2}m\,(v^2 - u^2)$

$$= \tfrac{1}{2}m\,(v + u)(v - u)$$

But $m\,(v - u) = I$ the impulse of the force of the sphere on the plane.

Therefore change in K.E. $= \frac{1}{2}I\,(v + u)$.

§5·7. Loss of kinetic energy due to direct impact between two smooth spheres: By §5·4, the impulse of the blow I on the sphere of mass m_1 is in the direction O_1O_2 while the impulse of the blow on m_2 is also I but is in the direction O_2O_1.

The change in K.E. of $m_1 = \frac{1}{2}m_1\,(v_1^2 - u_1^2)$

$$= \tfrac{1}{2}m_1\,(v_1 - u_1)(v_1 + u_1)$$

$$\text{But} \quad I = m_1\,(v_1 - u_1)$$

Therefore change in K.E. of $m_1 = \frac{1}{2}I\,(v_1 + u_1)$

The change in the K.E. of the sphere m_2

$$= \tfrac{1}{2}m_2\,(v_2^2 - u_2^2) = \tfrac{1}{2}m_2\,(v_2 - u_2)(v_2 + u_2)$$

$$\text{But} \quad m_2\,(v_2 - u_2) = -I.$$

Therefore change in K.E. of $m_2 = -\frac{1}{2}I\,(v_2 + u_2)$

Total change in K.E. $= \frac{1}{2}I\,(v_1 - u_1) - \frac{1}{2}I\,(v_2 - u_2)$.

$$= \tfrac{1}{2}I\,[(v_1 - v_2) - (u_1 - u_2)]$$

$$= \tfrac{1}{2} I [(v_1 - v_2) + (u_1 - u_2)]$$

$$= \frac{m_1 m_2 (1 - e^2)(u_1 - u_2)^2}{2(m_1 + m_2)}.$$

Loss K.E. due to direct impact between the spheres

$$= \frac{m_1 m_2 (1 - e^2)(u_1 - u_2)^2}{2(m_1 + m_2)} = \tfrac{1}{2} I (u_1 - u_2)(1 - e^2)$$

COROLLARY 1. If the spheres are perfectly elastic $e = 1$, the loss in K.E. is zero.

COROLLARY 2. If the spheres are inelastic $e = 0$.

$$\text{Loss in K.E.} = \frac{m_1 m_2 (u_1 - u_2)^2}{2(m_1 + m_2)}$$

5·8. Loss of kinetic energy due to oblique impact between two spheres: Since the velocities of the spheres perpendicular to the common normal remain unaltered due to the oblique impact between the two spheres, there can be no loss in K.E. perpendicular to the common normal. The only change in K.E. will be along the common normal. An expression for the loss of K.E. due to oblique impact between the two spheres is obtained by substituting $u_1 \cos \alpha$ for u_1 and $u_2 \cos \beta$ for u_2 in the expression obtained in §8.7. Loss of K.E. due to oblique impact.

$$= \frac{m_1 m_2 (1 - e^2)(u_1 \cos \alpha - u_2 \cos \beta)^2}{2(m_1 + m_2)}$$

6·2. Centripetal and centrifugal forces: In order to enable a particle of mass m to describe a circle of radius r with uniform speed v, a force is required to impart the normal acceleration. The magnitude of this force is $\frac{mv^2}{r}$ or $mr\omega^2$. This force should be directed towards the centre of the circle and is known as centripetal force. This force can be produced in a variety of ways. For example, when a particle tied to one end of a string is whirled round, the centripetal force is supplied by the tension of the string. In the case of a cyclist riding with uniform speed along a circular road, the necessary centripetal force is provided by the force of friction between the tyres of the wheels and the road. In the case of a planet moving round the sun in an approximately circular orbit, the centripetal force is provided by the gravitational force exerted by the sun on the planet.

By Newton's third law of motion, for every action there must be an equal and opposite reaction. Hence there must also be acting on the particle describing uniform circular motion, an equal and opposite force. This force is known as centrifugal reaction and it is always directed away from the centre. For a stone tied to one end of a string and whirled in a circle with uniform speed, the stone in turn exerts an equal and opposite force on the hand. It is on account of the centrifugal reaction, the string is kept taut. The centripetal force and centrifugal reaction are equal in magnitude but opposite in direction.

6·3. Hodograph: Let a particle P be moving along any curved path. If from a fixed point O in the same plane a line OQ is drawn parallel and proportional to the speed of P, then the curve traced out by Q, as P moves in the path is called the hodograph of the particle P.

The hodograph of a particle moving with uniform velocity along a straight path is fixed point Q at a distance v from 0.

The hodograph of a particle moving along a curve with speed v is another curve whose radius is proportional to the magnitude of the velocity of P.

It can be shown that the velocity of the point Q in the hodograph at any instant, represents in magnitude and direction, the acceleration of P in its path at the same instant. Let P_1 and P_2 be the positions of the particle P in a short interval of time δt. Let OQ_1 and OQ_2 be the velocities v_1 and v_2 of the particle P at P_1 and P_2. Then the curve Q_1Q_2 is the hodograph of the particle P in its path. In time δt, the velocity of the particle P changes from OQ_1 to OQ_2. Now Q_1Q_2 represents the change of velocity of the particle Q in the hodograph in the time δt. Hence the acceleration of P in its path is Lt $\delta t \to 0$ $\frac{Q_1Q_2}{\delta t}$ = velocity of Q in the hodograph. Therefore the velocity of Q in the hodograph represents the acceleration of P in its path.

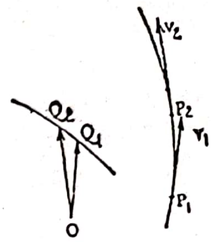

Fig. 39

6·4. Expression for normal acceleration by the hodograph method: Let a particle P move along a circle with centre O and radius r with uniform speed v. Let P_1 and P_2 represent the position of P before and after a short interval of time δt. Let O_1Q_1 and O_1Q_2 be drawn from O_1 parallel and proportional to the velocities of the particle at P_1 and P_2 respectively. Then Q_1Q_2 is the hodograph of the particle P. Q_1Q_2 is an arc of a circle of radius v. The velocity of Q in the hodograph is $\frac{Q_1Q_2}{\delta t}$. If the angle $P_1OP_2 = \delta\theta$, the angle $Q_1O_1Q_2$ is also $\delta\theta$.

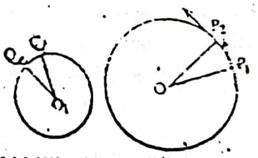

Fig. 40

Now arc $P_1P_2 = r\delta\theta$ and arc $Q_1Q_2 = v\delta\theta$

$$\frac{\text{arc } Q_1Q_2}{\text{arc } P_1P_2} = \frac{v\delta\theta}{r\delta\theta} = \frac{v}{r} \qquad \text{......(1)}$$

Therefore $$\frac{\text{arc } Q_1Q_2}{\delta t} \Big/ \frac{\text{arc } P_1P_2}{\delta t} = \frac{v}{r} \qquad \text{......(2)}$$

When the speed of rotation of the shaft increases, the weight and B rise moving the collar D upwards. This closes the valve admitting steam to the cylinders partially and thus reduces the supply of steam and therefore lowers the speed of the engine until it regains its normal value.

Similarly, when the speed decreases, the weights A and B descend lowering D. This opens the valve so that more steam admitted to the cylinders until the speed is brought to the normal value. The governor of a steam engine may be regarded as a double conical pendulum.

6·7. Motion of a cyclist along a curved path: If a cyclist is to negotiate a circular path, he invariably leans from the vertical towards the centre of the circular path and thus presses the ground in an inclined position. The horizontal component of reaction of the ground supplies the centripetal force necessary for circular motion.

AB represents a section of the cycle with the cyclist, D the centre of the circular path, mg the total weight of the circle and the cyclist, R the reaction of the ground and θ the inclination of the cycle to the vertical.

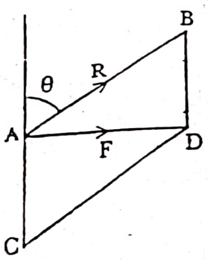

Fig. 43

The vertical component $R \cos \theta$ of the reaction balances mg the weight of the cycle and cyclist while the horizontal component of the reaction $R \sin \theta$ supplies the centripetal force needed for circular motion.

Therefore $R \cos \theta = mg$ ......(1)

and $R \sin \theta = \frac{mv^2}{r}$ ......(2)

where v is the velocity of the cyclist and r the radius of the circular path. Dividing equation (2) by equation (1)

$$\tan \theta = \frac{v^2}{rg} \quad ......(3)$$

Equation (3) gives the inclination of the plane of the cycle to the vertical in order that the cyclist may describe a circular path of radius r with a uniform speed v.

From equation (3) we find that as v increases and r decreases, θ increases and the cyclist runs the risk of falling to the ground, if he takes a sharp turn while moving with a great speed.

6·8. Motion of a Railway Carriage Round a Curved Track: When a railway carriage moves round a horizontal circular track, the necessary centripetal force for executing circular motion is supplied by the pressure exerted by the rails on the flanges of the wheels. Let $ABCD$ represent a vertical section of a railway carriage through the line joining the centre of gravity G of the carriage and the centre O of the circular track of radius r. Let A and B be the points where the wheels of the carriage touch the rails. R_1 and R_2 are the vertical reactions and X_1 and X_2 the pressures of the rails on the flanges at A and B respectively fig. 44.) Let v be the speed of the carriage. Now, $R_1 + R_2$ balances the weight of the carriage Mg and $X_1 + X_2$ supplies the centripetal force required for circular motion.

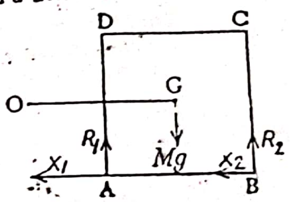

Fig. 44

Hence $R_1 + R_2 = Mg$ ......(1)

$X_1 + X_2 = \frac{Mv^2}{r}$ ......(2)

The equation holds good also in the case of a motor car running along a circular road-way with a velocity v. In this case X_1 and X_2 represent the forces of friction between the tyres and the ground.

6·9. Upsetting of a Carriage on a Curved Level Track: The centripetal force necessary for circular motion $\left(\frac{Mv^2}{r}\right)$ should act through the centre of gravity G of the carriage. This is not the case in practice as this force acts at the points of contact of the wheels on the rails. Consequently, the moment of the centripetal force about G has a tendency to upset the carriage while negotiating the curve at high speed. Let X be the resultant of the pressures of the rails at A and B. The resultant force is equal to a single force:

X at G in a parallel direction together with a couple which tends to rotate the carriage in the direction $ABCD$. If h be the height of G above AB and $2a$ the distance between the rails, the moment of the couple that tends to upset the carriage is

$$X:h=\frac{Mv^2}{r}\cdot h.$$

But the moment of the weight Mg of the carriage about $A=Mg\cdot a$. counteracts this tendency of the carriage to upset. Hence the preventing the upsetting of the carriage, we must have

$$Mag>\frac{Mv^2}{r}\cdot h$$

$$v^2<\frac{agr}{h} \quad \text{or} \quad v<\sqrt{\left(\frac{agr}{h}\right)}$$

The tendency to upset is least, if a is large and h is small. For given values of a and h, the tendency to upset is further reduced by making the speed of the carriage v small and the radius r of the circular track large.

Since there is no vertical motion

$$R_1+R_2=Mg \qquad \text{......(1)}$$

Taking moments about the point where the vertical through G meets the ground, we have

$$R_1a-R_2a=\frac{Mv^2}{r}\cdot h \qquad \text{......(2)}$$

Therefore $R_1-R_2=\frac{Mv^2}{ar}\cdot h$ ......(3)

From equations (1) and (3),

$$R_1=\frac{M}{2}\left(g+\frac{v^2h}{ra}\right) \qquad \text{......(4)}$$

and $R_2=\frac{M}{2}\left(g-\frac{v^2h}{ra}\right)$ ......(5)

From equation (5) we find that if

$$v=\sqrt{\frac{gra}{h}} \quad R_2=0.$$

This means the inner wheels do not exert any pressure on the rail. Consequently, if $v>\sqrt{\frac{gra}{g}}$, the carriage will have a tendency to upset towards the outside of curved railway.

6·10. Motion of a carriage on a banked up curve: If the rails are laid along a curve at the same horizontal level, the centripetal force required for circular motion is supplied by the pressure exerted by the rails on the flanges of the wheels. By Newton's third law, the flanges of the wheels exert equal and opposite pressure on the rails. This would result in the wearing out of rails due to the large amount of friction that it is called into play.

To avoid this wearing out of the rails, the plane of the track is tilted suitably so as to completely eliminate the flange pressure on the rails. This is done by tilting the sleepers up so that the outer rail is raised above the inner one, so that the floor of the carriage is inclined to the horizontal. The normal reactions in this case will be inclined to the vertical so that their vertical components balance the weight of the carriage, while the horizontal components supply the necessary force for circular motion.

Let $ABCD$ be a vertical section of the carriage through the line joining the centre of gravity G and the centre O of the circular track. Let the outer rail be raised over the inner, so that the floor of the carriage AB is inclined at an angle θ to the horizontal, and there is no lateral pressure exerted by the flanges of the wheels on the rails.

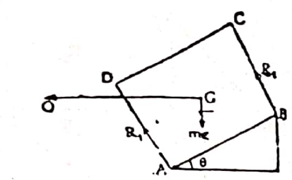

Fig. 45

If R_1 and R_2 be the normal reactions at the inner and outer rails.

Resolving vertically, we have

$$(R_1+R_2)\cos\theta=mg \qquad \text{......(1)}$$

Resolving horizontally, we have

$$(R_1+R_2)\sin\theta=mv^2/r \qquad \text{......(2)}$$

where v is the velocity of the carriage and r the radius of circular path.

Dividing equation (2) by equation (1), we have

$$\tan\theta=v^2/rg \qquad \text{......(3)}$$

Equation (3) gives the angle through which the sleepers are to be tilted from the horizontal so that there is no lateral flange pressure on the rails.

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If, however, a carriage moving with a different velocity has to pass round the curve, it is not possible to eliminate completely the lateral pressure exerted by the flanges on the rails.

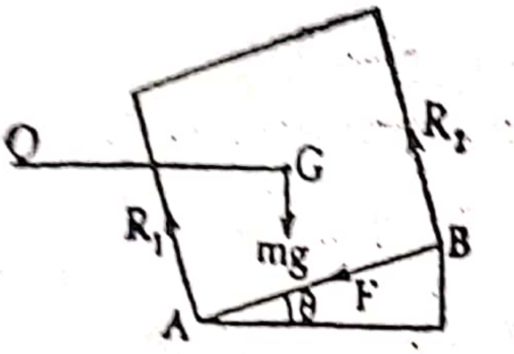

Fig. 46

Assuming that the height of the rail over the inner is adjusted so that there is no flange pressure for a critical speed v, let F be the additional lateral flange pressure acting from B to A for a carriage moving along the curve with a velocity V.

Then, resolving vertically and horizontally, we have

$$(R_1+R_2)\cos\theta - F\sin\theta = mg \qquad \text{......(4)}$$

and $$(R_1+R_2)\sin\theta + F\cos\theta = mV^2/r \qquad \text{......(5)}$$

Multiplying equation (4) by $\sin\theta$ and equation (5) by $\cos\theta$ and subtracting, $F=\dfrac{mV^2}{r}\cos\theta - mg\sin\theta = \dfrac{m\cos\theta}{r}(V^2-rg\tan\theta)$.

But $\tan\theta = \dfrac{v^2}{rg}$. Hence, $F = \dfrac{m\cos\theta}{r}(V^2-v^2)$.

If $V > v$, F is positive and the additional lateral pressure acts along BA, *i.e.*, the pressure is exerted at the outer rail.

If $V < v$, F is negative and therefore acts along AB, the flange pressure in this case is exerted at the inner rail.

Therefore $T - mg\cos\theta = \frac{mv^2}{l}$

or $T = \frac{mv^2}{l} + mg\cos\theta$ ......(2)

Substituting the value of v given by equation (1) and writing $\cos\theta = \frac{l-h}{l}$ we have

$$T = \frac{m}{l}\,[u^2 + g(l - 3h)]$$ ......(3)

Equation (1) and (3) give the velocity of the particle and the tension of the string at P.

From equation (1) we find that v decreases as h increases and attains a minimum value at the highest point B of the circle. Denoting the minimum value of v by v_B, we have

$$v_B = \sqrt{(u^2 - 4gl)}$$ ......(4)

Also from equation (3) we find that as h increases T decreases and attains a minimum value at B. If we denote this value by T_B we have

$$T_B = \frac{m}{l}(u^2 - 5gl)$$ ......(5)

(b) If the particle is to perform complete revolutions, both v and T should not vanish anywhere in the circular path from A to B.

For v not to vanish, the condition is

$$u^2 - 4gl > 0$$
or $u^2 > 4gl$
or $u > \sqrt{4gl}$ ......(6)

For T not to vanish

$$\frac{m}{l}(u^2 - 5gl) > 0$$
or $u^2 > 5gl$
or $u > \sqrt{5gl}$ ......(7)

Since the first condition is covered by the second, the particle will be able to describe complete revolutions if

$$u^2 > 5gl$$
or $u > \sqrt{5gl}$

If $u = \sqrt{5gl}$, the particle just performs a complete revolution.

(c) If $u < \sqrt{5gl}$, the particle will either oscillate about the lowest point A or leave the circular path altogether.

Let the velocity of the particle vanish at a height h_1 then

$$0 = u^2 - 2gh_1$$
or $h_1 = \frac{u^2}{2g}$

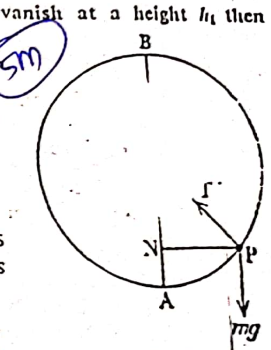

Fig. 50

The tension of the string vanishes at a height h_2 given by

$$h_2 = \frac{u^2 + gl}{3g}$$

The particle performs oscillations about A if v vanishes while T remains positive i.e., if $h_1 < h_2$

i.e., if $\frac{u^2}{2g} < \frac{u^2 + gl}{3g}$

or $u^2 > 2gl$

or $u > \sqrt{2gl}$ ......(8)

The particle will leave the circular path, if T vanishes while v is positive i.e., if $h_2 < h_1$

i.e., if $\frac{u^2 + gl}{3g} < \frac{u^2}{2g}$

i.e., if $u^2 < 2gl$

or $u < \sqrt{2gl}$ ......(9)

To summarise, if $u > \sqrt{5gl}$, the particle will execute complete revolutions.

If $u > \sqrt{2gl}$ and $< \sqrt{5gl}$, the particle will oscillate about A.

If $u < \sqrt{2gl}$ the particle will leave the circular path.

6·15. Effect of the Earth's rotation on the value of the acceleration due to gravity: Let OA and OB represent the equatorial and polar radii of the earth respectively. Let P be a point on the earth's surface whose latitude.

$\angle POA = \lambda$.

Consider a particle of mass m situated at P at latitude λ. $\angle POA = \lambda$. The gravitational pull acts along PO. Let this force be represented by PD. As the earth rotates about its polar axis with angular velocity ω, the particle which shares the earth's rotation about its axis describes a circle with C as centre and PC as radius. If the radius of the earth is R,

PC – R cos λ.

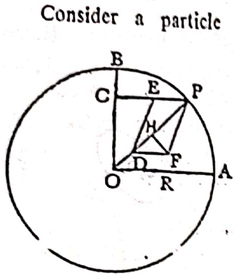

Fig. 51

To enable the particle at P to execute circular motion with angular velocity ω, the centripetal force necessary is $mR\cos\lambda \cdot \omega^2$ along PC. Let this be represented by PE. This centripetal force on the particle is supplied by the earth's pull mg on the particle. Complete the parallelogram $PEDF$. Now PF represents the effective pull of the earth. Let this cause an acceleration g' along PF. Resolve mg along PO into two components, one $mg\cos\lambda$ along PC and $mg\sin\lambda$ perpendicular to PC. Out of $mg\cos\lambda$ a part of it namely $m\omega^2 R\cos\lambda$ to produce centripetal force and the rest force along PC is

$$mg\cos\lambda - m\omega^2 R\cos\lambda.$$

The component $mg\sin\lambda$ is not affected by rotation.

Therefore, the effective weight of the body mg' along PF is

$$mg' = [(mg\cos\lambda - m\omega^2 R\cos\lambda)^2 + (mg\sin\lambda)^2]^{\frac{1}{2}}$$
$$= m\,[g^2\cos^2\lambda + \omega^4 R^2\cos^2\lambda - 2g\omega^2 R\cos^2\lambda + g^2\sin^2\lambda]^{\frac{1}{2}}$$

or $$g' = g\left[1 - \frac{2\omega^2 R\cos^2\lambda}{g}\right]^{\frac{1}{2}}$$

$$\simeq g\left[1 - \frac{\omega^2 R\cos^2\lambda}{g}\right] \text{ neglecting } \omega^4 \text{ term} \quad \ldots\ldots(10)$$

From equation (10), it is easily seen that, if $R\omega^2 = g$, $g' = 0$. In this case, the entire force of gravity on the particle will be used up in providing for the particle centripetal force to execute circular motion and nothing is left to overcome the weight of the particle. Thus the particle on the equator will fly off from the earth's surface. It can be shown that the angular velocity with which the earth should rotate round its axis in order that a particle on the equator

may fly off, is about 17 times the normal angular velocity of the earth.

(5m) **1·16. Variation of g with altitude:** Consider a unit mass on the surface of the earth of radius R, the mass of the earth being M. Let g be the acceleration due to gravity on the surface of the earth. The gravitational force on the unit mass due to the mass M acting at the centre,

$$g = \frac{GM}{R^2} \quad \ldots\ldots(1)$$

Consider the same unit mass at an altitude h where acceleration due to gravity is g

$$g_1 = \frac{GM}{(R+h)^2} \quad \ldots\ldots(2)$$

$$\frac{g_1}{g} = \frac{R^2}{(R+h)^2} = \frac{R^2}{R^2(1+h/R)^2}$$

$$\frac{g_1}{g} = \left(1 + \frac{h}{R}\right)^{-2}$$

$$g_1 = g\left(1 - \frac{2h}{R}\right) \text{ when } h \leqslant R$$

g decreases as altitude increases.

Example 1. *A body of mass 4 kg rests on a smooth horizontal plane and is connected by a rope of length 1 metre to a fixed peg on the plane. If it is whirled so as to execute circular motion on the table making 240 r.p.m., find the tension of the rope.*

No. of rev. per second $= \frac{240}{60} = 4$

Angular velocity of the body

$$= 2\pi \times 4 = 8\pi \text{ radians/sec}$$

Centripetal force required for circular motion

$$= mr\omega^2 = 4 \times 1 \times 64\pi^2 = 256\pi^2\ N$$

This force is obviously provided by the tension of the rope.

Therefore $T = 256\pi^2\ N$

Example 2. *A cord 2 metre long can just support without snapping a weight of 10 kg. If one end of the cord is attached to a fixed peg on a horizontal table and a mass of 4 kg attached to the other end and is made to revolve in a circle on the table with uniform angular velocity about the fixed point, find the maximum possible velocity for the mass.*

$$\therefore \quad v = \sqrt{\frac{2}{3} gR} = \sqrt{\frac{2}{3} (9.8)(6.4 \times 10^6)} = 6469 \text{ ms}^{-1}.$$

6.7. VARIATION OF g WITH LATITUDE OR ROTATION OF THE EARTH

Let us assume that the earth is a uniform sphere of radius R revolving about its polar diameter NS (Fig. 6.5). Consider a particle of mass m on the surface of the earth at a latitude λ. If the earth were at rest, a particle of mass m placed at P will experience a force mg along the radius PO towards O.

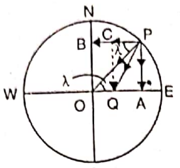

Fig. 6.5

Let ω be the angular velocity of the earth. As the earth revolves, the particle at P will execute circular motion with B as centre and BP as radius. A centrifugal force will develop and the centrifugal force acting on P along BP, away from $B = mBP.\ \omega^2$.

$$= m (R \cos \lambda)\, \omega^2 \qquad (\because BP = R \cos \lambda)$$

$$= m R\, \omega^2 \cos \lambda.$$

Force mg acts along PO. Resolve mg into two rectangular components (*i*) $mg \sin \lambda$ along PA and (*ii*) $mg \cos \lambda$ along PB. Out of the resolved component along PB, a portion $m R\, \omega^2 \cos \lambda$ is used in overcoming centrifugal force.

Let the net force be represented by PC. Then

$$PC = mg \cos \lambda - mR\, \omega^2 \cos \lambda \text{ and } PA = mg \sin \lambda.$$

The resultant force (mg') experienced by P is along PQ, such that

$$(PQ)^2 = (PC)^2 + (PA)^2 \text{ or } PQ = [(PC)^2 + (PA)^2]^{1/2}$$

i.e.,

$$mg' = [(mg \cos \lambda - mR\, \omega^2 \cos \lambda)^2 + (mg \sin \lambda)^2]^{1/2}$$

$$= mg \left[1 + \frac{R^2 \omega^4}{g^2} \cos^2 \lambda - \frac{2R\omega^2}{g} \cos^2 \lambda\right]^{\frac{1}{2}}$$

$$\therefore \quad mg' = mg \left[1 - \frac{2R\omega^2}{g} \cos^2 \lambda\right]^{\frac{1}{2}} \qquad \left(\text{neglecting } \frac{R^2\omega^4}{g^2} \cos^2 \lambda\right)$$

$$= mg \left[1 - \frac{R\omega^2 \cos^2 \lambda}{g}\right]$$

($\because R\omega^2/g$ is small, its higher powers can be neglected)

$$\therefore \quad g' = g \left[1 - \frac{R\, \omega^2 \cos^2 \lambda}{g}\right]$$

Example 6: *How many times faster than the present speed would the earth have to rotate about its axis, in order that the apparent weight of bodies at equator be zero ? What should be the new period of rotation ?*

We have $\quad g' = g\left(1 - \frac{R\omega^2 \cos^2 \lambda}{g}\right)$

where $\quad g'$ = value of acceleration due to gravity at latitude λ.

At the equator, $\lambda = 0$ and $\therefore \cos^2 \lambda = 1$.

Hence, $\quad g' = g\left(1 - \frac{R\omega^2}{g}\right)$

Now, $$\frac{R\omega^2}{g} = \frac{(6.37 \times 10^6)}{9.78} \times \left(\frac{2\pi}{86164}\right)^2 = \frac{1}{289} \qquad \text{...(1)}$$

In order that the weight of a body at equator may be zero, the value of g should be zero.

If the new angular speed of earth were ω', then $\frac{R(\omega')^2}{g} = 1 \qquad$ ...(2)

Dividing (2) by (1), $\left(\frac{\omega'}{\omega}\right)^2 = 289$

or $\quad \left(\frac{\omega'}{\omega}\right) = \sqrt{289} = 17$ or $\omega' = 17\omega$.

Hence the earth should have about seventeen times the present angular velocity in order that apparent weight of bodies at equator be zero.

Now the earth makes one complete revolution in 86164 seconds. When the earth rotates 17 times faster, its new period will be 86164/17 = 5069 seconds = 1 h 24 m 29 s.

Example 7 : *If the earth were to cease rotating about its axis, what will be the change in the value of g at a place of latitude 45°, assuming the earth to be a sphere of radius 6.38 × 10⁶ metres.*

We have, $\quad g' = g\left(1 - \frac{R\omega^2 \cos^2 \lambda}{g}\right)$

[where g' = value of acceleration due to gravity at latitude λ] *i.e.*,

$$g' = g - R\omega^2 \cos^2 \lambda$$

[where g = value of acceleration due to gravity, if the earth were at rest].

Hence, $\quad g - g'$ = change in the value of $g = R\omega^2 \cos^2 \lambda$.

Here, $\quad R = 6.38 \times 10^6$ m ; $\omega = \frac{2\pi}{24 \times 60 \times 60}$ rad s^{-1};

$\lambda = 45°$ and $\therefore \cos^2 \lambda = \frac{1}{2}$. $(g - g') = ?$

$$\therefore \quad g - g' = (6.38 \times 10^6) \times \left(\frac{2\pi}{24 \times 60 \times 60}\right)^2 \times \frac{1}{2} = 0.0169 \text{ ms}^{-2}.$$

6.8. VARIATION OF WITH g ALTITUDE

Let P be a point on the surface of the earth and Q another point at an altitude h (Fig. 6.6). Mass of the earth is M and radius of the earth is R. Let g be the acceleration due to gravity on the surface of the earth. Then

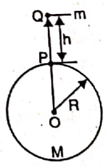

Fig. 6.6

The force experienced by a body of mass m at P $\Big\}$ $= mg = \frac{GMm}{R^2} \qquad$ ...(*i*)

The force experienced by a body of mass m at Q $\Big\}$ $= mg' = \frac{GMm}{(R+h)^2} \qquad$ ...(*ii*)

where g' is the acceleration due to gravity at an altitude h.

Dividing (*ii*) by (*i*),

$$\frac{g'}{g} = \frac{R^2}{(R+h)^2} = \frac{R^2}{R^2\,[1+(h/R)]^2} = \left(1+\frac{h}{R}\right)^{-2}$$

$$= \left(1-\frac{2h}{R}\right) \qquad \text{[neglecting higher powers of } h/R]$$

or $$g' = g\left(1-\frac{2h}{R}\right).$$

This shows that the acceleration due to gravity decreases with increase in altitude.

Example 8 : *How far away from earth does acceleration due to gravity become one percent of its value at the earth's surface ? Assume that the earth is a sphere of radius 6.38 × 10⁶ metres.*

Acceleration due to gravity on the earth's surface $= g = \dfrac{GM}{R^2}$

Acceleration due to gravity at a height $h = g' = \dfrac{GM}{(R+h)^2}$

Here, $R = 6.38 \times 10^6$ m ; $g' = (1/100)g$

or $\dfrac{g'}{g} = \dfrac{1}{100}$; $h = ?$

$$\frac{g'}{g} = \frac{R^2}{(R+h)^2} \text{ i.e., } \frac{1}{100} = \frac{R^2}{(R+h)^2} \text{ or } \frac{1}{10} = \frac{R}{R+h}$$

or $h = 9R = 9 \times 6.38 \times 10^6 = 5.742 \times 10^7$ m.

Example 9 : *A pendulum that beats seconds on the surface of the earth, is found to lose 10.8 seconds per day, when taken to the top of a hill 800 m high. What is the radius of the earth ?*

Let M and R be the mass and radius of the earth. Let g be the acceleration due to gravity on the surface of the earth. Then,

$$g = \frac{GM}{R^2}$$

Let g' be the acceleration due to gravity on the top of the hill. Then,

$$g' = \frac{GM}{(R+h)^2}.$$

Hence, $\dfrac{g}{g'} = \left(\dfrac{R+h}{R}\right)^2$. Let T and T' be the periods on the earth and on the top of the hill.

Then, $\dfrac{g}{g'} = \dfrac{(T')^2}{T^2}$

Here, $T = 2$ s ; $T' = \dfrac{86400}{86400-10.8} \times 2 = \dfrac{86400}{86389.2} \times 2.$

$$\therefore \quad \frac{(T')^2}{T^2} = \left(\frac{86400}{86389.2}\right)^2$$

Hence, $\dfrac{R+h}{R} = \dfrac{86400}{86389.2} = 1.00012$

or $\dfrac{R+800}{R} = 1.00012$ or $R = 6.666 \times 10^6$ m.

6.9 VARIATION OF g WITH DEPTH

Let g and g' be the values of acceleration due to gravity at P and Q respectively (Fig. 6.7). At P, the whole mass of the earth attracts the body.

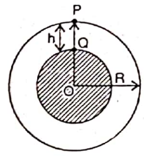

Fig. 6.7

$$\therefore \quad mg = \frac{GMm}{R^2} \qquad ...(1)$$

where m = mass of the body,
M = mass of the earth and
R = Radius of the earth

At Q, the body is attracted by the mass of the earth of radius $(R-h)$.

$$\therefore \quad mg' = \frac{GM'm}{(R-h)^2} \qquad ...(2)$$

Here, $M = \dfrac{4}{3}\pi R^3 \rho$ and $M' = \dfrac{4}{3}\pi (R-h)^3 \rho$

where ρ is the mean density of the earth.

Dividing (2) by (1), $\dfrac{g'}{g} = \dfrac{M'}{M}\dfrac{R^2}{(R-h)^2}$

$$= \frac{\frac{4}{3}\pi(R-h)^3\rho}{\frac{4}{3}\pi R^3\rho} \times \frac{R^2}{(R-h)^2} = \frac{(R-h)}{R} = \left(1-\frac{h}{R}\right)$$

$$\therefore \quad g' = g\left(1-\frac{h}{R}\right)$$

Therefore, the acceleration due to gravity decreases with increase of depth.

6.10. THE COMPOUND PENDULUM

Any rigid body capable of oscillating freely about a horizontal axis passing through it is a compound pendulum.

To find the period of oscillation of a compound pendulum :

Let O be the point of suspension and G the centre of mass (Fig. 6.8). In the equilibrium position, OG is vertical. $OG = h$. Suppose the body is given a small angular displacement about O and let go. The centre of mass G is displaced to G'. The body oscillates about the equilibrium position. It can be shown that the motion is simple harmonic. Let M be the mass of the pendulum. The restoring couple due to gravity = $Mgh \sin\theta$. The couple is also equal to $I\,(d^2\theta/dt^2)$ where I = M.I. of the body about the axis of rotation and $(d^2\theta/dt^2)$ = angular acceleration.

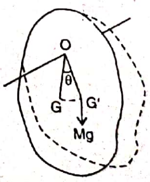

Fig. 6.8

The equation of motion of the body is

6.1. NEWTON'S LAW OF GRAVITATION

Statement : *Every particle of matter in the universe attracts every other particle with a force which is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.*

Explanation : If m_1 and m_2 are the masses of two particles situated at a distance r apart, the force of attraction between them is given by

$$F \propto \frac{m_1 m_2}{r^2} \text{ or } F = \frac{G m_1 m_2}{r^2}$$

where G is a universal constant, called the *Universal gravitational constant.* The law of gravitation is universal. It holds from huge interplanetary distances to extremely small distances. The law does not hold good for interatomic distances, which are as small as 10^{-9} m. The force of attraction between any two bodies is not affected by the intervening medium. This force is also not affected by the nature, state or chemical structure of the bodies involved but depends only on their masses. Even temperature has no appreciable effect on gravitation. $G = 6.673 \times 10^{-11}$ N m^2 kg^{-2}

Definition of G. If $m_1 = m_2 = 1$ kg and $r = 1$ m, then $F = G$. Thus, *the Gravitational constant is equal to the force of attraction between two unit masses of matter unit distance apart.*

Dimensions of G. $G = \dfrac{Fr^2}{m_1 m_2}$.

Dimensions of G are given by $[G] = \dfrac{MLT^{-2}\, L^2}{M^2} = M^{-1} L^3 T^{-2}$.

Mass and Density of earth : If m is the mass of a body and g the acceleration due to gravity, the force of attraction of the earth on the mass $m = mg$.

Let M = mass of the earth ; R = radius of the earth.

Gravitational force of attraction between a body of mass m and earth $= \dfrac{GMm}{R^2}$

$$\therefore \quad \frac{GMm}{R^2} = m \cdot g \text{ or } M = \frac{R^2 \cdot g}{G}$$

Volume of the earth $= V = \dfrac{4}{3}\pi R^3$.

$$\therefore \quad \text{Density of the earth} = \rho = \frac{M}{V} = \frac{(R^2 g/G)}{\frac{4}{3}\pi R^3} = \frac{3g}{4\pi RG}$$

Inertial mass : The mass of a body may be determined by measuring the acceleration a [pro]duced on it by a known force F.

Thus, $m = F/a$. The mass m thus determined is called inertial mass.

Gravitational mass : The mass of a body may also be determined by measuring the gravitational [forc]e exerted on it by earth.

$$F = \frac{GMm}{R^2} \text{ or } m = \frac{FR^2}{GM}.$$

The mass m thus determined is called gravitational mass.

Example 1 : *Estimate the mass of the Sun, assuming the orbit of the earth round the Sun to be [a cir]cle. The distance between the Sun and the earth is 1.49×10^{11} m, and $G = 6.66 \times 10^{-11}$ Nm^2 kg^{-2}.*

$$\left.\begin{array}{l}\text{Force of attraction between}\\ \text{the sun and the earth}\end{array}\right\} = \frac{GMm}{R^2}$$

Here, M = mass of the Sun ; m = mass of the earth, and R = Radius of the earth's orbit round sun.

This clearly supplies the centripetal force to the earth as it goes round the Sun in its orbit.

The centripetal force = mv^2/R.

Here, v = velocity of the earth in its circular orbit.

Now $v = 2\pi R/T$

where T = Time period of the earth's motion round the Sun.

$$\therefore \quad \text{Centripetal force} = \frac{m(2\pi R/T)^2}{R} = \frac{m4\pi^2 R}{T^2}$$

$$\text{For equilibrium,} \quad \frac{GMm}{R^2} = \frac{m4\pi^2 R}{T^2} \text{ or } M = \frac{4\pi^2 R^3}{T^2 . G}$$

Here, $R = 1.49 \times 10^{11}$ m, $T = 365$ days $= 365 \times 24 \times 60 \times 60$ s, $G = 6.66 \times 10^{-11}$ Nm^2 kg^{-2}

$$\therefore \quad M = \frac{4\pi^2 (1.49 \times 10^{11})^3}{(365 \times 24 \times 60 \times 60)^2 (6.66 \times 10^{-11})} = 1.971 \times 10^{30} \text{ kg}$$

Example 2 : *Calculate the mass of the earth and the mean density of the earth from the [foll]owing data :*

Radius of the earth = 6.4×10^6 m ; $g = 9.8$ ms^{-2} ; $G = 6.67 \times 10^{-11}$ SI units.

We have, mass of the earth = $M = R^2 . g/G$.

Here, $R = 6.4 \times 10^6$ m ; $g = 9.8$ ms^{-2} ; $G = 6.67 \times 10^{-11}$ SI units

$$\therefore \quad M = \frac{(6.4 \times 10^6)^2 (9.8)}{(6.67 \times 10^{-11})} = 6.017 \times 10^{24} \text{ kg}$$

Density of the earth = $\rho = 3g/(4\pi RG)$

$$= \frac{3 \times 9.8}{4\pi (6.4 \times 10^6)(6.67 \times 10^{-11})} = 5480 \text{ kg m}^{-3}$$

Example 3 : *A body weighs 900 kg on the surface of the earth. How much will it weigh on the [sur]face of Mars whose mass is one-ninth and radius one-half that of the earth ?*

Let M and R be the mass and radius of the earth. Let m be the mass of the body. Then, the force [of] attraction which the earth exerts on this body

$$F = \frac{GMm}{R^2} \text{ or } 900 = \frac{GMm}{R^2} \quad ...(1)$$

Let W be the weight of the body on the surface of Mars.

$$\text{Then,} \quad W = \frac{G.(M/9)m}{(R/2)^2} \text{ or } W = \frac{GMm}{R^2} \times \frac{4}{9} \quad ...(2)$$

Dividing (2) by (1), $\frac{W}{900} = \frac{4}{9}$ or $W = 400$ kgf.

Example 4 : *Show how the mass of the earth may be compared with that of the sun from a knowledge of the time-periods of the earth round the sun and of the moon round the earth, together with the radii of their respective orbits, (taken to be circular).*

Let M_s and M_e be the masses of the sun and the earth respectively. Let R_e be the radius of the earth's orbit round the sun. Let ω_e be the angular velocity of the earth.

$$\text{Then,} \quad \frac{G.M_s M_e}{R_e^2} = M_e R_e \omega_e^2$$

$$\text{or} \quad GM_s = R_e^3 \omega_e^2 \quad ...(i)$$

$$\text{Similarly,} \quad \frac{G M_e M_m}{R_m^2} = M_m R_m \omega_m^2$$

where M_m = mass of the moon, R_m = radius of moon's orbit round the earth and ω_m = angular velocity of the moon.

$$\therefore \quad G.M_e = R_m^3 . \omega_m^2 \quad ...(ii)$$

$$\text{From } (i) \text{ and } (ii), \quad \frac{M_e}{M_s} = \left(\frac{R_m}{R_e}\right)^3 \left(\frac{\omega_m}{\omega_e}\right)^2 \quad ...(iii)$$

Let T_e and T_m be the time periods of the earth and the moon respectively. Then

$$\left(\frac{\omega_m}{\omega_e}\right)^2 = \left(\frac{T_e}{T_m}\right)^2 \qquad \left(\because \omega = \frac{2\pi}{T}\right)$$

$$\therefore \quad \text{From } (iii), \quad \frac{M_e}{M_s} = \left(\frac{R_m}{R_e}\right)^3 \left(\frac{T_e}{T_m}\right)^2$$

Thus knowing R_m, R_e, T_e and T_m, the mass of the earth can be compared with that of the sun.

6.2. KEPLER'S LAWS OF PLANETARY MOTION

(1) *Every planet moves in an elliptical orbit around the sun, the sun being at one of the foci.*

(2) *The radius vector, drawn from the sun to a planet sweeps out equal areas in equal times i.e., the areal velocity of the radius vector is constant (dA/dt = constant).*

(3) *The square of the period of revolution of the planet around the sun is proportional to the cube of the semi-major axis of the ellipse ($T^2 \propto a^3$).*

Deduction of Newton's Law of Gravitation from Kepler's Laws

Consider two planets of masses m_1 and m_2. Let r_1 and r_2 be the radii of their circular orbits. Let T_1 and T_2 be their periods of revolution round the sun.

The centrifugal force acting on the first planet,

$$F_1 = m_1 r_1 . \omega^2 = m_1 r_1 \left(\frac{2\pi}{T_1}\right)^2$$

Similarly, the centrifugal force acting on the second planet

$$F_2 = m_2 r_2 \left(\frac{2\pi}{T_2}\right)^2$$

Now, $$\frac{F_1}{F_2} = \frac{m_1 r_1}{m_2 r_2}\left(\frac{T_2}{T_1}\right)^2$$

But according to Kepler's third law, $$\left(\frac{T_2}{T_1}\right)^2 = \left(\frac{r_2}{r_1}\right)^3$$

or $$\frac{F_1}{F_2} = \frac{m_1 . r_1}{m_2 . r_2}\left(\frac{r_2}{r_1}\right)^3 = \frac{m_1 . r_2^2}{m_2 . r_1^2}$$

i.e., the force on the planet is directly proportional to $\frac{m}{r^2}$ or $F \propto \frac{m}{r^2}$. Therefore, the force is proportional to the mass of the planet. Since the attraction is mutual, the force is also proportional to the mass of the sun *M*. Hence $F \propto \frac{Mm}{r^2}$ or $F = \frac{GMm}{r^2}$ which is Newton's Law of Gravitation.

6.3. DETERMINATION OF G–BOYS' EXPERIMENT

The apparatus consists of two co-axial glass tubes T_1 and T_2 mounted on a platform provided with levelling screws (Fig. 6.1). The inner tube T_1 is fixed, while the outer tube T_2 can be rotated about the common axis. A small mirror, *RS*, is suspended in the inner tube by a fine quartz fibre *f* from a torsion head *H*. From the two ends of the mirror, two gold spheres *A* and *B* are suspended, such that the spheres are at different depths below the mirror. In the outer co-axial tube T_2, two large lead balls *C* and *D* are suspended from its revolving lid such that the centre of *C* is in level with that of *A*, the centre of *D* is in level with that of *B* and the distance *AC* = *BD*. Two rubber pads P_1 and P_2 are placed below the two lead spheres, as a safeguard against damage, in case they should fall accidentally.

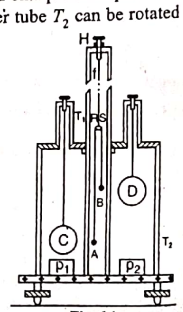

Fig. 6.1

The experiment is performed by rotating the outer glass tube until the large lead spheres lie on the opposite sides of the two gold balls, so as to exert the maximum moment on the suspended system. In this position, the angle through which the mirror (*RS*) turns is maximum. The outer glass tube is then rotated so that the lead spheres now lie on the other sides of the gold balls, in an exactly similar position, producing the greatest deflection. The mean of these two observations gives the deflection of the mirror θ.

A lamp and scale arrangement is used to measure θ.

Force of attraction between spheres *A* and *C* = $GMm/(AC)^2$

Force of attraction between spheres *B* and *D* = $GMm/(BD)^2$

Since *AC* = *BD*, the two forces are equal, parallel and act in opposite directions, thus constituting a couple.

∴ The moment of the deflecting couple $= \frac{GMm}{(AC)^2} \times 2l = \frac{GMm}{d^2} \times 2l$

(where 2 *l* = the length of the mirror strip *RS* and *AC* = *d*).

The deflection of the mirror strip under this couple is resisted by the torsion or twist set up in the suspension fibre. The mirror strip comes to rest when the deflecting couple due to gravitational pull is balanced by the restoring torsional couple set up in the suspension fibre. Now, if *c* be the torsional couple per unit twist, then for angular deflection θ, the total restoring couple is *c* . θ.

∴ In equilibrium position, $\frac{GMm}{d^2} \times 2l = c\,\theta$.

From this, the value of *G* can be calculated. Using the arrangement of the quartz fibre and the mirror strip with gold balls as a torsion pendulum, the period *T* is found. Then $T = 2\pi\sqrt{I/c}$ where *I* = moment of inertia of the suspended system. From this *c* can be calculated.

The results obtained by him are very accurate. The value obtained for *G* by Boys is 6.6576×10^{-11} Nm² kg⁻².

Advantages : (1) The size of the apparatus is very much reduced. The disturbances due to convection currents are therefore almost negligible.

(2) By arranging the masses at different levels, the effect of the attraction of the heavier mass on the remote smaller mass is very much reduced.

(3) By the lamp and scale arrangement, very small deflections can be measured accurately.

(4) The use of a quartz fibre has made the apparatus very sensitive and accurate.

6.4. GRAVITATIONAL FIELD AND GRAVITATIONAL POTENTIAL

Gravitational Field : *The space around a body within which its gravitational force of attraction is perceptible is called its gravitational field.*

The gravitational field is an example of a vector field. Each point in this field has a vector associated with it. The intensity of the gravitational field at a point due to a body is the force experienced by a unit mass placed at that point.

Gravitational Potential : *The work done in moving a unit mass from infinity to a point in a gravitational field is called the gravitational potential at that point.*

Gravitational potential is always negative in sign, its highest value being zero at infinity. It is a scalar quantity.

Intensity of gravitational field at a point : *It is defined as the space rate of change of gravitational potential at the point. i.e.*, $F = -dV/dr$ where dV is the small change of gravitational potential for a small distance dr.

Gravitational potential due to a point mass. Consider a point *A* at a distance *x* from a particle of mass *m* (Fig. 6.2).

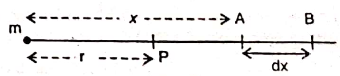

Fig. 6.2

Force of attraction experienced by a unit mass at *A* $= \frac{G.m}{x^2}$.

Work done in displacing the unit mass from *A* to *B* through a distance *dx* $= \frac{G.m}{x^2}dx$.

∴ The potential difference between *A* and *B* = $\delta V = \frac{G.m}{x^2}dx$

Hence the total work done in moving the unit mass from infinity to *P* or

The potential at $P = V = \int \delta V = \int_{\infty}^{r} \frac{G.m}{x^2}dx = -\frac{G.m}{r}$

Thus the gravitational potential has the maximum value of zero at infinity and decreases as distance is decreased.

Equipotential Surface : *A surface at all the points of which the gravitational potential is the ne is called an equipotential surface.*

For example, a spherical surface around a point mass with the mass as centre, is an equipotential face. Since the potential on this surface is constant, no work is done against the gravitational force moving a unit (or any other) mass along it.

5. GRAVITATIONAL POTENTIAL AND FIELD DUE TO A SPHERICAL SHELL

(*i*) **Point outside the shell :** Consider a point P outside : spherical shell at a distance r from its centre O (Fig. 6.3). t a be the radius of the shell, ρ the mass per unit area of : surface of the shell, and M its total mass. Join OP and let $= r$. Consider a thin slice of the shell contained between o planes AB and CD drawn close to each other at right gles to OP. Join O and A, O and C and A and P.

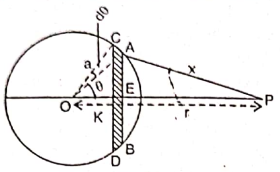

Fig. 6.3

Let $\angle AOP = \theta$, and $\angle AOC = d\theta$.

Now, AE = Radius of the slice $= a \sin\theta$

Circumference of the slice $= 2\pi \times AE = 2\pi a \sin\theta$.

Width of the slice $= CA = a\, d\theta$.

Hence, surface area of the slice $= 2\pi a \sin\theta \times a\, d\theta$

$= 2\pi a^2 \sin\theta\, d\theta$.

$\therefore$ Mass of the slice $= 2\pi a^2 \rho \sin\theta\, d\theta$.

Let $PA = x$. Every point on the slice may be taken to be practically equidistant from P.

$$\therefore \left.\begin{array}{l}\text{Potential at } P\\ \text{due to the ring}\end{array}\right\} = dV = \frac{-G\, 2\pi a^2 \rho \sin\theta\, d\theta}{x} \quad \ldots(1)$$

To find the value of x, consider the triangle OAP.

$$x^2 = a^2 + r^2 - 2ar\cos\theta$$

Differentiating, $2x\, dx = 2ar \sin\theta\, d\theta$ [$\because$ a and r are constants]

or
$$x = \frac{a \cdot r \sin\theta\, d\theta}{dx}$$

Substituting this value of x in (1),

$$dV = \frac{-G\, 2\pi a^2 \rho \sin\theta\, d\theta\, dx}{a \cdot r \sin\theta\, d\theta} = \frac{-2\pi a \cdot \rho G}{r}\, dx$$

If the entire shell is split up into slices of this kind, the value of PA will vary from $(r - a)$ to $(r + a)$. Hence,

$$\left.\begin{array}{l}\text{the potential at } P \text{ due}\\ \text{to the entire shell}\end{array}\right\} = V = \int_{r-a}^{r+a} \frac{-2\pi a \rho G}{r}\, dx.$$

$$= \frac{-2\pi a \cdot \rho G}{r}[x]_{r-a}^{r+a} = \frac{-2\pi a \rho G}{r} \cdot 2a = -4\pi a^2 \rho \frac{G}{r}$$

Now $4\pi a^2 \rho$ = Mass of the whole shell.

$$V = -\frac{G \cdot M}{r}.$$

This potential is the same as due to a mass M at O. Hence, the mass of the shell behaves as though it were concentrated at its centre.

(*ii*) **Point on the surface of the shell.** Let us consider a point which lies on the surface of the shell itself. The limits for the value of x will be 0 and $2a$. Hence

$$\left.\begin{array}{l}\text{Potential at a point on}\\ \text{the surface of the shell}\end{array}\right\} = V = \int_0^{2a} \frac{-2\pi a \rho G}{r}\, dx$$

$$= \frac{-2\pi \cdot a \cdot \rho \cdot G}{r}[x]_0^{2a} = \frac{-4 \cdot \pi \cdot a^2 \rho \cdot G}{r} = \frac{-G \cdot M}{r} = \frac{-G \cdot M}{a}; (\because r = a)$$

$$\therefore \quad V = \frac{-G \cdot M}{a}$$

(*iii*) **Point inside the shell.** Let the point P be situated at K inside the shell, such that $OK = r$. The limits for the value of x will be $(a - r)$ and $(a + r)$.

$$\therefore \left.\begin{array}{l}\text{Potential at a point } (K)\\ \text{inside the shell}\end{array}\right\} = V = \int_{a-r}^{a+r} \frac{-2\pi a \rho \cdot G}{r}\, dx$$

$$= \frac{-2\pi a \rho \cdot G}{r}[x]_{a-r}^{a+r} = -4\pi a \rho \cdot G.$$

Multiplying and dividing by a, $V = \dfrac{-4\pi a^2 \rho \cdot G}{a} = \dfrac{-G \cdot M}{a}$

$$\therefore \quad V = -\frac{G \cdot M}{a}$$

Hence the potential at all points inside a spherical shell is the same and is equal to the value of the gravitational potential on the surface.

GRAVITATIONAL FIELD. The intensity of the gravitational field F is given by $F = -dV/dr$.

(*i*) **At a point outside the shell :** $V = \dfrac{-GM}{r}$

$$\therefore \quad F = \frac{-dV}{dr} = -\frac{d}{dr}\left[\frac{-G \cdot M}{r}\right] = \frac{-G \cdot M}{r^2} \quad \ldots(i)$$

The negative sign indicates that the force is towards the centre O.

(*ii*) **At a point on the outer surface of the shell :** Putting $r = a$ in the expression (*i*), we get the intensity of the gravitational field at a point on the surface of the shell.

$$F = -\frac{GM}{a^2}$$

(*iii*) **At a point inside the shell.**

Potential $V = \dfrac{-GM}{a}$ = a constant. $\quad \therefore F = \dfrac{-dV}{dr} = 0.$

Hence there is no gravitational field inside a spherical shell.

6.6 GRAVITATIONAL POTENTIAL AND FIELD DUE TO A SOLID SPHERE

(*i*) **Point outside the sphere. :** Let P be a point outside the sphere at a distance r from the centre O [Fig. 6.4(*a*)]. Let M be the mass of the sphere, a its radius and ρ its density. A solid sphere may be imagined to be made up of a large number of concentric shells. Each one of the shells produces potential at the point P outside the shell, as if its entire mass is concentrated at the centre O.

Thus if m is the mass of one such shell,

The potential at P due to the shell $= \frac{-Gm}{r}$

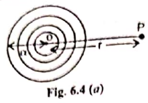

Fig. 6.4 (a)

Potential due to the whole sphere $V = -\Sigma \frac{Gm}{r} = -\frac{G}{r}\Sigma m$

Clearly, $\Sigma m = M =$ Mass of the solid sphere.

$$\therefore \quad V = \frac{-GM}{r} \qquad ...(i)$$

(ii) **Point on the surface :** If the point P lies on the surface of the solid sphere, we have $r = a$. Putting $r = a$ in *(i)*, we get,

The potential at a point on the surface $= \frac{-GM}{a}$

(iii) **Point inside the sphere :** Let the point now lie inside the solid sphere at a distance r from the centre O.

The solid sphere may be imagined to be made up of *(i)* an inner solid sphere of radius r and *(ii)* a hollow sphere of internal radius r and external radius a. The hollow sphere may be imagined to be made up of concentric shells with radii ranging from r to a.

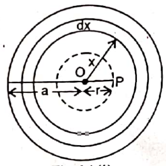

Fig. 6.4 (b)

Potential at P due to the whole solid sphere is

$V =$ Potential at P due to the inner solid sphere (V_1) + Potential at P due to all the shells (V_2)

(a) To determine the potential at P due to the inner solid sphere

The point P lies on the surface of the inner solid sphere of radius r [Fig. 6.4 (b)].

Mass of the inner sphere $= \frac{4}{3}\pi r^3 \rho.$

Potential at P due to the inner sphere

$$= V_1 = \frac{-G\frac{4}{3}\pi r^3 \rho}{r} = -G\frac{4}{3}\pi r^2 \rho \qquad ...(i)$$

(b) To determine the potential at P due to all the outer shells

Consider one such shell of radius x and thickness dx. The point P lies inside the spherical shell.

Mass of the shell $= 4\pi x^2\, dx\, \rho.$

$\therefore$ Potential at P due to this shell $= \frac{-G4\pi x^2\, dx\rho}{x} = -G\,4\pi x\, dx\rho$

$$\therefore \quad \left.\begin{matrix}\text{Potential at } P \text{ due} \\ \text{to all shells}\end{matrix}\right\} = V_2 = \int_r^a -G\,4\pi\, x\, dx\rho$$

$$= -G\,4\pi\rho \int_r^a x\,dx = -G\,4\pi\rho\left[\frac{x^2}{2}\right]_r^a = -G\,4\pi\rho\left(\frac{a^2 - r^2}{2}\right)$$

$\therefore$ Total potential at $P = V = V_1 + V_2$

$$= -G\frac{4}{3}\pi r^2\rho - G\,4\pi\rho\left(\frac{a^2 - r^2}{2}\right) = -G\frac{4}{3}\pi\rho\left[r^2 + \frac{(3a^2 - 3r^2)}{2}\right]$$

$$= -G\frac{4}{3}\pi\rho\left[\frac{2r^2 + 3a^2 - 3r^2}{2}\right] = -G\frac{4}{3}\pi\rho\left[\frac{3a^2 - r^2}{2}\right]$$

$$= -G\frac{4}{3}\pi a^3\rho\left[\frac{3a^2 - r^2}{2a^3}\right] \qquad \text{(multiplying and dividing by } a^3\text{)}$$

$$\therefore \quad V = -GM\left[\frac{3a^2 - r^2}{2a^3}\right] \qquad \left(\because \frac{4}{3}\pi a^3\rho = M\right)$$

GRAVITATIONAL FIELD

(i) **Point outside the sphere.** Potential $V = \frac{-GM}{r}$

$$\therefore \quad \text{intensity } F = \frac{-dV}{dr} = \frac{-d}{dr}\left[\frac{-G.M}{r}\right] = \frac{-G.M}{r^2} \qquad ...(i)$$

(ii) **Point on the surface of the sphere.** For a point on the surface of the solid sphere, $r = a$, and therefore

$$F = \frac{-G.M}{a^2} \qquad \text{[putting } r = a\text{, in } (i)\text{]}$$

(iii) **Point inside the sphere.** Potential at a point inside the solid sphere at a distance r from the centre O,

$$V = -G.M\left[\frac{3a^2 - r^2}{2a^3}\right]$$

$$\therefore \quad \left.\begin{matrix}\text{Intensity of the} \\ \text{field at } P\end{matrix}\right\} = F = -\frac{dV}{dr} = -\frac{d}{dr}\left[-GM\left(\frac{3a^2 - r^2}{2a^3}\right)\right]$$

$$= -\frac{GM}{a^3}r$$

Thus, the intensity of the gravitational field at a point inside a solid sphere is directly proportional to the distance of the point from the centre of the sphere.

Example 5 : *With what velocity should a body be projected vertically upwards from the surface of the earth so that it may first attain a height of R/2 where R is the radius of the earth ?* ($R = 6.4 \times 10^6$ *m*).

At the surface of the earth *i.e.*, at a distance R from its centre,

P.E. of the body $= -m\,MG/R = -mg\,R^2/R = -mgR$ $\qquad (\because MG = gR^2)$

At a distance $\left(\frac{R}{2} + R\right)$ or $\frac{3}{2}R$ from the centre of the earth

$$\text{P.E. of the body} = -MmG/\left(\frac{3}{2}R\right) = -\frac{2}{3}mg\,R$$

$$\therefore \quad \left.\begin{matrix}\text{Increase in P.E.} \\ \text{of the body}\end{matrix}\right\} = -\frac{2}{3}mgR - (-mgR) = \frac{1}{3}mgR$$

If v be its velocity of projection, its K.E. $= \frac{1}{2}mv^2 = \frac{1}{3}mgR.$

9·1. Translatory and Rotatory motions of a rigid body: A rigid body may be defined as one whose size and shape is invariable so that the distance between any two parts of it is always unaltered. The motion of a rigid body is said to be translatory if each particle of the body undergoes the same displacement in the same direction in a given interval of time *i.e.,* all the particles of the body have the same velocity. The instantaneous velocity of each particle in this kind of motion is given by dx/dt and the instantaneous acceleration of any particle is given by d^2x/dt^2.

The motion of a rigid body is said to be rotational, if each particle of the body rotates in a circle, the locus of the centres of all these circles is a straight line called the axis of rotation perpendicular to the plane of rotation.

Consider a rigid body rotating about a fixed axis through O perpendicular to the plane of the paper. It is easily seen that the linear velocities of particles like P and Q at different distances from the axis of rotation are different. Since P and Q describe the arcs PP_1 and QQ_1 in the same time, it follows that, the angular velocity of each particle of the rigid body about the fixed axis has the same value.

9·2. Moment of Inertia: A rigid body rotating about an axis has always a tendency to oppose its state of rotation exactly in the same way as the mass of a particle opposes the tendency to its state of translatory motion. This property of a rotating body is called its Moment of inertia.

A particle of mass m situated at a distance r from a given axis, the product mr^2 is called the moment of inertia of the particle about the given axis. If a system of particles of masses m_1 m_2, m_3 .... comprising a body are at distances r_1 r_2 r_3 .. from a given axis, then the moment of inertia of the system about the given axis is given by

$$I=m_1r_1^2+m_2r_2^2+m_3r_3^2+ \quad ...=\sum mr^2.$$

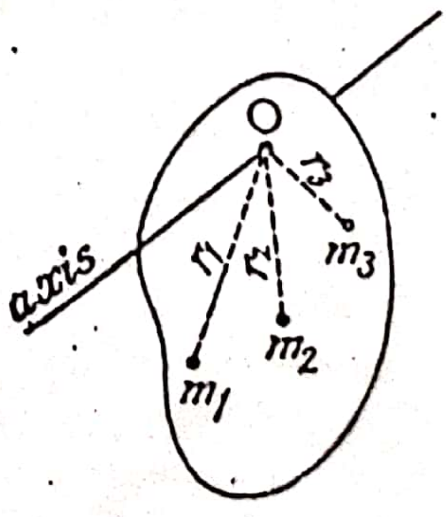

Fig. 79

In the case of a rigid body where there is a continuous distribution of matter the moment of inertia about a given axis is obtained by integration. If dm be the mass of a particle of the rigid body at a distance r from the given axis, the moment of inertia of the body about the given axis is given by

$$I = \int r^2\, dm.$$

Hence the moment of inertia of rigid body rotating about a fixed axis depends not only on the mass of the body but also on the manner in which the mass is distributed with respect to the axis.

If we imagine the entire mass of a rigid body to be concentrated at some point in the body whose distance from the given axis is k and the product $Mk^2 = \sum mr^2$, then we may write

$$I = Mk^2.$$

Here k is called the radius of gyration of the body about the given axis

$$k = \sqrt{\frac{I}{M}}.$$

9·3. Kinetic energy of a body rotating about a fixed axis: Consider a rigid body of mass M rotating uniformly about an axis through a point O perpendicular to the plane of the paper. Let m_1, m_2, m_3 ... be the masses of particles of the body at distances r_1, r_2, r_3......from the axis of rotation.

If ω be the angular velocity of the body and v_1, v_2, v_3......the linear velocities of the particles of mass m_1, m_2, m_3......at that instant, then $v_1 = r_1\omega$; $v_2 = r_2\omega$; $v_3 = r_3\omega$ and so on.

K.E. of the particles of mass m_1

$$= \tfrac{1}{2}m_1v_1^2 = \tfrac{1}{2}m_1r_1^2\omega^2$$

K.E. of the particles of mass m_2

$$= \tfrac{1}{2}m_2v_2^2 = \tfrac{1}{2}m_2r_2^2\omega^2$$

K.E. of the particle of mass m_3

$$= \tfrac{1}{2}m_3v_3^2 = \tfrac{1}{2}m_3r_3^2\omega^2$$

The K.E. of rotation of the whole body about the given axis is given by the sum of the kinetic energies of the several particles that constitute the body.

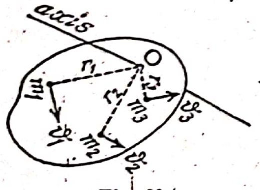

Fig. 80

K.E. of the whole body

$$= \tfrac{1}{2}m_1r_1^2\omega^2 + \tfrac{1}{2}m_2r_2^2\omega^2 + \tfrac{1}{2}m_3r_3^2\omega^2 + \ldots\ldots$$

$$= \tfrac{1}{2}\omega^2(m_1r_1^2 + m_2r_2^2 + m_3r_3^2 + \ldots)$$

$$= \tfrac{1}{2}\omega^2 \sum mr^2$$

$$= \tfrac{1}{2}I\omega^2.$$

Therefore K.E. of rotation of a rigid body

$$= \tfrac{1}{2}I\omega^2.$$

9·4. Angular momentum of a rotating body: Consider a rigid body of mass M rotating about an axis through O perpendicular to the plane of the paper with uniform angular velocity ω. Let m_1, m_2, m_3 be the masses of particles of the body at distances r_1, r_2, r_3 .. from the axis of rotation and let v_1, v_2, v_3.. ...be the linear velocities of these particles.

$$v_1 = r_1\,\omega,\; v_2 = r_2\,\omega,\; v_3 = r_3\,\omega \text{ and so on.}$$

Momentum of particle of mass m_1 is $m_1v_1 = m_1r_1\,\omega$

Moment of the momentum of the particle m_1 about the axis through O $= m_1r_1^2\,\omega$

Momentum of the particle of mass m_2

$$= m_2r_2\,\omega$$

Moment of momentum of the particle m_2 about the axis through O

$$= m_2r_2^2\,\omega$$

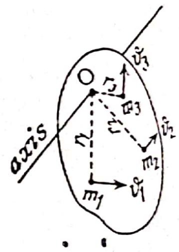

Fig. 81

and so on. The sum of the moments of momentum of all the particles of the body about the axis through O is called the angular momentum of the body.

Angular momentum

$$= m_1r_1^2\omega + m_2r_2^2\omega + m_3r_3^2\omega + \ldots\ldots$$

$$= \omega \sum mr^2 = I\omega$$

Angular momentum of a rotating body is a vector quantity and the vector representing the angular momentum is drawn along the axis of rotation of the body.

9·5. Relation between the Torque and Angular acceleration of a rigid body: Consider a rigid body of mass M rotating about a fixed axis through O perpendicular to the plane of the paper. Let P be a particle of the body of mass m at a distance r from the axis. Let the body rotate through a small angle $d\theta$ in a very small interval of time dt about the axis. Let the linear displacement of P perpendicular to OP in this short time dt be dx.

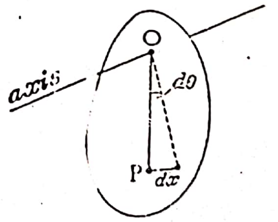

Fig. 82

measure of the angular impulse during the time dt. The total angular impulse in a finite time t is given by,

$$\int_0^t C\,dt = \int_0^t Fr\,dt = \int_0^t F\,dt \times r$$

= Moment of the linear impulse.

Also $$\int_0^t C\,dt = \int_0^t I\frac{d\omega}{dt} = \int_{\omega_0}^{\omega} I\,d\omega$$

Angular Impulse $= I(\omega - \omega_2)$.

9·7. Theorem of Perpendicular axes: If I_x and I_y be the moments of inertia of a plane lamina of mass M about two axes OX and OY at right angles to each other in its plane, then the moment of inertia I_z of the lamina about the axes OZ perpendicular to the plane of the lamina is given by $I_z = I_x + I_y$.

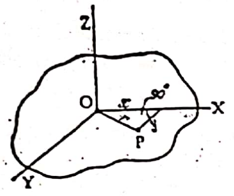

Fig. 83

Let P be a particle of mass dm in the plane of the lamina whose distances from OX, OY and OZ are y, x and r respectively.

Moment of inertia of the particle about OZ

$$= r^2 dm = dm\,(x^2 + y^2).$$

Moment of inertia of the whole lamina about OZ

$$I_z = \int (x^2 + y^2)\,dm$$
$$= M(x^2 + y^2) = Mx^2 + My^2 = I_y + I_x.$$

9·8. Theorem of Parallel axes: If I be the moment of inertia of a body of mass M about any axis CD and I_0, its moment of inertia about a parallel axis AB passing through the C.G. of the body and a the distance between the two axes, then

$$I = I_0 + Ma^2.$$

Let P be a particle of mass m at a distance x from the axis AB.

Moment of inertia of the particle about $AB = mx^2$.

Moment of inertia of the particle about $CD = m(a + x)^2$.

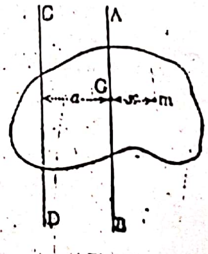

Fig. 84

$$I = \sum m(a + x)^2 = \sum ma^2 + \sum mx^2 + 2a \sum mx$$
$$= I_0 + Ma^2.$$

$\sum mx = 0$ since the body balances about a knife edge placed below the C.G.

9·9. Moment of inertia of a uniform rod:

(a) *About an axis passing through one end perpendicular to its length.* Let OA be a thin uniform rod of length l and mass M and YY_1 an axis through O perpendicular to OA.

Mass per unit length of the rod $= \frac{M}{l}$. Consider an element of the rod of length dx at a distance x from the axis YOY_1.

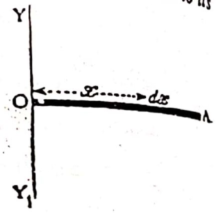

Fig. 85

Mass of the element $= \frac{M}{l}\,dx$.

Moment of inertia of the element about the axis YOY_1

$$= \frac{M}{l}\,dx \times x^2.$$

Moment of inertia of the whole rod about YOY_1

$$= \int_0^l \frac{M}{l} x^2 dx = \frac{M}{l}\left[\frac{x^2}{3}\right]_0^l = \frac{Ml^2}{3}$$

(b) *About an axis through the C.G. of the rod perpendicular to its length.*

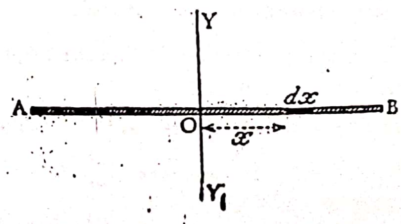

Fig. 86

9·17. Acceleration of a body rolling down an inclined plane without slipping: A body rolling down an inclined plane without slipping has two motions: (*a*) a rotational motion about a horizontal axis through its centre of mass and (*b*) a translatory motion down the plane.

Let a solid sphere of mass M roll from rest at A down a plane inclined at an angle θ to the horizontal without slipping to B, a distance x along the plane. Let the body acquire a linear velocity v and an angular velocity ω about the axis of rotation. The vertical distance travelled by the body in moving from A to B is given by $h = x \sin \theta$.

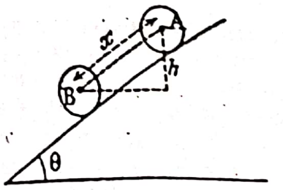

Fig. 96

The loss in potential energy in moving from A to B

$= Mgx \sin \theta$.

If I be the moment of inertia and k the radius of gyration of the body about the axis of rotation, the gain in the kinetic energy of rotation in moving from A to B

$$= \tfrac{1}{2} I\omega^2 = \tfrac{1}{2} Mk^2\omega^2 = \tfrac{1}{2} Mk^2 \frac{v^2}{r^2} \text{ since } \omega = \frac{v}{r}.$$

The gain of kinetic energy of translation $= \tfrac{1}{2} Mv^2$.

By the principle of conservation of energy,

$$\tfrac{1}{2} Mk^2 \frac{v^2}{r^2} + \tfrac{1}{2} Mv^2 = Mgx \sin \theta$$

$$\tfrac{1}{2} Mv^2 \left(1 + \frac{k^2}{r^2}\right) = Mgx \sin \theta$$

$$\text{Therefore} \quad v^2 = 2\left(\frac{g \sin \theta}{\frac{k^2}{r^2} + 1}\right) x$$

This is of the form $v^2 = 2as$.

Therefore the acceleration of the body rolling down an inclined plane without slipping is given by

$$a = \frac{g \sin \theta}{\frac{k^2}{r^2} + 1}$$

We shall find the value of this expression for various regular bodies.

(*a*) *Solid sphere*: For a solid sphere $k^2 = \tfrac{2}{5} r^2$.

$$\text{Therefore} \quad a = \frac{g \sin \theta}{\frac{2r^2}{5r^2} + 1} = \frac{g \sin \theta}{\frac{2}{5} + 1} = \tfrac{5}{7} g \sin \theta$$

(*b*) *Spherical shell*: In this case $k^2 = \tfrac{2}{3} r^2$.

$$\text{Therefore} \quad a = \frac{g \sin \theta}{\frac{2r^2}{3r^2} + 1} = \frac{g \sin \theta}{\frac{2}{3} + 1} = \tfrac{3}{5} g \sin \theta$$

(*c*) *A disc*: Here $k^2 = \dfrac{r^2}{2}$

$$\text{Therefore} \quad a = \frac{g \sin \theta}{\frac{r^2}{2r^2} + 1} = \frac{g \sin \theta}{\frac{1}{2} + 1} = \tfrac{2}{3} g \sin \theta$$

(*d*) *Solid cylinder*: $k^2 = \dfrac{r^2}{2}$

Therefore $a = \tfrac{2}{3} g \sin \theta$

9·18. Oscillations of a small sphere on a large concave smooth surface: Let m be the mass of a small sphere of radius r oscillating on a large smooth concave smooth surface of radius R. Let A be the position of the centre of the ball in its equilibrium position. A will now be vertically below the centre of the concave surface.

Let B be the position of the centre of the sphere at an instant of time t after it has passed the equilibrium position. Let $\angle AOB = \theta$ (small) and the AB which is also small $= x$.

The potential energy of the sphere at B

$= Mg \cdot AD$

Now $AD = OA - OD$

$= (R - r)(1 - \cos \theta)$

$= 2(R - r) \sin^2 \tfrac{1}{2}\theta$

$= 2(R - r) \tfrac{1}{4}\theta^2$

(when θ is small)

P.E. at the instant the sphere is at B

$= Mg \times 2(R - r) \tfrac{1}{4}\theta^2$

$= \tfrac{1}{2} Mg (R - r) \theta^2$

But $\theta = \dfrac{x}{R - r}$

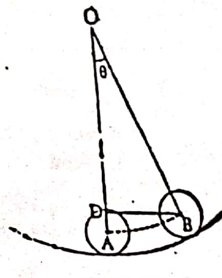

Fig. 97

Therefore P.E. at B

$$\tfrac{1}{2}Mg\,(R-r)\,\frac{x^2}{(R-r)^2}=\frac{Mg}{2\,(R-r)}\,x^2.$$

If v and ω represent the linear and angular velocities of the sphere at the instant it is at B, kinetic energy of rotation of the sphere at B

$$=\tfrac{1}{2}\,I\omega^2=\tfrac{1}{2}\cdot\tfrac{2}{5}\,.\,Mr^2\,.\,\frac{v^2}{r^2}$$

[because $v=r\omega$ and $I=\tfrac{2}{5}Mr^2$]

$$=\tfrac{1}{5}Mv^2.$$

K.E. of translation at $B=\tfrac{1}{2}Mv^2$.

Total K.E. at B

$$=\tfrac{1}{2}Mv^2+\tfrac{1}{5}Mv^2=\tfrac{7}{10}Mv^2.$$

$$\text{But } v=\frac{dx}{dt}$$

$$\text{Total K.E. at } B=\frac{7}{10}M\left(\frac{dx}{dt}\right)^2$$

By the principle of conservation of energy

K.E. + P.E. at B = a constant.

$$\text{Therefore } \frac{7}{10}M\left(\frac{dx}{dt}\right)^2+\frac{Mg}{2\,(R-r)}\,x^2=0$$

a constant. Differentiating w.r. to t

$$\frac{7}{10}M\,.\,2\frac{d^2x}{dt^2}\,.\,\frac{dx}{dt}+\frac{Mg}{2\,(R-r)}\times 2x\,\frac{dx}{dt}=0$$

$$\text{or }\ \frac{d^2x}{dt^2}+\frac{5g}{7\,(R-r)}\,.\,x=0\ \text{ or }\ \frac{d^2x}{dt^2}=-\frac{5g}{7\,(R-r)}\,.\,x$$

Since $\dfrac{5g}{7(R-r)}$ is a constant, the acceleration of the ball is directly proportional to its displacement. The oscillations of the ball on the concave surface are simple harmonic for small oscillations.

The period of oscillation is given by

$$T=\frac{2\pi}{\sqrt{\dfrac{2\pi}{7\,(R-r)}}}=2\pi\sqrt{\frac{7\,(R-r)}{5g}}$$

9·20. The Compound Pendulum : A compound pendulum consists of a rigid body capable of rotation about a fixed horizontal axis under gravity. Let the axis of rotation pass through the point O in a vertical section of the body taken through the centre of gravity G of the body. In the equilibrium position OG will be vertical. Let $OG = h$. If θ is the small angular displacement of the body from the equilibrium position in time t and M the mass of the body, the couple tending to restore the body to its equilibrium position is $Mgh \sin\theta$. The couple will produce an angular acceleration $\frac{d^2\theta}{dt^2}$. If I be the moment of inertal of the body about the axis of rotation, the product of moment of inertia and the angular acceleration is also equal to the couple acting. Therefore

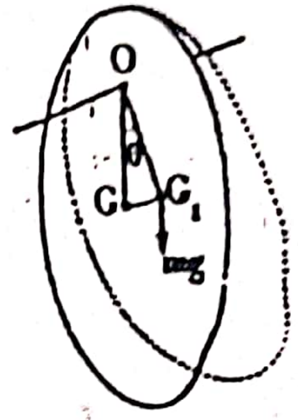

Fig. 99

$$I \frac{d^2\theta}{dt^2} = -Mgh \sin\theta \quad ......(1)$$

The significance of the negative sign is that the angular acceleration and the angular displacement are oppositely directed.

When θ is small, $\sin\theta = \theta$.

Therefore $I \frac{d^2\theta}{dt^2} = -Mgh\,\theta$

$$\text{or} \quad \frac{d^2\theta}{dt^2} = -\frac{Mgh}{I} \cdot \theta \quad ......(2)$$

If k be the radius of gyration about the axis of rotation then $I = Mk^2$.

$$\text{Therefore} \quad \frac{d^2\theta}{dt^2} = -\frac{gh}{k^2} \cdot \theta \quad ......(3)$$

This represents a simple harmonic oscillation of period

$$T = \frac{2\pi}{\sqrt{\frac{gh}{k^2}}} = 2\pi \sqrt{\frac{k^2}{gh}} \quad ......(4)$$

If K be the radius of gyration about an axis through G, parallel to the axis of rotation, then by parallel axes theorem we have

$$k^2 = K^2 + Mh^2 \quad \text{or}, \quad k^2 = K^2 + h^2. \quad ......(5)$$

$$\text{Therefore} \quad T = 2\pi \sqrt{\frac{K^2 + h^2}{hg}} \quad ......(6)$$

$$\text{Hence} \quad T = 2\pi \sqrt{\frac{K^2 + h^2}{hg}}$$

9·21. **Centre of Suspension and Centre of Oscillation :** The point O where the axis of rotation meets the vertical plane through the centre of gravity G of the rigid body is called the centre of suspension.

A simple pendulum which has the same period as the given component pendulum is called the equivalent simple pendulum. The equivalent simple pendulum $L = \frac{k^2}{h}$ or $\frac{K^2+h^2}{h}$.

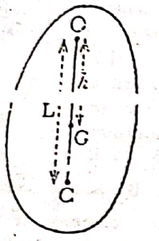

Fig. 100

... to a point C such that $OC = L$ the length of the equivalent simple pendulum, the point C is called the centre of oscillation. The centre of oscillation is obviously a point at which the mass of the body may be considered to be concentrated without any change in the periodic time.

If the body is suspended about a parallel axis through C, we have $CG = L - h$. The length of the equivalent simple pendulum will be

$$L_1 = \frac{K^2 + (L-h)^2}{L-h}$$

But $L = \frac{K^2+h^2}{h}$

we have $K^2 = Lh - h^2$

$$\text{or} \quad L_1 = \frac{Lh - h^2 + L^2 - 2Lh + h^2}{L-h} = \frac{L(L-h)}{L-h} = L.$$

Hence the centres of suspension and oscillation are interchangeable.

9·22. **Centre of Percussion :** When a body capable of rotation about a fixed axis is given a blow at a suitable point such that there is no impulsive force exerted on the fixed axis, that point is known as the *Centre of Percussion* of the body with respect to the axis.

If a pendulum supported on the axis through O is given a blow at the centre of oscillation C, it will rotate about O without any jar on the axis of rotation. The centre of oscillation C on account of the reason is also called the centre of percussion.

9·23. **Minimum periods of a compound pendulum :** From the expression $T = 2\pi\sqrt{\left(\frac{K^2+h^2}{hg}\right)}$ we find that the value of the period T depends on the length of the equivalent simple pendulum namely $\frac{K^2+h^2}{h}$.

If T is minimum, $\frac{dT}{dh} = 0$

$$i.e., \quad \frac{d}{dh}\left(\frac{K^2}{h} + h\right) = 0 \quad i.e., \quad 1 - \frac{K^2}{h^2} = 0$$

$$\text{or} \quad K^2 = h^2 \quad \text{or} \quad K = \pm h.$$

A compound pendulum will have its period a minimum when the depth of the centre of gravity of the pendulum below the centre of suspension is equal in magnitude to the radius of gyration about an axis through the centre of gravity parallel to the axis of rotation.

9·24. **Kater's Pendulum :** The fact that the centres of suspension and oscillation of a compound pendulum are interchangeable and their distance apart is equal to the length of the equivalent simple pendulum is used by Kater in the construction of a reversible pendulum which could be used to determine accurately the value of g at a place.

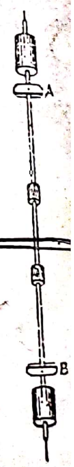

Fig. 1[illegible]

Kater's pendulum consists of a long rod of metal provided with two fixed knife edges A and B on each side of the centre of gravity at unequal distances from it. The rod is fitted at each end with cylinder one of which is of boxwood and the other of metal. Two adjustable masses are also attached on the rod between the knife edges. One of these masses is of wood and the other of metal. Their adjustment enables the C.G. of the pendulum to shift to such a position as to make the periods of oscillation of the pendulum about either knife edge to be the same. Then the distance between the two knife edges is equal to the length L of the equivalent simple pendulum. If T be the equal period about either knife edge then

$$T = 2\pi\sqrt{\frac{L}{g}}.$$

5·1. Force of friction : If two bodies which are perfectly smooth rest against each other, the only force between them is along the common normal at the point of contact. In practice it is not possible to have two perfectly smooth surfaces in contact, and so there will always exist a force between them which tends to resist the sliding of one surface over another. This force is called the force of friction. Friction plays a prominent role in everyday life. For example, a railway engine cannot move over the rails unless a force is exerted in the forward direction. If the wheels and the rails are perfectly smooth, the wheels rotate without moving forwards. The slipping will be prevented only if there is a force of friction between the driving wheels and the rails.

If a block of metal placed on a horizontal table is pulled by a string with a very small force, the force of friction is called into play between the block and the table and prevents the motion of the block. If the pulling force is gradually increased, the force of friction also gradually increases to such a value so as to be just sufficient to prevent the motion of the block. For a certain value of the pulling force, the frictional force attains a maximum value. If the pulling force is increased further, the block begins to move on the table. The maximum value of the force of friction which just prevents motion is called *limiting friction*.

5·2. Laws of friction : (1) When two bodies are in contact, the direction of the frictional force between them is always opposite to the direction in which one body tends to slide over the other.

(2) The magnitude of the force of friction between two bodies in equilibrium is just sufficient to prevent one body sliding over the other. It attains a maximum value, when one body is just on the point of sliding over the other.

(3) The force of limiting friction always bears a constant ratio to the normal reaction and this ratio is denoted by the letter μ and is called the coefficient of friction. The value of μ depends on the nature of the substance of which the bodies are composed.

(4) The force of limiting friction is independent of the extent or the shape of the surfaces in contact, provided the normal reaction is unaltered.

(5) When one body moves over another, the force of friction still exists between them opposing motion, but its value is slightly less than force of limiting friction and it is independent of the velocity of the body. But the ratio of the force of friction to the normal reaction is slightly less than that when the body is just on the point of motion.

5·3. Angle of friction, resultant reaction and cone of friction : When a body is just on the point of moving over another if the force of limiting friction F and the normal reaction R between them are compounded into a single force S, then the angle between this force S and the normal reaction R is called the *angle of* friction. It is denoted by the letter λ. The single force S is called the *resultant reaction* .

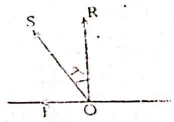

Fig. 81

$$\tan \lambda = \frac{F}{R}$$

But $$\frac{F}{R} = \mu$$

Therefore $$\tan \lambda = \mu.$$

The tangent of the angle of friction is equal to the coefficient of friction.

Also $$S^2 = F^2 + R^2 \text{ or } S = \sqrt{(F^2 + R^2)}.$$

Since the maximum value of the force of friction is $F = \mu R$, the greatest angle which the resultant reaction can make with the normal reaction is the angle of friction $\lambda = \tan^{-1} \mu$.

When the equilibrium between two bodies is limiting, if we imagine a cone with the point of contact between the bodies as the vertex, the normal reaction as axis and semi-vertical angle equal to the angle of friction, it is possible for the resultant reaction to lie on the surface of the cone or inside the cone but not outside it. Such an imaginary cone is called the *cone of friction*. (Fig. 81)

5·4. Equilibrium of a body on a rough plane inclined to the horizontal : Consider a body of weight W placed on a rough plane whose inclination to the horizontal is graduall

...ced. Let α be the inclination of the plane to the horizontal ...en the equilibrium is limiting.

The forces acting on the body are (1) its weight W vertically downwards (2) the force of limiting friction μR up the plane and (3) the normal reaction R perpendicular to the plane. Resolving the weight W into $W \sin \alpha$ down the plane and $W \cos \alpha$ perpendicular to the plane we have

$$W \sin \alpha = \mu R$$
$$W \cos \alpha = R$$

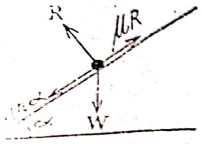

Fig. 82

Therefore $\tan \alpha = \mu = \tan \lambda$

$$\alpha = \lambda$$

Hence a body placed on a rough inclined plane will be just ...e point of sliding down the plane when the inclination of the ...e to the horizontal becomes equal to the angle of friction.

5·5. Equilibrium of a body on a rough plane under the ...ion of a force when the inclination of the plane with the ...izontal is greater than the angle of friction: Suppose a ...y of weight W rests on a rough ...ined plane inclined at an angle ... α with the horizontal being ...ported by a force P acting at an ... θ with the inclined plane.

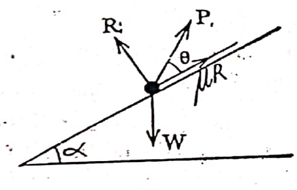

Fig. 83

When the body is just on the point ...oving down the plane, the forces ...ng on the body are the weight W ...ically downwards, the normal ...ion R at right angles to the inclined plane, the force of friction ... up the plane and force P inclined at angle θ with the plane.

Resolving the forces parallel and perpendicular to the plane,

$$P \cos \theta + \mu R = W \sin \alpha \quad \ldots\ldots(1)$$
$$P \sin \theta + R = W \cos \alpha \quad \ldots\ldots(2)$$

From equation (2)

$$R = W \cos \alpha - P \sin \theta \quad \ldots\ldots(3)$$

Substituting the value of R in equation (1), we have

$$P \cos \theta + \mu W \cos \alpha - \mu P \sin \theta = W \sin \alpha$$
$$P (\cos \theta - \mu \sin \theta) = W (\sin \alpha - \mu \cos \alpha)$$

Substituting $\mu = \tan \lambda$

$$P\,[\cos \theta - \tan \lambda \sin \theta] = W (\sin \alpha - \tan \lambda \cos \alpha)$$
$$\frac{P}{\cos \lambda}\left[\cos \theta \cos \lambda - \sin \theta \sin \lambda\right] = \frac{W}{\cos \lambda} \times (\sin \alpha \cos \lambda - \cos \alpha \sin \lambda)$$
$$P = W \frac{\sin (\alpha - \lambda)}{\cos (\theta + \lambda)} \quad \ldots\ldots(4)$$

Let P_1 be the magnitude of the external force when the body is just on the point of moving up the plane. In this case μR acts down the plane.

Resolving parallel and perpendicular to the plane

$$P_1 \cos \theta = W \sin \alpha + \mu R \quad \ldots\ldots(5)$$
$$P_1 \sin \theta + R = W \cos \alpha \quad \ldots\ldots(6)$$

From equations (5) and (6) we have

$$P_1 = W \frac{\sin (\alpha - \lambda)}{\cos (\theta - \lambda)} \quad \ldots\ldots(7)$$

The force P_1 is minimum if $\cos (\theta - \lambda) = 1$ *i.e.,* $\theta - \lambda = 0$ or $\theta = \lambda$.

Special case

If the force P is parallel to the plane, the value of P for the body to be just on the point of sliding down the plane is $P = W \frac{\sin (\alpha - \lambda)}{\cos \lambda}$

The value of P when the body is just on the point of moving up the plane is $P_1 = W \frac{\sin (\alpha + \lambda)}{\cos \lambda}$

For all values between P_1 and P_2 the body will be in equilibrium which is not limiting.

5·6. The friction dynamometer: The fact that a rope or belt coiled round a cylinder is capable of exerting a great couple on the cylinder on account of the friction between the surfaces is made use of in the construction of the *friction dynamometer*, which is used for the measurement of power.

In this device, a large pulley A is rigidly fixed to the shaft B ... a motor whose power is to be measured. A flexible cord or belt ...ing wooden blocks on its underside is passed round the pulley. ... end of the cord is attached to the hook of a spring balance ...ched to a rigid support on the ground. To the other end of the ...d is attached a loaded bucket. The load in the bucket may be ...usted until it remains at rest without rising or falling. The total ...eight of the bucket W and the reading of the spring balance S are ...d.

Let a be the radius of the pulley and b the outer radius of the ... belt.

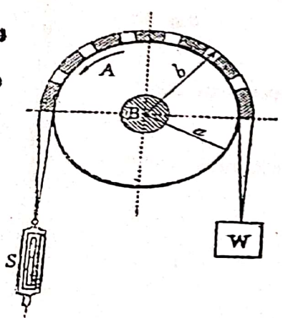

Fig. 84

Taking moments about the axis of the shaft, we have the moment of the resultant of forces due to W and S

$$= (W-S)\left(\frac{a+b}{2}\right)$$

If F be the force of friction, the moment of the frictional force about the axis of the shaft $= F \times a$

When there is equilibrium,

$$Fa = (W-S)\left(\frac{a+b}{2}\right)$$

or
$$F = \frac{(W-S)(a+b)}{2a}$$

If the shaft is making n revolution per second, the distance through ...hich the edge of the pulley moves against the frictional force per ...econd $= 2\pi an$. Therefore the work done per sec. in overcoming ...iction $= 2\pi an \times F$. Substituting the value for F, we have work ...one per second

$$= 2\pi an \times \frac{(W-S)(a+b)}{2a}$$

$$= \pi n(a+b)(W-S),$$

But the work done per sec. is the power of the engine. Hence, power of the engine $= \pi n(a+b)(W-S)$.

5·7. **The friction clutch:** A clutch is a mechanism by which ...tatory motion of one shaft can be transmitted to another shaft, shafts being mounted coaxially. In one type of clutch known as

the *gradual engagement clutch*, one of the shafts rotates rapidly while the other is either stationary or moving with a low speed. As the engagement of the clutch proceeds, the rapidly moving shaft is retarded while the slowly moving shaft is accelerated. The process goes on until the two shafts rotate as one with the same speed. The clutch is now said to be fully engaged. The clutch used in motor car between the engine and the gear box is based on the action of the frictional force that is called into play between two rotating bodies when they are pressed together. This type of clutch is known as *friction clutch*.

To understand the action of the friction clutch, let us consider two shafts C and D (Fig. 85) supported on bearings A and B so that they are free to rotate about a common axis PQ. E and F are two circular discs which face each other and which are keyed to the ends of the shafts. Suppose the shaft C with its discs E is rotating rapidly while the shaft D with its disc F is stationary; the two shafts being pressed together endways when the faces of the discs come into contact the force of friction between them tends to retard the speed of the disc E. As the force with which the discs press on each other gradually increases, the frictional force between them also gradually increases and at a certain stage it becomes sufficiently great to overcome the resistance between the discs. Thereafter the disc F begins to rotate with its speed gradually increasing. This goes on until the two discs move with the same speed. At this stage there is no slip between the discs and the clutch is fully engaged.

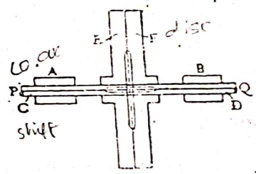

Fig. 85

In the motor car clutches, the discs are kept pressed against each other by means of a spring. The elasticity of the spring always keep the clutch in engagement. Whenever it is required to disengage the clutch, one of the discs is pulled back against the pressure of the spring.

Example 1 *Two weights each equal to W rest on the faces of a double inclined plane whose inclinations with the horizontal are θ and θ' and are connected by a light inextensible string after passing over a light smooth pulley fixed at the common vertex. Find the value of coefficient of friction, if the equilibrium of the weights is limiting*

3.1 Introduction.

Definition : *The centre of gravity of a body is the point at which the resultant of the weights of all the particles of the body acts, whatever may be the orientation of the body. The total weight of the body may be supposed to act at its centre of gravity.*

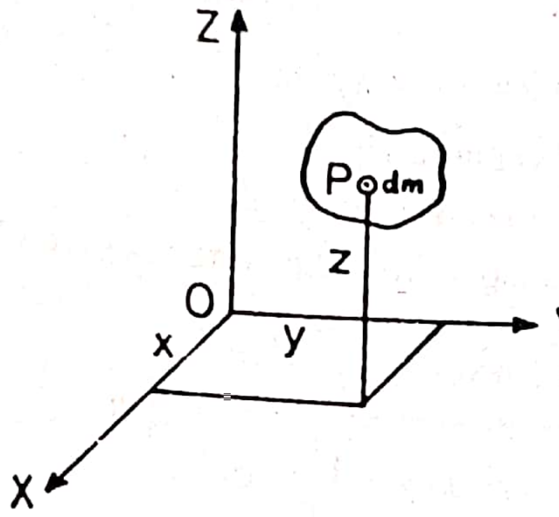

Fig. 3.1

Suppose the particles $A, B, C, ...$ of a body have masses $m_1, m_2, m_3, ...$ Let their coordinates in a rectangular cartesian coordinate system be $(x_1, y_1, z_1), (x_2, y_2, z_2), ... (x_n, y_n, z_n)$.

Then, the coordinates of the centre of gravity G of the body are

$$\bar{x} = \frac{\sum m_n x_n}{\sum m_n};$$

$$\bar{y} = \frac{\sum m_n y_n}{\sum m_n}; \qquad \bar{z} = \frac{m_n z_n}{\sum m_n};$$

Suppose an element P of the body has a mass dm (Fig. 3.1) and its coordinates are x, y, z. Then,

$$\bar{x} = \frac{\int x\, dm}{\int dm} = \frac{1}{M}\int x\, dm \; ; \; \bar{y} = \frac{1}{M}\int y\, dm \; ; \; \bar{z} = \frac{1}{M}\int z\, dm$$

Here, the integrals extend over all elements of the body, and $M = \int dm =$ Total mass of the body.

Distinction between C. G and C. M.

1. Now weights of the different particles constituting the body are proportional to the respective masses. Hence, C.G.., if it exists is the same as the C.M..

2. If the body be removed to an infinite distance in space where the attracting force of the earth is inoperative or if it be imagined to be taken to the centre of the earth, the force of gravity there will be zero. The body will lose its weight. Hence, there arises no question of centre of gravity. But the body will have centre of mass as it will retain its mass which is independent of gravity and is an inherent property of matter. Thus a body *may not have a centre of gravity but it has a centre of mass.*

18. Centre of gravity of a trepezoidal Lamina : Let $ABCD$ be a trepezoidal lamina where the lengths of the parallel sides AB and CD are $2a$ and $2b$ respectively. Let F and E be the mid-points of AB and CD. Join AE and BE. The trepezoidal lamina is divided into three triangular laminae ADE, AEB and BCE.

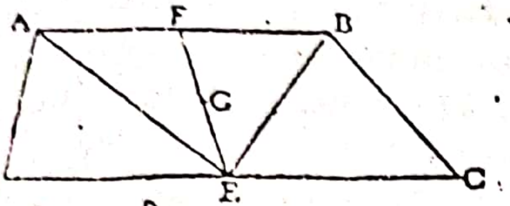

Fig. 56

The weights of the triangular laminae ADE, AEB and BEC are proportional to their areas which in turn are proportional to their bases b, $2a$ and b respectively since the altitudes of the triangles are equal.

The wt of AEB is proportional to $2a$ and is equivalent to $\frac{2a}{3}$, $\frac{2a}{3}$ and $\frac{2a}{3}$ at A, E and B respectively. The weight of ADE is proportional to b and is equivalent to $\frac{1}{3}b$, $\frac{1}{3}b$ and $\frac{1}{3}b$ at A, D and E respectively. The weight of BEC is proportional to b and is equivalent to $\frac{1}{3}b$, $b/3$ and $\frac{1}{3}b$ at B, E and C respectively.

The weights $\frac{2a}{3} + \frac{b}{3}$ at A and B are equivalent to $\frac{4a+2b}{3}$ at F.

Similarly, $\frac{1}{3}b$ at D and $\frac{1}{3}b$ at C are equivalent to $\frac{2b}{3}$ at E.

The total weight at E is proportional to $\frac{2a}{3} + \frac{4b}{3}$ or $\frac{2a+4b}{3}$ at E.

The resultant of $\frac{4a+2b}{3}$ at F and $\frac{2a+4b}{3}$ at E will act at G such that

$$\left(\frac{4a+2b}{3}\right) FG = \left(\frac{2a+4b}{3}\right) GE$$

$$\frac{FG}{GE} = \frac{2a+4b}{4a+2b} = \frac{a+2b}{2a+b}.$$

3.2. Centre of gravity of a right solid cone

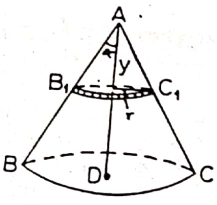

Fig. 3.2

Let ABC represent a solid cone of height h and semi-vertical angle α (Fig. 3.2). The cone may be considered to be made up of a large number of circular discs parallel to the base. The centre of gravity of each disc lies at its centre. Therefore, the C. G., of the cone should lie along the axis AD of the cone.

Consider a disc B_1C_1 of thickness dy at a distance y below the vertex A. If r is the radius of the disc, then

$$r = y \tan\alpha$$

Volume of the disc = Area × thickness = $\pi y^2 \tan^2\alpha \, dy$

Mass of the disc $= dm = \pi y^2 \rho \tan^2\alpha \, dy$.

where ρ = density of the cone.

The distance of the $C.\ G.$ of the cone from the vertex is given by

$$\bar{y} = \frac{\int y\, dm}{\int dm} = \frac{\int_0^h \pi y^3 \rho \tan^2\alpha \, dy}{\int_0^h \pi y^2 \rho \tan^2\alpha \, dy} = \frac{\int_0^h y^3\, dy}{\int_0^h y^2\, dy} = \frac{3}{4} h.$$

Therefore, the $C.\ G.$, of the cone is along its axis at a distance of $\frac{3}{4} h$ from the vertex.

3.3. Centre of gravity of a hollow right circular cone (without base)

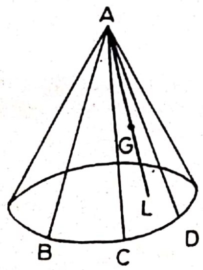

Fig. 3.3

Let h be the height of the cone. The slant surface of the cone may be divided into an infinite number of triangles ABC, ACD, ...etc., by joining the vertex A to the points on the edge of the base (Fig. 3.3). The centre of gravity of each such triangular area is at its centroid. It is at a height of $h/3$ above the circular base of the cone. Hence, the $C.\ G.$, of the whole cone must lie on a plane parallel to the base at a height $h/3$ from it. By symmetry, the $C.\ G.$, must also lie on the axis of the cone AL. Hence, the $C.\ G.$, of the hollow cone is at G such that $\frac{GL}{AL} = \frac{1}{3}$.

3.4. Centre of gravity of a solid hemisphere :

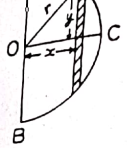

Fig. 3.4

Let ABC represent a solid hemisphere of radius r, centre O and density ρ (Fig. 3.4). Consider an elementary slice of the hemisphere with radius y and thickness dx, at a distance x from O.

Volume of the slice $= \pi y^2 dx = \pi (r^2 - x^2)\, dx$.

Mass of the slice $= dm = \rho\pi (r^2 - x^2)\, dx$.

The distance of the $C.\ G.$, of the hemisphere from O is given by

$$\bar{x} = \frac{\int x\, dm}{\int dm} = \frac{\int_0^r x \rho\pi (r^2 - x^2)\, dx}{\int_0^r \rho\pi (r^2 - x^2)\, dx} = \frac{\int_0^r (r^2 x - x^3)\, dx}{\int_0^r (r^2 - x^2)\, dx}$$

$$\bar{x} = \frac{3}{8} r.$$

Hence, the $C.\ G.$, of the solid hemisphere is on its axis at a distance $\frac{3}{8} r$ from the centre.

3.5. Centre of gravity of a hollow hemisphere

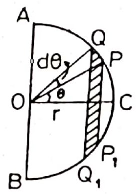

Fig. 3.5

Let ACB be a section of a hemisphere of radius r, centre O and surface density ρ [Fig. 3.5]. Imagine the surface of the hemisphere to be divided into slices like PQQ_1P_1 by planes parallel to AB. If $\angle POC = \theta$ and $\angle POQ = d\theta$, then

radius of the ring $= r \sin\theta$

width of the ring $= r\, d\theta$

Area of the ring $= 2\pi r^2 \sin\theta \, d\theta$

$\therefore$ mass of the ring

$= dm = 2\pi r^2 \rho \sin\theta \, d\theta$.

The $C.\ G.$, of this ring is at the centre of the ring at a distance $r\cos\theta$ from O.

The distance of the $C.\ G.$, of the hollow hemisphere from O is given by

$$\bar{x} = \frac{\int x\, dm}{\int dm} = \frac{\int_0^{\pi/2} (r\cos\theta)\, 2\pi r^2 \rho \sin\theta \, d\theta}{\int_0^{\pi/2} 2\pi r^2 \rho \sin\theta \, d\theta} = \frac{\int_0^{\pi/2} r \sin\theta \cos\theta \, d\theta}{\int_0^{\pi/2} \sin\theta \, d\theta}$$

$\therefore \quad \bar{x} = r/2$.

The C. G. ,of a hollow hemisphere is on its axis at a distance $r/2$ from the centre, i. e., the centre of gravity is at the mid point of the radius OC.

3.6. Centre of gravity of a solid tetrahedron

Let $ABCD$ be the tetrahedron and G_1 the centre of gravity of the base BCD (Fig. 3.6). Let h be the altitude of the tetrahedron and ρ its density. Suppose the tetrahedron is divided into thin slices by planes parallel to the base BCD. Consider one such slice $B_1C_1D_1$ of thickness dx at a depth x below A. Let S be the area of the triangular base BCD. Then we have, $\frac{B_1C_1}{BC} = \frac{x}{h}$.

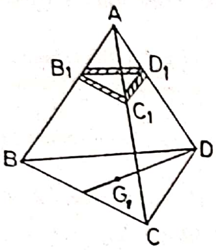

Fig. 3.6

If a_1 and a are the altitudes of triangles $B_1C_1D_1$ and BCD respectively,

$$\frac{a_1}{a} = \frac{x}{h}.$$

Now, area of $\Delta B_1C_1D_1 = \frac{1}{2} B_1C_1 \times a_1$

Area of $\Delta BCD = \frac{1}{2} BC \times a = S.$

Hence, $\frac{\text{Area of } \Delta B_1C_1D_1}{S} = \frac{B_1C_1}{BC} \times \frac{a_1}{a} = \frac{x^2}{h^2}$

∴ Area of $\Delta B_1C_1D_1 = Sx^2/h^2$

Volume of the slice $B_1C_1D_1 = Sx^2 dx/h^2$

Mass of the slice $= dm = \rho S x^2\, dx/h^2$

The distance of the centre of gravity of the tetrahedron from A is given by

$$\bar{x} = \frac{\int x\, dm}{\int dm} = \frac{\int_0^h x \rho\, Sx^2\, dx/h^2}{\int_0^h \rho\, S\, x^2\, dx/h^2} = \frac{\int_0^h x^3\, dx}{\int_0^h x^2\, dx} = \frac{3}{4} h$$

Hence, the $C.G.$,of a uniform tetrahedron lies at a point G on the line AH such that $AG : GH = 3 : 1$

3.7. Centre of gravity of a compound body

Let G_1, G_2 be the centres of gravity of the two bodies A and B. Their weights W_1 and W_2 are like parallel forces acting vertically downwards at

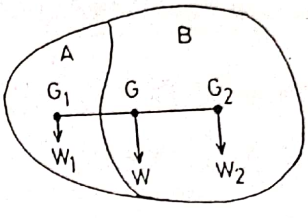

Fig. 3.7

G_1 and G_2 [Fig. 3.7]. Their resultant is $W_1 + W_2$ and acts at a point G in $G_1 G_2$ such that

$$W_1 \times G_1G = W_2 \times G_2G \quad \text{or} \quad G_1 G = \frac{W_2}{W_1 + W_2} G_1 G_2.$$

This gives the position of the centre of gravity of the whole body.

Example 1. *A solid homogenous body consists of a cylinder and a cone having their common bases joined together. If the centre of gravity of the body is at the centre of the common base, find the ratio of the heights of the cone and the cylinder.*

Sol. Let h_1 and h_2 be the heights of the cone and the cylinder. Let G_1 and G_2 be the centres of gravity of the cone and the cylinder [Fig. 3.8]. The weight of the cone $\frac{1}{3}\pi r^2 h_1 \rho$ acts at G_1 such that $AG_1 = \frac{3}{4} h_1$ or $GG_1 = \frac{1}{4} h_1$. The weight of the cylinder $\pi r^2 h_2 \rho$ acts at G_2 such that $GG_2 = h_2/2$.

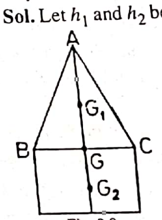

Fig. 3.8

$$\therefore \frac{1}{3}\pi r^2 h_1 \rho \times \frac{1}{4} h_1 = \pi r^2 h_2 \rho \times \frac{h_2}{2} \text{ or } \frac{h_1}{h_2} = \sqrt{6}$$

Example 2. *A solid cone and a solid hemisphere of the same material have a common base. Find the ratio of the height of the cone to the radius of the hemisphere, if the C. G., of the combination coincides with the centre of the common base.*

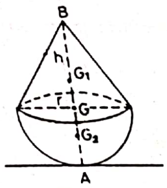

Fig. 3.9

Sol. Let h be the height of the cone and r the radius of the hemisphere. Let G_1 and G_2 be the centres of gravity of the cone and hemisphere [Fig. 3.9]. The C. G., of the combination is at G, the centre

Let the horizontal line $B'C'$ divide the area into two parts, so that the thrusts on these portions are equal. Let this line be at a distance x above the vertex A.

Thrust on $AB'C'$

$$= \text{pressure at its C.G.} \times \text{area} = \left(h - \tfrac{2}{3}x\right)\rho g \times \tfrac{1}{2}x.B'C'$$

$$= \left(h - \tfrac{2}{3}x\right)\rho g \times \tfrac{1}{2}x \cdot \frac{ax}{h} \qquad \left(\because B'C' = \frac{ax}{h}\right)$$

Thrust on $AB'C' = \frac{1}{2} \times$ Thrust on the whole triangle.

$$\therefore \quad \left(h - \tfrac{2}{3}x\right)\rho g \times \tfrac{1}{2}x \cdot \frac{ax}{h} = \frac{1}{2} \times \frac{1}{6}h^2 a\rho g$$

or $$\left(h - \tfrac{2}{3}x\right)x^2 = \frac{1}{6}h^3$$

or $$4x^3 - 6x^2h + h^3 = 0$$

or $$(2x - h)(2x^2 - 2xh - h^2) = 0$$

$$\therefore \quad 2x - h = 0 \quad \text{or} \quad x = \frac{h}{2}.$$

4.3 Centre of pressure

We know that the liquid pressure acts *normally* at every point of the immersed area. The force acting on an elementary area like dS is $h\rho g dS$. The thrusts on different elements of the plane form a set of *like parallel forces*. All these parallel forces can be compounded into a resultant acting at some definite point on the plane area. This point is called the centre of pressure.

The centre of pressure of a plane surface in contact with a fluid is the point on the surface through which the line of action of the resultant of the thrusts on the various elements of the area passes.

Determination of Centre of pressure—General case -

Consider a plane surface of area S immersed vertically in a liquid of density ρ. Let XY be the surface of the liquid (Fig. 4.7).

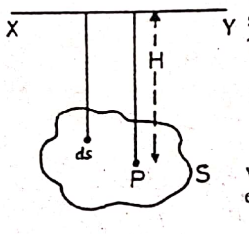

Fig. 4.7

Thrust on an elementary area dS at a depth $h = h\rho g dS$

Moment of this thrust about XY

$$= (h\rho g dS) \times h = h^2 \rho g dS$$

Resultant moment of all thrusts $= \int h^2 \rho g dS$

where the integration is carried over all the elements of the plane area.

Resultant thrust on the plane area $= \int h\rho g dS$

Let the centre of pressure of the plane area be at the point P. Let the distance of P from XY be H.

Moment of the resultant thrust about $XY = H\int h\rho g\, dS$.

By definition of the resultant of several forces, we get

Moment of resultant force = resultant of the moments of the forces .

or $$H\int h\rho g\, dS = \int h^2 \rho g\, dS$$

or $$H = \frac{\int h^2\, dS}{\int h\, dS}$$

The result holds good for any inclined position of the plane also.

4.4 Centre of pressure of a rectangular lamina immersed vertically in a liquid with one edge in the surface of the liquid.

Let $ABCD$ be a plane rectangular lamina immersed vertically in a liquid of density ρ with one edge AB in the surface XY of the liquid (Fig. 4.8). Let $AB = a$ and $AD = b$. Divide the rectangle into a number of narrow strips parallel to AB. Consider one such strip of width dx at a depth x below the surface of the liquid.

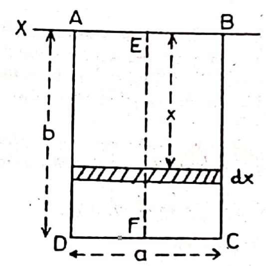

Fig. 4.8

The thrust acting on the strip

$$= (x\rho g) \times (a dx) = x\rho g a\, dx$$

Moment of this thrust about AB

$$= (x\rho g a\, dx) \times x = x^2 \rho g a\, dx$$

Sum of the moments of the thrusts on all the strips $= \int_0^b x^2 \rho g a\, dx$

Resultant of the thrusts on all the strips $= \int_0^b x\rho g a\, dx$

Moment of the resultant thrust about $AB = H\int_0^b x\rho g a\, dx$

where H = depth of the centre of pressure below AB.

$$H\int_0^b x\rho g a\, dx = \int_0^b x^2 \rho g a\, dx$$

or $$H\rho g a \frac{b^2}{2} = \rho g a \frac{b^3}{3} \quad \text{or} \quad H = \frac{2}{3}b.$$

The thrust on every elementary strip acts through its midpoint. Hence the centre of pressure will lie on EF where E and F are the mid-points of AB and DC.

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4.5. Centre of pressure of a triangular lamina immersed vertically in a liquid with its vertex in the surface of the liquid and its base horizontal.

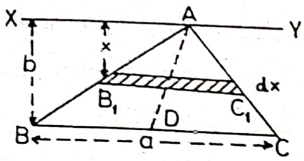

Fig.4.12

Sol. Let ABC be a triangular lamina immersed vertically in a liquid with its vertex A in the surface XY of the liquid and with its base BC horizontal (Fig. 4.12). $BC = a$. Let the depth of the base of the lamina be b from the free surface of the liquid. Divide the triangle into a number of elementary strips of width dx parallel to the base BC. Consider one such strip B_1C_1 of width dx at a depth x below the surface XY.

Area of the strip $B_1C_1 = B_1C_1\, dx = (ax/b)dx$

Thrust on the strip $B_1C_1 = (x\rho g) \times (ax/b)\, dx$

Moment of this thrust about $XY = \left(\frac{ax^3\rho g}{b}dx\right)$

Total moment due to all the strips $= \int_0^b \frac{a\rho g}{b} x^3\, dx.$

Resultant of the thrusts on all the strips $= \int_0^b \frac{a\rho g}{b} x^2 dx.$

Moment of the resultant thrust about $XY = H\int_0^b \frac{a\rho g}{b} x^2 dx.$

Here H= the depth of the centre of pressure below XY.

Since the two moments are equal,

$$\int_0^b \frac{a\rho g}{b} x^3\, dx = H\int_0^b \frac{a\rho g}{b} x^2\, dx.$$

or $$\frac{a\rho g}{b}\left(\frac{b^4}{4}\right) = H\frac{a\rho g}{b}\left(\frac{b^3}{3}\right).$$

or $$H = \frac{3}{4}b.$$

The centre of pressure lies on the line joining the mid-points of the strips. i.e., lies on the median AD at a depth $3b/4$ below the surface XY.

4.6 Centre of pressure of a triangular lamina immersed in a liquid with one side in the surface, when there is no external pressure.

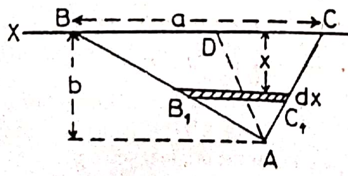

Fig. 4.13

Let ABC be a triangular lamina immersed in a liquid with its base BC = a in the surface XY of the liquid (Fig. 4.13). Let AD be a median of the triangle. Let b be the depth of the vertex A below the surface XY. Divide the triangle into a number of elementary strips of width dx parallel to the base BC. Consider one such strip B_1C_1 at a depth x below BC.

Area of the strip $B_1C_1 = B_1C_1\, dx = \frac{a(b-x)}{b}dx$

$$\left(\text{Since } \frac{B_1C_1}{a} = \frac{b-x}{b}\right)$$

Thrust on the strip $B_1C_1 = x\rho g\frac{a(b-x)}{b}dx$

Moment of this thrust about $XY = x^2\rho g\frac{a(b-x)}{b}dx.$

Total moment due to all the strips $= \int_0^b x^2\rho g\frac{a(b-x)}{b}dx.$

Resultant of the thrusts on all the strips $= \int_0^b x\rho g\frac{a(b-x)}{b}dx.$

Let H be the depth of the centre of pressure below XY.

Moment of the resultant thrust about $XY = H\int_0^b x\rho g\frac{a(b-x)}{b}dx.$

$$\therefore \quad \int_0^b x^2\rho g\frac{a(b-x)}{b}dx = H\int_0^b x\rho g\frac{a(b-x)}{b}dx.$$

or $$H\int_0^b x(b-x)dx = \int_0^b x^2(b-x)dx$$

or $$H\left[\frac{b^3}{2} - \frac{b^3}{3}\right] = \left[\frac{b^4}{3} - \frac{b^4}{4}\right]$$

or $$H = \frac{b}{2}$$

The centre of pressure is on the median AD at a depth $b/2$ below XY.

Example 1. *A triangle is wholly immersed in a liquid with its base in the surface. Show that a horizontal straight line drawn through the centre of pressure of the triangle divides it into two parts, the pressures on which are equal.*

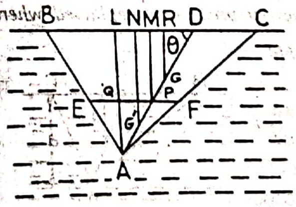

Fig. 4.14

Sol. We know that the depth of C.P. is $\frac{1}{2}$ of the height of the triangle. P is the centre of pressure. P is at a depth $AL/2$ below BC. EF is a line drawn through the C.P ∥ to BC (Fig. 4.14). ∆s ABC and AEF are similar.

$$\therefore \quad \frac{EF}{BC} = \frac{AP}{AD} = \frac{1}{2} \quad (\because AP = \tfrac{1}{2}AD)$$

$$\therefore \quad EF = \frac{1}{2}BC \qquad \text{... (1)}$$

Let $BC = a$ and $\rho =$ the density of the liquid. Then, area of the triangle is $\frac{1}{2}ad$.

Now, let us take into account the atmospheric pressure. The atmospheric pressure is equivalent to a height h of water. Now, the thrusts acting on the triangle are :

(*i*) the pressure $\frac{1}{2}d\rho g \times \frac{1}{2}ad$ acting at P at a depth $\frac{1}{2}d$ below BC,

(*ii*) the additional thrust $h\rho g \times \frac{1}{2}ad$ due to the atmospheric pressure acting at the centre of gravity which is at a depth $\frac{1}{2}d$ from B C.

Let P' be the new position of the centre of pressure. It is at a depth H from BC. Taking moments about BC,

$$H\left(\frac{1}{2}d\rho g \times \frac{1}{2}ad + h\rho g \times \frac{1}{2}ad\right) = \frac{1}{2}d\rho g \times \frac{1}{2}ad \times \frac{1}{2}d + h\rho g \times \frac{1}{2}ad \times \frac{1}{3}d.$$

$$\therefore \quad H = \frac{\frac{1}{8}d^3a\rho g + \frac{1}{6}d^2a\rho gh}{\frac{1}{4}d^2\rho ga + \frac{1}{2}d\rho gha}$$

$$= \frac{1}{6}d\left[\frac{3d+4h}{d+2h}\right]$$

Hence the vertical distance between P and $P' = \frac{1}{2}d - H$

$$= \frac{1}{2}d - \frac{1}{6}d\left[\frac{3d+4h}{d+2h}\right] = \frac{1}{3}\left[\frac{hd}{d+2h}\right].$$

4.7 Floating Bodies

Laws of Floatation : (1) *The weight of the floating body is equal to the weight of the liquid displaced by it.*

(2) *The centre of gravity of the floating body and the centre of gravity of the liquid displaced (i.e., the centre of buoyancy) are in the same vertical line.*

Stability of Floating bodies :

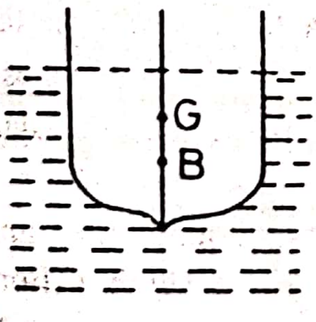

Fig. 4.16

The equilibrium of a freely floating body is said to be stable, if on being slightly displaced, the body returns to the original equilibrium position.

Consider a floating body in equilibrium. G is the centre of gravity of the floating body and B is the centre of buoyancy. The line BG is vertical (Fig. 4.16).

When the floating body is slightly displaced (Fig. 4.17) C is the new centre of buoyancy. The vertical line through C meets the original vertical line BG at M. M is called the '*metacentre*' of the floating body. GM is called the *metacentric height*. The weight of the body W acts vertically downwards through G. The upthrust of value W acts vertically upwards through C. If the metacentre is above G, the couple due to the forces at G and C is anticlockwise and brings the floating body back to its original position. Hence in this case the equilibrium is stable.

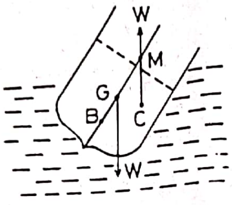

Fig. 4.17

But if M lies below G (Fig. 4.18), the couple due to the forces at G and C is clock-wise and the couple tends to turn the body away from the equilibrium position. Hence this equilibrium is unstable. Hence *for a floating body to be in stable equilibrium, the metacentre must be always above the centre of gravity of the body.*

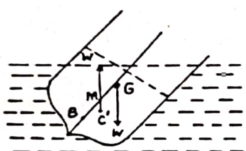

Fig. 4.18

Note : In the case of a sphere floating in a liquid, a tilt one way or other does not change the shape of the displaced liquid. Hence M coincides with G all the time. Therefore, it is said to be in *neutral equilibrium* and it continues to float in all positions.

Experimental determination of the metacentric height of a ship.

The weight of the ship W is determined by the displacement method. Two identical boats are attached one on each side of the ship. In Fig. 4.19, A and B represent the boats at a distance l apart on the deck. Filling A and B alternately with water is equivalent to moving a known weight w from A to B across the deck. Filling the boat B with the same mass of water as in A, turns the ship through an angle θ. The tilt θ is determined by means of a plumb line suspended in the ship.

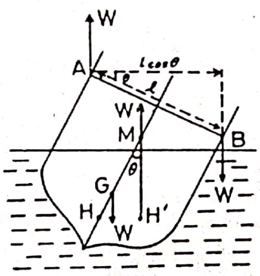

Fig. 4.19

Now, this shift of weight w from A to B is equivalent to a downward force w at B and an upward force w at A constituting a couple of moment $wl\cos\theta$. Let H and H' be the original and altered positions of centres of buoyancy, G the centre of gravity of the ship and GM the metacentric height.

The weight W of the ship acting downwards at G and an equivalent upward thrust at the new centre of buoyancy H' form a couple with an opposing moment $W \times GM \sin\theta$. For equilibrium in the tilted position of ship,

$$W \times GM \sin\theta = w \times l \cos\theta \text{ or } GM = \frac{wl}{W\tan\theta}$$

or

$$GM = \frac{wl}{W\theta} \qquad [\theta \text{ being small, } \tan\theta = \theta].$$

Thus knowing W, w, l and θ, we can easily calculate the metacentric height of the ship.

Example 1 . *A ship is of 20000 tons displacement. A load of 30 tons moved 50 metres across the deck makes the ship tilt through $(\frac{3}{4})°$. Calculate the metacentric height.*

Sol. Here, $w = 30$ tons, $l = 50$m, $W = 20000$ tons,

$$\theta = (\tfrac{3}{4})° = \tfrac{3}{4} \times \pi/180 \text{ radians.}$$

$$\therefore \quad GM = \frac{30 \times 50}{20000 \times \left(\frac{3}{4}\right) \times \pi/180} = 5.79\text{m.}$$

Example 2. *Calculate the metacentric height and determine the necessary condition for the stable equilibrium of a cylinder of length l, radius r and density ρ, floating vertically in a liquid of density σ.*

Sol. Let x be the height of the immersed part of the cylinder (Fig. 4.20). By the law of floatation, Weight of the floating body = Weight of the liquid displaced by it $\pi r^2 l\rho = \pi r^2 x\sigma$ or $x = l\rho/\sigma$.

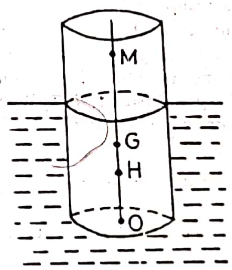

Fig. 4.20

Let O be the centre of the bottom face of the cylinder. Let H, G and M be the centre of buoyancy, centre of gravity and metacentre respectively. Now,

$$OH = x/2 = l\rho/(2\sigma). \; OG = l/2$$

$$\therefore \quad HG = OG - OH = \frac{l}{2} - \frac{l\rho}{2\sigma} = \frac{l(\sigma - \rho)}{2\sigma}.$$

We know that the distance between the centre of buoyancy of the displaced liquid and the metacentre is AK^2/V where AK^2 is the M.I. of the surface-plane of the cylinder about its diameter. $\therefore K^2 = r^2/4$, K being the radius of gyration of the plane about the surface-line or the diameter of the cylinder. V is the volume of the immersed part of the body.

$$\therefore \quad HM = \frac{\pi r^2 (r^2/4)}{\pi r^2 x} = \frac{r^2}{4x}$$

For stability, $HM > HG$.

$$\frac{r^2}{4x} > \frac{l(\sigma - \rho)}{2\sigma}$$

or
$$\frac{r^2}{4l\,\rho/\sigma} > \frac{l\,(\sigma-\rho)}{2\sigma} \qquad [\because x = l\rho/\sigma]$$

$$\frac{r^2}{l^2} > \frac{2\rho}{\sigma}\left(1 - \frac{\rho}{\sigma}\right)$$

This is, therefore, the necessary condition for stable equilibrium.

4.8 Atmospheric pressure

Air is a mixture of gases like oxygen, nitrogen, carbon dioxide etc. It envelops the earth and this envelope is called the *atmosphere.* Since air has weight, it exerts pressure on all the surfaces in contact with it. The thrust exerted by the atmosphere on unit area of the earth's surface is called the *atmospheric pressure.* The atmospheric pressure is greatest at the surface of the earth. The normal atmospheric pressure may be taken as the pressure exerted by a column of mercury at 0°C and height 0·76 m. The atmospheric pressure decreases as we go higher and higher above the earth's surface. The atmospheric pressure at any place can be measured by a barometer.

Variation of atmospheric pressure with altitude :

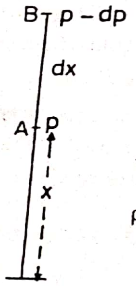

Fig. 4. 21

Let A and B be two points at heights x and $x + dx$ above the earth's surface. Let p be the pressure at A (Fig. 4.21). Since the density of air and, therefore, its pressure decreases with altitude, the pressure at B will be $p - dp$. Let ρ be the density of air between A and B.

Then $\qquad -dp = \rho\, g\, dx \qquad$ ... (1)

The negative sign indicates that the pressure decreases with height.

If the temperature of the air is assumed to be constant, $\rho \propto p$ or $\rho = kp$ where k is a constant.

Substituting this value of ρ in Eq. (1),

$$-dp = kpg\,dx \text{ or } \frac{-dp}{p} = kg\,dx$$

or
$$\frac{dp}{p} = -kg\,dx$$

Integrating, $\quad \log_e p = -kgx + C \qquad$ ... (2)

where C is the constant of integration.

Let p_0 be the pressure at sea level. Then, when $x = 0, p = p_0$. Hence $C = \log_e p_0$.

Substituting for C in Eq. (2), $\qquad \log_e\left(\frac{p}{p_0}\right) = -kgx$

$$\therefore \frac{p}{p_0} = e^{-kgx} \quad \text{or} \quad p = p_0 e^{-kgx}$$

Example 1. *Show that if the altitude increases in arithmetical progression, the pressure decreases in geometrical progression.*

Suppose we have a number of heights $x_1, x_2, x_3, \ldots$ in AP. Then, $x_2 - x_1 = x_3 - x_2 = x_4 - x_3$ and so on. Let $p_1, p_2, p_3, \ldots$ be the pressures at these heights.

Then, $\quad p_1 = p_0 e^{-kgx_1}, \quad p_2 = p_0 e^{-kgx_2}, \quad p_3 = p_0 e^{-kgx_3} \ldots$

Now, $\log_e\left(\frac{p_1}{p_2}\right) = kg\,(x_2 - x_1)$ and $\log_e\left(\frac{p_2}{p_3}\right) = kg\,(x_3 - x_2)$

Since $x_2 - x_1 = x_3 - x_2$, we have $\log_e\left(\frac{p_1}{p_2}\right) = \log_e\left(\frac{p_2}{p_3}\right)$

$$\therefore \quad p_1/p_2 = p_2/p_3.$$

i.e., $p_1, p_2, p_3, \ldots$ are in $G.P.$

Example 2. *What is meant by the height of the homogeneous atmosphere ? Find its value assuming the normal pressure is 0·76 metres of mercury and density of air and mercury to be 1·293 and 13600 kg/m³ respectively.*

Sol. The pressure of air and hence its density decreases with increase of altitude. Hence atmospheric air is not of uniform density. Suppose the atmospheric air were of uniform density ρ extending to a height H above the surface of the earth. Then the pressure exerted by this air column is $H\rho g$. If this pressure $H\rho g$ is equal to the standard atmospheric pressure, the height H is called the *height of the homogeneous atmosphere.*

$$H\rho g = 0{\cdot}76 \times 13600 \times 9{\cdot}8$$

or
$$H \times 1{\cdot}293 \times 9{\cdot}8 = 0{\cdot}76 \times 13600 \times 9{\cdot}8$$

or
$$H = 7990\text{m}.$$

Hence, the height of the homogeneous atmosphere is nearly 8 kilometres.

Example 3. *Calculate the difference in height between two stations from the barometric heights.*

Sol. Let H_1 and H_2 be the barometric heights at altitudes h_1 and h_2 and p_1 and p_2 the corresponding atmospheric pressures. $p_1/p_2 = H_1/H_2$.

$$p_1 = p_0 e^{-kgh_1} \quad \text{and} \quad p_2 = p_0 e^{-kgh_2}$$

$$\log_e (p_1/p_2) = kg\,(h_2 - h_1) \text{ or } h_2 - h_1 = \frac{\log_e (p_1/p_2)}{kg}.$$

or
$$h_2 - h_1 = \frac{\log_e (H_1/H_2)}{kg}$$