I. INTRODUCTION

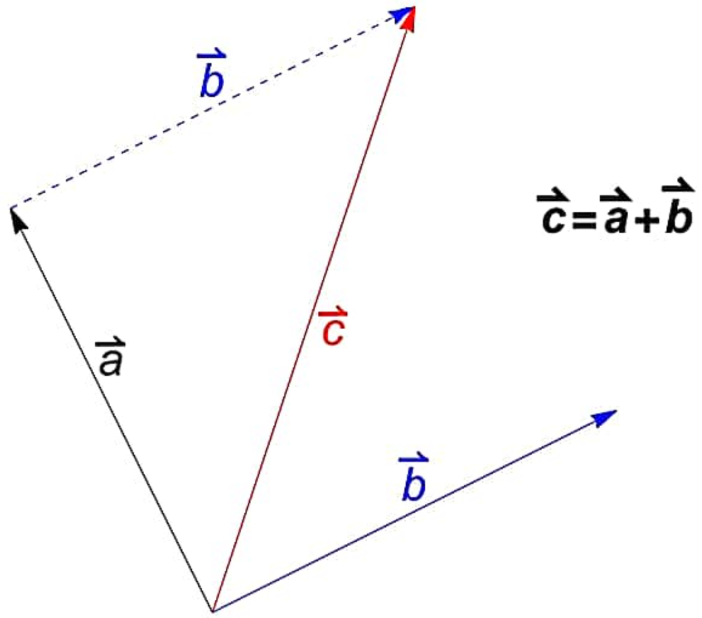
A. Vectors

A *vector* is characterized by the following *three* properties:

- has a magnitude
- has direction (Equivalently, has several components in a selected system of coordinates).
- obeys certain addition rules ("rule of parallelogram"). (Equivalently, components of a vector are transformed according to certain rules if the system of coordinates is rotated).

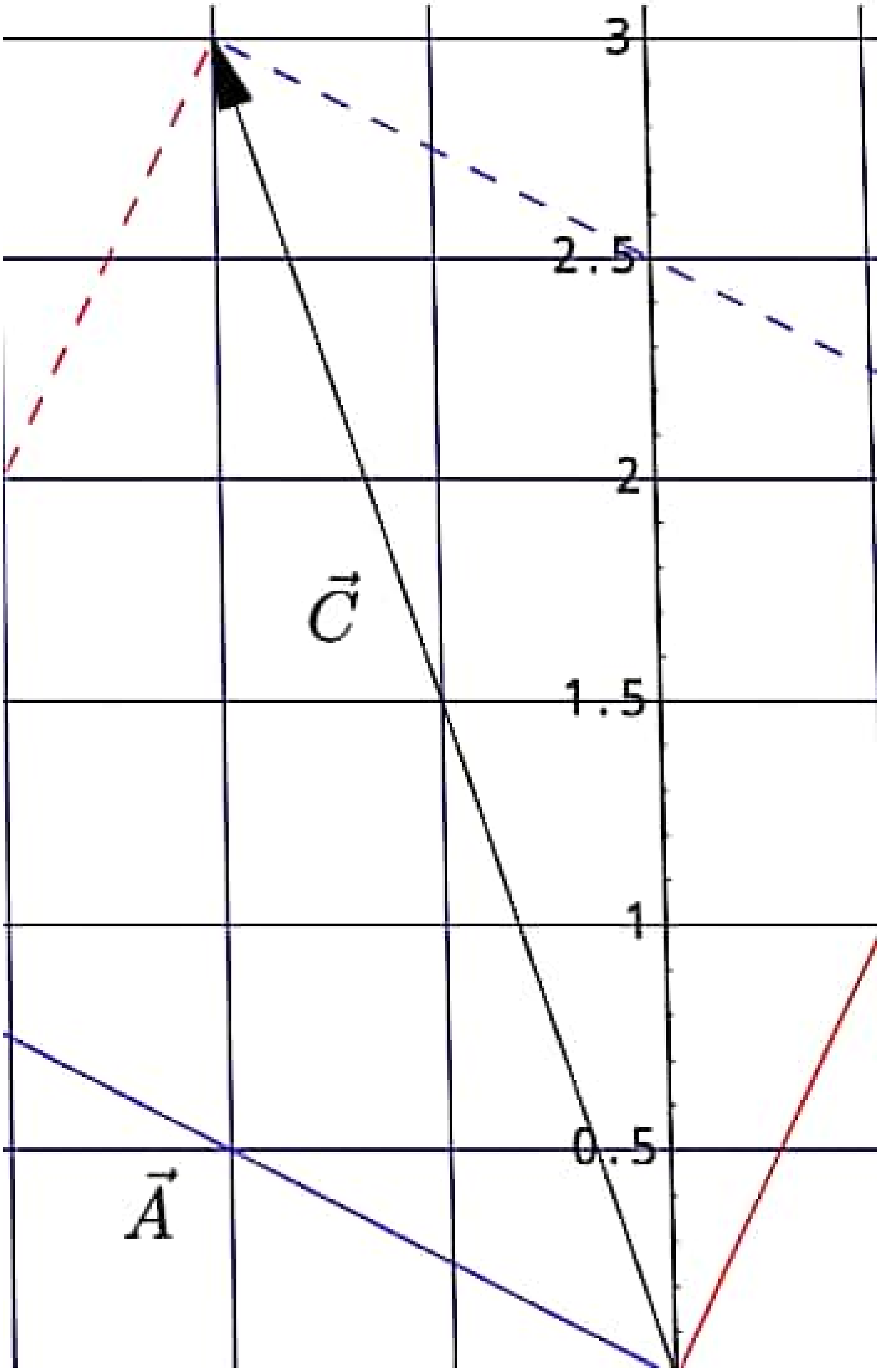
This is in contrast to a *scalar*, which has only magnitude and which is *not* changed when a system of coordinates is rotated.

How do we know which physical quantity is a vector, which is a scalar and which is neither? From experiment (of course). Examples of scalars are mass, kinetic energy and (the forthcoming) charge. Examples of vectors are the displacement, velocity and force.



Tail-to-Head addition rule.

f the resulting vector can be diffe



For two vectors, $\vec{a}$ and $\vec{b}$ one can define their sum $\vec{c} = \vec{a} + \vec{b}$ with components

$$c_x = a_x + b_x \ , \quad c_y = a_y + b_y \tag{2}$$

The magnitude of $\vec{c}$ then follows from eq. (1). Note that physical dimensions of $\vec{a}$ and $\vec{b}$ must be identical.

Preview. Addition of vectors plays a key role in E&M in that it enters the so-called "superposition principle".

3. Two vectors: scalar (dot) product

If $\vec{a}$ and $\vec{b}$ make an angle ϕ with each other, their scalar (dot) product is defined as

$$\vec{a} \cdot \vec{b} = ab \cos(\phi)$$

or in components

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y \tag{3}$$

Example. See Fig. 1.

$$\vec{A} = (-2, 1),\ \vec{B} = (1, 2) \ \Rightarrow\ \vec{A} \cdot \vec{B} = (-2)1 + 1 \cdot 2 = 0$$

(thus angle is 90^o).

Example Find angle between 2 vectors $\vec{B}$ and $\vec{C}$ in Fig. 1.

$$\text{General:} \quad \cos\theta = \frac{\vec{a} \cdot \vec{b}}{ab} \tag{4}$$

In Fig. 1:

$$\vec{B} = (1, 2),\ \vec{C} = (-1, 3) \ \Rightarrow\ B = \sqrt{1^2 + 2^2} = \sqrt{5},\ C = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$$

$$\cos\theta = \frac{(-1) \cdot 1 + 3 \cdot 2}{\sqrt{5}\sqrt{10}} = \frac{1}{\sqrt{2}},\ \theta = 45^o$$

A different system of coordinates can be used to evaluate $\vec{a} \cdot \vec{b}$, with different individual components but with the same result. For two orthogonal vectors $\vec{a} \cdot \vec{b} = 0$ in any system of coordinates. The main application of the scalar product is the concept of work $\Delta W = \vec{F} \cdot \Delta\vec{r}$, with $\Delta\vec{r}$ being the displacement. Force which is perpendicular to displacement does not work!

Preview. We will learn that magnetic force on a moving particle is always perpendicular to velocity. Thus, this force makes no work, and the kinetic energy of such a particle is conserved.

Example: Prove the Pythagorean theorem $c^2 = a^2 + b^2$.

4. Two vectors: vector product

At this point we must proceed to the 3D space. Important here is the correct system of coordinates, as in Fig. 2. You can rotate the system of coordinates any way you like, but you cannot reflect it in a mirror (which would switch right and left hands). If $\vec{a}$ and $\vec{b}$ make

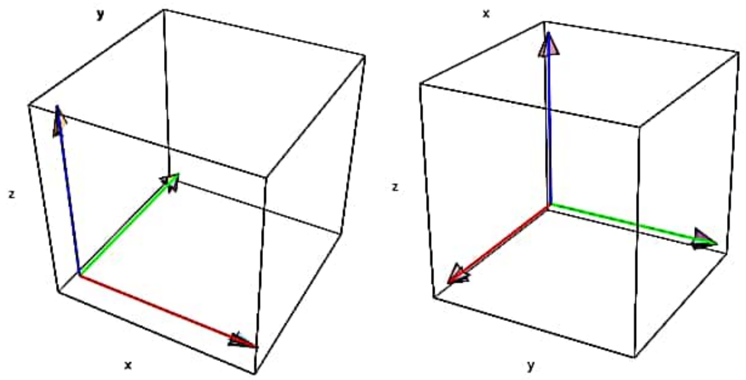


FIG. 2: The correct, "right-hand" systems of coordinates. Checkpoint - curl fingers of the RIGHT hand from x (red) to y (green), then the thumb should point into the z direction (blue). (Note that axes labeling of the figures is outside of the boxes, not necessarily near the corresponding axes; also, for the figure on the right the origin of coordinates is at the *far* end of the box, if it is hard to see in your printout).

an angle $\phi \leq 180^o$ with each other, their vector (cross) product $\vec{c} = \vec{a} \times \vec{b}$ has a magnitude

$$c = ab\sin(\phi)$$

The direction is defined as perpendicular to both $\vec{a}$ and $\vec{b}$ using the following rule: curl the fingers of the right hand from $\vec{a}$ to $\vec{b}$ in the shortest direction (i.e., the angle must be smaller than 180^o). Then the thumb points in the $\vec{c}$ direction. Check with Fig. 3.

Changing the order changes the sign, $\vec{b} \times \vec{a} = -\vec{a} \times \vec{b}$. In particular, $\vec{a} \times \vec{a} = \vec{0}$. More generally, the cross product is zero for any two parallel vectors.

Ring Diagram:

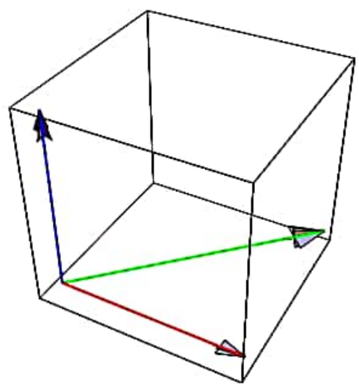
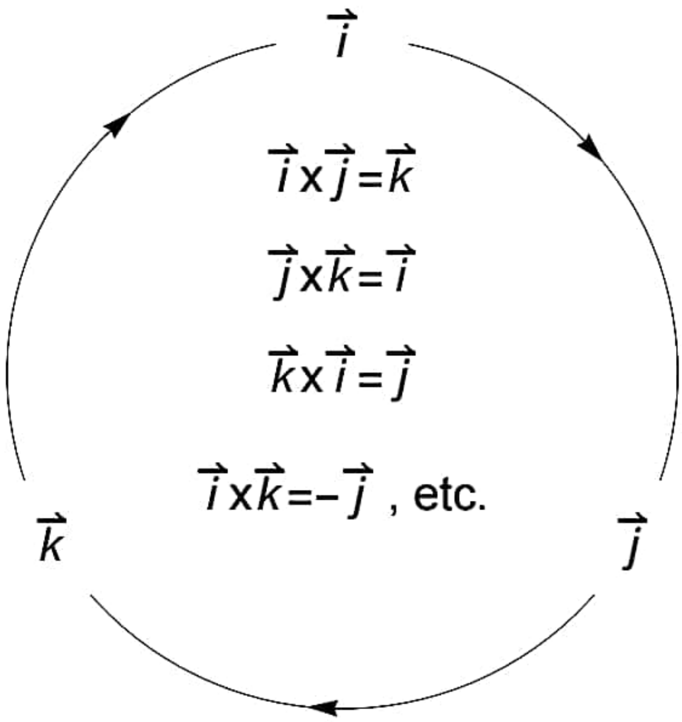


FIG. 3: Example of a cross product $\vec{c}$ (blue) $= \vec{a}$ (red) $\times \vec{b}$ (green). (If you have no colors, $\vec{c}$ is vertical in the example, $\vec{a}$ is along the front edge to lower right, $\vec{b}$ is diagonal).



Suppose now a system of coordinates is introduced with unit vectors $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$ pointing in the x, y and z directions, respectively. First of all, if $\hat{\imath}$, $\hat{\jmath}, \hat{k}$ are written "in a ring", the cross product of any two of them equals in clockwise direction the third one, i.e.

$$\hat{\imath} \times \hat{\jmath} = \hat{k} \, , \, \hat{\jmath} \times \hat{k} = \hat{\imath} \, , \, \hat{k} \times \hat{\imath} = \hat{\jmath}$$

etc.

Example. Fig. 1:

$$\vec{A} = -2\hat{\imath} + \hat{\jmath} \, ; \, \vec{B} = \hat{\imath} + 2\hat{\jmath}$$

$$\vec{A} \times \vec{B} = (-2\hat{\imath} + \hat{\jmath}) \times (\hat{\imath} + 2\hat{\jmath}) = (-2) \cdot 2\hat{\imath} \times \hat{\jmath} + \hat{\jmath} \times \hat{\imath} =$$

$$= -4\hat{k} - \hat{k} = -5\hat{k}$$

(Note: in Fig. 1 $\hat{k}$ goes out of the page; the cross product $\vec{A} \times \vec{B}$ goes into the page, as indicated by "-".)

More generally, the cross product is expressed as a 3-by-3 determinant

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \hat{\imath} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{\jmath} \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \hat{k} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \quad (5)$$

The two-by-two determinants can be easily expanded. In practice, there will be many zeroes, so calculations are not too hard.

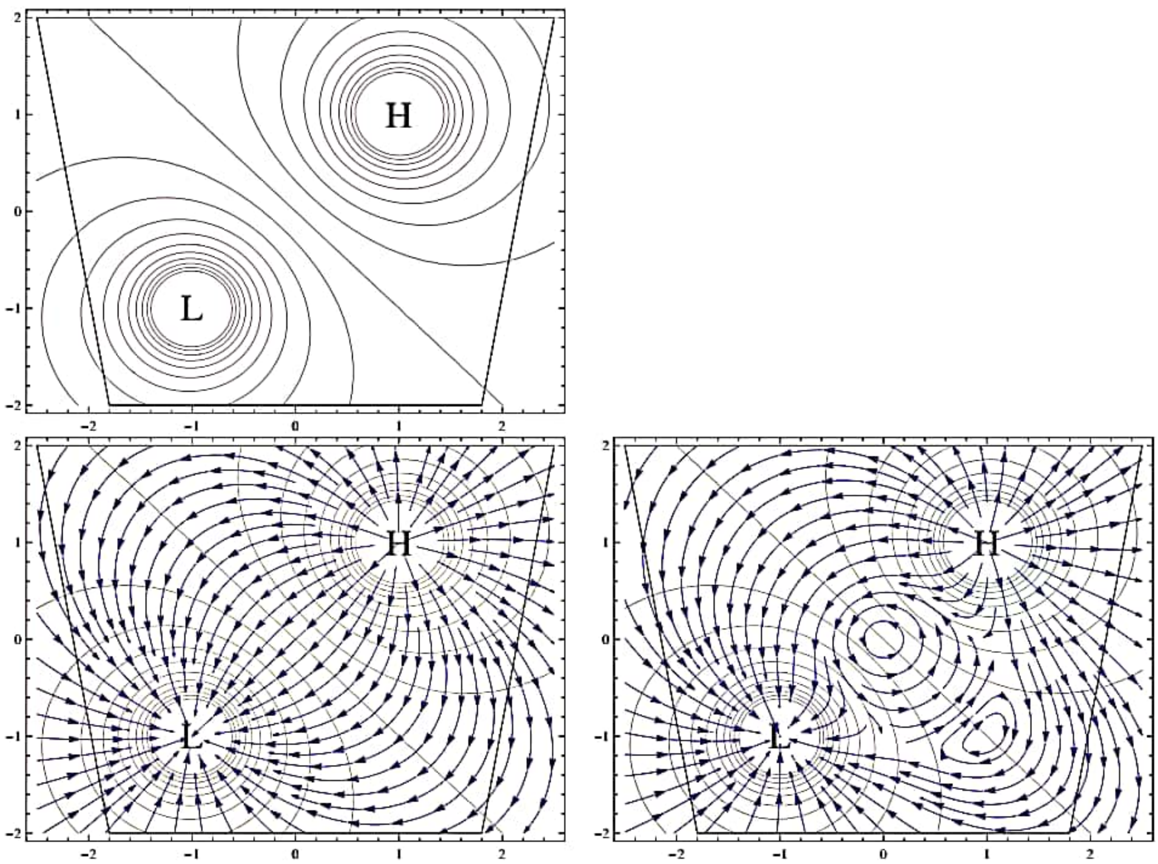
Preview. Vector product is most relevant to magnetism; it determines, e.g. the magnetic force on a particle in a field, $\vec{F} = q\vec{v} \times \vec{B}$ with q being the charge, $\vec{v}$ the velocity, and $\vec{B}$ the intensity of magnetic field at the location of the particle.

Example. See Fig. 1.

$$\vec{A} \times \vec{B} = \hat{k}((-2)2 - 1 \cdot 1) = -5\hat{k}$$

B. Fields

So far we were dealing with scalars or vectors attributed to a *single* particle (or a single point, if you prefer). Consider now a much more general situation when a scalar or a vector is attributed to *every* point in space. This brings us to a concept of a *field*, scalar or vector, respectively. Field can also depend on time. A good example of a scalar field is the temperature (or pressure) map which you see in the weather forecast. Similarly, the velocities of the air flow (usually superimposed on the same map) give a vector field.



Examples of scalar and vector fields: weather maps. Top - pressure field (scalar); lines connect points with identical pressure. Lower: wind velocity fields; left - regular flow from high to lower pressure, right - turbulent flow (note regions with non-zero circulation, "tornadoes"). The left maps are similar to those for potential V and electrostatic field $\vec{E}$ of an electric dipole. The type of the map on the right is encountered in time dependent fields, such as those which lead to electromagnetic radiation.

1. *Representation of a field; field lines*

How to represent a field in a picture? For a scalar field the best way is to draw lines of a constant level, e.g. lines with constant temperature every 10^oC (another good example is a topographic map which indicates levels of constant height. Try to sketch maps of a hill top, of a crest and of a "saddle").

For a vector field graphical representation can be harder. The easiest approach would be to select a large number of points in space and to draw vectors from each of them (see, e.g., the example of gravitational field later in these notes). You might not always enjoy the picture, however, since it will look too "discrete", while one feels that field should be continuous. A much better way is to draw the "field lines" - see Fig. 4. They give information about both magnitude and direction of the vector field. Many non-trivial mathematical theorems about the field are easily justified in terms of such pictures. Field lines also provide an enormous boost for physical intuition since rather abstract vector constructions are replaced by simple, easy to understand pictures.

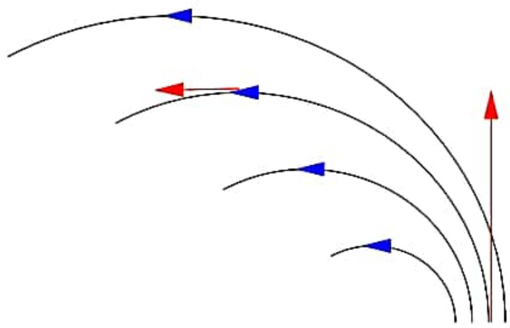


FIG. 4: Example of vector field lines. At each point the direction of vector field is tangent to the line. The magnitude of the vector field at a given point is proportional to the density of lines.

2. *Properties of field lines and related definitions*

The condition that the magnitude of the vector field at a given point is proportional to the density of lines, generally speaking, would require that some lines should be added or removed at various places in the picture. Remarkably, however, for the fields we are going to consider this happens only at some special points, and otherwise field lines run continuously. Points from which lines start are often called "sources", and points where they vanish are "sinks".

Preview. For electrostatic field $\vec{E}$ sources and sinks for field lines are positive and negative charges, respectively. Only there the lines can start or interrupt. (See the gravitational example below, which is similar to a negative charge; a positive charge will have lines going out). There are *no* magnetic charges in Nature, and thus magnetic field lines never start or end, but either loop (around currents) or come and go to infinity.

Example. Gravitational field at any point $\vec{r}$ outside of a planet is defined as the ratio of a force $\vec{F}$ on a probe to the mass of that probe, m. Show that this equals the gravitational acceleration $\vec{g}\,(\vec{r})$. Sketch the vector field lines for the field $\vec{g}$ - see Fig. 5.

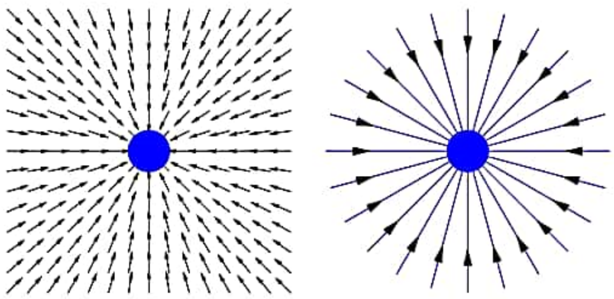


FIG. 5: Gravitational field around a planet. Left - representation by vectors, right - representation by field lines. Since the density of lines determines the magnitude of field, the latter decays inversely proportional to square of the distance from the center. The structure of this field is very similar to the electrostatic field outside a negatively charged sphere.

Gravitational field is detected by a probe, but we need a quantitative feature which is independent of the actual probe m:

$$\vec{F}_g = -G\frac{Mm}{r^3}\vec{r}, \;\; \frac{\vec{F}_g}{m} = -G\frac{M}{r^3}\vec{r} = \vec{g}$$

Here $\vec{r}$ is from the center of the planet to the observation point (do not need the probe anymore). Similarly, can construct a scalar function, the *gravitational potential.*

$$V_g \equiv U_g/m = -G\frac{Mm}{r}/m = -GM/r$$

Note

$$|V_g| = \frac{1}{2}v_{esc}^2\,, \;\; \text{and} \;\; |V_g| \ll c^2$$

II. ELECTRIC CHARGE

A. Notations and units

Notations: q, Q or (special) e for the charge of an electron.

Units: C (coulombs). Very large! (Historically, C was introduced as $A \cdot s$, with A being the ampere, for current. Today it is more common to treat C as another fundamental unit, which together with kg (kilogram), m (meter) and s (second) determines the SI system of units. The ampere A is then derived as C/s).

Charge of an electron

$$\boxed{e \simeq -1.6 \cdot 10^{-19}\, C}$$

In fact, this charge is quite appreciable and can be directly measured in the lab.

B. Superposition of charges

If several charges, positive or negative q_1, q_2, ... etc., are placed on a small particle, at large distances that particle will act as a single charge with

$$Q_{tot} = q_1 + q_2 + \ldots \tag{6}$$

C. Quantization of charge

The smallest charge is the charge of an electron, i.e. for any observable charge Q one should have

$$Q/e = 0\ ,\ \ \pm 1\ ,\ \ \pm 2\ , \ldots$$

D. Charge conservation

In a closed system

$$\boxed{Q_{tot} = const} \tag{7}$$

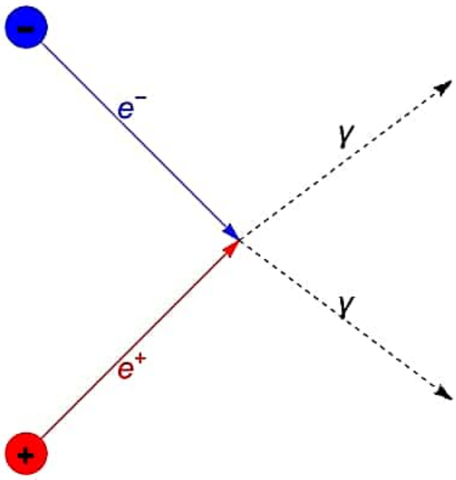
This is a fundamental Law of Nature, which is valid even if the number of elementary particles is not conserved (as in nuclear reactions)!

Examples. Decay of a neutron into a proton and an electron (+ some kind of neutrino which has no charge and is of little interest here):

$$n^0 \to p^+ + e^- + \nu^0$$

Example Annihilation of the electron e^- and a positron e^+:

$$e^- + e^+ = 2\gamma^0$$



E. The Coulomb's Law

If two charges q_1, q_2 are separated by a distance r, the force between them is

$$\boxed{F = k\frac{q_1 q_2}{r^2}\ , \quad k \simeq 9 \cdot 10^9 N \cdot m^2/C^2} \tag{8}$$

with positive sign referring to repulsion and negative to attraction. The force acts along the line connecting the two charges - see Fig. 6.

(some books write the product of *absolute* values of charges, to emphasize that F is the magnitude of force, which is always positive. However, the form given by eq. (8) is correct, and has more information as long as you know what it means).

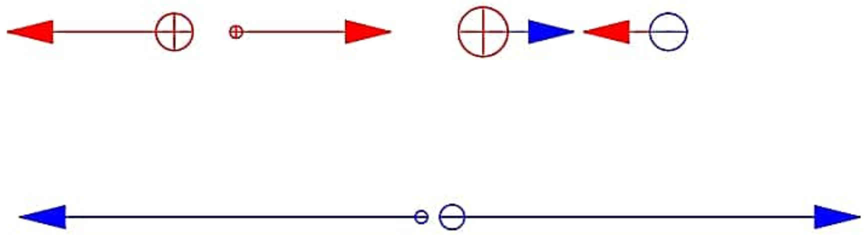


FIG. 6: The Coulomb interaction between charges. Figures are drawn to scale, with radii of charges being proportional to their magnitudes, and forces being proportional to predictions of the Coulomb Law. Positive and negative charges are indicated by red and blue, respectively. Note the following: (a) same charges repel each other, while opposite charges are attracted. (b) Forces acting on each of the two interacting charge are the *same* in magnitude, even if charges are different (otherwise the 3rd Law of Newton would be violated). (c) Forces become extremely large if the two charges are very close to each other, even if both charges are small

If one really wants to be pedantic (e.g., when dealing with a computer which has a poor sense of humor), the Coulomb's law can be formulated in a vector form: If $\vec{r}_{12}$ is the vector which points from charge 1 to charge 2 (with $r = |\vec{r}_{12}|$, as before), then the vector of force $\vec{F}_{21}$ which acts on charge 2 (and is due to interaction with charge 1) is given by

$$\vec{F}_{21} = k\frac{q_1 q_2}{r^3}\vec{r}_{12} \tag{9}$$

Example: check the above equation for a pair of charges from Fig. 6) [in fact, those pictures were generated by a computer using eq. (9)].

The vector version of Coulombs Law is more convenient in large formal calculations with many charges.

F. Superposition of forces

Consider a charge, let's call it q_0 which interacts with many other charges in the system, q_1, q_2, ..., etc. Then the *total* force which acts on q_0 is the vector superposition of individual forces, i.e.

$$\vec{F}_{0,\,net} = \vec{F}_{01} + \vec{F}_{02} + \ldots = \sum_{i=1}^{n} k \frac{q_i q_0}{r_{i0}^3} \vec{r}_{i0} \tag{10}$$

This is illustrated in Fig. 7 where the charge of interest, q_0 is the one in lower right.

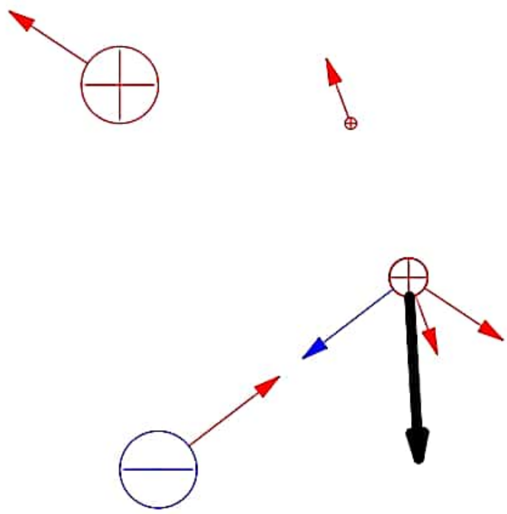
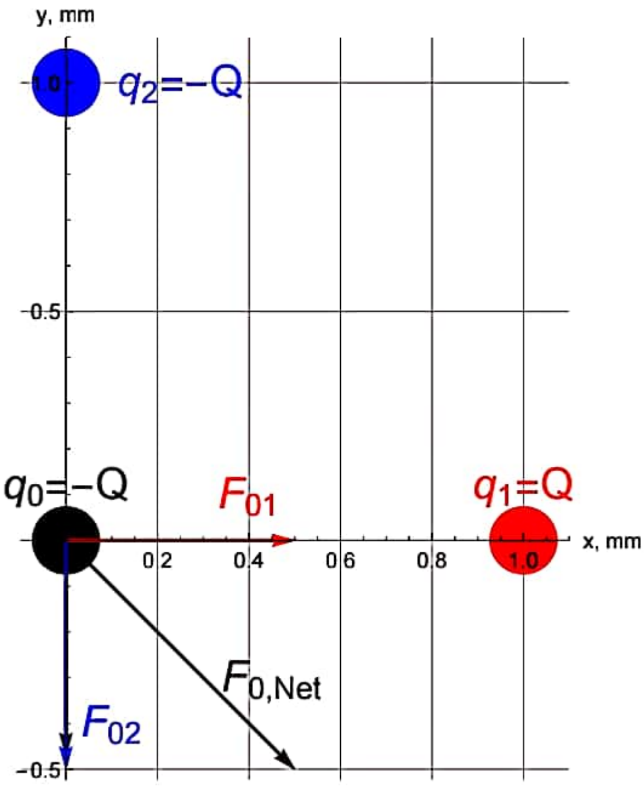


FIG. 7: The principle of superposition. The total force (black arrow in the picture) acting on a given charge equals the *vector* sum of all three individual forces which act on this charge due to its pairwise interaction with every other charge present in the system.

Example: $Q = 2\,\mu C$, $a = 1\,mm$. Find the force on the charge at the origin.



$$F_0 = \sqrt{F_{01}^2 + F_{02}^2} = \sqrt{2} \cdot F_{01}$$

$$F_{01} = k\frac{Q^2}{a^2} \approx 9 \cdot 10^9 \left(2 \cdot 10^{-6}\right)^2 / (10^{-3})^2 = 3.6 \cdot 10^4\,N\,,\; F_0 = \sqrt{2} \times 3.6 \cdot 10^4 = \ldots$$

Continuous q_i: sum over "all other charges" q_i is replaced by a corresponding integral (volume, surface or linear integral depending on the actual charge distribution).

$$\sum \to \int dV\,,\ \text{or} \int dA\,,\ \text{or} \int dl$$

$$q_i \to \rho dV\,,\ \text{or}\ \sigma dA\,,\ \text{or}\ \lambda dl$$

Here ρ, σ and λ are the volume charge density, surface charge density and linear charge density, respectively, with units

$$[\rho] = C/m^3\,,\ [\sigma] = C/m^2\,,\ [\lambda] = C/m$$

G. Reaction of a charge to electrostatic and other forces

Recall that the 2nd Law of Newton

$$\vec{F} = m\vec{a}\ , \quad or \quad \vec{F} = d\vec{p}/dt \tag{11}$$

is valid for *any* force, whatever its origin. So, if m is the charge q_0 and $\vec{F}_{0,\,net}$ is the total electrostatic force acting on that charge, as in eq. (10), then the 2nd Law allows one to find the acceleration $\vec{a}$, as for any other particle. If other, non-electrostatic forces also act on the charge, they should be just added to give the *total* force, and the 2nd Law will allow to find acceleration.

Advanced: although we are talking about electrostatics, particles are permitted to move, albeit not too fast. If they do move fast, with speeds comparable to the speed of light, the 2nd Law in the above version need correction, and Coulomb's also needs to be modified to account for retardation. (Equivalently, magnetic fields due to particle motion must be included). In addition, rapidly accelerating charges will emit electromagnetic waves, which are not part of the story (yet).

Example: Estimate the speed of an electron in a hydrogen atom with radius about $0.53 \cdot 10^{-10}\, m$.

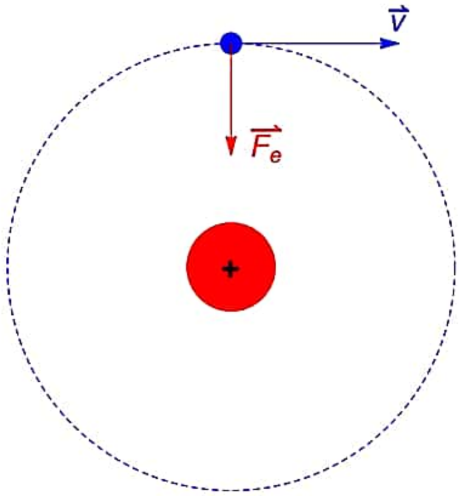


FIG. 8:

Solution: the centripetal acceleration $a = v^2/r$ is due to coulomb interaction between the electron and the proton. Thus,

$$F = k\frac{e^2}{r^2} \simeq 9 \cdot 10^9 \frac{(1.6 \cdot 10^{-19})^2}{(0.53 \cdot 10^{-10})^2} \approx 8.2\, 10^{-8}\, N$$

From 2nd Law find the acceleration of the electron:

$$a_e = F/m_e \approx 8.2\, 10^{-8}/(9.1 \cdot 10^{-31}) = \ldots$$

with m_e being the mass of electron.

To find speed v use $F = m_e a_c$

$$m\frac{v^2}{r} = k\frac{e^2}{r^2}$$

(the heavy proton practically does not move). Or,

$$v = \sqrt{ke^2/(m_e \cdot r)} = \sqrt{\frac{9 \cdot 10^9\, (1.6 \cdot 10^{-19})^2}{9.1 \cdot 10^{-31} \times 0.53 \cdot 10^{-10}}} \ldots$$

(Check that it does not exceed speed of light!).

Acceleration of the proton:

$$a_p = F/m_p = a\frac{m_e}{m_p}$$

with $m_p \sim 1.67 \cdot 10^{-27}\, kg$. Note: F - same (3rd Law !).

What other forces can act on a charge? The answer depends whether we consider an elementary charge or just a charged "macroscopic" particle (which can be tiny on a human scale, like a fine dust particle).

If the charge is elementary, there is *only one* other long range force which can act on it. This is the force of gravity, $F_g = m\vec{g}$ with $\vec{g}$ being the gravitational acceleration. (Nuclear "forces" which can act on protons are of very short range, about $10^{-14}\ m$, not of human scale at all. They are also not "forces" in the strict meaning of word, since they do not lead to anything like the 2nd Law).

The gravitational interaction between 2 elementary charges is negligibly small (estimate!), but if a charge interacts with a huge body, like a planet, the electrostatic and gravitational forces can be comparable, as in the Millican experiment.

Discussion. Relation between the Coulomb's Law and the Newton's Law of gravitation

$$F_G = -G\frac{m_1 m_2}{r^2}$$

with $G \simeq 6.7 \cdot 10^{-11}\, N\, m^2/kg^2$.

Compare to Coulomb's law:

r^{-2} - same!

$m_{1,2}$ - analogous to $q_{1,2}$

BUT:

"-" in the formula AND $m_{1,2} > 0$

Compare forces between two electrons:

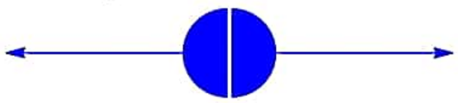
$$F_G = -G\frac{m_e^2}{r^2}\,;\ F_e = k\frac{e^2}{r^2}$$

$$\frac{F_G}{F_e} \sim \frac{G m_e^2}{k e^2}$$

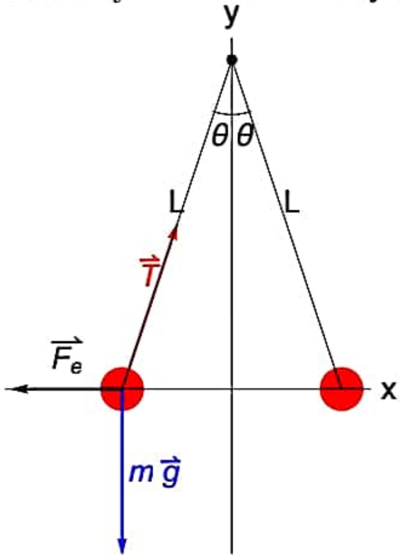
$$\frac{F_G}{F_e} \sim \frac{10^{-10-60}}{10^{10-38}} \sim 10^{-42}$$

For a non-elementary charge one can introduce other forces, similarly to what is commonly done in regular mechanics. For example, for two suspended light charged pit balls one can discuss the tension force $\vec{T}$ as the third force which equilibrates the gravitational $\vec{F}_g$ and the electric F_e forces (i.e., $\vec{F}_e + \vec{F}_g + \vec{T} = 0$ if the system is in equilibrium - see example below. In principle, tension is not a fundamental force but is also of electromagnetic origin, but this is only in principle. In reality, one cannot predict the value of T from considering interactions of elementary charges in the thread, and T must be deduced from measurements.

Advanced: There is a fundamental difficulty in E&M, What is the size of an electron? If it is finite, there are enormous forces trying to break it apart (see Coulomb's Law). Which forces prevent it from breaking? (we do not know, and at the moment it seems impossible to introduce such forces consistently, so that they satisfy relativity, conservation of energy and momentum, etc.). The other option is that electron is an infinitesimal point, but then one encounters INFINITY(!) when the center of the electron is approached. The latter is very hard to deal with, both mathematically and conceptionally, but seems to remain the only option which is currently available.



Example: In a Lab demo two light balls with $m = 1$ milli-gram each are suspended on two massless threads with $L = 1\,m$. When charged with equal negative charges Q the balls separated by $r = 2\,cm$. Find Q and the number of extra electrons on each ball.



$$\vec{T} + m\vec{g} + \vec{F_e} = 0$$

Let $\sin\theta = r/2L \approx \tan\theta$:

$$T\sin\theta - F_e = 0$$

$$T\cos\theta - mg = 0$$

Thus,

$$F_e = mg\tan\theta \;=\; k\frac{Q^2}{r^2}$$

$$Q \approx -\left(\frac{mgr^3}{2kL}\right)^{1/2} \sim \left(0.5 \cdot 8 \cdot 10^{-6+1-6-10}\right)^{1/2}$$

Advanced. Insufficiency of classical mechanics to get the size of an atom

Have $[k] = N \cdot m^2/C^2$, $[e] = C$, $[m] = kg$. *Let us try to construct length:*

$$[m] = [kg \cdot m^3/s^2C^2]^\alpha [C]^\beta [kg]^\gamma$$

No solution! What to do? Need a new fundamental constant (Bohr). It is $\hbar \sim 10^{-34} J \cdot s$ *(Plank's constant).*

Extra credit (optional): estimate the size of an atom by adding $\hbar$ *to previous dimensions.*

III. ELECTRIC FIELD

A. Field due to a point charge

1. Definition and units

Consider the Coulomb's law, eq. (9), but now we treat the charges unequally. The 1st charge is the primary charge, just q, the second charge is a *probe* , a small charge with a value q_0. The law can now be written as

$$\vec{F}_0 = k\frac{q \cdot q_0}{r^3}\vec{r}$$

with F_0 being the force which acts on the probe and $\vec{r}$ pointing from the primary charge towards the location of the probe.

Now consider the following ratio

$$\vec{F}_0/q_0 = k\frac{q}{r^3}\vec{r}$$

The most remarkable fact about this expression is that it *does not depend on the probe*! Thus, the ratio is a characteristic of the charge q only, but not of q_0. It deserves a name - *the electric field at point* $\vec{r}$ and a standard notation $\vec{E}(\vec{r})$. The units however, are derived from the known ones: $\left[\vec{E}\right] = N/C$ (and later we learn that this is the same as V/m, *volts per meter*). Explicitly, one has for a field due to a point charge q

$$\vec{E} = k\frac{q}{r^3}\vec{r} \tag{12}$$

or, without vectors

$$\boxed{E = k\frac{q}{r^2}} \tag{13}$$

with positive sign indicating that field goes *away* from the charge and negative sign indicating a field going *towards* the charge, if it happens to be negative. r is just the distance from charge q to the observation point, and we do not need the probe at this point anymore(!)

The vector $\vec{E}(\vec{r})$ is defined for *any* point in space around q. Instead of showing the vectors, however, it is much more convenient to depict the field lines (see the Introduction). Such lines have the property that their tangent coincides with the direction of a vector at a given point. Since $\vec{E}$ always points away from the positive charge (towards a negative charge), for a single charge the field lines will be just straight lines, as in Fig. 9. Note that positive and negative charges serve, respectively, as "sources" and "sinks" for the field lines.

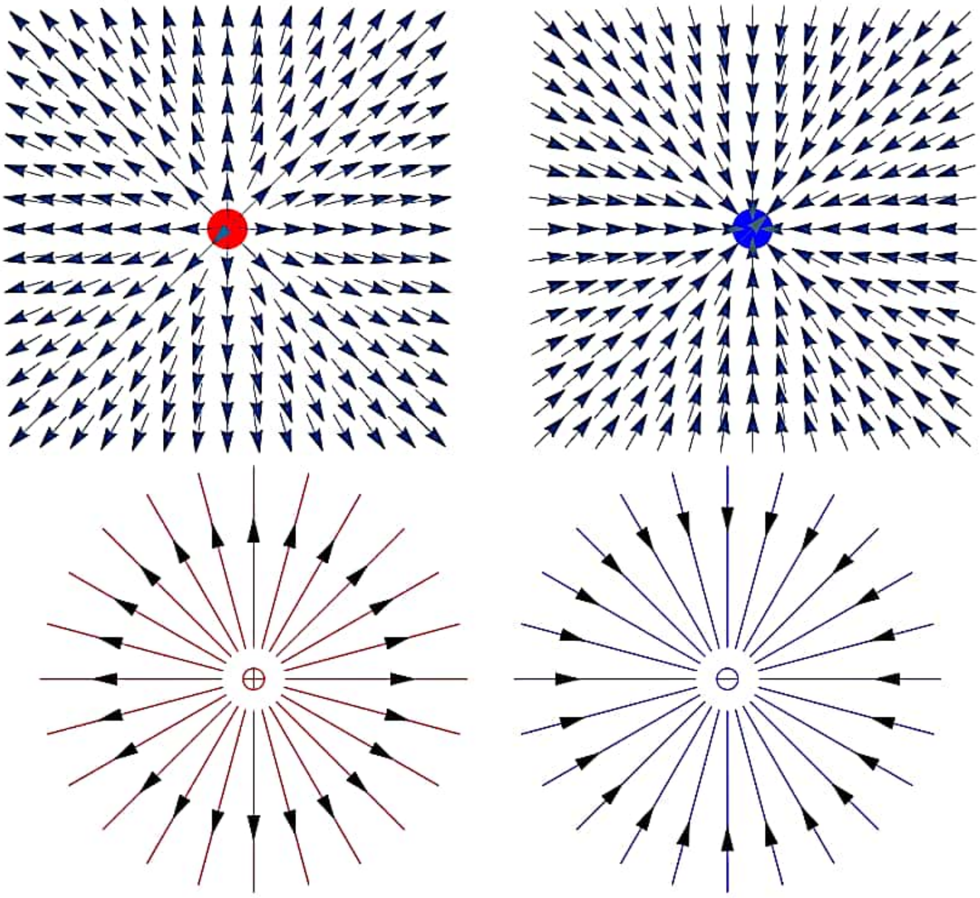


FIG. 9: Vector fields (upper row) and electric field lines (lower row) due to single point charges. Note that the field becomes infinitely strong when a charge is approached.

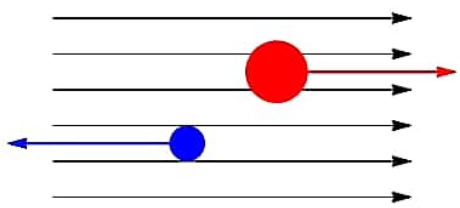
B. Field due to several charges

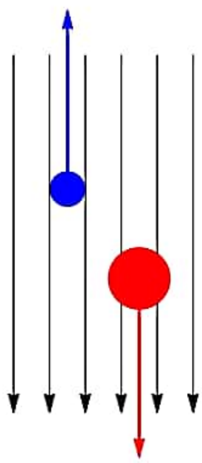
1. *Definition and force on a charge in a field*

Similarly to the field of a single charge, in a general case one can introduce field $\vec{E}(\vec{r})$ as a ratio of the force which acts on a small probe placed at $\vec{r}$ to the magnitude of the probe. (After that, the probe does not matter).

In practice, this definition is often reversed. Field $\vec{E}$ is assumed to be known at a given point, and one is asked to find the force on a charge q which is placed there (the charge may or may not be called "probe" in this case). From the definition one has

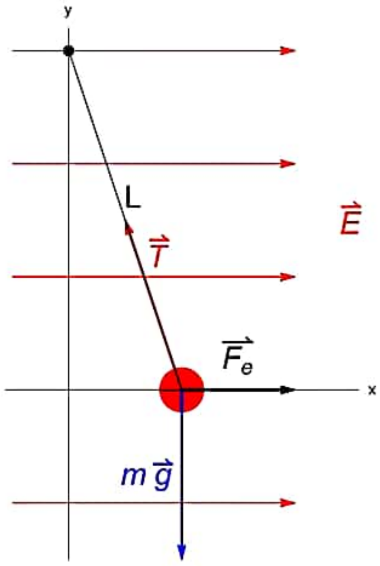
$$\boxed{\vec{F} = q\vec{E}} \tag{14}$$





Note that if the charge is negative (blue), the force is *opposite* to the field. If the blue object has mass m and is to be balanced against force of gravity:

$$qE = mg$$



$$\vec{T} + m\vec{g} + \vec{F}_e = 0$$

$$-T \sin\alpha + F_e = 0$$

$$T \cos\alpha - mg = 0$$

Thus,

$$mg \tan\alpha = F_e = qE$$

2. Superposition of fields

Since the force obeys the *superposition principle*, the latter is also valid for the fields. The total field $\vec{E}$ at a given point is determined by a vector sum of contributions of individual charges

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \dots \tag{15}$$

The fields $\vec{E}_1$, $\vec{E}_2$, etc. are determined by eq. (12) with $\vec{r}$ replaced by a vector pointing from a corresponding charge to the observation point.

Example Field due to a dipole. We will consider the observation point equally distanced from both charges, as in fig. 10. The distance between charges is d and the distance from each charge to the observation point is L. Both charges are identical in magnitude and equal $\pm q$, respectively.

Let the two charges have respective coordinates $\vec{r}_1 = (-d/2, 0)$ and $\vec{r}_2 = (d/2, 0)$; the observation point is then located at $\vec{r}_0 = (0, h)$, with $h = \sqrt{L^2 - d^2/4}$. Let $\vec{E}_1$ be the field

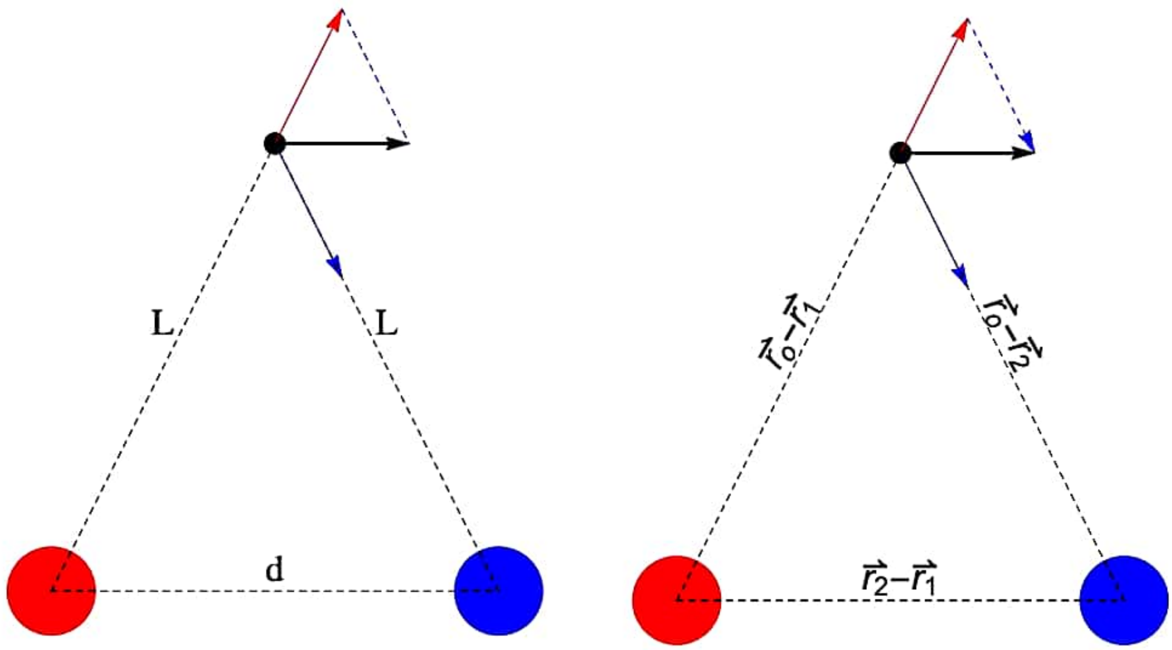


FIG. 10: Evaluation of a field due to a dipole. Left - from similar triangles. Right - from vectors. As a reminder, a tiny probe is shown at the observation point, equally distanced from both charges. In reality, there is nothing present at that point, just field.

from positive (red) charge and $\vec{E}_2$ field from the negative (blue) charge. The black horizontal field is their resultant $\vec{E}_{dip} = \vec{E}_1 + \vec{E}_2$. From similar triangles

$$E_{dip}/E_1 = d/L \;\Rightarrow\; E_{dip} = E_1 \frac{d}{L}$$

$$\text{From } E_1 = kq/L^2\text{:} \quad \boxed{E_{dip} = \frac{kqd}{L^3}} \tag{16}$$

Advanced. Alternatively, we can use vectors and the superposition principle:

$$\vec{E}_{dip} = \vec{E}_1 + \vec{E}_2 = kq\frac{\vec{r}_0 - \vec{r}_1}{L^3} - kq\frac{\vec{r}_0 - \vec{r}_2}{L^3} = \frac{kq}{L^3}\left\{\vec{r}_0 - \vec{r}_1 - \vec{r}_0 + \vec{r}_2\right\} = \frac{kq}{L^3}\left\{\vec{r}_2 - \vec{r}_1\right\}$$

which is a vector pointing to the right (from positive to negative, parallel to the dipole) with the same magnitude.

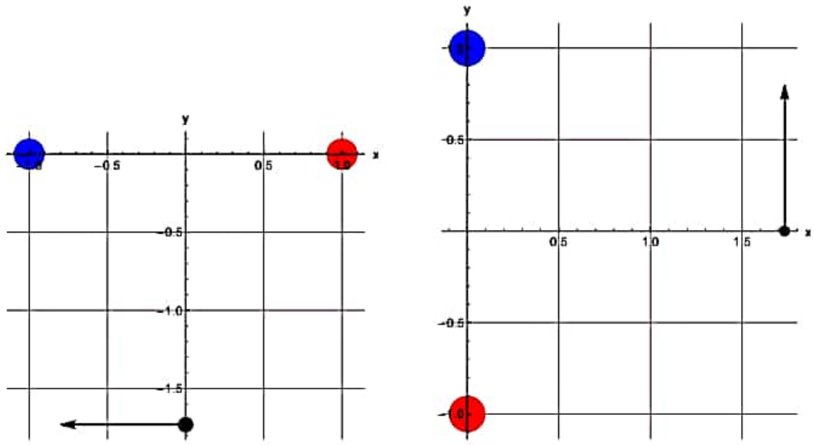


FIG. 11: Picture of the dipole can be rotated.

Example. Find the field from a dipole if the observation point and the charges form an equilatteral triangle with side a.
Direction - see Fig. 11. Magnitude: $d = L = a$ and

$$E_{dip} = kqd/L^3 = kq/a^2$$

Another example. Same arrangement, but both charges are positive - Fig. 12.

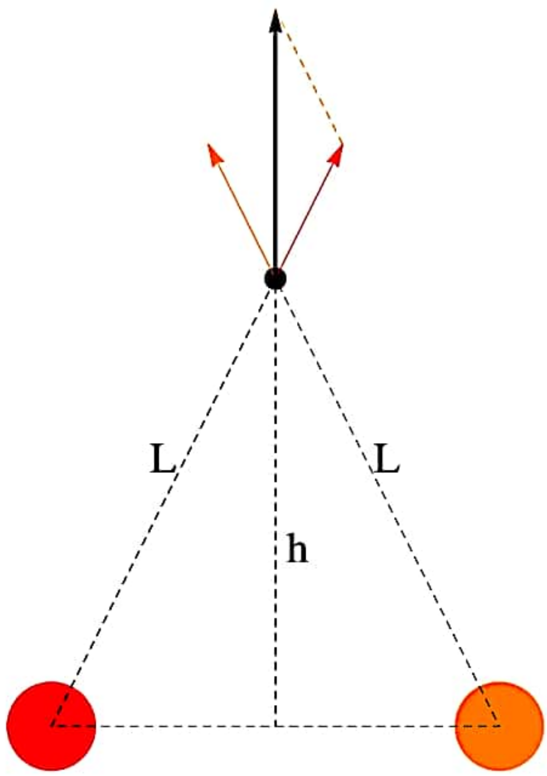


FIG. 12: Example of evaluation of a field due to two identical positive charges.

Now

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = kq\frac{\vec{r}_0 - \vec{r}_1}{L^3} + kq\frac{\vec{r}_0 - \vec{r}_2}{L^3} = \frac{kq}{L^3}\left\{\vec{r}_0 - \vec{r}_1 + \vec{r}_0 - \vec{r}_2\right\}$$

or

$$\vec{E} = \frac{kq}{L^3}\left\{2\vec{r}_0 - \vec{r}_2 - \vec{r}_1\right\} = \frac{kq}{L^3}(0, 2h) = \frac{2kqh}{L^3}(0, 1)$$

which is a vector pointing up.

In principle, the superposition principle allows one to reconstruct field due to any known charge distribution. If charges are distributed continuously, one just needs to break the distributed charge into small individual domains, and threat each of the as a point charge. This leads to an integral instead of a sum in eq. (15), but otherwise it is the same idea. We will later see how it works on examples.

C. Electrostatic Field Lines (EFL)

In a general case the structure of field lines is more complex than for a single charge; in particular they are not straight lines anymore. Nevertheless, some general properties can be established:

- tangent to the EFL determines the direction of the electric field $\vec{E}$
- density of EFL determines the magnitude of E
- EFL originate on positive charges
- EFL terminate on negative charges
- EFL can come and go to infinity
- EFL CANNOT start or end in empty space
- EFL CANNOT loop
- as a rule, EFL CANNOT cross

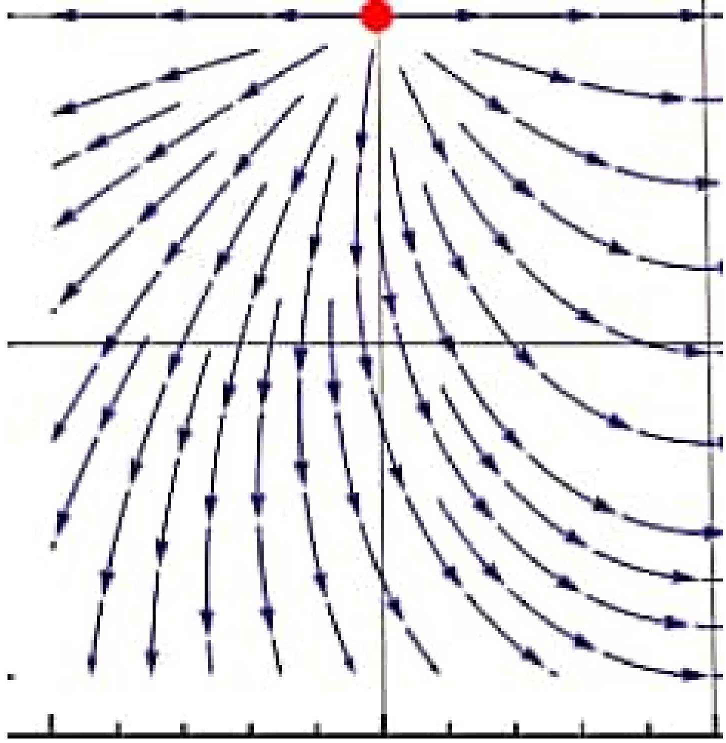
Looping is not allowed since it would contradict conservation of energy. At the point of crossing of two lines it would be impossible to determine the direction of the field. (A special case is the point of zero field; such points however, are extremely rare since all *three* components of $\vec{E}$ must go to zero at the same time).

1. Field lines due to a dipole

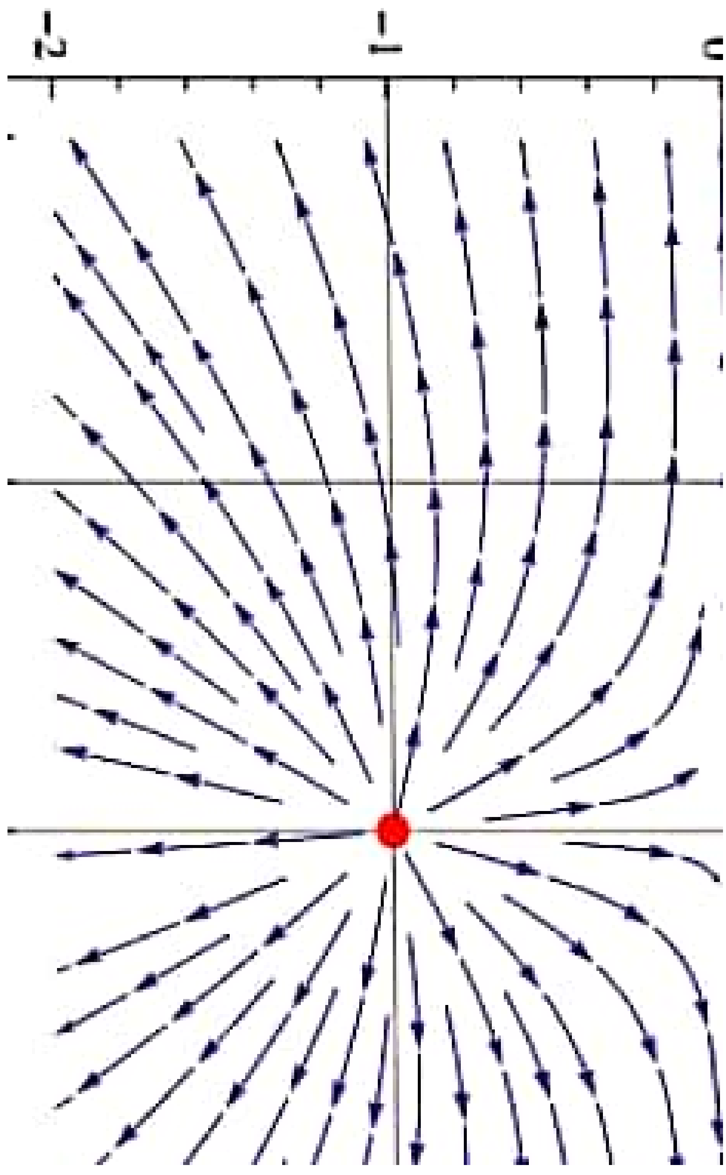
Generally, plotting field lines for several charges is not easy. Two things help. First, directly near charges fields are so strong that other charges do not matter. It is a good start. Second, in many problems there is some special symmetry which helps to understand the structure of field.

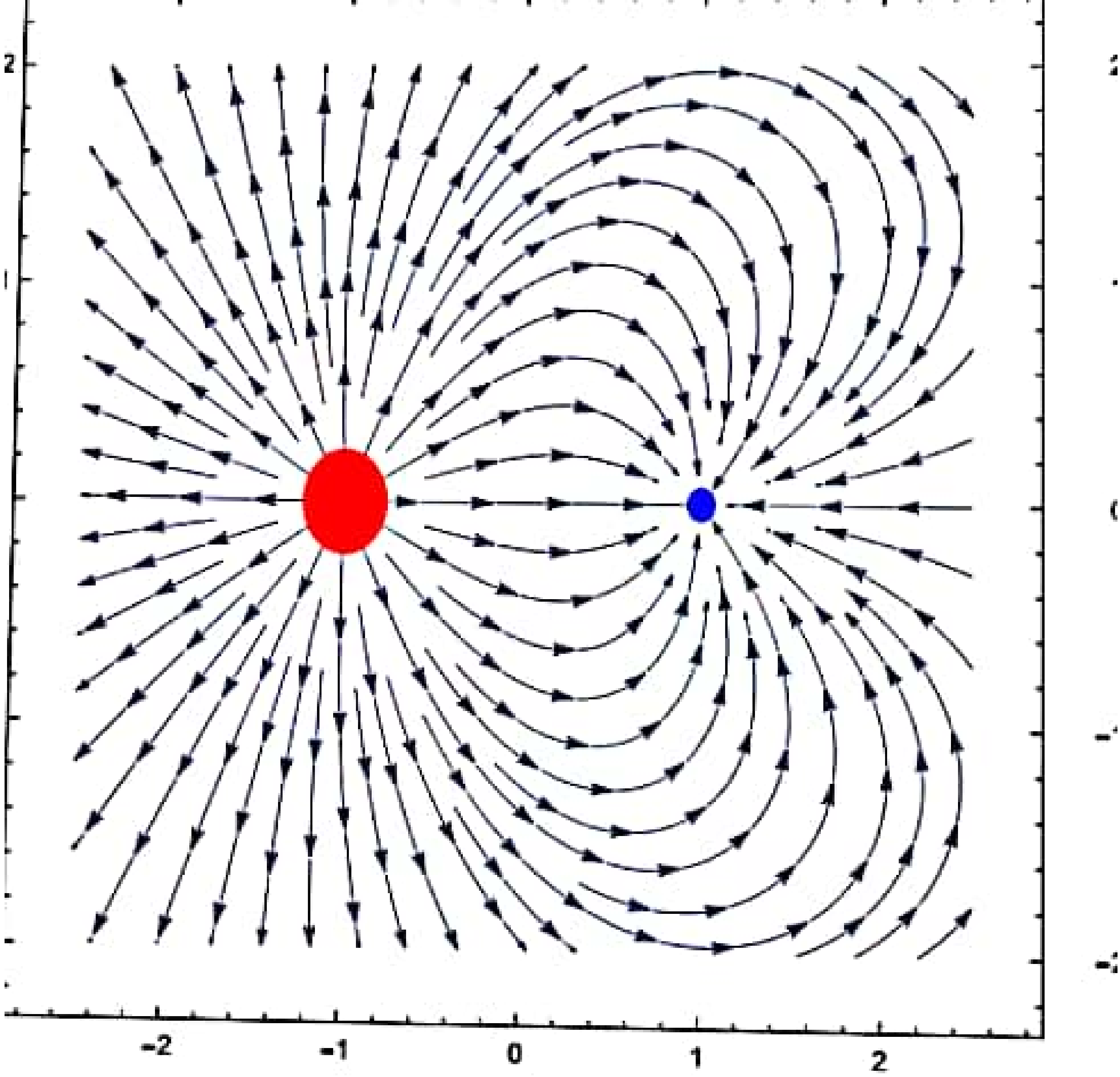
Field due to a dipole - Fig. 13: Note that there are no points with zero field.

Field due to two identical charges - Fig. 14: There is one point where the field is zero.



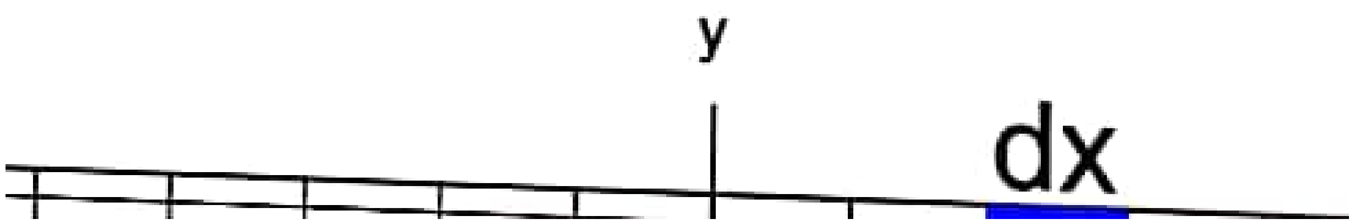
IG. 13: Electric field li



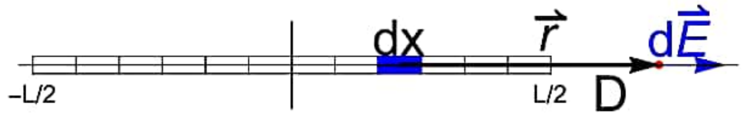


15: Electric field lines due to two non-equal ch
ıes larger. The smaller charge is negative (left

xample. Field from a uniformly charged li
nds



Example. Field from a uniformly charged line at a point (red) with distance D from the end, along the rod



Introduce $X = L/2 + D$-distance of the red point from the center. Contribution of the selected (blue) fragment

$$dE = k\lambda dx/(X - x)^2$$

$$E = \int_{-L/2}^{L/2} dx\, k\lambda/(X - x)^2 = k\lambda \frac{1}{X - x}\Big|_{-L/2}^{L/2}$$

$$= k\lambda \left(\frac{1}{D} - \frac{1}{D + L}\right) = \frac{k\lambda L}{D(D + L)}$$

IV. GAUSS THEOREM

A. Quantification of the number of lines

The electric field lines give a good qualitative picture of the field, So far, however, we did not specify the exact *number* of lines to draw, so that the field intensity intensity was only proportional to their density. As long as the number lines is our choice, let us try to determine the number of lines, Φ, in such a manner that the density of lines will be *exactly equal* to the intensity of field. We will do that first for a single charge where we know the field

$$E = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2} \tag{17}$$

where ϵ_0 is just another coefficient, related to $k \simeq 9 \cdot 10^9\ N \cdot m^2/C^2$ which we used before by $k = 1/(4\pi\epsilon_0)$.

Now let us surround a single charge by a sphere with radius r, as in Fig. 16. The area of the sphere is $4\pi r^2$, and if the charge is at the center of the sphere, the density of lines is

$$\frac{\Phi}{4\pi r^2}$$

Comparing this to eq. (17), one determines the number of lines as

$$\Phi = q/\epsilon_0 \tag{18}$$

B. Deformations of the Gaussian surface

The sphere in Fig. 16 is often called a *Gaussian* surface. Note that once the number of lines which emerge from a charge, Φ, is selected the number of lines which cross the surface does not depend either on its shape or on its size, as long as the charge remains inside - see Fig. 17.

We are almost ready to prove the Gauss theorem, although some formalities are still required.

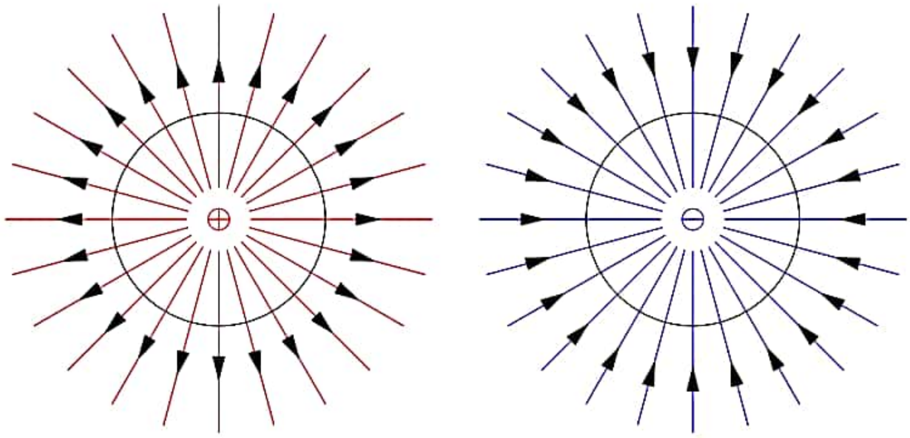


FIG. 16: A charge is surrounded by a sphere with radius r. If Φ is the number of lines which emerge from the charge, their density will be given by $\Phi/\left(4\pi r^2\right)$. The number of lines is considered *positive* if they go out of the surface (picture on left); if the lines go into the surface (picture on right) their contribution is *negative*. For a properly selected Φ, as in eq. (18), the density of lines will exactly equal to the magnitude of electric field at any distance from the charge.

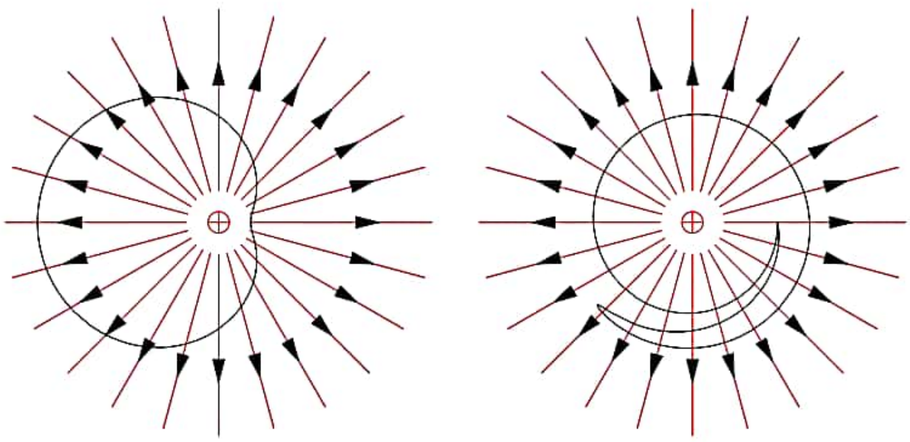
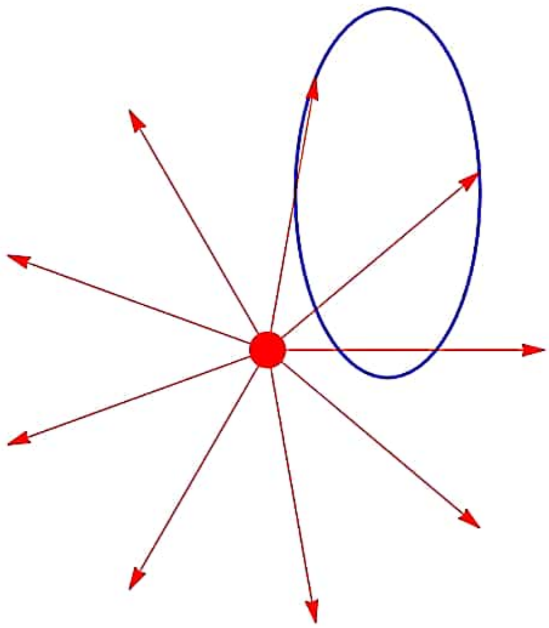


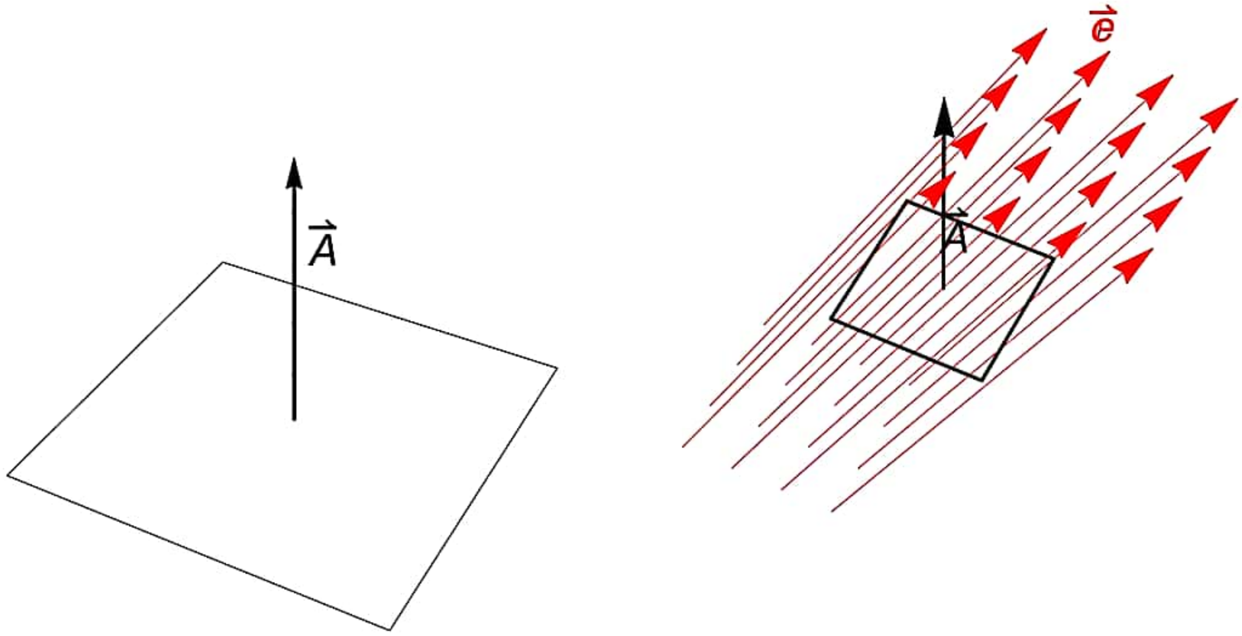
FIG. 17: Deformations of the Gauss surface. Note that the total number of lines which cross the surface (with account for sign) does not change as long as the charge remains inside.



The outside charge does not contribute to the flux.

C. Definition of the flux

The more controlled definition of the "number of lines which cross a given surface" is via the electric flux.



Vector of the surface area (left) and the flux $\Phi = \vec{E} \cdot \vec{A}$. Density of field lines is proportional ("equal") to $|\vec{E}|$.

Consider a small surface element with area ΔA and let us characterize it by a vector $\Delta\vec{A}$ which points in the direction of the normal to the surface. The number of lines which cross this surface, $\Delta\Phi$, is determined from the condition that $\left|\vec{E}\right|$ coincides with the density of lines. Thus, one obtains

$$\Delta\Phi = \vec{E} \cdot \Delta\vec{A} \tag{19}$$

This is called the *flux* through the surface element. Note that this flux is a scalar (see Introduction for pictures).

Similarly, flux can be introduced for any surface, not only small. That surface can be partitioned in small elements, $\Delta\vec{A}_i$ each characterized by its own $\Delta\Phi_i$ (positive or negative), and individual contributions should be just added together. In the limit, this leads to an integral $\vec{E} \cdot d\vec{A}$ over the surface. The most interesting case is when the surface is closed, so that

$$\Phi = \oint \vec{E} \cdot d\vec{A} \tag{20}$$

This serves as a formal definition of the flux. Check that it indeed coincides with the number of lines in a simple configuration of Fig. 16.

Since the field $\vec{E}$ obeys the superposition principle, so does the flux. I.e., fluxes due to several charges just add up (as scalars!). This conclusion is the main reason for the detour from the more narrative, field line description. (It is not easy to justify from the start the superposition principle in terms of the field lines since adding an extra charge will dramatically modify the structure of the field - see the previous lecture).

D. Gauss theorem

Since we have the principle of superposition, and since an individual charge produces a flux given by eq. (18), one has

$$\Phi = q_{enc}/\epsilon_0 \tag{21}$$

where q_{enc} is the net charge enclosed inside the surface. The shape of the surface does not matter, and so does not matter any outside charge.

E. Gauss Theorem (GT) and Coulomb's law

Equation (17) from which we started when deriving the GT is a direct consequence of the Coulomb's law. So is the GT itself. Conversely, it could be possible to postulate the GT as a fundamental law, and then derive eq. (17) from it. The only thing which should be added here are considerations of symmetry (which are important in almost every practical application of the GT - see below).

Consider again Fig. 16, but imagine now that you do not know the magnitude of the electric field. You do know, however, that the field is radial (from symmetry!), and the flux

Φ through the Gaussian surface centered at the charge is given by

$$\Phi = 4\pi r^2 E$$

The field E is yet unknown, but it follows from the GT, eq. (21) with q_{enc} being the single charge q:

$$E = \frac{q}{4\pi\epsilon_0 r^2}$$

This is exactly eq. (17).

F. Applications of the GT

There are two major applications of the GT. The first is finding the field for some symmetric, usually continuous distribution of charges. The second type is finding the charge once something is known about the field, as in the case of the conductor.

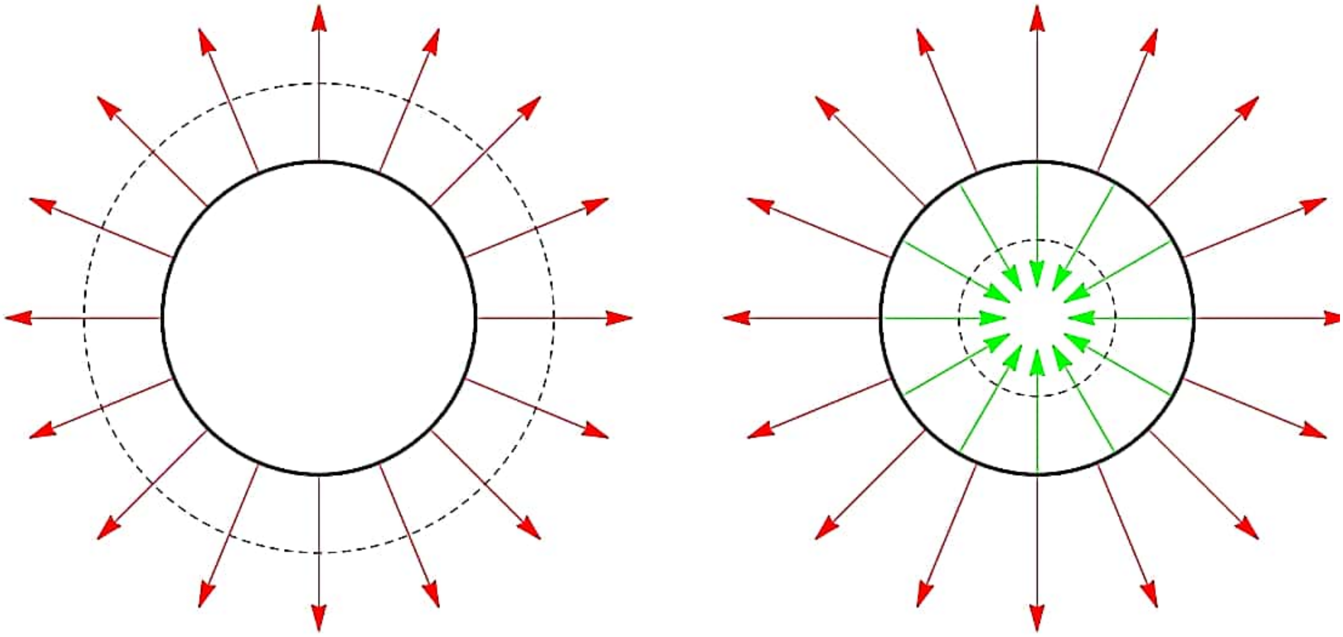
In the first group of applications the key point is selection of a "good' Gaussian surface, which is consistent with symmetry. Then, the flux Φ can be evaluated, and field E found from the GT. Let's see how it works.

1. Charged spherical shell

Consider a shell with radius R charged with a charge Q uniformly distributed over its surface. From symmetry, field lines are radial. For an arbitrary radius of the Gaussian surface r the flux is given by

$$\Phi = \oint E \cdot dA = E \oint dA = E \cdot 4\pi r^2$$

although the field E is yet unknown, and we find it from the GT. The result strongly depends on whether we are inside or outside the real shell. One has:



The Gauss surface (dashed) with radius r outside the shell (left) and inside the shell (right). Outside, $r > R$: $q_{enc} = Q$; inside, $r < R$: $q_{enc} = 0$. Thus

$$\text{GT (outside): } \Phi = \frac{Q}{\epsilon_0} \;\Rightarrow\; E(r > R) = \frac{Q}{4\pi\epsilon_0 r^2} = k\frac{Q}{r^2} \text{ as if charge } Q \text{ was the } \textit{center}$$

$$\text{GT (inside): } \Phi = 0 \;\Rightarrow\; E(r < R) = 0 \text{ everywhere inside, not only at the center}$$

Example. A spherical shell with $R = 5\,m$ has a net charge of $Q = 1\,\mu C$ uniformly distributed over the surface. What is the magnitude of the electric field at (a) a distance $r = 1\,m$ from the *center* of the sphere and (b) a distance $d = 1\,m$ from the *surface* of the sphere?

$$\text{(a): } r < R \;\Rightarrow\; E(r) = 0$$

$$\text{(b): } r = R + d > R \;\Rightarrow\; E(r) = k\frac{Q}{(R+d)^2} = 9 \cdot 10^9 \frac{1 \cdot 10^{-6}}{(5+1)^2} = 2.5 \cdot 10^2\,\frac{N}{C}$$

2. Uniformly charged sphere

Let ρ be the charge density inside a sphere with radius R. The above relation $\Phi = 4\pi r^2 E$ is still valid. For q_{enc} in the GT one has:
Outside, $r > R$: $q_{enc} = Q$, with $Q = 4/3 \cdot \pi R^3 \rho$ being the total charge, and $E = Q/\left(4\pi\epsilon_0 r^2\right)$, exactly like before. Inside, $r < R$: $q_{enc} = 4/3 \cdot \pi r^3 \rho$, and the GT gives

$$4\pi r^2 E = q_{enc}/\epsilon_0\ , \quad E(r) = \frac{\rho}{3\epsilon_0} r$$

At the surface the field is given by $E_0 = \rho R/3\epsilon_0$ and this result can be approached from the outside as well (show this!). The structure of field is shown in Fig. 18. This problem

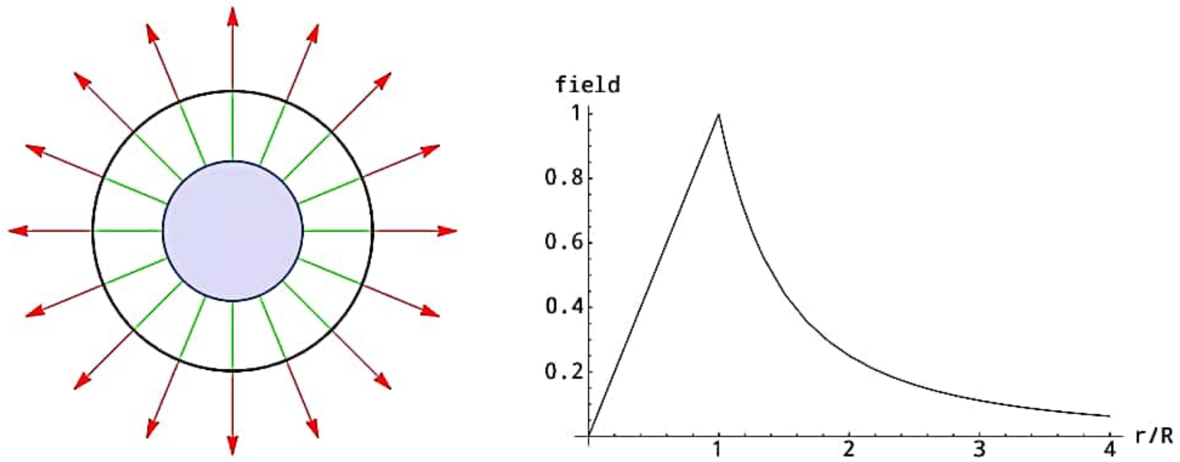


FIG. 18: The field of a uniformly charged sphere. The field is given in scaled units, E/E_0 with E_0 being the field at the surface (see text).

is also of interest in gravitational context, representing, e.g. the gravitational acceleration both inside and outside Earth.

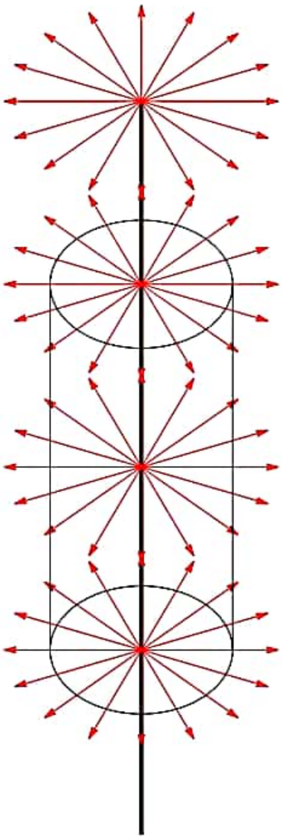
Other types of symmetry will be discussed in class. Here is the summary only:

3. Uniformly charged infinite line

Units: $[\lambda] = C/m$

Symmetry: cylindrical

Gaussian surface: cylinder coaxial with the line, with radius r and arbitrary length L.



Flux (only side surface contributes) $\Phi = 2\pi r L E$, charge inside $q_{enc} = \lambda L$

$$\text{From GT: } 2\pi r L E = \lambda L/\epsilon_0 \,, \quad E = \frac{\lambda}{2\pi\epsilon_0 r} \tag{22}$$

4. Uniformly charged non-conducting plane

Units: $[\sigma] = C/m^2$

Symmetry: planar -see Fig. 19

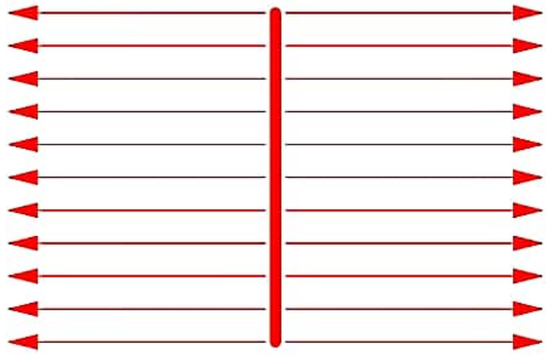


FIG. 19: Electric field due to an infinite positively charged plane. Since field lines are parallel to each other, their density remains constant and the magnitude of the field is *independent* of the distance from the plane and is given by eq. (23). For a negatively charged plane the picture would be similar, with lines going *into* the plane.

Gaussian surface: rectangular box with one face (with some area A) parallel to the plane. The charged plane cuts the box in the middle.

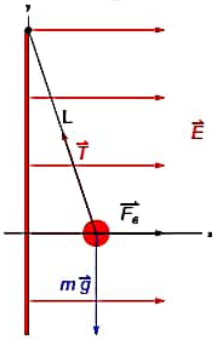
Flux and enclosed charge:

$$\Phi = 2AE\,,\ q_{enc} = A\sigma$$

Result (after GT is applied):

$$\boxed{E = \frac{\sigma}{2\epsilon_0}} \tag{23}$$

Example. For$Q = 2\,\mu C$; $\sigma = 1\,\mu C/m^2$ and $m = 10^{-4}\,kg$ find the angle with vertical.

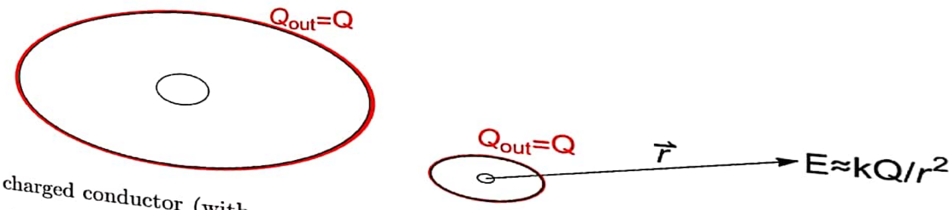


$$\text{x:}\quad -T\sin\theta + F_e = 0\,,\;\; \text{y:}\;\; T\cos\theta = mg\;\;\Rightarrow$$

$$mg\tan\theta = F_e = QE = Q\frac{\sigma}{2\epsilon_0} = 2\cdot 10^{-6}\frac{10^{-6}}{2\cdot 8.85\cdot 10^{-12}} = \ldots$$

$$\tan\theta = F_e/(10^{-4}\cdot 9.8) = \ldots$$

Impossibility of electric field inside metal (left) - otherwisde current, and impossibility of a charge inside (right) - would violate Gauss theorem.



A charged conductor (with a cavity). Left: all extra charge Q goes to the *outside* surface; inside no charge/no field. Right: far away outside acts as a point charge Q regardless of actual shape.

V. ELECTROSTATIC POTENTIAL (EP)

A. Definitions, units, etc.

The EP at a given point in the electric field (which is created by all other charges in the Universe) is defined as the potential energy of a unit positive charge if it were placed at that point. In other words, if you place a small probe q_0 at a given point $\vec{r}$, and that probe has a potential energy $U(\vec{r})$, the potential $V(\vec{r})$ is defined as

$$V(\vec{r}) = U(\vec{r})/q_0 \tag{24}$$

The actual value of q_0, or even its sign, do not matter - we shall see it later.

Units: V (volts); $V = J/C$

Major application: In practice, often the EP can be calculated (or measured) first. Then, potential energy of a given charge q which is placed at a given point in the field (created by other charges) is given by

$$U(\vec{r}) = qV(\vec{r}) \tag{25}$$

If a charge is being moved from one point (A) to another point (B), the work done by the field on that charge is given by

$$W_{AB} = U_A - U_B = -\Delta U = q(V_A - V_B) = \boxed{-q\Delta V} \tag{26}$$

Conventions. The direct physical meaning is given to the *difference* of potentials. Adding a constant to V will not matter. By convention, potential at infinity is selected as zero (if not stated otherwise). If there are grounded conductors in the problem, zero potential is associated with them.

Example. A positively charged particle with charge q and mass m is placed at a point with potential V_1. Find the speed of the particle when it reaches a point with $V_2 < V_1$. Solution: from energy conservation

$$\frac{1}{2}mv^2 + qV_2 = 0 + qV_1\,;\ \frac{1}{2}mv^2 = q(V_1 - V_2) > 0$$

$$v = \sqrt{2(q/m)\cdot(V_1 - V_2)}$$

Outlook. We will do the following. First, I will remind you for which type of forces it makes sense to talk about the potential energy, and how it is related to work. Next, I will show that the Coulomb's force is of such type. Next, we will calculate the potential energy of one point charge in the field of another point charge, and use the original definition to find the EP. Then, we will examine the connection between the EP and the field $\vec{E}$, and consider the EP in and around conductors.

B. Work and energy in electrostatic field

1. *Conservative forces*

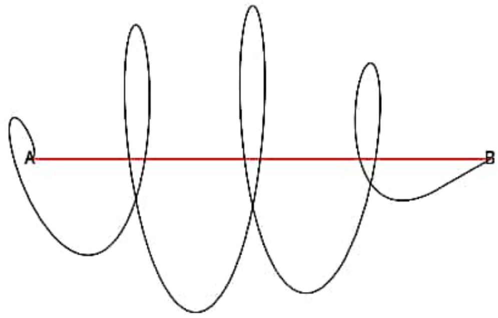


FIG. 20: Path-independence of the work done by a force. For any force the work is given by eq. (27), but for special "conservative" forces, the work does not depend on the actual path which connects points A and B. For such forces one can talk about potential energy, $U(\vec{r})$ and use eq. (28) to determine the work on a path. The electrostatic force is of that kind.

For any force $\vec{F}$ the work along a selected path can be obtained as

$$W_{AB} = \int_A^B \vec{F} \cdot d\vec{s} \tag{27}$$

For some fundamental forces, however, the actual path does not matter - see Fig. 20. Then, one can introduce *potential energy*, and not care about the path:

$$W_{AB} = U_A - U_B \tag{28}$$

The gravitational force was of such kind, so is the Coulomb's force (we show that below). Such forces are known as "conservative", to emphasize that the full mechanical energy (potential plus kinetic) is conserved.

With the convention that potential energy is taken as zero at infinity, one has

$$U_A = \int_A^\infty \vec{F} \cdot d\vec{s} \tag{29}$$

This gives us a prescription how to calculate U if the force is known.

C. Interaction of two charges

From eq. (29) one has

$$U(r_A) = \int_{r_A}^{\infty} \vec{F} \cdot d\vec{s} = \int_{r_A}^{\infty} F(r)dr = \int_{r_A}^{\infty} \frac{kq_1q_2}{r^2}dr = kq_1q_2 \int_{r_A}^{\infty} \frac{1}{r^2}dr =$$

$$= kq_1q_2 \left(-\frac{1}{r}\right)\Bigg|_{r_A}^{\infty} = kq_1q_2\frac{1}{r_A}$$

Thus, with $r = r_A$, the distance between charges, the interaction energy is given by

$$\boxed{U(r) = k\frac{q_1q_2}{r}} \tag{30}$$

This can be positive (repulsion) or negative (attraction). (The latter means that field performs negative work when the charge is dragged to infinity).

For more than two charges, q_1, q_2, q_3, etc. multiple pairwise interactions must be considered (with r_{ik} being the distance between charges i and k), and the total energy is just the sum

$$U(r) = k\frac{q_1q_2}{r_{12}} + k\frac{q_1q_3}{r_{13}} + k\frac{q_2q_3}{r_{23}} + \dots \tag{31}$$

D. Potential due to a point charge

We start with eq. (30) but treat one charge as the "primary" charge q, and the other as the probe q_0. Then

$$U = k\frac{qq_0}{r}$$

by definition, V is U/q_0, thus

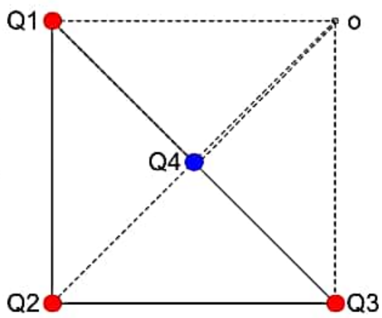
$$\boxed{V(r) = \frac{kq}{r}} \tag{32}$$

This is a potential of electric field due to a point charge at a distance r from that charge.

If there are several charges, potentials just add up (as scalars!, which is much easier)

$$V(r) = \frac{kq_1}{r_1} + \frac{kq_2}{r_2} + \frac{kq_3}{r_3} + \dots \tag{33}$$

Here r_1, r_2, etc. are the distances from each charge to the observation point.



Example. 3 identical charges $Q_1 = Q_2 = Q_3 = Q$ are placed at the vertexes of a right triangle with side a. A 4th charge $Q_4 = q$ is placed at the center of hypotenuse. Find the potential at the point "o" in the figure. Solution:

$$V_o = kQ_1/r_{1o} + kQ_2/r_{2o} + kQ_3/r_{3o} + kQ_4/r_{4o} =$$

$$= kQ(1/a + 1/a\sqrt{2} + 1/a) + kq/(a\sqrt{2}/2)$$

For a continuous charge distribution the generalization is straightforward. The distribution is broken in tiny fragments with charges $dq = \rho(\vec{r})\, dv$ with ρ being the charge density and dv the elementary volume. Each fragment is then treated as a point charge, and the sum becomes an integral

$$V = k \int \frac{\rho dv}{r} \tag{34}$$

The integration is carried out over the entire volume where the charge is distributed, and r is the distance from the integration point to the observation point (where the potential is measured). Note that the observation point can be both outside or inside the charge distribution. (In the latter case the distance r can go to zero when the integration and observation points coincide, but the integral still converges - this is an advantage of living in a 3-dimensional space).

Example. Find the potential of a charge $Q = 1\, nC$ uniformly distributed over the volume of a cube with side $a = 1\, cm$ at a distance $d = 100\, m$ from center. Solution. Exact integration VERY hard, but note $d \gg a$ thus

$$V \approx kQ/d \simeq 9 \cdot 10^9 \cdot (1 \cdot 10^{-9})/100 = \ldots$$

Example. A charged rod with (possibly non-uniform) linear density λ. Find V at a distance D from the end of the rod (red dot). Use $L = 1\,m$, $D = 9\,cm$ for calculations. Explore the cases $\lambda = const = 1\mu C/m$ and $\lambda = \alpha x$ with $\alpha = 1\mu C/m^2$

$$dV = k\frac{\lambda\,dx}{r}\,,\ r = x + D\ \Rightarrow\ V = k\int_0^L \frac{\lambda(x)\,dx}{x+D}$$

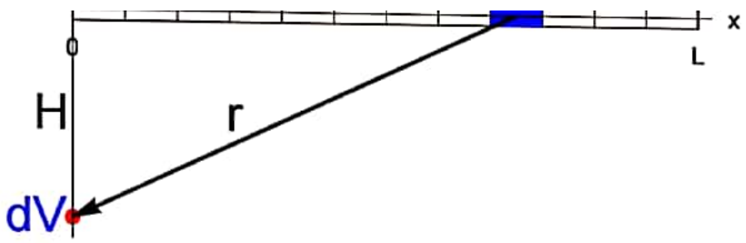
1) $\lambda = const$

$$V = k\lambda\int_0^L \frac{dx}{x+D} = k\lambda\ln(x+D)|_0^L = k\lambda(\ln(L+D) - \ln D) = k\lambda\ln\frac{L+D}{D}$$

Note: cannot take the limit $L \to \infty$ (unlike the case of the field E).

2) $\lambda = \alpha x$

$$V = k\alpha\int_0^L \frac{x\,dx}{x+D} = k\alpha\int_0^L \frac{(x+D)-D}{x+D}dx = k\alpha\int_0^L \left(1 - D\frac{1}{x+D}\right)dx = k\alpha\left(L - D\ln\frac{L+D}{D}\right)$$



$$dV = k\frac{\lambda\, dx}{r}\ ;\ r^2 = x^2 + H^2 \ \Rightarrow\ V = k\int_0^L \frac{\lambda(x)\, dx}{\sqrt{x^2+H^2}}$$

1) $\lambda = const$

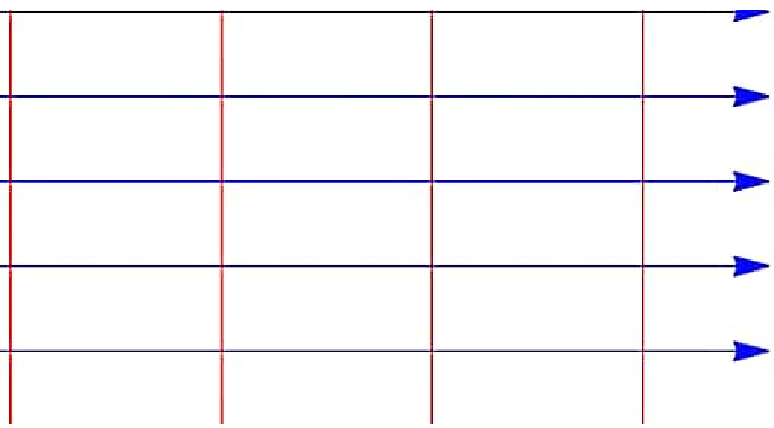
$$V = k\lambda \int_0^L \frac{dx}{\sqrt{x^2+H^2}} = k\lambda \ln(x+\sqrt{x^2+H^2})\Big|_0^L = k\lambda(\ln(L+\sqrt{L^2+H^2}) - \ln H)$$

Note: cannot take the limit $L \to \infty$ (unlike the case of the field E).

2) $\lambda = \alpha x$

$$V = k\alpha \int_0^L \frac{x\, dx}{\sqrt{x^2+H^2}} = k\alpha \int_0^L \frac{d(x^2)/2}{\sqrt{x^2+H^2}} = k\alpha\frac{1}{2}\int_0^{L^2} \frac{dz}{\sqrt{z+H^2}} =$$

$$= \frac{1}{2}k\alpha \cdot 2\sqrt{x^2+H^2}\Big|_0^L = k\alpha(\sqrt{L^2+H^2} - H)$$



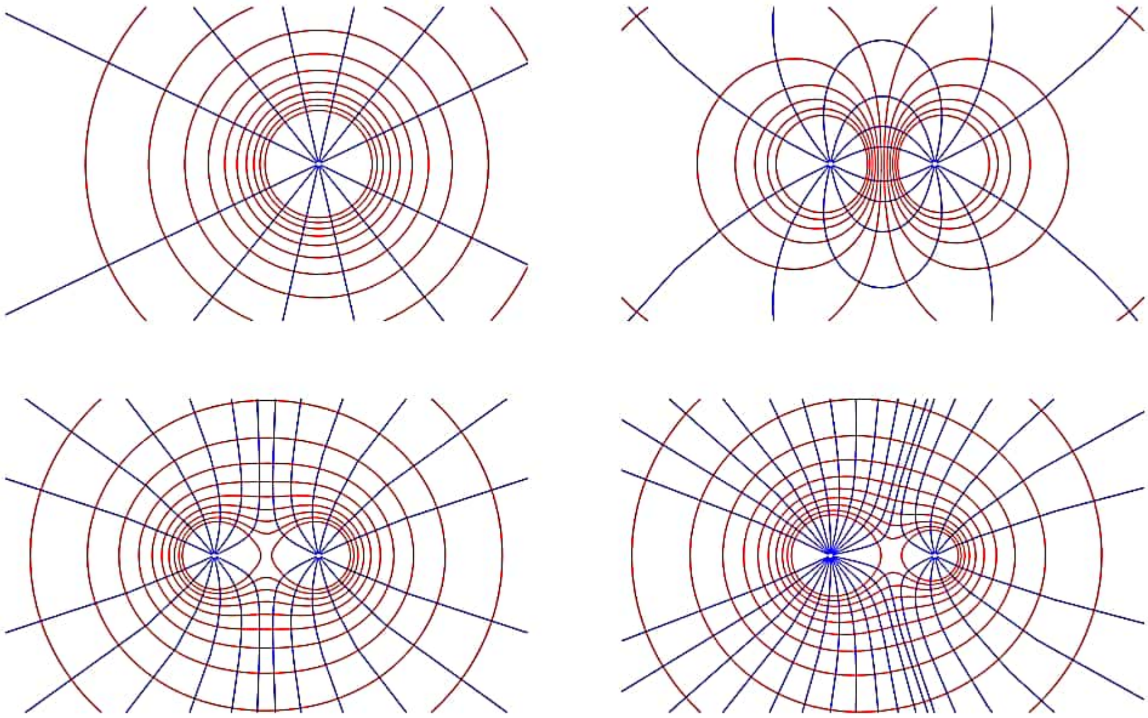


FIG. 21: Field lines (blue) and equipotential surfaces (red) for a single charge, a dipole, two equal charges of the same sign, and two unequal charges of the same sign. For clarity of the figure, equipotential surfaces are not shown in the immediate vicinity of the charges (where they become very close to each other); also direction of the field lines is not indicated since it depends on the actual signs of the charges. Note that at the intersection points the equipotential surfaces and the field lines are orthogonal to each other.

2. Field from potential

In the simplest example of a constant field,

$$E = -\frac{\Delta V}{\Delta s}$$

Δs being the distance between the equipotential surfaces. In a more general case, this relation also can be used, only approximately for small Δs; it does, however, become exact in the limit $\Delta s \to 0$, so that

$$E = -\frac{\partial V}{\partial s}$$

The derivative is taken in the direction of the *fastest* change in V.

To get individual components of $\vec{E}$ consider eq. (35) with $d\vec{s} = dx\vec{i} + dy\vec{j} + dz\vec{k}$. Now consider only a displacement in the x-direction (so that $dy = 0$ and $dz = 0$), and take a derivative to get rid of the integral. One has

$$E_x = -\frac{\partial V}{\partial x} \tag{38}$$

and similarly for other components.

Example. For $V(x,y) = x^2 + x - 3xy^2 - y^5 + 7$ find $\vec{E}$. Solution

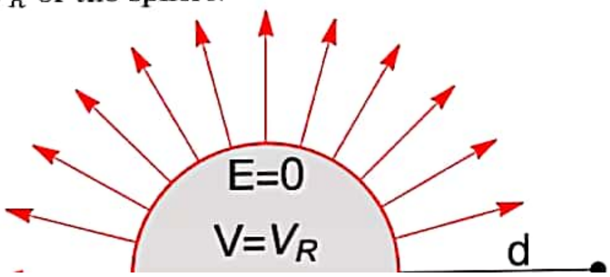
$$E_x = -dV/dx = -2x - 1 + 3y^2\,,\ E_y = -dV/dy = 6xy + 5y^4$$

Question. It looks that the scalar quantity V contains as much information as the vector $\vec{E}$. However, $\vec{E}$ is characterized by 3 numbers, while V by just one?? The reason is that the 3 components of $\vec{E}$ are not independent, but obey special relations (which actually allow us to introduce V), so that everything is consistent.

Left: since field inside a conductor is always zero (otherwise current!), the potential is constant.

Right: The field inside an empty cavity must be zero - otherwise $V_A \neq V_B$, which is impossible.

Example. A conducting sphere has a charge $Q = 1\,\mu C$ and radius $R = 2\,m$. (a) Find the field E and the potential V at a distance $d = 1\,m$ from the surface. (b) Find the potential V_R of the sphere.



VI. PROPERTIES OF A CONDUCTOR IN ELECTROSTATICS

Field:

- inside: electric field is zero (otherwise - current!)
- outside: field lines approach the surface at 90^o (otherwise - surface current).

Charge:

- inside: no free charge (follows from $\vec{E} = 0$ and the GT)
- surface: there can be surface charge (see below) .
- Extra charge: when paced on conductor always goes to external surface.
- external electric field: additional charges, positive and negative, will appear on the surfaces. The amount and distributions of such charges will be to ensure $\vec{E} = 0$ inside the conductor and, at the same time to satisfy the conservation of charge.

Potential:

- inside: constant
- surface: constant (same as inside)
- (by convention) zero at infinity or at a grounded conductor

Example: charge q at the center of an uncharged conducting spherical shell with a and b the external and external radii, respectively.

Field:

- $0 < r < a$: $E = kq/r^2$ (unmodified, from symmetry); if the sphere was initially charged with Q, this result would not change
- $a < r < b$: $E = 0$ (as inside any conductor)
- $r > b$: $E = kq/r^2$ (unmodified, from GT and symmetry); if the sphere was initially charged with Q, this result would change to $E = k(q + Q)/r^2$

Charge:

- inner surface: $-q$ (to ensure $E = 0$ inside)
- outer surface: $+q$ (from charge conservation); if the sphere was initially charged with Q, this result would change to $q + Q$.

If the charge q would be not in the center, properties which rely on symmetry would be lost. However, the results which rely only on the GT and charge conservation will hold.

Potential (uncharged sphere):

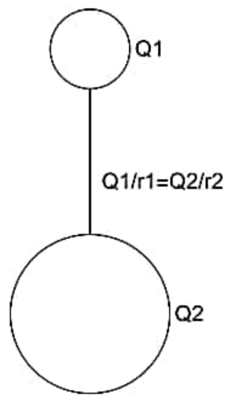
- Outside: field outside is unmodified by the shell, thus the same potential, kq/r with $r > b$
- outer surface: same, with $r = b$ (potential is continuous!)
- inside the body of the metal shell: same constant kq/b
- inside the cavity: same field as from a free charge, thus the potential can differ only by a constant,

$$V(r) = \frac{kq}{r} + const \ , \quad r < a$$

const from the condition

$$V(a) = \frac{kq}{b}$$

which gives $const = kq(1/b - 1/a) < 0$.



Example. Find Q_1 and Q_2

$$Q_1 = Q\frac{r_1}{r_1 + r_2} \ , \ Q_2 = Q\frac{r_2}{r_1 + r_2}$$

Since there is no field inside conductors, the entire conductor has the same potential. When a conductor is brought into electric field, it strongly affects both the field lines and the equipotential surfaces - see Fig. 22

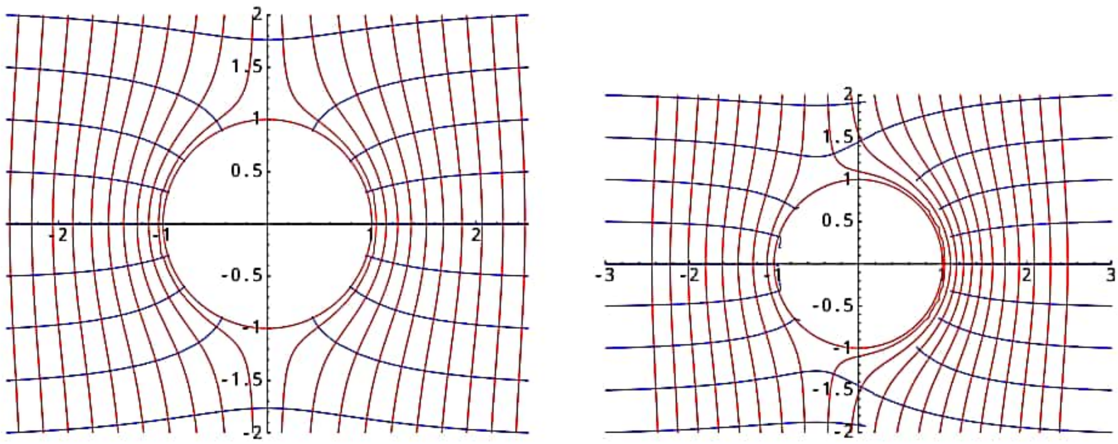
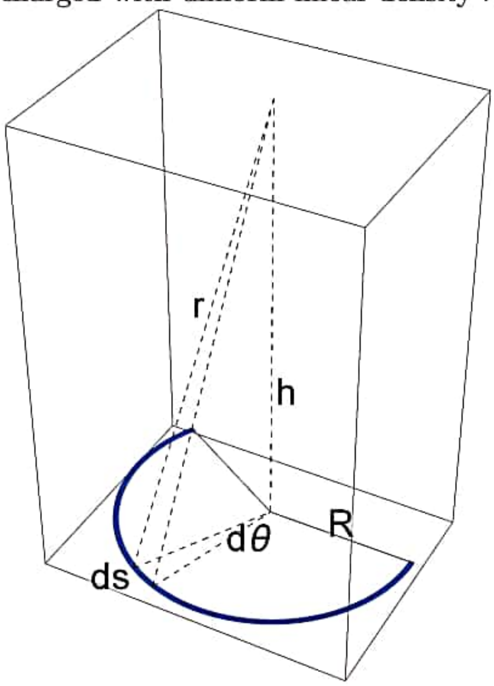


FIG. 22: The structure of equipotential surfaces (red) and electric field lines (blue) around a conducting sphere placed in a uniform electric field. The sphere is uncharged (left figure) or carries and extra charge in the figure on the right. Note that field lines terminate on the surface of a conductor, approaching it at a right angle. The entire conductor has the same potential.

Example. V from a continuous charge distribution. Consider a circular "horseshoe" in the $x - y$ plane, subtended by angle θ with z axis passing through center. The horseshoe is charged with uniform linear density λ. Find V at $z = h$. Discuss limits.



$$\text{formally } dV = k\lambda dx/r = k\lambda R d\theta/r \,,\; V = \int dV = \int_0^\theta d\theta \ldots$$

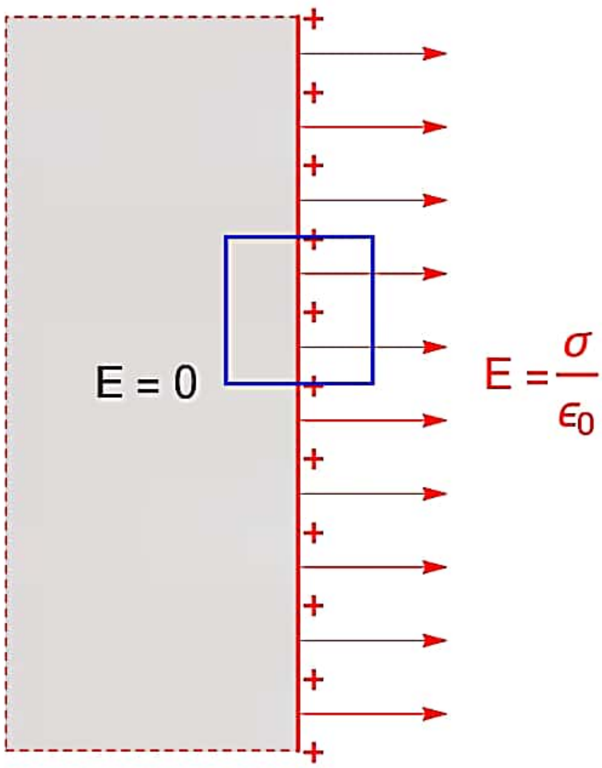
However, since $r = \sqrt{R^2 + h^2} = const$ every element will give same contribution. Thus

$$V = \frac{kQ}{r} \text{ with } Q = \lambda R\theta \ \ (\theta \text{ in radians})$$

For $\theta = 2\pi$ one has $Q = 2\pi R\lambda$ and $V \simeq kQ/h$ for $h \gg R$.

For $h = 0$ (center) $V = kQ/R$.

1. Field near the surface of a conductor



Field is now 'one-sided' (compare with fig. 19 for an insulator). For a coductor, Gauss theorem gives

$$\boxed{E = \frac{\sigma}{\epsilon_0}} \text{ (near conducting surface)} \qquad (39)$$

VII. CAPACITANCE

A. Definitions, units, etc.

1. *Definition*

Consider an isolated conductor. Let us place on it some charge Q. The conductor will acquire some voltage V. By definition, the *capacitance*

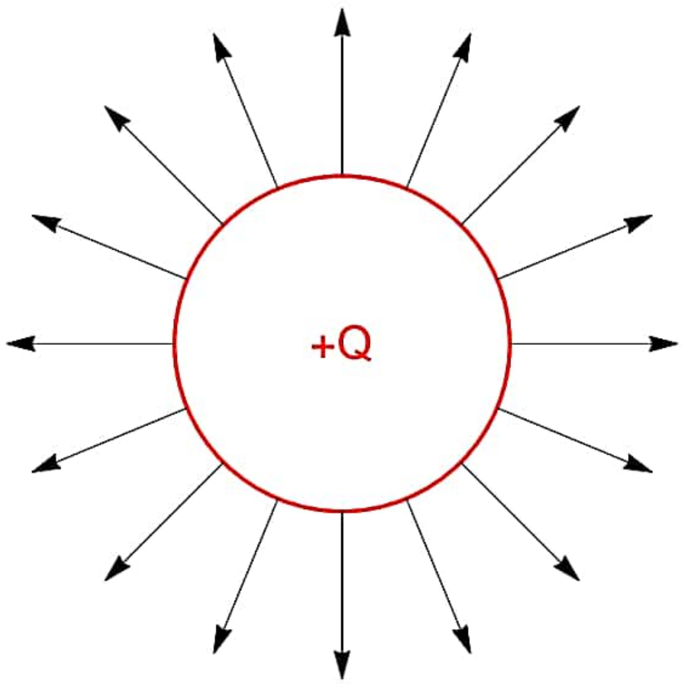
$$C = Q/V \tag{40}$$

In most cases it does not depend on Q or V, but is determined solely by the geometry of the conductor.

Units $[C] = C/V = F$ (farads)

Capacitors. We will see that the capacitance of a single conductor is usually very small. A simple *capacitor* is a combination of two conductors, one charged with a charge $+Q$ and the other $-Q$. The same definition for C is used, only now V is understood as the absolute value of the *difference* of potentials between the conductors. For a clever arrangement, capacitance of a combination can be billion times larger than capacitances of individual conductors.

B. An isolated sphere



Consider a conducting sphere with radius R. To find C we place on it a charge Q and find V.

Outside of the sphere the field is the same as from a point charge at it's center

$$E(r) = \frac{kQ}{r^2}\ , \quad r > R$$

(we had this result before from the Gauss theorem). If the field is the same, so will be the potential

$$V(r) = \frac{kQ}{r}\ , \quad r \geq R$$

The surface, and thus the entire sphere will have a potential

$$V_R = \frac{kQ}{R}$$

One thus has

$$C_R = Q/V = R/k$$

which is more common to write as

$$C_R = 4\pi\epsilon_0 R \tag{41}$$

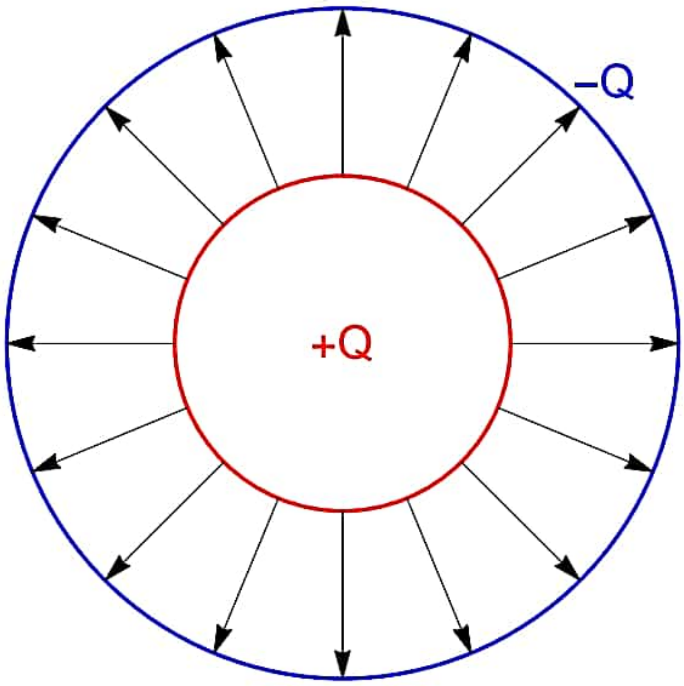
Since ϵ_0 is very small so is the typical capacitance.

Example. Find the capacitance of a conducting sphere of the size of Earth, $R \simeq 6400\,km$.

$$C \simeq 6.4 \cdot 10^6/(9 \cdot 10^9) \approx 0.7\,mF$$

In practice small C means that an attempt to store larger charge will be accompanied by a dangerously high voltage $V = Q/C$.

C. A spherical capacitor



Let us surround the inner sphere (radius R_1) by a larger thin spherical shell with radius R_2. The charge Q is placed on the inner sphere, and the charge $-Q$ on the outer one. The field is now given by

$$E = kQ/r^2 \; , \; R_1 < r < R_2 \;\; \text{and} \;\; E = 0 \; , \; r > R_2$$

Since between the spheres the field is the same as from a single, the potential is almost as before

$$V(r) = kQ/r + const$$

. The *const*, however, does not matter as long as one need the *difference*

$$\Delta V = V_{R_1} - V_{R_2} = kQ\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = kQ\frac{R_2 - R_1}{R_1 R_2}$$

From here one gets

$$C = Q/\Delta V = \frac{1}{k}\frac{R_1 R_2}{R_2 - R_1}$$

which is usually written as

$$C_{sph} = 4\pi\epsilon_0 \frac{R_1 R_2}{R_2 - R_1} \tag{42}$$

Note that for R_2 close to R_1 the capacitance C_{sph} can be very LARGE compared to the one of a single sphere of the same size.

D. Parallel-plate capacitor

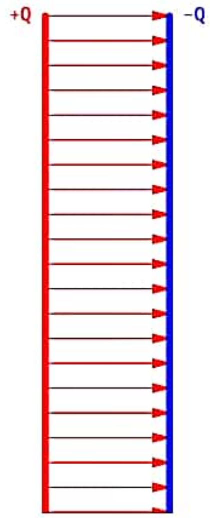


FIG. 23: Structure of electric field inside of a parallel-plate capacitor (edge effects are neglected). Note the following: (a) there is no field outside the capacitor; (b) charges on the plates are always the same in magnitude and opposite in signs (and $+Q$ is "the charge of a capacitor"); (c) field is directed from positive plate to negative, which has a lower potential; the field has a magnitude given by eq. (43) if the space between plates is empty and is *always* related to potential difference by eq. (44).

The field of a single large charged plane is given by

$$\frac{\sigma}{2\epsilon_0}$$

(we had this from the Gauss theorem; $\sigma = Q/A$ is the charge density of the plate with area A). For two plates charged with opposite signs, the field between the planes will double, so that

$$E = \frac{\sigma}{\epsilon_0} = \frac{Q}{A\epsilon_0} \tag{43}$$

Here A is the area of each plate. The field E is uniform, thus

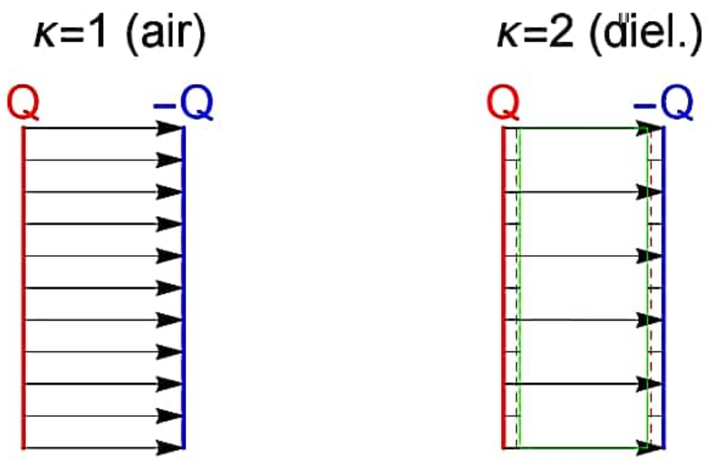
$$|\Delta V| = E \cdot d \tag{44}$$

(d being the distance between the plates). For C one has

$$\boxed{C = Q/|\Delta V| = \frac{A\epsilon_0}{d}} \tag{45}$$

This also can be derived from the formula for a spherical capacitor for $R_2 \approx R_1 = R$ (and with $4\pi R^2$ being the area A) and with a small difference $R_2 - R_1 = d$.

E. Capacitor with a dielectric



Physics of a dielectric is hard and will be discussed separately. At the moment, we just use the formal property that if field E is created by some fixed charges, placing of a dielectric plate perpendicularly to the field will reduce that field inside the plate in accordance with

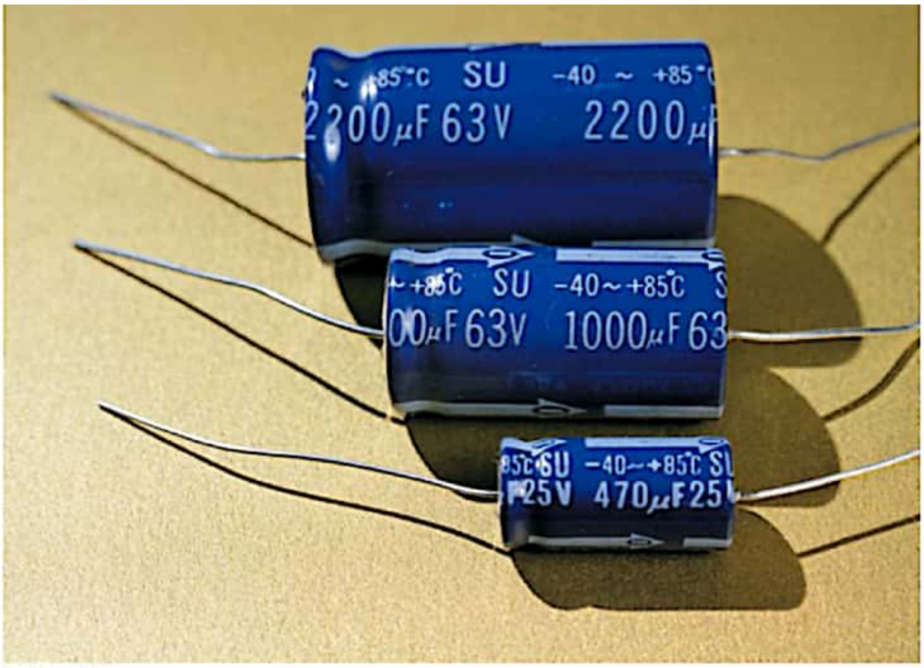
$$E \to E/\kappa$$

$\kappa > 1$ is known as the *dielectric constant*, and is listed in Tables for most of the common insulating materials. Consider now the entire space between the plates of a parallel-plate capacitor filled with a dielectric. The charge Q is fixed. The formula for the voltage difference, $\Delta V = -Ed$, remains the same, but the field is κ times smaller, and so is ΔV. The capacitance, $C = Q/\Delta V$ will be increased

$$C = \frac{A\kappa\epsilon_0}{d} \tag{46}$$

Example Given $C = 1\,pF$, $A = 50\,cm^2$, $\kappa = 2$ find d in mm. Solution:

$$C = \frac{A\epsilon_0\kappa}{d} \Rightarrow d = \frac{A\epsilon_0\kappa}{C} = \frac{50 \cdot 10^{-4} \times 8.85 \cdot 10^{-12} \times 2}{1 \cdot 10^{-12}} = \ldots \times 1000\,mm$$


~ +85°C SU -40 ~ +85°C
2200μF 63V 2200μF
~+85°C SU -40~+85°C
00μF 63V 1000μF 63
85°C SU -40~+85°C SU
F25V 470μF25

F. Capacitor and a battery

Schematics is shown in Fig. 24. Note that as long as the capacitor is connected to the

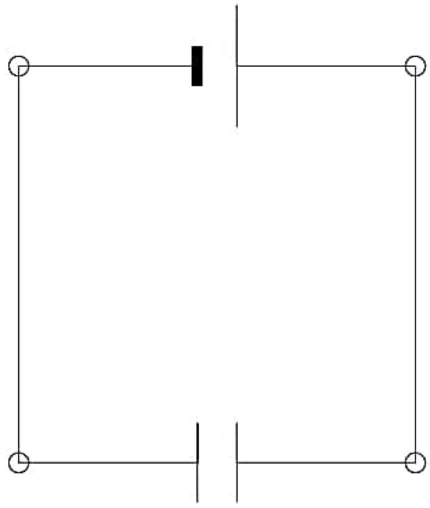


FIG. 24: Schematic representation of a capacitor and a battery. The short fat terminal is negative. The capacitor acquires the same voltage V as between the terminals of the battery. The charge taken by the capacitor is $Q = CV$. If disconnected from the battery, the capacitor will keep this charge.

battery it has the same voltage V, regardless of other things (e.g. if we move the plates apart or if a we insert a dielectric). If C is changed, extra charge will be supplied (or taken away) by the battery in order to keep the same V. Once disconnected, the capacitor will keep the charge. Voltage can change if C is changed.

G. Energy

Let q the the current value of the charge (which is smaller than Q). Bringing an extra charge dq from the negative plate to the positive plate requires work dW with

$$dW = V(q)dq = \frac{1}{C}q \cdot dq \, ; \; \Rightarrow \; W = \frac{1}{C}\int_0^Q q \cdot dq = \frac{1}{C}\frac{Q^2}{2}$$

The total work is the energy U_C stored in the capacitor, i.e.

$$\boxed{U_C = \frac{Q^2}{2C} = \frac{1}{2}V^2C} \tag{47}$$

Example. Given $A = 50\,cm^2$, $\kappa = 2$, $d = 0.3\,mm$, $Q = 1\,nC$. Find U. Solution:

$$C = \frac{A\epsilon_0\kappa}{d} = \frac{50 \cdot 10^{-4} \times 8.85 \cdot 10^{-12} \times 2}{0.3 \cdot 10^{-3}} = \ldots, \; U = \frac{Q^2}{2C} = \ldots$$

Question. A capacitor is connected to a battery. What happens to the energy when a dielectric plate is inserted? Where does the energy come from?

Question. A capacitor is charged and disconnected from a battery. What happens to the energy when a dielectric plate is inserted?

H. Connections of several capacitors

1. *Parallel*

See Fig. 25 (left).

Same: Voltage

Add up: charges

$$Q = q_1 + q_2 + \ldots = C_1 V + C_2 V + \ldots = V\,(C_1 + C_2 + \ldots)$$

Thus,

$$C_{eq} = C_1 + C_2 + \ldots \tag{48}$$

2. *Series*

Fig. 25 (right).

Same - charge.

Add up - voltages.

$$V = V_1 + V_2 + \ldots = \frac{q}{C_1} + \frac{q}{C_2} + \ldots = q\left(\frac{1}{C_1} + \frac{1}{C_2} + \ldots\right)$$

Or,

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \ldots \tag{49}$$

In many cases, more complex circuits can be analyzed using the so-called reduction method. Examples will be given in class.

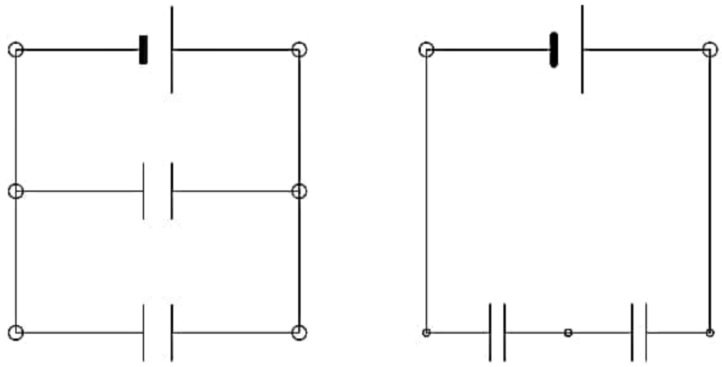
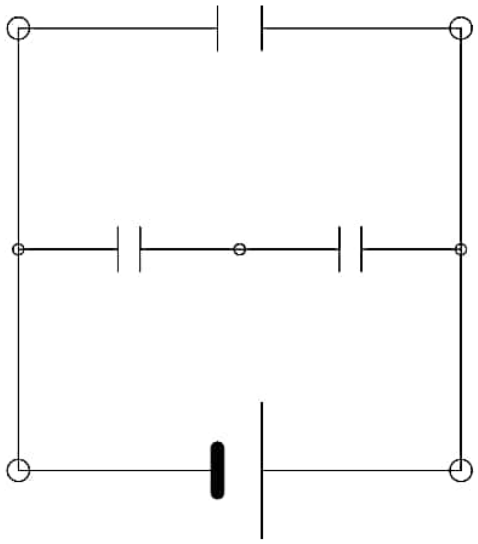


FIG. 25: Parallel (left) and series (right) connections. If each pair of the capacitors will be replaced by a single equivalent capacitance, as in Fig. 24, with the values of C_{eq} given by eq. (48) or eq. (49), the battery will "not know" about the change. In particular, the total charge taken from the battery will be $Q = C_{eq}V$, the energy stored will be $(1/2)V^2C_{eq}$, etc.

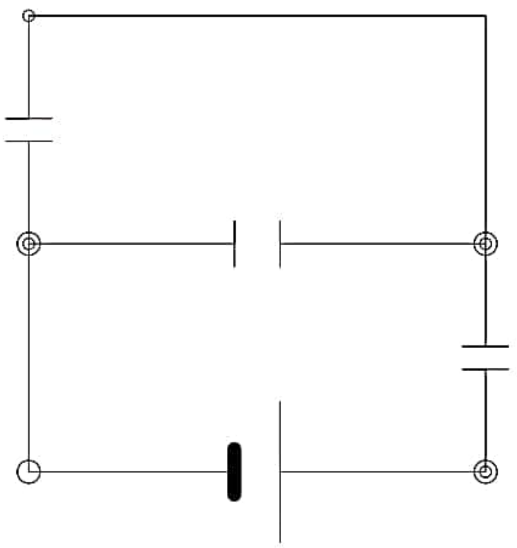
Example. All C's equal $1\,\mu F$. Find C_{eq}



Solution. Let C_1 - upper; C_2 and C_3 - middle.

$$C_{23} = \frac{C_2C_3}{C_2 + C_3} = \frac{C}{2}, \; C_{eq} = C_{23} + C_1 = \frac{3}{2}\mu F$$

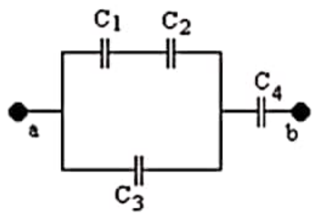
Example. All C's equal $1\,\mu F$. Find C_{eq}



Solution. Let C_1 - upper; C_2 and C_3 - middle and lower right.

$$C_{12} = C_1 + C_2 = 2C\ ,\ C_{eq} = \frac{C_{12}C_3}{C_{12}+C_3} = \frac{2\cdot 1}{3}\,\mu F$$

Example. All C's equal $1\,\mu F$ and $V_{ab} = 1\,V$. Find C_{eq} , all voltages and charges



Solution.

$$C_{12} = \frac{C_1C_2}{C_1+C_2} = 0.5\,\mu F\ ;\ C_{123} = C_{12} + C_3 = 1.5\,\mu F$$

$$C_{eq} = \frac{C_{123}C_4}{C_{123}+C_4} = 1.5/2.5 = 0.6\,\mu F\ ;\ Q = VC_{eq}$$

$$q_4 = Q \quad \text{since in series with "battery"}; V_4 = q_4/C_4$$

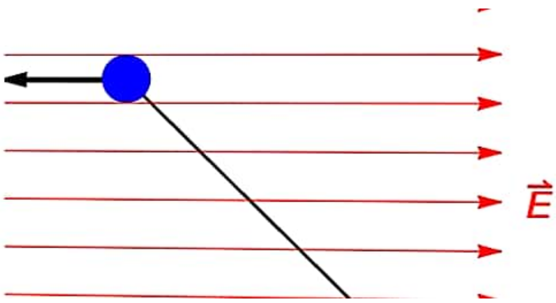
On the 1-2-3 combination

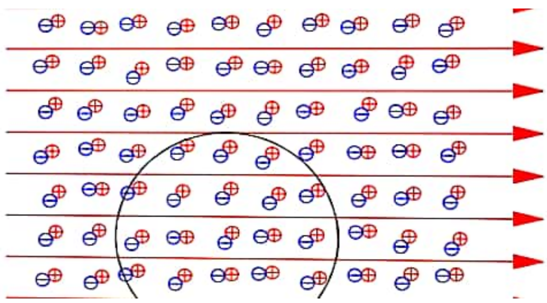
$$q_{123} = Q \quad \text{since in series with "battery"}; V_{123} = q_{123}/C_{123}$$

(equivalently, $V_{123} = V_{ab} - V_4$)

$$V_3 = V_{123}\ ,\ q_3 = V_3C_3$$

$$q_1 = q_2 = V_{123}C_{12}\ ,\ V_2 = \frac{q_1}{C_1}\ ,\ V_2 = \frac{q_2}{V_2}$$


E



VIII. CURRENT

A. Definitions and units

Current:

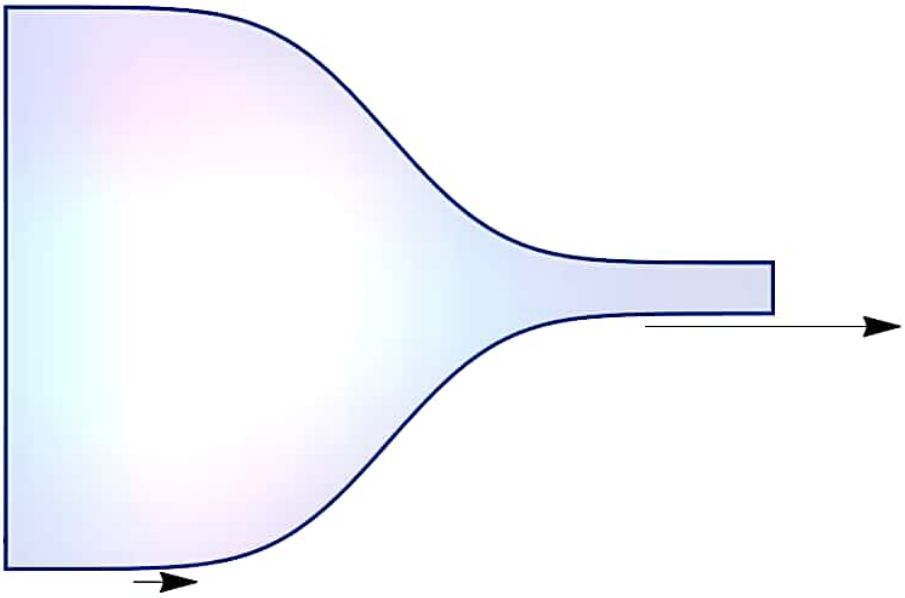
$$i = \frac{dq}{dt} \simeq \frac{\Delta q}{\Delta t} \tag{50}$$

Units: $[i] = \mathcal{A} = C/s$ (amperes)

Density of current:

$$J = i/A \tag{51}$$

Units: $[J] = \mathcal{A}/m^2$



J - vector, i -scalar; $i_1 = i_2$ and if $A_1 > A_2$ then $J_1 < J_2$

Major dependence ("Ohm's rule"):

$$i = V/R \tag{52}$$

R-resistance; $[R] = \Omega = V/\mathcal{A}$ (ohms)


T, Kelvin

D. Power

Units: $[P] = J/s = W$ (watts)

1. Single resistor

$$P = iV = \frac{V^2}{R} = i^2 R \tag{54}$$

2. Simple connections

In parallel:

$$P = P_1 + P_2$$

If $R_1 > R_2$,

$$P_1 = \frac{V^2}{R_1} < P_2 = \frac{V^2}{R_2}$$

In series:

$$P = P_1 + P_2$$

If $R_1 > R_2$,

$$P_1 = i^2 R_1 > P_2 = i^2 R_2$$

Example. How many meters of wire ($d = 2\,mm$, $\rho = 10^{-6}\,ohm - m$) is needed to construct a 100 volt, 3.1 kW heater?

$$P = V^2/R \Rightarrow R = V^2/P = \ldots$$
$$R = \rho L/A \Rightarrow L = RA/\rho = \frac{V^2}{P}\frac{\pi d^2/4}{\rho} = \frac{100^2}{3100}\frac{\pi 10^{-6}}{10^{-6}} = \ldots$$

E. Series and parallel connections

See Fig. 27.

Parallel:

$$\frac{1}{R_e} = \frac{1}{R_1} + \frac{1}{R_2} + \ldots \tag{55}$$

Series:

$$R_e = R_1 + R_2 + \dots \tag{56}$$

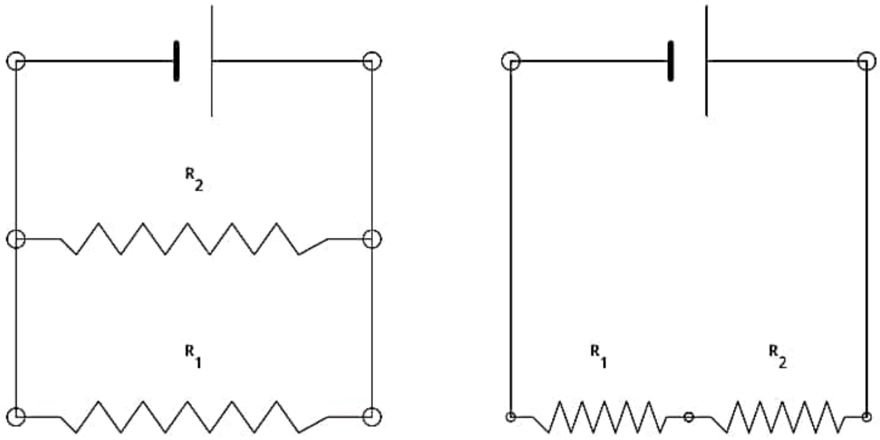


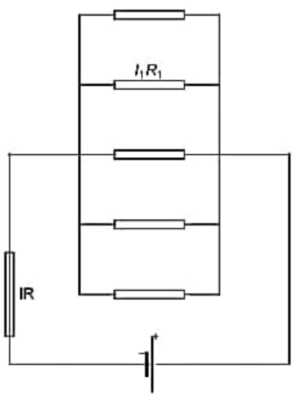
FIG. 27: The parallel (left) and the series (right) connections. See eqs. (55) and (56), respectively.

A large number (N) of identical resistors R.

$$\text{series:}\ \ R_{eq} = NR$$

$$\text{parallel:}\ \ R_{eq} = R/N$$

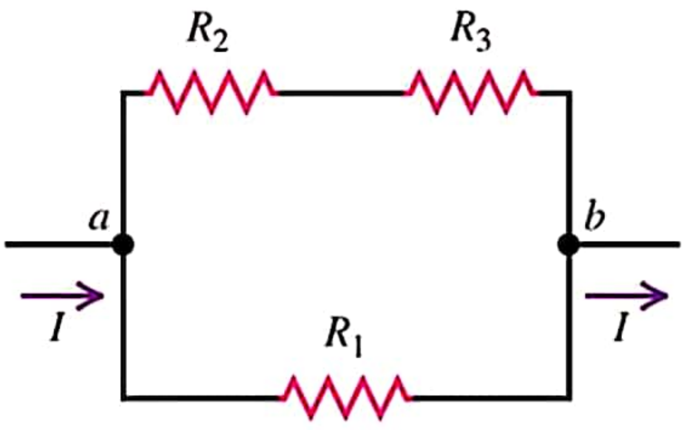
Example: all resistors are identical, $R_1 = \dots = R$ and the voltage on the battery is V. Find R_{eq} and currents.



$$R_{eq} = \frac{R}{5} + R = \frac{6}{5}R\,,\ I = \frac{V}{R_{eq}} = \frac{5V}{6R}\,,\ I_1 = \dots = \frac{I}{5} = \frac{V}{6R}$$

Example. Compare powers released on R_1, R_2, R_3 if $R_1 = R_2 = R_3 = R$.

(d) R_1 in parallel with series combination of R_2 and R_3

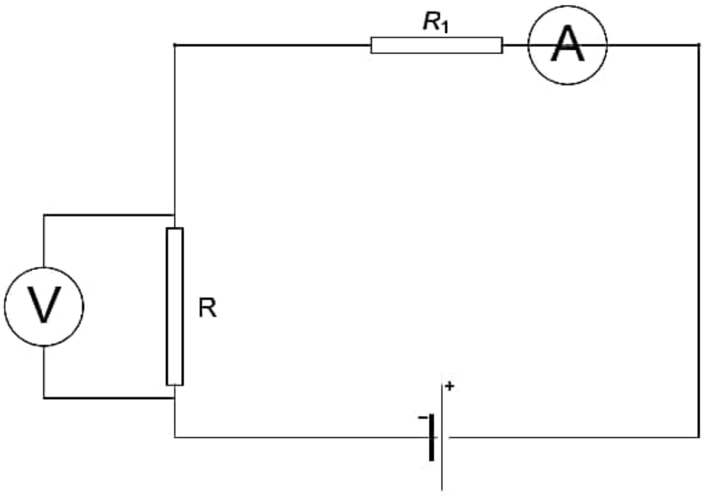


Assume the leads from points **a** and **b** go directly to the terminals of a battery V. Then

$$I_1 = \frac{V}{R_1}, \; I_2 = I_3 = \frac{V}{R_2 + R_3} = I_1/2$$

$$P_1 = I_1^2 R_1, \; P_2 = I_2^2 R_2 = P_1/4, \ldots$$

F. Ampmeter and voltmeter



ampmeter - in series with resistor, $R_A \approx 0$ ("ideal ampmeter", $R_1 + R_A \approx R_1$)

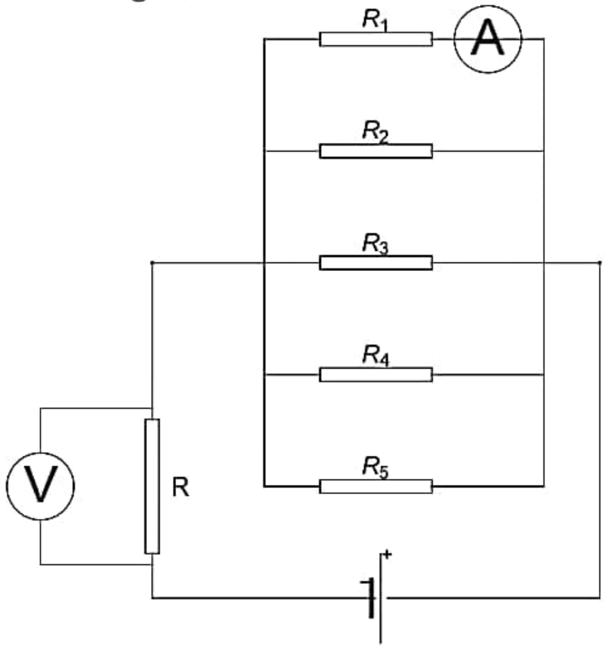
voltmeter - in parallel with resistor, $R_V \approx \infty$ ("ideal voltmeter", $RR_V/(R + R_V) \approx R$)

Consider a resistor R_1 in parallel with resistor R_2, with currents I_1 and I_2 and $I = I_1 + I_2$ the total current. Since voltage is the same

$$IR_{eq} = I_1 R_1 = I_2 R_2 \text{ or } \frac{I_1}{I_2} = \frac{R_2}{R_1}$$

If $R_1 \approx 0$ ("wire"), $I_2 \approx 0$ and $I_1 \approx I$ ("path of smallest resistance")

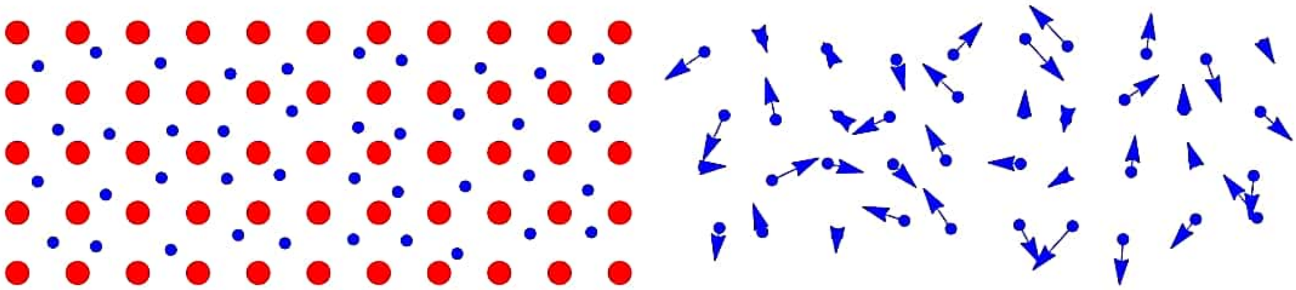
Example. $R = R_1 = 1\,\Omega$ and $R_2 = 2$, $R_3 = 3$, $R_4 = 4$, $R_5 = 5\,\Omega$. If $I_1 = 1\,A$. What is the reading of the voltmeter?



$$I_2 = I_1 \frac{R_1}{R_2} = \frac{1}{2} A,\ I_3 = I_1 \frac{R_1}{R_3} = \frac{1}{3} A,\ I_4 = I_1 \frac{R_1}{R_4} = \frac{1}{4} A,\ I_5 = I_1 \frac{R_1}{R_5} = \frac{1}{5} A$$

$$I = I_1 + I_2 + \ldots + I_5 = \frac{137}{60} A,\ V = IR = 1 \cdot \frac{137}{60} \simeq 2.28\,V$$

G. Microscopic picture of conductivity



Let n be the density of electrons (or, other charge carriers), v_d - their drift velocity. Then, density of current

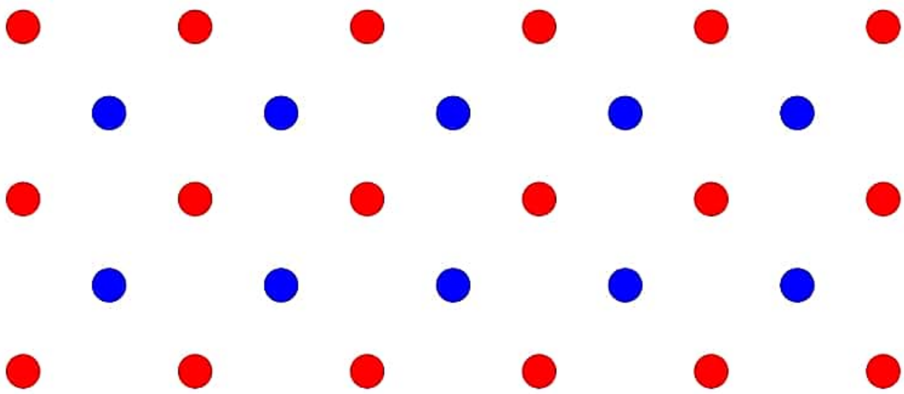
$$J = env_d \tag{57}$$

If n is very large (as in metals), v_d is rather small, a few mm/s.

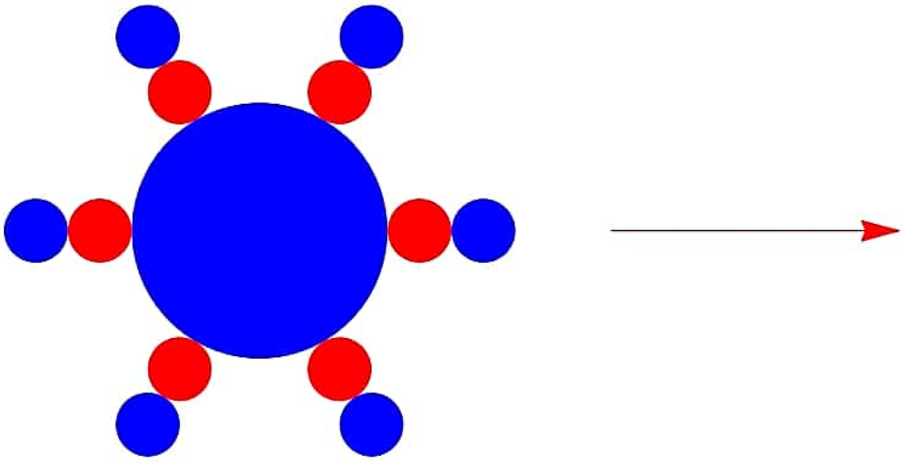
Example. For $n \sim 10^{29}\, m^{-3}$ and $J \sim 10^6\, \mathcal{A}/m^2$ estimate v_d . Solution.

$$J = |e|nv_d \;\Rightarrow\; v_d = J/(|e|n) \sim \frac{10^6}{10^{-19} \times 10^{29}} \sim 10^{-4}\,\frac{m}{s}$$

H. Dielectric



I. Liquids (electrolytes)



IX. CIRCUITS

A. The reduction method

See the example in Fig. 28, with $R_{14} = R_1 + R_4$ and $R_{23} = R_2 R_3 / (R_2 + R_3)$.

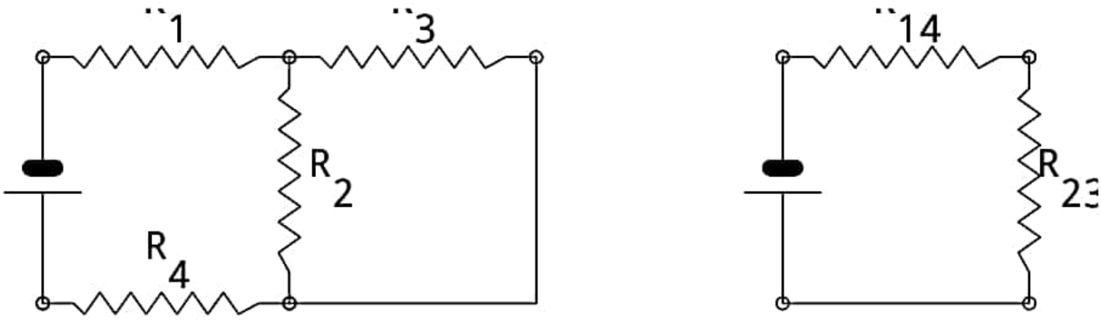
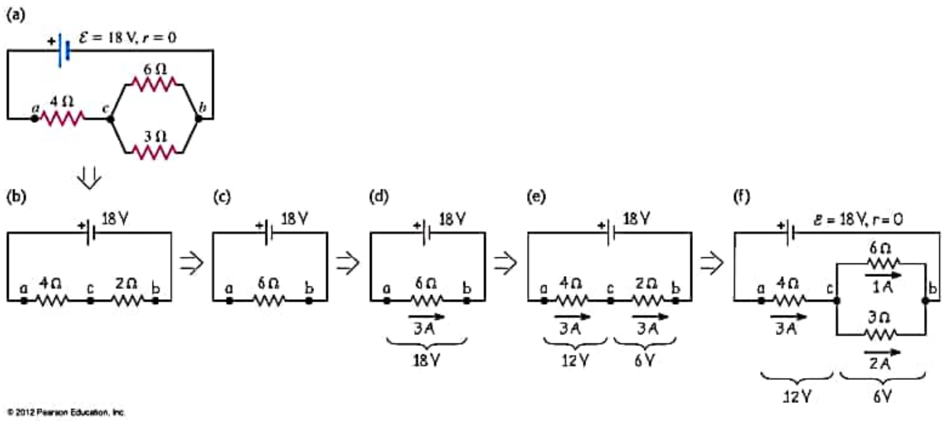


FIG. 28: Example of the reduction method



Not all circuits can be solved by the reduction method - see Fig. 29.

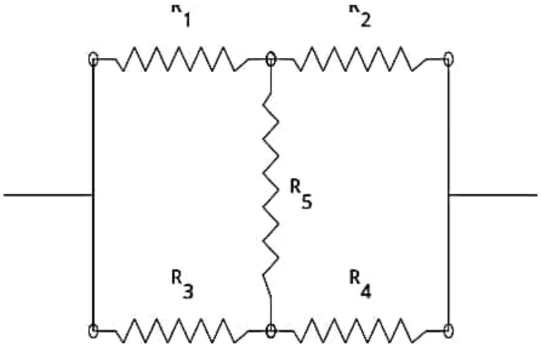


FIG. 29: Irreducible circuit

B. The real battery

See Fig. 30

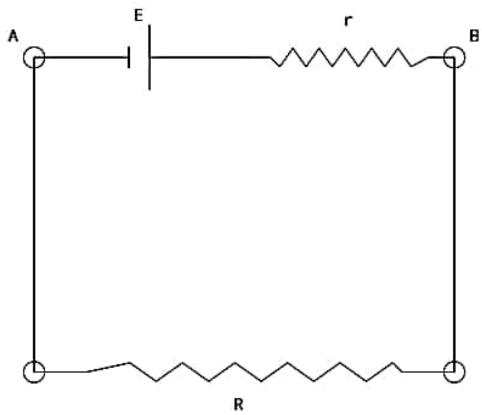


FIG. 30: The real battery. R is the external load. The voltage on the terminals A and B is smaller than $\mathcal{E}$ - see text.

Voltage on the terminals

$$V_{AB} = iR = \frac{\mathcal{E}}{r+R}R = \mathcal{E}\frac{R}{R+r} < \mathcal{E}$$

Note that $V_{AB} = \mathcal{E}$ only for $R \to \infty$, i.e for an open circuit.

C. The potential method

See Fig. 31. One has

$$V_B = V_A - I_1R_1 + \mathcal{E}_1 + I_2R_2 - \mathcal{E}_2$$

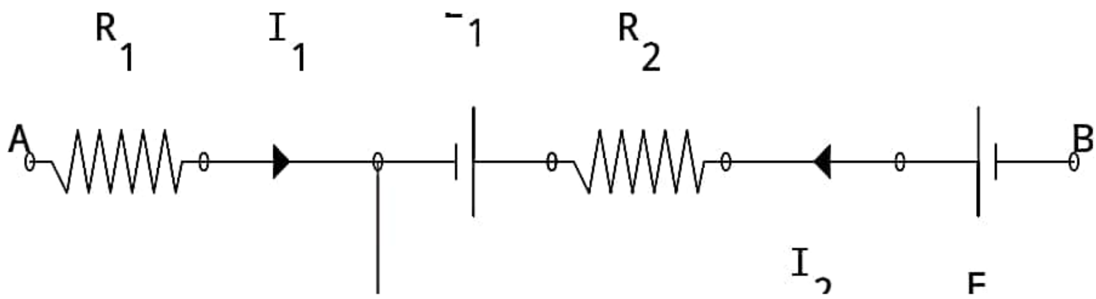
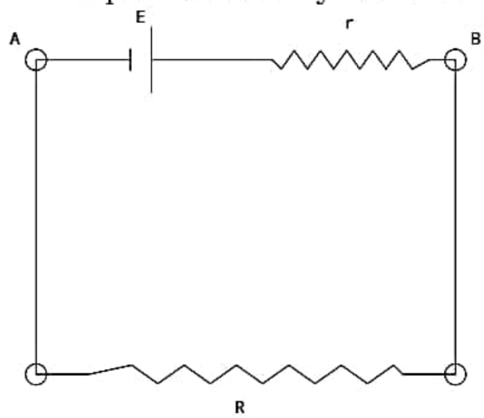


FIG. 31: The potential method

For a closed loop, $V_B = V_A$ and the total potential drop is zero. This is the *loop rule*.

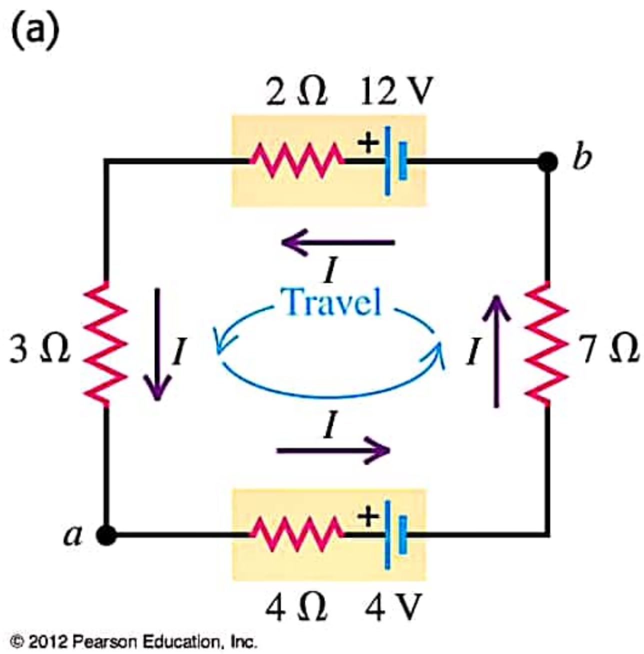
Example. Real battery revisited.



The real battery. R is the external load, r is the internal resistance. Find the voltage between the terminals A and B.

$$\text{start from A: } \mathcal{E} - Ir - IR = 0 \Rightarrow I = \frac{\mathcal{E}}{R+r}, \; V_{AB} = V_R = IR = \mathcal{E}\frac{R}{R+r}$$

$$\text{equivalently: } V_B = V_A + \mathcal{E} - Ir = V_A + \mathcal{E}\frac{R}{R+r}$$

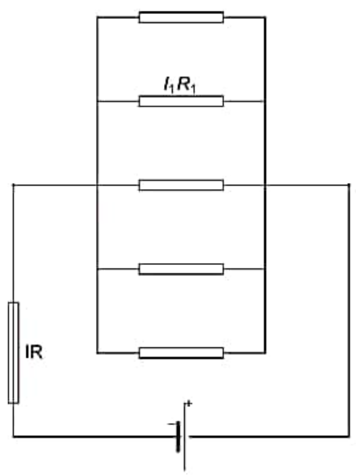


Start from (a), CCW:

$$-4I - 4 - 7I + 12 - 2I - 3I = 0\,,\; I = \frac{8}{16} = \frac{1}{2}A$$

$$P_{12} = \frac{1}{2} \cdot 12 = 6\,W\,,\; P_4 = -\frac{1}{2} \cdot 4 = -2\,W$$

Example. Isolated loop.



CCW current and loop with R_1, R and E: $+E - I_1R_1 - IR = 0$

D. Multiloop circuits and the Kirchoff's equations

See Fig. 32.

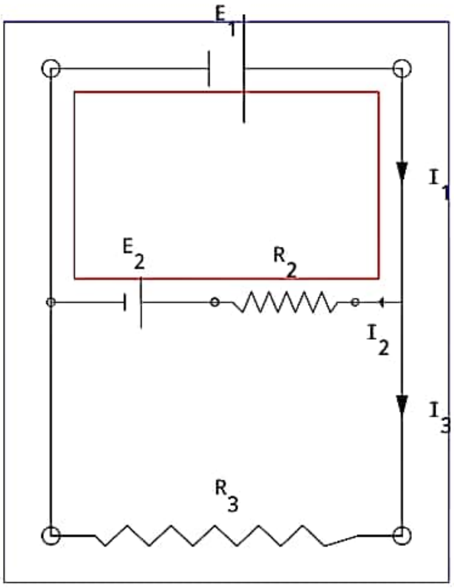


FIG. 32: Example of a two-loop circuit

Red loop:

$$\mathcal{E}_1 - I_2 R_2 - \mathcal{E}_2 = 0$$

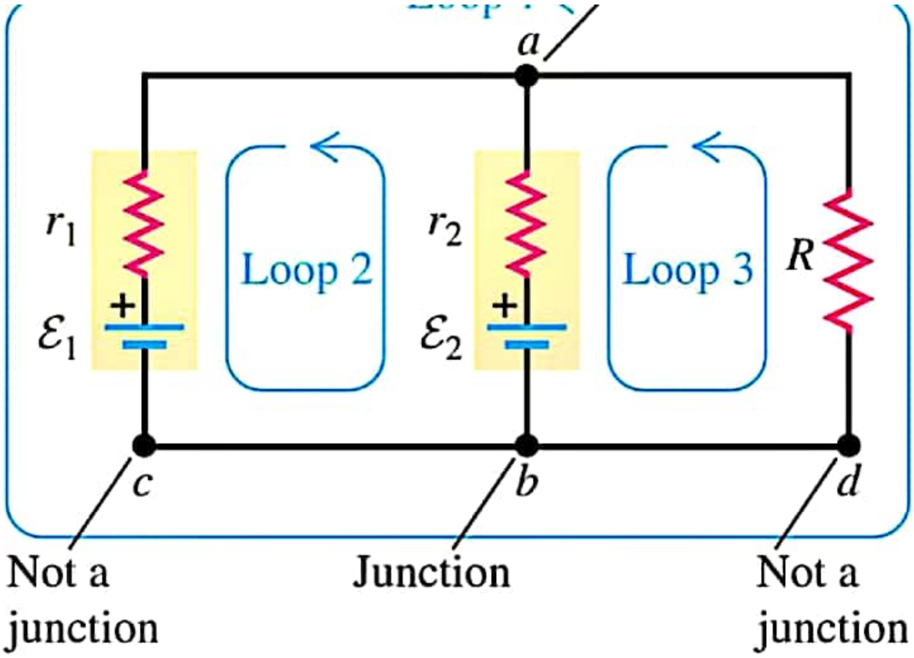
Blue loop:

$$\mathcal{E}_1 - I_3 R_3 = 0$$

(Note, always try to select a loop with a single resistor, even if with many batteries).

Junction rule:

$$I_1 = I_2 + I_3$$



Assume currents i_1, i_2, i, all up.

Loop 1 (start from c:

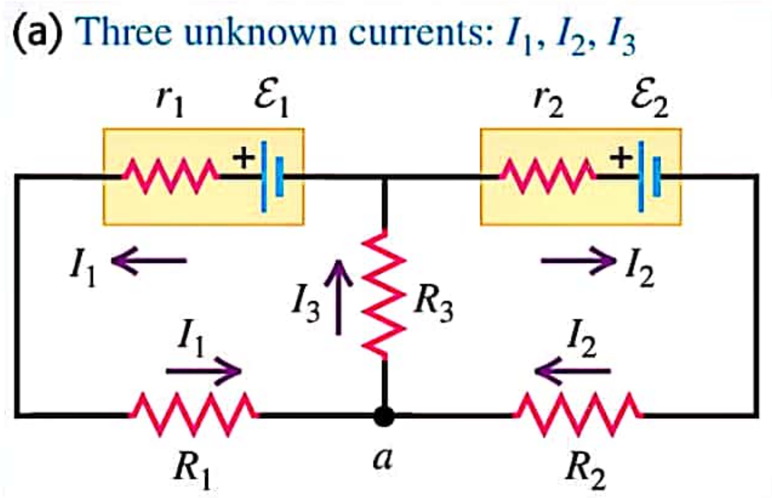
$$-iR + i_1 r_1 - \mathcal{E}_1 = 0$$

Loop 2:

$$+\mathcal{E}_2 - i_2 r_2 + i_1 r_1 - \mathcal{E}_1 = 0$$

Junction a:

$$i_1 + i_2 + i = 0$$



left loop, CCW

$$-I_3R_3 + E_1 - I_1(r_1 + R_1) = 0$$

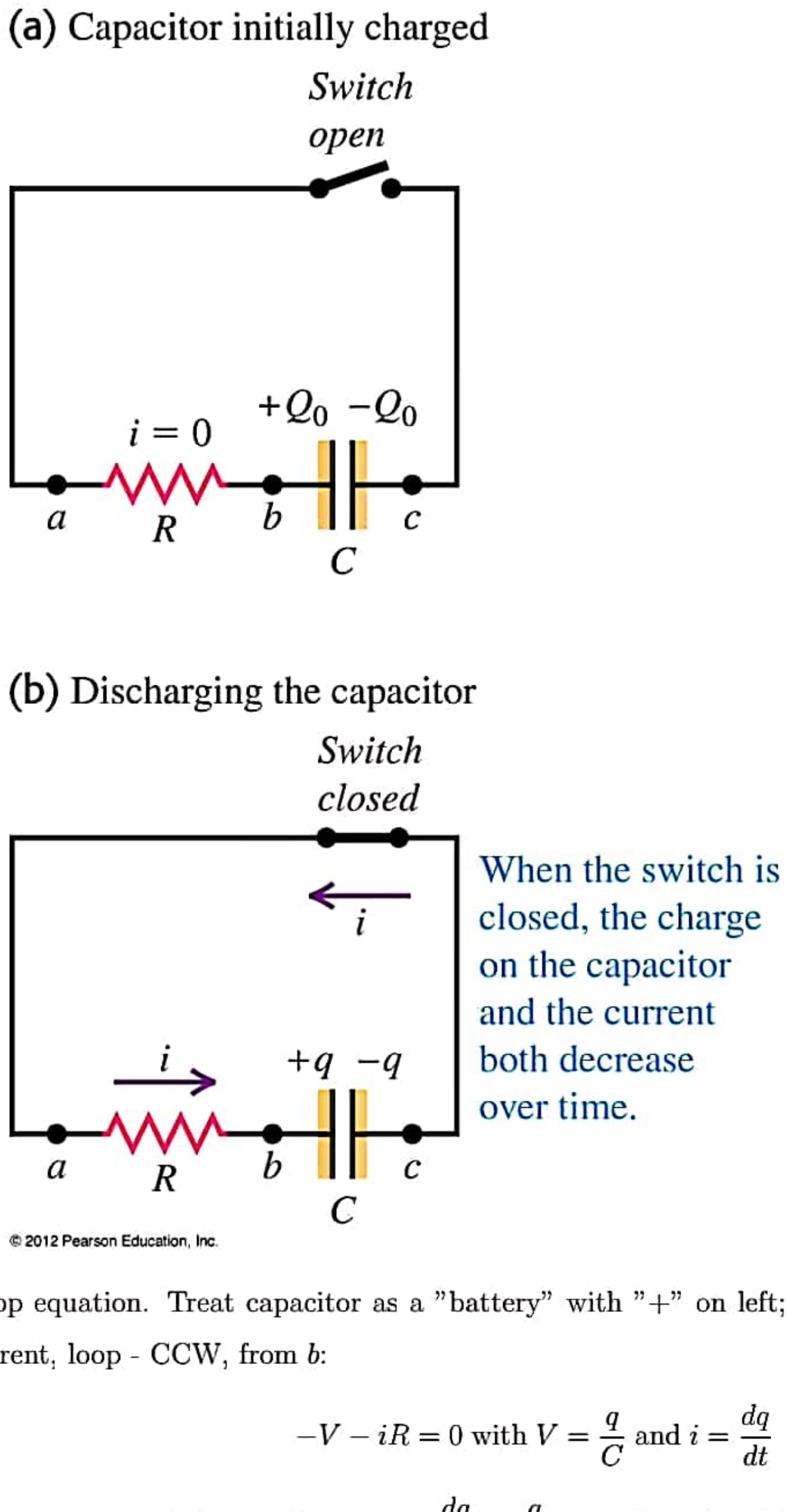
right loop, CCW

$$+I_2R_2 + E_2 - I_2r_2 + I_3R_3 = 0$$

Junction a:

$$I_1 + I_2 = I_3$$

E. ***RC* circuits**



Loop equation. Treat capacitor as a "battery" with "+" on left; assume counterclockwise current, loop - CCW, from b:

$$-V - iR = 0 \text{ with } V = \frac{q}{C} \text{ and } i = \frac{dq}{dt}$$

$$\text{differential equation: } \frac{dq}{dt} + \frac{q}{RC} = 0\,, \text{ look for } q(t) = Q_0 e^{\lambda t}$$

$$\text{differential equation: } \frac{dq}{dt} + \frac{q}{RC} = 0\,, \text{ look for } q(t) = Q_0 e^{\lambda t} \tag{58}$$

$$\frac{dq}{dt} = \lambda Q_0 e^{\lambda t} = \lambda q(t) \ \Rightarrow\ \lambda q(t) + \frac{q(t)}{RC} = 0 \ \Rightarrow\ \lambda = -\frac{1}{RC}$$

$$\boxed{q(t) = Q_0 e^{-t/\tau}\,, \text{ with } \tau = RC} \tag{59}$$

$$\text{Current: } i(t) = \frac{dq}{dt} = -\frac{Q_0}{RC} e^{-t/\tau}\,; \text{ voltage: } V(t) = \frac{q(t)}{C} = V_0 e^{-t/\tau}\,,\ V_0 = \frac{Q_0}{C} \tag{60}$$

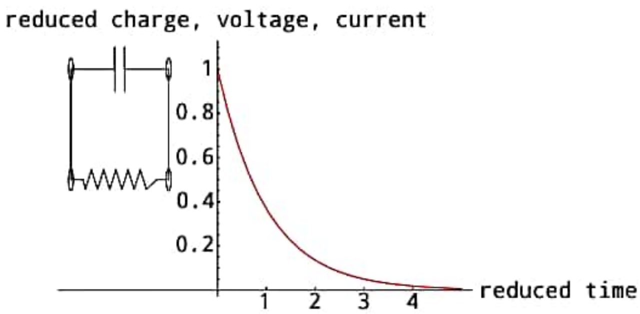
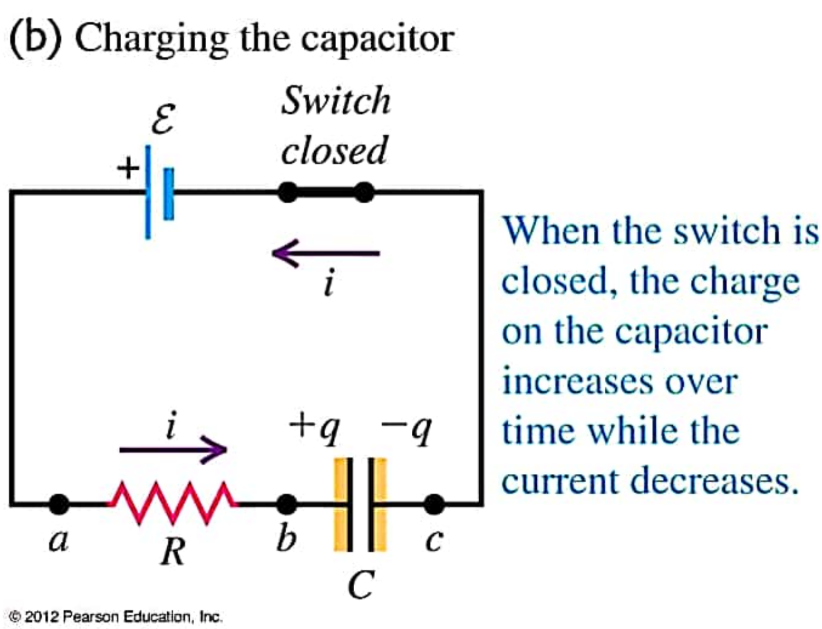


FIG. 33: Discharging a capacitor. Reduced time is t/τ, with $\tau = RC$. Reduced charge, voltage, current are q/Q_0, V/V_0 or i/i_0, respectively, with $V_0 = Q_0/C$ and $i_0 = -V_0/R$.



Treat capacitor as another "battery" with "+" on left. Loop:

$$\mathcal{E} - V - iR = 0 \text{ with } V = \frac{q}{C},\ i = \frac{dq}{dt} \quad \Rightarrow$$

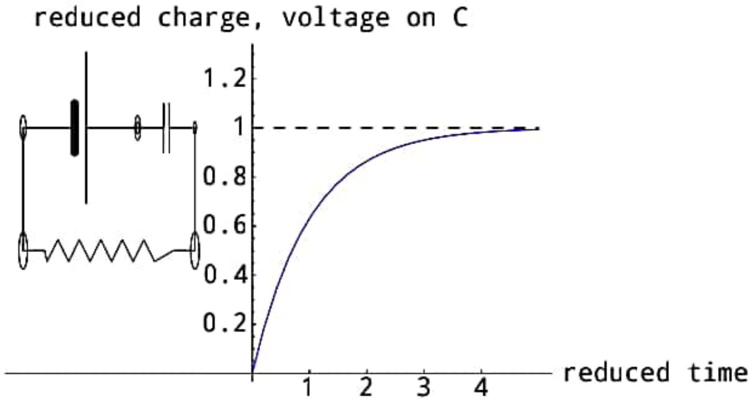
$$\text{differential equation: } \frac{dq}{dt} + \frac{q}{RC} = \frac{\mathcal{E}}{R}$$

$$\text{new variable: } \tilde{q}(t) = q(t) - \mathcal{E}C \text{ with } \tilde{q}(0) = -\mathcal{E}C$$

$$\frac{d\tilde{q}}{dt} + \frac{\tilde{q}}{RC} = 0 \ \Rightarrow\ \tilde{q}(t) = \tilde{q}(0)e^{-t/\tau} = -\mathcal{E}Ce^{-t/\tau} \text{ and } q(t) = \mathcal{E}C + \tilde{q}(t) \text{ or}$$

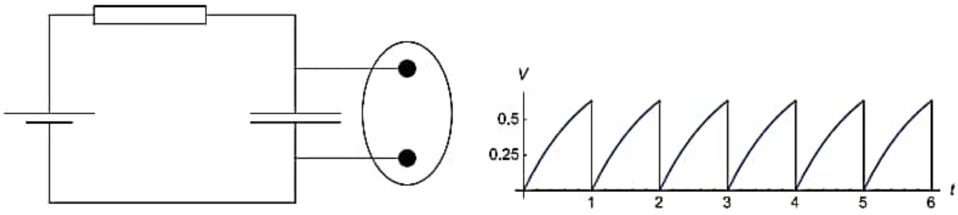
$$\boxed{q(t) = \mathcal{E}C\left(1 - e^{-t/\tau}\right)},\quad \text{same } \tau = RC \tag{61}$$

$$V_C(t) = \frac{q}{C} = \mathcal{E}\left(1 - e^{-t/\tau}\right),\ i(t) = \frac{dq}{dt} = i_{\max}e^{-t/\tau},\ i_{\max} = \frac{\mathcal{E}}{R} \tag{62}$$



Charging a capacitor. Reduced time is t/τ; reduced charge and voltage are $q/\mathcal{E}C$ and $V/\mathcal{E}$, respectively. Current follows a decay exponential, as in Fig. 33.

Applications (e.g. "timer") will be discussed in class.



Examples RC - time-dependent

For an RC charging circuit, how many time constants elapse for the capacitor to charge up to 95% of its final value?

$$Q = Q_{\max}\left(1 - e^{-t/\tau}\right)\ ,\ Q = 0.95 Q_{\max}$$

$$e^{-t/\tau} = 1 - 0.95 = 0.05\ ,\ \frac{t}{\tau} = -\ln 0.05 \approx 3$$

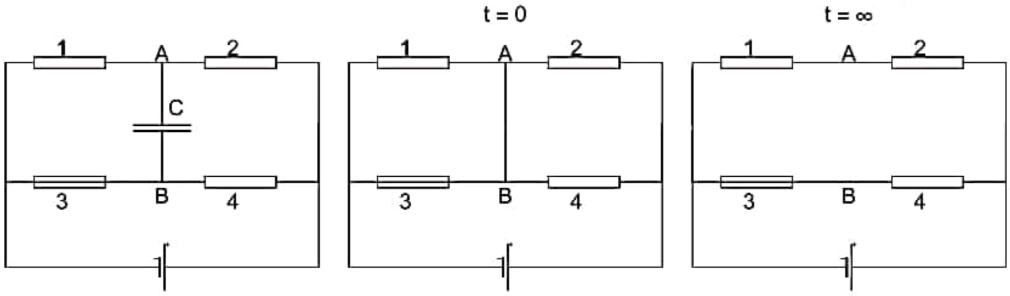
A $C = 0.1\,\mu F$ capacitor is connected is series with an $R = 400\,k\Omega$ resistor, and this combination is connected across an ideal $V = 12$ volt battery. What is the current in the circuit when the capacitor has reached 40% of its maximum charge?

$$Q_{\max} = VC\,,\ Q(t) = Q_{\max}\left(1 - e^{-t/\tau}\right)\ ,\ Q = 0.4 Q_{\max}$$

$$e^{-t/\tau} = 1 - 0.4 = 0.6\ ,\ i(t) = \frac{dQ}{dt} = \frac{Q_{\max}}{\tau} e^{-t/\tau} = \frac{V}{R} \times 0.6 = \ldots$$

Resistors-Capacitors:

Example. Find all currents at $t = 0^+$ and at $t \to \infty$. Find $Q_C(\infty)$.



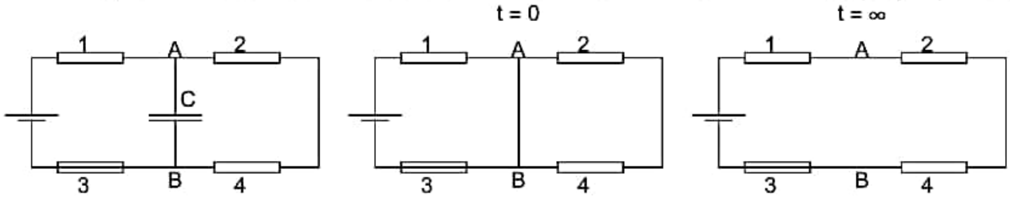
$$t = 0: \; R_{13} = \frac{1 \cdot 3}{1+3}\,; \; R_{24} = \frac{2 \cdot 4}{2+4}\,; R_e = R_{13} + R_{24}\,, \; I = \frac{E}{R_e}$$

$$V_{13} = IR_{13}\,, \; I_1 = V_{13}/R_1\,, \; I_3 = V_{13}/R_3\,, \; \ldots$$

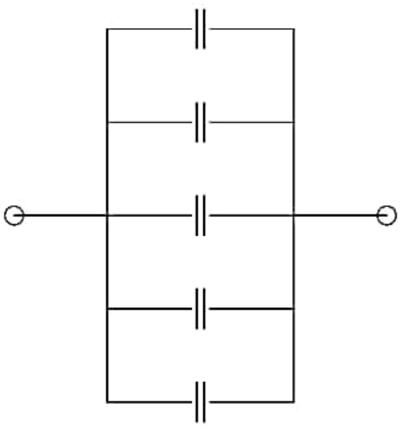
$$t \to \infty: \; R_e = \frac{(1+2)(3+4)}{(1+2)+(3+4)}\,; \; I_1 = I_2 = \frac{E}{1+2}\,; \; I_3 = I_4 = \frac{E}{3+4}$$

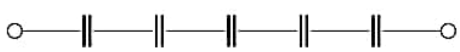
$$V_A = 0 + I_1 \cdot 1\,; \; V_B = 0 + I_3 \cdot 3 \;\Rightarrow\; V_{AB} = I_1 \cdot 1 - I_3 \cdot 3\,, \; Q = CV_{AB}$$

Example. Find all currents at $t = 0^+$ and at $t \to \infty$. Find $Q_C(\infty)$. (in class)



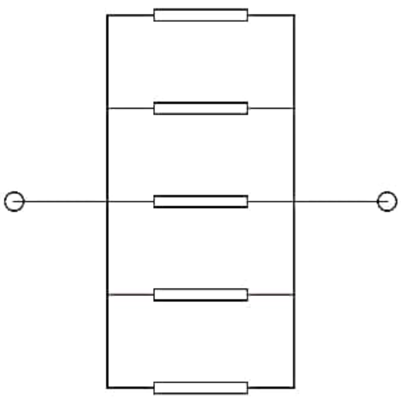
Identical C:





Parallel: $C_{eq} = n\,C$. Series: $C_{eq} = C/n$

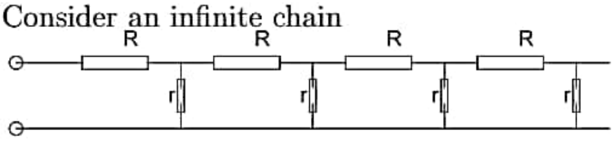
Identical R:



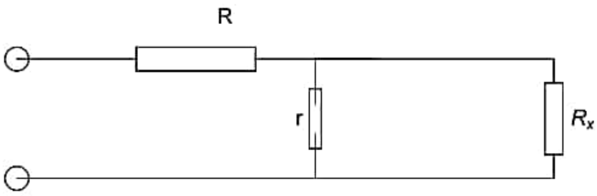
Parallel: $R_{eq} = R/n$. Series: $R_{eq} = Rn$

Advanced. Transmission lines.

Consider an infinite chain



Idea: remove one repeating link and nothing changes! ("Hilbert Hotel"):



The resistance between the terminals is the same R_x. Thus get an equation

$$R_x = R + \frac{rR_x}{r + R_x}$$

with one positive root

$$R_x = \frac{1}{2}R + \sqrt{R(r + R/4)}$$

Approximate solutions. One can just truncate the chain after some finite number of links, n. E.g., if only 1 link (2 resistors)

$$R_x^{(1)} = R + r$$

If 2 links (4 resistors)

$$R^{(2)} = R + \frac{r(R+r)}{r + (R+r)}$$