UNIT- I

Elementary point set topology :

i) Sets and Elements :

A collection of well defined object is called a set

Ex : $A = \{a, e, i, o, u\}$ (1)

$B = \{A, B \ldots\ldots z\}$

ii) Equal :

Two sets are equal if they have the same element

$X = \{1, 2, 3, 4, 5\}$

$Y = \{5, 4, 1, 2, 3\}$

iii) Subset :

Let X, Y be two sets then X is a subset of Y if every element of X is also an element of Y and this relationship is indicated by $X \subset Y$ (or) $Y \subset X$

Ex : $X = \{1, 2, 3\}$

$Y = \{1, 2, 3, 4\}$ (i.e) $X \subset Y$

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iv) Empty set :

A set is said to be an empty set if it has no element it is denoted by ϕ (or) $\{ \}$

v) Union :

Let x, y be two sets then the union $\{x \cup y\}$ consists of all points including x or y

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Ex: $A = \{0, 1, 2\}$

$B = \{3, 4\}$

$A \cup B = \{0, 1, 2, 3, 4\}$

$x \cup y = \{x \cup y \mid x \in x$ (or) $y \in y\}$

vi) Intersection :

Let x, y be two sets the intersection of two sets is formed by all points which are both x and y and it is denoted by

$x \cap y = \{x \cap y \mid x \in x$ (or) $y \in y\}$

Ex: $A = \{1, 2, 3\}$

$B = \{2, 4, 5\}$

$A \cap B = \{2\}$

$x \cap y = \{x \cap y \mid x \in x$ (or) $y \in y\}$

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vii) complement :

The complement of a set x consists of all points which are not in x it is denoted by $\sim x$

Ex: $x = \{a, c, d\}$

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$A = \{c\}$

$\sim A = \{a, d\}$

viii) Relative complement :

Let x, y be two sets, the relative complement consists of all points that are in y but not in x it is denoted by $y \sim x$ (y relative to x)

ix) Distributive laws :

$$x \cup (y \cap z) = (x \cup y) \cap (x \cup z)$$

$$x \cap (y \cup z) = (x \cap y) \cup (x \cap z)$$

x) De-Morgan's law :

$$\sim (x \cup y) = \sim x \cap \sim y$$

$$\sim (x \cap y) = \sim x \cup \sim y.$$

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Metric Space :

Definition :

A set is called a metric space if there is defined for every pair $x \in S$, $y \in S$ a non-negative real number $d(x,y)$ in such a way that the following conditions are

i) $d(x,y) = 0 \iff x = y$

ii) $d(y,x) = d(x,y)$ ④

iii) $d(x,z) \leq d(x,y) + d(y,z)$

Note :

R and C are metric space when $d(x,y) = |x-y|$. Then n dimensional Euclidean space R^n is the set of real n.

$$x = x_1, x_2, \ldots\ldots x_n$$

with the distance defined by $d(x,y)^2 = \sum_1^n |x_i - y_i|^2$

In the extended complex plane we defined

$$d(z,z') = \frac{2|z-z'|}{\sqrt{(1+|z|^2)(1+|z'|^2)}}$$

Neighbourhood :

A set $N \subset S$ is called a neighbourhood of $y \in S$ if it contains a ball $B(y, \delta)$.

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In other words, the neighbourhood of y is a set which contains all points sufficiently near to y.

Open :

A set is open if it is a neighbourhood each of its element

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closed :

The complement of an open set is said to be closed.

Fundamental properties of open and closed set

i) The intersection of a finite number of open set is open

ii) The union of any collection of open set is open.

iii) The union of finite number of closed set is closed.

iv) The intersection of any collection of closed set is closed

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Interior :

The interior of a set x is the largest open set contained in x, then there exists the union of all open set contained in x. It can also described as the set of all points of which x is a neighbourhood it is denoted by Int x.

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Closure :

The closure of x is the smallest closed set which contains x are the intersection of all closed set ($\supset x$) a point belongs to the closure of x if and if all it is neighbourhood intersect x, it is denoted by cl x (or) $\bar{x}$

Boundary :

The boundary of x is the closure minus the interior, a point belongs to the boundary if and if all its neighbourhood intersect both x and $\sim x$, it is denoted by Bd $\cdot x$ (or) ∂x.

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Exterior :

The exterior of x is the interior of $\sim x$, it is also the complement of closure it is denoted by $\sim x^{-}$

Observe that

$Int\ X \subset X \subset X^{-}$ and that X is open

if $Int\ X = X$

closed if $X^{-} = X$

$X \subset Y \Rightarrow Int\ X \subset Int\ Y$

$\bar{X} \subset Y^{-}$

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Isolated points :

we say that $x \in X$ is an isolated point of X, if x has a neighbourhood whose intersection with X reduces to the point x

Connectedness :

If E is any non-empty subset of metric space S. conside E as a metric space with the same distance $d(x,y)$ as on all of S neighbourhood and open sets on E

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are defined as any metric space but open set on E not be open.

Region :

A non-empty connected open set is called a region

(8)

Theorem 1 :

The non-empty connected subsets of the real line are intervals.

Proof :

Suppose that,

The Real line R is represented as the union $R = A \cup B$ of the two disjoint closed set.

If neither empty, we can find $a_1 \in A$ $b_1 \in B$. we may assume that $a_1 < b_1$, we bisect the interval (a_1, b_1) and note one of the two halves has its left end point in A and its right end point in B.

we denotes its interval by (a_2, b_2) and continuous the process infinite in this way.

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we obtain a sequence of nested

interval (a_n, b_n) with $a_n \in A$, $b_n \in B$

The sequence $\{a_n\}$ and $\{b_n\}$ have a

common limit c. since A and B are closed

c would have to be common point of A

and B.

This is contradiction.

show that either A (or) B are empty and

hence R is connected.

(9)

Theorem 2 :

A non-empty open set in a place is

connected iff and if any two points can

be join by a polygon which lies in the set.

Proof :

First we have to prove that,

condition is Necessary.

Let A be a open connected set and

choose a point $a \in A$.

we denoted by A. the subset of A,

whose point can be joint to 'a' by polygon

is in A_1 and by A_2.

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The subset whose points can't be joined.

Let us prove that,

A_1 and A_2 are open first order $a_1 \in A_1$

There exists a neighbourhood $|z-a_1| < \epsilon_1$ containing A. All points in the neighbourhood can be joined to a_1 by a line segment and from there to by a polygon.

Hence, whole neighbourhood is contained in A_1, and A_1 is open.

(10)

secondary if $a_2 \in A_2$, Let $|z-a_2| < \epsilon_1$ be a neighbourhood contained in A.

If a point in this neighbourhood in could be joined to a small by a. line segment and from there to "a".

This is a contrary to the definition of A_2 and we conclude that A_2 is open

since A was connected either A_1 or A_2 must be empty but A_1 contains the point 'a'.

Hence A_2 is empty and all the points

can be joined to "a".

Finally any two its in A can be joined by a way of a.

Thus we have proved the condition is necessary. (11)

In order to prove that, sufficient part,

we assume that, A has a representation as the union of the two disjoint $A = A_1 \cup A_2$ open set.

choose $a_1 \in A_1$, $a_2 \in A_2$ and suppose that those point can be joined by polygon in A_1.

one of the sides of the polygon must than, join a point in A_1 to a point in A_2 and for this reason it is sufficient to consider the case,

where a_1 and a_2 are joint by a line segment. This segment as a parametric representation $z = a_1 + t(a_2 - a_1)$ where "t" runs through the interval $0 \le t \le 1$. The

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subset of the interval $0 < t < 1$.

which corresponds to point in A_1 and A_2 respectively are evidently open disjoint and non-void.

(12)

These contradiction,

The connectedness of interval and we have proved the condition of therm is sufficient.

The theorem generalized easily to R^n and C^n.

Closed Region :

A region is the more dimensional analogue of an open interval the closure of a closed region.

Theorem 3 :

Every set has a unique decomposition into components.

proof :

If E is the given set consider,

The point $a \in E$ and let $C(a)$ denote the union of all connected subset of E. That contain 'a' the c(a) is sure to contain 'a' for the set consisting of the single point 'a' is connected.

If we can show that c(a) is connected. Then it is a maximal that connected set in other words a component. (13)

(i.e) To prove that

The components are either disjoint (or) identical.

If $c \in C(a) \cap C(b)$. then $C(a) \subset C(c)$ by the definition of c(c) connectedness of c(a). Hence $a \in c(c)$ and by the same reasoning $c(c) \subset c(a)$. so that infact $c(a) = c(c)$.

Similarly,

$c(b) = c(c)$ and consequently $c(a) = c(b)$

we call c(a) the component of 'a'.

Suppose that,

c(a) were not connected then we would find relatively open sets

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$A, B \neq \phi$, $C(a) = A \cup B$, $A \cap B = \phi$.

we may assume that,

$a \in A$. while 'B' contains a point 'b'.

since $b \in C(a)$. There is a connected set

$E_0 \subset E$, which contains a and b representation.

$E_0 = (E_0 \cap A) \cup (E_0 \cap B)$ would be a decomposition

into relatively open subsets and since $a \in E_0 \cap A$

be $b \in E_0 \cap B$.

Neither part would be empty. This is

a contradiction. we conclude $C(a)$ is a

connected.

Hence proved.

5m **Theorem 4 :**

In R^n the components of any

open set are open.

proof :

The δ neighbourhood in R^n are

connected.

consider,

$a \in C(a) \subset E$.

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If E is open it contains $B(a,\delta)$ and

because $B(a,\delta)$ is connected. $B(a,\delta) \subset C(a)$.

Hence $C(a)$ is open.

There is true for any space S which

is locally connected.

Note :

Every open set in R is countable union

of disjoint open intervals.

Dense :

A set E is dense in S if $S = \bar{E}$. A

metric space is separable if there exists

a connected set which is dense in S.

compactness:

A metric space is said to be complete

is every cauchy sequence is convergent.

open covering :

A collecting of open sets in an

open covering of a set x, if ∞ is contain

in the union of open sets.

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i.e) $X = \bigcup_{n=1}^{\infty} I_n$.

A subcovering is a sub collection with the same property.

A finite covering is one that consists of a finite number of sets.

(16)

compact:

A set X is compact if and if every open covering of X contains a finite subcovering

Totally bounded:

Apr-17 2m (X.) A set X is totally bounded, if for every $\epsilon > 0$, X can be covered by finitely many balls of radius ϵ.

Theorem 5:

Apr-17 (X) 5m. A set is compact if and if it is complete and totally bounded.

proof:

Let us assume that the set is compact

To prove,

It is complete and totally bounded.

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Let the collection of all balls of radius 't' is an open covering and the compactness which implies that we can select finitely many balls of radius ε that cover X

(17)

i-e) The totally bounded set is necessary bounded, for it

$(X \subset B \subset X) \in U \ldots\ldots\ldots U B(x_m, \varepsilon)$

Then any points of X have a distance

~~$< \varepsilon$ max $d(x_i, x_j)$~~

i-e) X is complete and totally bounded

conversely,

Let us assume that, the metric space is complete and totally bounded.

Suppose that,

There exists open covering which does not contain a finite subcovering

Let $\varepsilon_n = 2^{-n}$.

W.K.T

$f\phi$ can be covered by finitely many ball

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$B(x, \varepsilon_1)$. If each had a finite subcovering because $B(x, \varepsilon_1)$ is itself totally bounded.

we can find an x_2

$B(x_1, \varepsilon_1)$ and $B(x_2, \varepsilon_2)$ has no finite subcover. [Any subset of a totally bounded set is totally bounded].

we obtain a sequence with the property that $B(x_n, \varepsilon_n)$ has no finite subcovering

(18)

and

$$x_{n+1} \in (B(x_n, \varepsilon_n))$$

The second property implies $d(x_n, x_{n+1}) < \varepsilon_n$

and hence,

$$d(x_n, x_{n+p}) \leq \varepsilon_n + \varepsilon_{n+1} + \cdots + \varepsilon_{n+p-1}$$

$$< 2^{-n+1}$$

It follows that x_n is a converge y sequence it converges to a limit y and this y belongs to one of the open set V is the given covering, because $y \in V$ is open it contains a Ball $B(y, \delta)$.

choose 'M' so large then,

$$d(x_n, y) < \delta/2 \text{ and } \varepsilon_n < \delta/2$$

then,

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$B(x_n, \xi_n) \subset B(y, \delta)$ for $d(x, x_n) < \xi_n$

$d(x, y) \leq d(x, x_n) + d(x_n, y) < \delta$

$B(x_n, \xi_n)$ admits a finite subcovering namely by the single set 'n'.

This is contradiction and we conclude that B is compact.

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Hence S is compact

Corollary :

A subset of R (or) C is compact iff if it closed and bounded.

Proof :

Assume that A subset of R (or) C is compact.

To prove :

It is closed and bounded.

we have to prove that.

Let subset of R (or) C is compact

(i.e) we have to show that,

Every bounded set in R (or) C is totally bounded.

Let us take,

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The case of c, If x is bounded. It is contained in a disks and here is a sequence the square with arbitary small size and the square can be turn.

By disks with alternatively small radius. This proves that x is totally bounded.

Let y is a limit point of the square $\{x_n\}$, if there exist a subsequence $\{x_{n_k}\}$ that converges to y.

(20)

If y is a limit point every neighbourhood of y contains infinitely many x_n.

The converse is true. suppose that $\epsilon_k \to 0$ if every $B(y, \epsilon_k)$ contains infinitely many x_n.

we can choose x_{n_k} is such a way that $x_{n_k} \in B(y, \epsilon_k)$ and $n_{k+1} > n_k$.

It is clear that x_{n_k} converges to y.

This is subset of R (or) C is compact iff it is closed and bounded.

Hence proved.

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Bolzano Weierstress theorem :

A metric space is compact iff every infinite sequence has a limit point.

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Proof :

Let us assume that

The metric space is compact.

To prove :

Every infinite sequence has a limit point.

Let $\{x_n\}$ be a infinite sequence

If y is not a limit point of $\{x_n\}$ it has a neighbourhood which contains infinitely many x_n.

If there were no limit point the open set containing only finitely many x_n would from an open covering.

If the compact case we would select a finite sub-covering and it would follow that the sequence is finite.

compact space is complete. we showed in essence that every has a

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limit point.

cauchys sequence with a limit point is necessarily convergent

which is contradiction to fact the sequence $\{x_n\}$ is finite.

y is the limit point of $\{x_n\}$

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conversely,

Assume that.

Every. infinite sequence has a limit point ω.

we have to prove that

A metric space is compact from the assumption.

we have,

The metric space is complete.

we have to prove that a metric spa is compact.

It is sufficient to prove that,

The space is not totally bounded

Then, there exis $\varepsilon > 0$. The space cannot be covered by finitely a many ε neighbourhood.

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We construct a sequence $\{x_n\}$ as follows.

x is arbitary when $x_1, x_2 \ldots\ldots x_n$ have been selected. we choose x_{n+1} if does not lying

$B(x, \epsilon) \cup \ldots\ldots \cup B(x_n, \epsilon)$

This is always possible because there neighbourhood do not cover the whole space.

But it is clear then $\{x_n\}$ has no convergent subsequence for $d(x_n, x_m) > \epsilon \; \forall$ $m \in n$.

(23)

which is contradiction.

Thus we have the metric space is compact the theorem is known as Bolzano weierstress.

Continuous function :

F is continuous at 'a' if be every $\epsilon > 0$ there exists $\delta > 0$ $d(x, a) < \delta$.

$$d'\{f(x), f(a)\} < \epsilon.$$

Note :

A function is continuous iff the inverse image of every open set is open. A function

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is continuous iff the inverse image of every closed set is closed.

Theorem 7 :

under a continuous mapping the image of every compact set is compact and consequently closed.

proof :

suppose that, F is defined and continuous on the compact set X. consider a open set covering of F(x) by open sets V and the inverse image $F^{-1}(V)$ are open and form a covering of X. Because X is compact we can select a finitely subcovering.

$X \subset F^{-1}(V_1) \cup \ldots\ldots \cup F^{-1}(V_m)$.

It follows that,

$F(X) \subset V_1 \cup \ldots\ldots \cup V_m$.

we have prove that f(x) is compact.

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Corollary :

A continuous real valued function on a compact set has a maximum and minimum

Theorem 8 :

under continuous mapping the image of any connect set is connected.

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Proof :

we assume that,

'F' is defined and continuous on the whole space 's' and the $F(s)$ is all of s.

suppose that

$s = A \cup B$, where A and B are open and disjoint.

Then,

$$s = F^{-1}(A) \cup F^{-1}(B)$$

i.e) s is a union of disjoint open sets

$f^{-1}(B) = 0$ and hence $A = 0$ (or) $B = 0$.

Thus we conclude that 's' is connected.

Uniformly continuous :

A function F is said to be uniformly

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continuous on x, if every $\varepsilon_1 > 0$.

$d'[f(x_1), f(x_2)] < \varepsilon_1$ for all points (x_1, x_2)

with $d(x_1, x_2) < \delta$.

Theorem 9:

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on a compact set every continuous fun is uniformly continuous.

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proof:

Suppose that,

F is continuous on a compact set "x"

for every $y \in x$. There is a ball $B(y, \rho)$ such that $d'[f(x), f(y)] < \varepsilon_1/2$ for $x \in B(y, \rho)$.

Here "ρ" may be depend only "y".

considering the covering of x be the smallest ball $B(y_1, \rho_1/2)$

There exists a finite subcovering

$x \subset [B(y_1, \rho_1/2) \cup \ldots\ldots \cup B(y_m, \rho_m/2)]$

let "δ" be the smallest of the numbers

$\rho_1/2, \rho_2/2 \ldots\ldots \rho_m/2$

and suppose that,

$d(x_1, x_2) < \delta$.

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There is a y_K with $d(x_1, y_K) < \rho_K/2$ and

obtain $d(x_2, y_2) < \delta + \rho_K/2 < \delta$.

Hence $d'[f(x_1), f(y_K)] < \epsilon/2$ &

$d'[f(x_2), f(y_K)] < \epsilon/2$

So that

$d'[f(x_1), f(x_2)] < \epsilon$ 27

$\therefore$ F is uniformly continuous.

Topological space :

A Topological space is a set "T" together with a collection of this subset called opensets. The following conditions are true

i) The empty set ϕ and the whole space T are open sets.

ii) The Intersection of any two open sets is an open sets.

iii) The union of an arbitrary collection of open sets is an open set.

Definition :

A topological space is called Hausdorff space, if any two distinct points are contained

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in disjoint open sets.

(i.e) $x \neq y$, $x \in U$ and $y \in V$ where U and V are open sets.

Then, $U \cap V = \phi$.

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Conformality :

Arcs and closed curves :

The derivatives $z'(t) = x'(t)$. It is (t) exists and is not equal to zero the arc ∞ has a tangent whose direction is determined by $\arg z'(t)$. We shall say that the arc is called curve if the points are co-insides.

(i.e) $z(\alpha) = z(\beta)$.

A constant function $z(t)$ defines a point.

* The arc is differentiable if $z'(t)$ exists and is continuous.

* The arc is said to regular if $z'(t) \neq 0$

* The arc is simple (or) Jordan arc if $z(t_1)$ & $z(t_2)$ only for $t_1 = t_2$.

* A arc is a closed curve if the end points coinside $z(\alpha) = z(\beta)$

* The opposite arc of $z = z(t)$, $\alpha \leq t \leq \beta$

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is the arc $z = x(-t)$, $-\beta \leq t \leq -\alpha$ and is denoted by γ and $-\gamma$. sometimes by γ and γ^{-1} depending on the connection.

Analytic function in Regions : (29)

Definition :

Halomorphic (or) complex analytic :

A complex valued function $f(z)$ defined on an open set Ω is said to be analytic in Ω if it has a derivatives at each point of Ω.

Definition :

A function $f(z)$ is analytic on an arbitary point set A, if it is restriction to "A" of a function which is analytic in some open set containing A.

Theorem 10 :

An Analytic function in a region Ω whose derivatives vanishes identically must reduce to a constant.

The same is true if either the real part, the imaginary part the modules

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UNIT-II

Complex integration :

Line integral :

In an interval line integral open interval (a,b). If $f(t) = u(t) + i\,v(t)$ is its function define in an interval (a,b). Then

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt$$

This is called a line integral is an interval (1) $[a,b]$.

Result : 1

If $c = \alpha + i\beta$ is a complex constant

Then $\int_a^b c\,f(t)\,dt = c\int_a^b f(t)\,dt$

Proof :

Given $c = \alpha + i\beta$

Let if $f(t) = u(t) + i\,v(t)$, $t \in (a,b)$

Now consider,

$$\int_a^b c\,f(t)\,dt = \int_a^b (\alpha + i\beta)(u(t) + i\,v(t))\,dt$$

$$\int_a^b c\,f(t)\,dt = \int_a^b \left[\alpha\,u(t) - \beta\,v(t)\right] + i\left[\alpha\,v(t) + \beta\,u(t)\right]dt$$

$$= \int_a^b \left[\alpha(u(t)) - \beta\,v(t)\right]dt + i\int_a^b \left[\alpha\,v(t) + \beta\,u(t)\right]dt \rightarrow (1)$$

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Now, consider

$$c\int_a^b f(t)\,dt = (\alpha+i\beta)\int_a^b [u(t)+iv(t)]\,dt$$

$$=\alpha\int_a^b [u(t)+iv(t)]\,dt + i\beta\int_a^b [u(t)+iv(t)]\,dt$$

$$=\alpha\int_a^b u(t)\,dt + i\alpha\int_a^b v(t)\,dt + i\beta\int_a^b u(t)\,dt - \beta\int_a^b v(t)\,dt$$

$$=\alpha\int_a^b u(t)\,dt - \beta\int_a^b v(t)\,dt + i\left[\alpha\int_a^b v(t)\,dt + \beta\, v(t)\right]dt \quad \to (2)$$

From (1) and (2)

$$\int_a^b c\,f(t)\,dt = c\int_a^b f(t)\,dt.$$

(2)

Result : 2

Fundamental inequality :

If $a \le b$ then the fundamental inequality $\left|\int_a^b f(t)\,dt\right| \le \int_a^b |f(t)|\,dt$ holds for arbitary complex function $f(t)$.

complex line integral of $f(z)$ (or) the integral of γ.

Let γ be piecewise differential arc with the equation $z = z(t)$, $a \le t \le b$. If the function $f(z)$ is defined and continuous on the arc γ Then $f(z(t))$ is also continuous

Then the complex line integral of $f(z)$ extended over the arc γ is given by

$$\int_\gamma f(z)\,dz = \int_a^b f(z(t))\,z'(t)\,dt$$

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Result-3 :

$$\int_a^b f(z)\,dz = -\int_a^b f(z)\,dz .$$

Note :

If $\gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$. Then

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz + \cdots + \int_{\gamma_n} f(z)\,dz .$$

Line Integral with respect to $\bar{z}$. The line Integral for the complex functions $f(z)$ with respect to $\bar{z}$ is defined by

$$\int_\gamma f\,d\bar{z} = \int_\gamma \bar{f}\,dz .$$

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(i.e) $\int_\gamma f(\bar{z})\,dz = \int_\gamma \overline{f(\bar{z})\,dz}$

Note :

1) $\int_\gamma f\,dz = \int_\gamma f\,|dz| = \int_\gamma f\,[z(t)]\,|z'(t)|\,dt$

2) $\int_{-\gamma} f\,|dz| = \int_\gamma f\,|dz|$

3) $\left|\int_\gamma f\,dz\right| \le \int_\gamma |f|\,|dz|$

1. Evaluate $\int_\gamma |dz|$ where γ is represent by the equation $|z-a| = \rho$ (or) Find the length of circle with centre a and the radius ρ.

Soln :

Given $|z-a| = \rho$

$z-a = \rho e^{i\theta}$, $0 \le \theta \le 2\pi$

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$z = a + \rho e^{i\theta}$ $\qquad e^{i\theta} = \cos\theta + i\sin\theta$

$dz = \rho e^{i\theta} i d\theta$ $\qquad ie^{i\theta} = i(\cos\theta + i\sin\theta)$

$= i\cos\theta - \sin\theta$

Now

$\int_\gamma |dz| = \int_0^{2\pi} |i\rho e^{i\theta}| d\theta$ $\qquad |ie^{i\theta}| = |x+iy|$

$= \sqrt{x^2+i^2y^2}$

$= \rho \int_0^{2\pi} d\theta$ $\qquad = \sqrt{-(\sin\theta+\cos\theta)}$

$= \rho (\theta)_0^{2\pi}$ $\qquad = 1$

$= \rho \cdot 2\pi$

Hence the length of the given circle is 2π.

2) compute $\int x\, dz$ where γ is the line segment from 0 to 1.

4

Soln :

$z = x+iy$

$dz = dx+idy$

Take represent the equation $y = x$

$dy = dx$

$dz = dx + idx$

$dz = (1+i)dx$

x varies from 0 to 1

$\int_0^1 x\, dz = \int_0^1 x(1+i)\, dx$

$= (1+i) \int_0^1 x\, dx$

$= (1+i) \left[\frac{x^2}{2}\right]_0^1$

$= \frac{1+i}{2}$

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Rectifiable arc :

The length of an arc can also be defined as the least upper bound of all sums

$|z(t_1) - z(t_0)| + |z(t_2) - z(t_1)| + \cdots + |z(t_n) - z(t_{n-1})|$

where $a = t_0 < t_1 < t_2 < \cdots < t_n = b$

If there least upper bound is finite we say that the arc is rectificable.

Functional (or) Functional arc γ :

The line integral of a form $\int_\gamma (p\,dx + q\,dy)$ is called functional (or) functional arc γ where p and q are define continuous cts in the region Ω and γ various in Ω.

(5)

Theorem :

The line integral $\int_\gamma (p\,dx + q\,dy)$ define in Ω depends only on the end points of γ iff $f > a$ function $u(x, y)$ in Ω with partial derivatives

$\frac{\partial u}{\partial x} = p, \quad \frac{\partial u}{\partial y} = q.$

Proof :

Necessary Part :

Let $z = z(t) = x(t) + iy(t)$ and $a \le t \le b$

To prove :

The line integral $\int_\gamma p\,dx + q\,dy$ depends

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only on the end point of γ.

Given a function $u(x,y)$ with condition

$\frac{\partial u}{\partial x} = p$ and $\frac{\partial u}{\partial y} = q$

$$\int_{\gamma} pdx + qdy = \int \left(\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy \right)$$

$$= \int_a^b \left(\frac{\partial u}{\partial x} x'(t) dt + \frac{\partial u}{\partial y} y'(t) dt \right)$$

$$= \int_a^b \left[\frac{\partial u}{\partial x} x'(t) + \frac{\partial u}{\partial y} y'(t) \right] dt$$

$$= \int_a^b \frac{d}{dt} [u(x(t))\ u'(y(t))] dt$$

$$\int_{\gamma} pdx + qdy = u\left[x(t) + y(t)\right]_a^b$$

$$= u[x(b), y(b)] - u[x(a), y(a)]$$

Hence the line integral the end points given function $u(x, y)$

(6)

Suffiecient part :

If $u(x,y) = \int_{\gamma} (pdx + qdy)$

There exists $\frac{\partial u}{\partial x} = p$; $\frac{\partial u}{\partial y} = q$

Given the integral depends only on end points choose the fixed points $(x_0, y_0) \in \Omega$ joining to any point (x, y) by the polygon γ contained in Ω whose sides are parallel to the co-ordinate axis.

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Define a function by

$$u(x,y) = \int_{\gamma} p\,dx + q\,dy \longrightarrow (1)$$

Thus integral is well defined. since the given integral depends only on end points. choose the least segment of γ horizontal we can keep y as constant and let x varies without c. hanging the other segment on least segment. we choose x for a parameter and equation takes from

$u(x,y) + \int q\,dx$ constant.

The lower limit of the integral being relevant $\frac{\partial u}{\partial x} = p$.

∴ In the same way by choosing the least segment vertical $\frac{\partial u}{\partial y} = q$.

7

Exact Differential equations :

Let $p(x,y)$ and $q(x,y)$ be two real valued function of x and y. Then the differential equation $p(x,y)\,dx + q(x,y)\,dy = 0$ is said to be exact if there exists a function $u(x,y)$ such that

$p = \frac{\partial u}{\partial x}$ and $q = \frac{\partial u}{\partial y}$.

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Corollary 1:

The integral depends only on end points iff the integral is exact differential i.e)

$$\int_{\gamma}(p\,dx+q\,dy) = \int \frac{\partial u}{\partial x}\,dx + \int \frac{\partial u}{\partial y}\,dy = \int_{\gamma} du$$

Proof:

Assume that the integral depends only on end points.

To prove :-

The integral is exact. By previous theorem, there exists a differential function $u(x,y)$ in Ω with partial derivative

(8)

$$\frac{\partial u}{\partial x} = p, \quad \frac{\partial u}{\partial y} = q.$$

$$\int_{\gamma} p\,dx + q\,dy = \int_{\gamma} \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy = \int_{\gamma} du.$$

conversely,

Assume that the integral exact we have

$$\int_{\gamma} (p\,dx + q\,dy) = \int_{\gamma} \frac{\partial u}{\partial x}\,dx + \int_{\gamma} \frac{\partial u}{\partial y}\,dy$$

$$= \int_a^b \left[\frac{\partial u}{\partial x} x'(t) + \frac{\partial u}{\partial y} y'(t)\right] dt$$

$$[\because z = z(t) = x(t) + i\,y(t), \quad a \le t \le b].$$

$$= \int_a^b \left[\frac{\partial u}{\partial x} \overset{\frac{dx}{dt}}{x'(t)} + \frac{\partial u}{\partial y} \overset{\frac{dy}{dt}}{y'(t)}\right] dt$$

$$= \int_a^b \frac{d}{dt}\left[u(x(t), y(t))\right] dt$$

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$= u[x(t), y(t)]_a^b$

$= u[x(b), y(b)] - u[x(a), y(a)]$

Hence the line integral depends on the end points.

corollary 2 :

The integral $\int_\gamma f\,dz$ with its function f depends only on end points of γ if f is the derivative of an analytic function.

9

proof :

consider $F(z) = \int_\gamma f(z)\,dz$

$= \int_\gamma f(z)\,(dx + i\,dy)$

$F(z) = \int_\gamma (f(z)\,dx + i f(z)\,dy)$

$= \int_\gamma f(z)\,dx + i \int_\gamma f(z)\,dy \longrightarrow (1)$

Diff w.r to "x" and "y" we get

$\frac{\partial F}{\partial x} = F(z) \longrightarrow (2)$

$\frac{\partial F}{\partial y} = i f(z) \Rightarrow \frac{\partial F/\partial y}{i} = f(z)$

(x) and (÷) by "i"

$-i \frac{\partial F}{\partial y} = F(z) \longrightarrow (3)$

From (2) = (3)

$-i \frac{\partial F}{\partial y} = \frac{\partial F}{\partial x}$

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$f_x = i f_y$

which is the complex from of cauchy Riemann equation (8) in F satisfy the C.R equation Then F is analytic. F is derivative of an analytic functions.

1. compute $\int_{\gamma} \bar{z}\, dz$ where γ is the directed line segment from "0" to 1+i.

soln:

(10)

$$\text{Given } \int_{\gamma} \bar{z}\, dz = \int_{\gamma} x(dx + i\,dy)$$

$$= \int_{\gamma} x\,dx + i \int_{\gamma} x\,dy$$

Now the equation of the line segment from the point 0 to 1+i is (0, 0) to (1, 1)

$$\left(\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}\right) = \frac{x - 0}{1 - 0} = \frac{y - 0}{1 - 0} \Rightarrow x = y$$

when $x = 0 \Rightarrow y = 0$

$x = 1 \Rightarrow y = 1$

(1) becomes

$$\int_{\gamma} \bar{z}\, dz = \int_0^1 x\,dx + i\int_0^1 y\,dy \Rightarrow \left(x^2/2\right)_0^1 + i\left(y^2/2\right)_0^1$$

$$= 1/2 + i/2 = (1+i)\,1/2.$$

5m **Problem:**

compute $\int_{\gamma} \bar{z}^2 dz$ where γ is the directed line segment 0 to 2+i.

soln:

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Given $\int_\gamma z^2 dz = \int_\gamma (x+iy)^2 (dx+idy)$ $\quad z=(x+iy)^2$

$= \int_\gamma (x^2+i^2y^2+2xiy)(dx+idy)$

$= \int_\gamma (x^2-y^2+i2xy)(dx+idy)$

$= \int_\gamma x^2dx - \int_\gamma y^2dx + i\int_\gamma 2xy\,dx + i\int_\gamma x^2dy - i\int_\gamma y^2dy - \int_\gamma 2xy\,dy \rightarrow (1)$

Now the equation of the line segment the point 0 to $2+i$ is $(0,0)$ to $(2,1)$

(11)

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1}$$

$$\frac{x-0}{2-0} = \frac{y-0}{1-0}$$

$x/2 = y/1 \Rightarrow x = 2y$

$\Rightarrow y = x/2$

where $x=0 \Rightarrow y=0$
$x=2 \Rightarrow y=1$

(1) $\Rightarrow \int_\gamma z^2dz = \int_0^2 x^2dx - \int_0^2 x^2/4\,dx + 2i\int_0^2 \frac{x^2}{2}dx + i\int_0^1 4y^2dy - i\int_0^1 y^2dy - 2\int_0^1 2y^2dy$

$= \left(x^3/3\right)_0^2 - \frac{1}{4}\left(\frac{x^3}{3}\right)_0^2 + i\left(\frac{x^3}{3}\right)_0^2 + 4i\left(\frac{y^3}{3}\right)_0^1 - i\left(\frac{y^3}{3}\right)_0^1 - 4\left(\frac{y^3}{3}\right)_0^1$

$= 8/3 - 2/3 + i\,8/3 + i\,4/3 - i/3 - 4/3$

$= (8/3 - 2/3 - 4/3) + i\,(8/3 + 4/3 - 1/3)$

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Complex Integration :

Let $u(t)$ and $v(t)$ be two real value co function defined on an interval (a, b). Let $f(t) = u(t) = u(t) + iv(t)$. Then $f(t)$ is complex valued continuous function defined in an interval (a, b).

In this case we defined.

$$\int_a^b f(t)\,dt = \int_a^b u(t)\,dt + i\int_a^b v(t)\,dt.$$

(12)

Cauchy Theorem for a Rectangle :

10m

If the function $f(z)$ is analytic on the rectangle R defined by $a \le x \le b$, $c \le y \le d$ Then

$$\int_{\delta R} f(z)\,dz = 0$$

where δR represent the perimeter of the rectangle.

Proof :

Let $\eta(R) = \int_{\delta R} f(z)\,dz$

Now subdivide R into 4 concurrent rectangle R^1, R^2, R^3, R^4 by joining the mid-points of the opposite sides by line segment as shown.

$$\eta(R) = \eta(R^{(1)}) + \eta(R^{(2)}) + \eta(R^{(3)}) + \eta(R^{(4)}) \longrightarrow (1)$$

Since the rectangle over the common side cancel hence atleast one of the rectangle $\eta(R^k)$ where $k = 1, 2, 3, 4$ must satisfy the condition.

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$$|\eta(R^k)| \geq \frac{1}{4}|\eta(R)| \longrightarrow (2)$$

for every $k = 1, 2, 3, 4$

suppose $|\eta(R^k)| < \frac{1}{4}|\eta(R)|$

$(1) \Rightarrow |\eta(R)| = |\eta(R^1) + \eta(R^2) + \eta(R^3) + \eta(R^4)|$

Triangle Inequality

$$|\eta(R)| \leq \frac{1}{4}|\eta(R)| + \frac{1}{4}|\eta(R)| + \frac{1}{4}|\eta(R)| + \frac{1}{4}|\eta(R)|$$

$$|\eta(R)| \leq |\eta(R_1)|$$

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which is contradiction

$\therefore$ There is atleast one rectangle satisfy the eqn (2).

we denote this rectangle by R

i.e) $R_1 \subset R$ such that $|\eta(R_1)| \geq \frac{1}{4}|\eta(R)|$

starting with R_1 again the method of bijection we can find $R_2 \subset R_1$.

such that $|\eta(R_2)| \geq \frac{1}{4}|\eta(R_1)|$

$$\geq \frac{1}{4} \cdot \frac{1}{4}|\eta(R)|$$

$$\geq \frac{1}{4^2}|\eta(R)|.$$

$$|\eta(R_2)| \geq 4^{-2}|\eta(R)|.$$

This process can be repeated infinitely sequence of nested rectangle $R \supset R_1 \supset R_2 \supset \dots \supset R_n$ with the property that

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$$|\eta(R_n)| \geq 4^{-n} |\eta(R)| \longrightarrow (3)$$

Let l and b be the length and breath of R. Then $l/2^n$ and $b/2^n$. Those of R^n.

If 'd' denote the length of the diagonal of R^n. Then

$$d_n = \sqrt{(l/2^n)^2 + (b/2^n)^2}$$

(14)

$$= \frac{1}{2^n} \sqrt{l^2+b^2}$$

$$= \frac{d}{2^n} \longrightarrow (4)$$

If 'λ' denote the length of the perimeter of R. Then $\lambda = 2(l+b)$. If 'λ_n' denote the length of the perimeter of R_n.

$$\text{Then } \lambda_n = 2(l_n+b_n) = 2\left(\frac{l}{2^n} + b/2^n\right)$$

$$= \frac{2(l+b)}{2^n} = \frac{\lambda}{2^n} \longrightarrow (5)$$

Since $R \supset R_1 \supset R_2 \supset \cdots \supset R_n$. Then by cantors' intersection theorem $\{R_n\}$ convergent to a point $z^* \in R$.

i.e) There is a point z^* is common to all these rectangle

Let $\varepsilon_1 > 0$ be given

Since $f(z)$ is analytic on R and hence z^*

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There exists $\delta > 0$ such that

$$\left| \frac{f(z) - f(z^*)}{z - z^*} - f'(z^*) \right| < \epsilon_1 \longrightarrow (6)$$

$|z - z^*| \leq \delta$

$$\left[\therefore f'(z^*) = \lim_{z \to z^*} \frac{f(z) - f(z^*)}{z - z^*} \right]$$

$$\left| \frac{f(z) - f(z^*) - f'(z^*)(z - z^*)}{z - z^*} \right| < \epsilon_1$$

i.e) $|f(z) - f(z^*) - f'(z^*)(z - z^*)| < \epsilon_1 |z - z^*| \longrightarrow (7)$

$$\eta(R) = \int_{\delta R} f(z)\, dz$$

(15)

$$\eta(R_n) = \int_{\delta R_n} f(z)\, dz$$

$$|\eta(R_n)| = \left| \int_{\delta R_n} f(z)\, dz \right|$$

$$\leq \int_{\delta R_n} |f(z)||dz|$$

$$\leq \int_{\delta R_n} |f(z) - f(z^*) - f'(z^*)(z - z^*)|\, dz$$

$$\leq \int_{\delta R_n} \epsilon_1 |z - z^*|\, dz \qquad [\therefore \text{by } (7)]$$

In the last integral $|z - z^*|$ is atmost equal to the length d_n of the diagonal of R_n

$|z - z^*| = d_n = d/2^n$ $\qquad [\therefore \text{by } (4) \Rightarrow d_n = \frac{d}{2^n}$.

$|\eta(R_n)| \leq \epsilon_1 d/2^n \int dz$ $\qquad [\therefore \text{by } (5) \Rightarrow \int_{\delta R_n} dz = l/2^n$

$\leq \epsilon_1 \frac{d}{2^n} \, l/2^n$ $\qquad [\therefore 4^n = (2^2)^n]$.

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$$\leq \epsilon \frac{d\ell}{4^n} \rightarrow (8)$$

From (3)

$$4^{-n} |\eta(R)| \leq |\eta(R_n)|$$

$$|\eta(R)| \leq 4^n |\eta(R_n)|$$

$$\leq 4^n \epsilon \frac{d\ell}{2^n 2^n} < d\ell\epsilon \quad \text{[by (8)]}.$$

$$\therefore |\eta(R)| < \epsilon d\ell$$

R.H.S is arbitary. L.H.S is small.

$$\int_{\delta R} f(z)\, dz = 0 \qquad (16)$$

Hence proved.

Theorem:

Let $f(z)$ be analytic on the set R' obtained from a rectangle R by omitting a finite number of ϵ_j such that $\lim_{z \to \epsilon_j} (z-\epsilon_j) f(z) = 0 \ \forall j$. Then

$$\int_{\delta R} f(z)\, dz = 0.$$

Proof:

Given R be a rectangle $R' \rightarrow$ Rectangle obtained from R by omitting a finite number of ϵ_j.

Let $f(z)$ be analytic on R'.

$$\lim_{z \to \epsilon_j} (z-\epsilon_j) f(z) = 0 \ \forall j$$

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To prove :

$$\int_{\delta R} f(z)\,dz = 0$$

Divide the rectangle R into 9 rectangle in the following manner. Apply cauchy's Theorem for all rectangles except R_0 which is the centre which contains by

$$\int_{\delta R} f(z)\,dz = \int_{\delta R_0} f(z)\,dz + \int_{\delta R_1} f(z)\,dz + \cdots\cdots + \int_{\delta R_8} f(z)\,dz$$

$$\int_{\delta R} f(z)\,dz = \int_{\delta R_0} f(z)\,dz$$

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$$\Rightarrow \left|\int_{\delta R} f(z)\,dz\right| = \left|\int_{\delta R_0} f(z)\,dz\right|$$

$$\left|\int_{\delta R} f(z)\,dz\right| \leq \int_{\delta R_0} |f(z)|\,dz \longrightarrow (1)$$

Since $\lim_{z \to \xi_j} (z-\xi_j)(f(z)) = 0$

$$\Rightarrow |(z-\xi_j)\,[f(z)-0]| < \xi$$

$$\Rightarrow |z-\xi_j|\,|f(z)| < \xi$$

$$|f(z)| < \frac{\xi}{|z-\xi_j|} \longrightarrow (2).$$

using (2) in (1) we get

$$\left|\int_{\delta R} f(z)\,dz\right| \leq \int_{\delta R_0} \frac{\xi}{|z-\xi_j|}\,|dz| \longrightarrow (3)$$

Suppose R_0 is a sequence with centre ξ

Then $|dz|$ is perimeter of the segments.

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i.e) $|dz| = 4a$, where "a" is side of the sequence

$$|z - \xi_j| > a/2$$

$$\frac{1}{z - \xi_j} < 2/a \longrightarrow (4)$$

sub (4) in (3) we get

$$\left|\int_{\delta R} f(z)dz\right| \le \xi \cdot 2/a\,(4a) = 8\xi$$

$$\left|\int_{\delta R} f(z)dz\right| < 8\xi \qquad \therefore \xi \text{ is arbitary.}$$

$$\left|\int_{\delta R} f(z)dz\right| = 0 \Rightarrow \int_{\delta R} f(z)dz = 0$$

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Hence the proof.

Cauchy Theorem :

If $f(z)$ is analytic in a open disc 's' then

$\int_{\gamma} f(z)dz = 0$ closed curve γ is Δ.

Proof :

Let $f(z)$ be analytic in an open disc

Let γ be a closed curve in Δ.

Define a function $f(z)$ by

$$f(z) = \int_{\gamma} f(z)\,dz$$

(i.e) $$f(z) = \int_{\gamma} f(z)\,(dx + i\,dy)$$

$$= \int_{\gamma} f(z)\,dx + i\int_{\gamma} f(z)\,dy. \longrightarrow (1)$$

where γ consists of the horizontal line segment from the centre (x_0, y_0) and the vertical

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line segment from (x_0, y_0) to (x, y)

Now consider the vertical segment y varies from y_0 to y and x fixed

$\therefore dx = 0$

eqn (1) becomes

$$f(z) = \int_{\sigma?} f(z)\,dx + i\int_{y_0}^{y} f(z)\,dy$$

(19)

$$\frac{d}{dy} f(z) = \frac{d}{dy} + i\int_{y_0}^{y} f(z)\,dy$$

$$\frac{d}{dy}[f(z)] = i\,f(z) \longrightarrow (2)$$

similarly along the horizontal line segment x varies x_0 to x and y fixed. (1) becomes,

$$dy = 0$$

$$f(z) = \int_{\sigma?} f(z)\,dz = \int_{x_0}^{x} f(z)\,dx$$

$$\frac{d}{dx} f(z) = \frac{d}{dx}\int_{x_0}^{x} f(z)\,dx$$

$$\frac{d}{dx}[f(z)] = f(z) \longrightarrow (3)$$

using (3) in (2) we get

$$\frac{d}{dy} f(z) = i\,\frac{d}{dx} f(z)$$

$$\Rightarrow \frac{d}{dx} f(z) = \frac{1}{i}\,\frac{d}{dy} f(z) = \frac{1}{i}\times\frac{i}{i}\cdot\frac{d}{dy} f(z)$$

$$\Rightarrow \frac{d}{dx} f(z) = -i\,\frac{d}{dy} f(z).$$

$\therefore$ F satisfies the cauchy's Riemann equations

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in complete form.

$\Rightarrow f(z)$ is analytic and $f'(z) = f(z)$.

$\Rightarrow f(z)\,dz$ is an exact differentiable.

$\Rightarrow \int_\gamma f(z)\,dz$ depends only on the end points

is closed curve

$$\int_\gamma f(z)\,dz = 0$$

Hence the proof..

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Theorem :

(7)

Let $f(z)$ be analytic in the region 'Δ''
finite
obtained by omitting a fixed ~~points~~ number of points ζ_j from an open disc Δ. If $f(z)$ satisfies the following conditions.

$$\lim_{z \to \zeta_j} (z-\zeta_j) f(z) = 0 \quad \forall j. \text{ Then}$$

$\int_\gamma f(z)\,dz = 0 \;\forall$ every closed curve γ in Δ.

Proof:

Given Δ be an open disc.

$\Delta \to$ obtained from by omitting ζ_j

$\gamma \to$ any closed curve in 'Δ''.

Given $f(z)$ is analytic in Δ and

$$\lim_{z \to \zeta_j} (z-\zeta_j) f(z) = 0$$

To prove :

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$\int_{\gamma} f(z)\,dz = 0$

Let open disc 'Δ' be $|z - z_0| < \delta$ and if exceed g_{ij} from Δ.

Assume that number of g_{ij} lies in Δ as shown in the following figure. Let us define

$$F(z) = \int_{\sigma} f(z)\,dz.$$

$$F(z) = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CD} f(z)(dx + i\,dy) \longrightarrow (1)$$

Suppose that the last segment is vertical i.e) AB and CD are vertical. We get along AB, CD.

x is fixed $\Rightarrow$ $\therefore dx = 0$

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Along BC, y is fixed

$\therefore dy = 0$

$(1) \Rightarrow dx = 0$

$$F(z) = i\int_{AB} f(z)\,dy + \int_{BC} f(z)\,dx + i\int_{CD} f(z)\,dy.$$

Now partially differentiating $F(z)$ w.r to "x" we get

$$\frac{\partial}{\partial x} F(z) = \frac{\partial}{\partial x} \int_{BC} f(z)\,dx = f(z) \longrightarrow (2)$$

Suppose that the last segment is horizontal we get

$$\frac{\partial}{\partial y} f(z) = i f(z) \longrightarrow (3)$$

From (2) & (3)

$$\frac{1}{i} \frac{\partial}{\partial y} f(z) = \frac{\partial}{\partial x} f(z)$$

$$-i \frac{\partial}{\partial y} f(z) = \partial/\partial x\, f(z)$$

$\therefore$ F satisfy the C.R equations

$\therefore$ $f(z)$ is analytic

It is partially derivatives are continuous and

hence we get,

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$$f'(z) = f(z)$$

$\therefore$ $f(z)$ is analytic

$\Rightarrow f(z)\,dz$ is an exact differentiable.

$\Rightarrow \int_{\gamma} f(z)\,dz$ depends only on the end points

by γ is called curve

$$\Rightarrow \int_{\gamma} f(z)\,dz = 0$$

Hence proved.

Theorem :

If the piecewise differentiable closed curve γ does not pass through the points 'a' then the value of Integrable

$\int_{\gamma} \frac{dz}{z-a}$ is multiple of $2\pi i$.

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UNIT - III.

Local Properties of Analytic functions :

Theorem :

Suppose that $f(z)$ is analytic in the region Ω obtained by omitting a point a from a region Ω. A necessary and sufficient conditions that there exists an analytic function in Ω which coincides with $f(z)$ in Ω is that $\lim_{z \to a} (z-a) f(z) = 0$. The extended function is uniquely determined. (1)

Proof :

Let a be the exponential point.

$$\lim_{z \to a} (z-a) f(z) = 0$$

Consider the circle c with centre a such that c and its interior lie inside Ω.

$$\text{when } z \neq a, \quad f(a) = \frac{1}{2\pi i} \int_c \frac{f(\zeta)\, d\zeta}{\zeta - a}$$

Then $f(z)$ is the extended function. This proof is the sufficient.

The necessary and uniqueness part follows from the fact that the extended function is continuous at $z = a$.

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Taylor's Theorem :

Theorem :

If $f(z)$ is analytic in a region Ω containing a it is possible to write

$$f(z) = f(a) + \frac{f'(a)}{1!}(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots + \frac{f^{n-1}(a)}{(n-1)!}(z-a)^{n-1} + \frac{f^{n}(z)}{n!}(z-a)^{n}$$

2

ξ ζ

where $f_n(z) = \frac{1}{2\pi i}\int_c \frac{f(\zeta)\,d\zeta}{(\zeta-a)^n(\zeta-z)}$ is analytic in Ω and c is a circle with centre 'a' and lie in Ω.

Proof :

⇒ consider the function $f_1(z) = \frac{f(z)-f(a)}{z-a}$

It is not defined at $z=a$ and

$$\lim_{z\to a}(z-a)f_1(z) = \lim_{z\to a}[f(z)-f(a)] \text{ and}$$

$$\lim_{z\to a} f(z) = f'(a)$$

There exists an analytic function say $f_1(z)$ such that by theorem.

1. $f_1(z) = f'(z)$, $z \neq a$

2. $f_1(a) = f'(a)$, $z = a$

Repeating this process we can defined an

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analytic function $f_2(z)$ as

$$f_2(z) = \frac{f_1(z) - f_1(a)}{z-a} \quad , z \neq a$$

and $f_2(a) = f_1'(a) \quad , z = a$

. . . .

. . . .

In this process $f_n(z)$ can be got from the following set of equations. (3)

$$f(z) = f(a) + (z-a) f_1(z)$$

$$f_1(z) = f_1(a) + (z-a) f_2(z)$$

$$f_2(z) = f_2(a) + (z-a) f_3(z)$$

. . . .

$$f_{n-1}(z) = f_{n-1}(a) + (z-a) f_n(z)$$

These are true for any z including $z = a$

Hence,

$$f(z) = f(a) + (z-a) f_1(a) + (z-a)^2 f_2(a) + \dots + (z-a)^{n-1} f_{n-1}(a) + (z-a)^n f_n(z) \longrightarrow (1)$$

Diff this n times and putting $z = a$ we get

$[\therefore z = a]$

$$= 0 + 0 + 0 + \dots \dots + n! \, f_n(a)$$

$$f^n(a) = \lfloor n \, f_n(a)$$

$$f_n(a) = \frac{1}{\lfloor n} f^n(a)$$

using this in eqn (1) we get

$$f(z) = f(a) + \frac{z-a}{\lfloor 1} f'(a) + \frac{(z-a)^2}{\lfloor 2} f''(a) + \dots + \frac{(z-a)^{n-1}}{\lfloor n-1} f^{n-1}(a) + (z-a)^n f_n(z) \longrightarrow (2)$$

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where $f_n(z)$ is analytic by our assumption

by cauchy's integral formula

$$f_n(z) = \frac{1}{2\pi i} \int_c \frac{f_n(\xi)}{(\xi - z)} \, d\xi$$

$$= \frac{1}{2\pi i} \int_c \frac{1}{(\xi - z)} \left[\frac{f(\xi)}{(\xi-a)^n} - \frac{f(a)}{(\xi-a)^n} - \frac{f'(a)}{\lfloor 1 (\xi-a)^{n-1}} \right.$$

$$\left. (4) \quad - \frac{f^{n-1}(a)}{\lfloor n-1 \, (\xi - a)} \right] d\xi$$

[using 2]

$$= \frac{1}{2\pi i} \int_c \frac{f(\xi)\, d\xi}{(\xi-a)^n (\xi - z)} - \frac{1}{2\pi i} \left(\sum_{m=1}^{n} \frac{f^{n-m}(a)}{\lfloor n-m} \int \frac{d\xi}{(\xi-a)^m (\xi - z)} \right) \rightarrow (3)$$

Let $f_m(a) = \int \frac{d\xi}{(\xi-a)^m (\xi - z)}$

$$f_1(a) = \int \frac{d\xi}{(\xi - a)(\xi - z)}$$

$$= \frac{1}{z-a} \int_c \left(\frac{1}{\xi - z} - \frac{1}{\xi - a} \right) d\xi$$

$$= \frac{1}{z-a} \left[\int \frac{d\xi}{\xi - z} - \frac{d\xi}{\xi - a} \right]$$

$$= \frac{1}{z-a} [2\pi i - 2\pi i]$$

$$= 0$$

But $f_1^n(z) = \lfloor n \, f_{n+1}(z)$ using lemma.

$$f_{n+1}(z) = \frac{1}{\lfloor n} f_1^n(z).$$

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$$f_{n+1}(a) = \frac{1}{\lfloor n} f_1^{n}(a)$$

where $n=0$ $\quad f_1(a) = f_1'(a) = 0$

$n=1$ $\quad f_2(a) = \frac{f_1'(a)}{\lfloor 2} = 0$

$$f_n(a) = 0$$

using this in (3) (5)

$$f_n(z) = \frac{1}{2\pi i} \int_c \frac{f(\xi)\,d\xi}{(\xi-a)^n(\xi-z)}$$

This representation is valid inside of c.

Zeros and poles:

Definition:

$z=a$ is said to be a zero of $f(z)$ if $f(a)=0$ and $f^n(a) \neq 0$. Then $z=a$ is said to be a zero of order n.

Example:

If $f(z) = (z-1)^5 (z+5)^2$ then $z=1$ is a zero of order 5 for $f(z)$ and $z=-5$ is a zero of order 2 for $f(z)$.

Singularities:

$z=a$ is said to be a singular point of $f(z)$ if $f(z)$ is not analytic (regular) at $z=a$. A singular point is also sometimes called as "exceptional points".

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Isolated singularities :

⇒ A singular point $z=a$ is said to be an isolated singularities if there is an neighbourhood of $z=a$ containing no other singularity of $f(z)$.

(6)

⇒ suppose in every neighbourhood of $z=a$. $f(z)$ has infinitely many singularities. Then $z=a$ is called a non-isolated singularity. In this case $z=a$ is a limit of the set of the singularities of $f(z)$.

Removable singularity :

An isolated singularity $z=a$ is said to be a removable singularity of $f(z)$ if

$$\lim_{z \to a} (z-a) f(z) = 0.$$

pole :

A singularity $z=a$ of $f(z)$ is said to be a pole of $f(z)$ if $\lim_{z \to a} f(z) = \infty$

A pole $z=a$ is said to be a order m if $\lim_{z \to a} (z-a)^m f(z) \neq 0$.

If $m=1$. Then the pole is called simple pole.

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Essential singularity :

A singularity $z = a$ of $f(z)$ is said to be an essential singularity if it is neither a we can have the following Laurent's expansion of $f(z)$ about the singular point. If $z = a$ is an isolated singularity of $f(z)$. Then $f(z)$ can be expanded above $z = a$ in the annular region.

$r_2 \leq |z-a| \leq r$ $(r_2 < r_1)$ so,

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=0}^{\infty} b_n (z-a)^{-n}.$$

where,

$$a_n = \frac{1}{2\pi i} \int_{c_1} \frac{f(\zeta)\,d\zeta}{(\zeta - a)^{n+1}}$$

$$b_n = \frac{1}{2\pi i} \int_{c_2} (\zeta - a)^{n-1} f(\zeta)\,d\zeta.$$

The second term of the RHS is called the principal part of $f(z)$ at $z = a$.

$\Rightarrow$ $z = a$ is a removable singularity if the principal part has no terms.

$\Rightarrow$ $z = a$ is a pole if the principal part has finite number of terms.

$\Rightarrow$ $z = a$ is an essential singularity if the principal part has an infinite number of terms.

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Note :

For finding the nature of a singularity the following two notes are useful.

1) The limit point of zeros of an analytic function $f(z)$ is an isolated essential singularity

2) The limit point of poles of $f(z)$ is a non - isolated essential singularity. (8)

Nature of singularity by use of limits :

A singularity at $z=a$ of $f(z)$ is called

a) Removable if $\lim_{z \to a} f(z)$ exists and finite.

b) Pole if $\lim_{z \to a} f(z)$ exists and infinite

c) Essential if $\lim_{z \to a} f(z)$ does not exists.

Example :

1. $f(z) = \frac{1}{z-1} + 4 + 5(z-1) + (z-1)^2$

$z=1$ is a simple pole.

2. $f(z) = \frac{z^2+7z}{(z+1)^2(z-2)^3}$

$z=-1$ is a pole of order 2.

$z=2$ is a pole of order 3.

3. Discuss the singularity at $z=0$

$f(z) = \frac{\sin z}{z}$

$z=0$ is a singularity.

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$$\frac{\sin z}{z} = \frac{1}{z}\left[z - \frac{z^3}{\underline{|3}} + \frac{z^5}{\underline{|5}} - \cdots\right]$$

$$= 1 - \frac{z^2}{\underline{|3}} + \frac{z^4}{\underline{|5}}$$

$$\lim_{z \to 0} \frac{\sin z}{z} = 1.$$

The limit exists and is finite.

$\therefore z = 0$ is a removable singularity

4. $f(z) = \frac{e^z}{z}$

$$= \frac{1}{z}\left[1 + \frac{z}{\underline{|1}} + \frac{z^2}{\underline{|2}} + \frac{z^3}{\underline{|3}} + \cdots\right]$$

$$= \frac{1}{z} + \frac{1}{\underline{|1}} + \frac{z}{\underline{|2}} + \cdots$$

There is only one negative power term of z

$\therefore z = 0$ is a simple pole.

5. $f(z) = \frac{e^z}{z^2}$

$$= \frac{1}{z^2}\left[1 + \frac{z}{\underline{|1}} + \frac{z^2}{\underline{|2}} + \cdots\right]$$

$$= \frac{1}{z^2} + \frac{1}{z} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \cdots$$

There are two negative power of z

$\therefore z = 0$ is a pole of order 2.

6. $f(z) = \cot z$

$$= \frac{\cos z}{\sin z}$$

$$\sin z = 0$$

$\Rightarrow z = n\pi \quad , n = 0, \pm 1, \pm 2, \ldots$

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The limit points of these pole is $z=\infty$

$\therefore z=\infty$ is a non-isolated essential singularity.

7. $f(z) = \sin(1/z)$

The zeros of $\sin(1/z)$ is given by

$\frac{1}{z} = n\pi, \quad n = 0, \pm 1, \pm 2 \ldots$ (10)

$\therefore z = \frac{1}{n\pi}, \quad n = 0, \pm 1, \pm 2 \ldots$

The limit point of these zeros is $z=0$.

$z=0$ is an isolated essential singularity.

8. $f(z) = \sin z - \cos z$ at $z=0$

zeros of $f(z)$ is given by

$$\sin z - \cos z = 0$$

$$\sin z = \cos z$$

$$\frac{\sin z}{\cos z} = 1$$

$$\tan z = 1$$

i-e) $z = \pi/4$, $z = n\pi \pm \pi/4 \quad n = 0, 1, 2 \ldots$

The limit point of these zeros is $z=\infty$

$\therefore z=\infty$ is a isolated essential singularity.

Weirstrass theorem:

An analytic function comes arbitarily close to any complex value in every

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neighbourhood of an essential singularity.

proof :

Let us prove this theorem by method of contradiction.

Assume that the theorem is not true.

Given $\epsilon, \delta > 0$, and for a complex number c

There exist a neighbourhood of a (except for z=a)

$|z-a| < \delta$ at which $|f(z)-c| < \epsilon$ (1)

$$\Rightarrow \frac{1}{|f(z)-c|} < \frac{1}{\epsilon}$$

W.K.T $z=a$, is an isolated singularity and if

$|f(z)|$ is bounded. Then f(z) has a removable

singularity at $z=a$.

Hence $\frac{1}{f(z)-c}$ has a removable singularity at

$z=a$

$\Rightarrow \frac{1}{f(z)-c}$ has no negative powers of $(z-a)$.

$$\therefore \frac{1}{f(z)-c} = \sum_{n=0}^{a} a_n (z-a)^n.$$

If $a_0 \neq 0$ define $\frac{1}{f(a)-c} = a_0$

$$f(a) - C = \frac{1}{a_0}$$

$$f(a) = c + \frac{1}{a_0}$$

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(i.e) $(f(z)-c)$ becomes analytic and non zero at $z=a$

There is a contradiction to one hypothesis. That $z=a$ is an essential singularity

If $a_n=0$, $n=0,1,2 \ldots m-1$ (12)

Then $\frac{1}{f(z)-c} = \sum_{n=m}^{\infty} a_m (z-a)^n$

$$= a_m (z-a)^m + a_{m+1}(z-a)^{m+1} + \ldots$$

$$= (z-a)^m [a_m + a_{m+1}(z-a) + \ldots]$$

$$= (z-a)^m \sum_{n=0}^{\infty} a_{m+n} (z-a)^n.$$

$\rightarrow (z=a)$ is a zero of order m to $\frac{1}{f(z)-c}$

$\rightarrow$ $z=a$ is a pole of order m to $f(z)-c$.

$\rightarrow$ $z=a$ is a pole of order m to $f(z)$.

This is also a contradiction to our hypothesis that a is an essential singularity.

$\therefore$ our assumption is wrong.

i.e) The theorem is true.

The local mapping :

"Determine the number of zeros of an analytic function".

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Statement :

Let z_j be the zeros of a function $f(z)$ which is analytic in a disc Δ and does not vanish identically each zero being counted as many times as its order indicate. For every closed curve γ in Δ which does not pass through a zero.

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$$\sum_j \eta(\gamma, z_j) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz$$ where the sum has only a finite number of terms $\neq 0$.

Proof :

Let $f(z)$ be analytic and not identically zero in Δ. consider a closed curve $\gamma \in \Delta$ not passing through a zero of $f(z)$

case (i) :

suppose that $f(z)$ has only a finite number of zeros in Δ.

Let $z_1, z_2, \ldots z_n$ be the zeros of $f(z)$ where each zero is repeated as many time as its order indicates

Then $f(z) = (z-z_1)(z-z_2) \ldots (z-z_n)\, g(z) \longrightarrow (1)$

where $g(z)$ is analytic and $\neq 0$ in Δ and $g(z) \neq 0$.

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(1) $\Rightarrow \log f(z) = \log [(z-z_1)(z-z_2)\cdots(z-z_n)\, g(z)]$

$$= \log z-z_1 + \log z-z_2 + \cdots + \log z-z_n + \log g(z)$$

Sing

$$\frac{1}{f(z)} f'(z) = \frac{1}{z-z_1} + \frac{1}{z-z_2} + \cdots + \frac{1}{z-z_n} + \frac{g'(z)}{g(z)}.$$

forming the log arithmetic derivative we obtain

$$\frac{f'(z)}{f(z)} = \frac{1}{z-z_1} + \frac{1}{z-z_2} + \cdots + \frac{1}{z-z_n} + \frac{g'(z)}{g(z)} \longrightarrow (2)$$

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For $z \neq z_j$

Now $\frac{g'(z)}{g(z)}$ is analytic in Δ and γ is a closed curve in Δ.

Hence by cauchy's theorem for a circular disc Δ,

$$\int_\gamma \frac{g'(z)}{g(z)}\, dz = 0$$

From (2) & (3) we get

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\, dz = \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_1} + \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_2} + \cdots + \frac{1}{2\pi i}\int_\gamma \frac{dz}{z-z_n}$$

$$= \eta(\gamma, z_1) + \eta(\gamma, z_2) + \cdots + \eta(\gamma, z_n)$$

$$= \sum_{j=1}^{n} \eta(\gamma, z_j)$$

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Case (ii) :

Let $f(z)$ has infinitely many zeros in Δ. Then there exists a closed disc Δ' concentric with Δ such that γ is contained in Δ'

In the compact set Δ' there are only finite number of zeros will have a limit and the function will be identically zero.

So for the infinite number of zeros which lie outside Δ'

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$$\eta(\gamma, z_j) = 0$$

This implies that their contribution to the infinite sum will be zero.

Hence the theorem.

corollary :

1. If $w = f(z)$ maps γ onto a closed curve Γ in the w-plane then $\eta(\Gamma, 0) = \sum_j \eta(\gamma, z_j)$

proof :

Let $w = f(z)$ maps γ onto Γ

$$\log w = \log f(z)$$

Diff w.r to "z" we get

$$\frac{1}{w}\, dw = \frac{1}{f(z)} f'(z)\, dz.$$

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ding and x^{1y} by $\frac{1}{2\pi i}$

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{dw}{w} = \frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}\,dz$$

$$\eta(\Gamma, 0) = \sum_j \eta(\gamma, z_j)$$

If $\eta(\gamma, z_j) = 0$ (or) 1 according as z_j lies outside (or) inside of γ.

$\therefore \frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}\,dz$ gives the total number of zeros enclosed by γ.

2) Let $z_j(a)$ be the roots of the equation $f(z)=a$

V.V. (*) where a is any complex number Then

$$\eta(\Gamma, a) = \sum_j \eta(\gamma, z_j(a))$$

Proof :

Let $w = f(z)$

$dw = f'(z)\,dz$

consider $\eta(\Gamma, a) = \frac{1}{2\pi i}\int_{\Gamma} \frac{dw}{w-a}$

$$= \frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}\,dz$$

$$\eta(\Gamma, a) = \sum_j \eta(\gamma, z_j(a)).$$

V.V. (*) 3. If γ is a circle then $f(z)$ takes the values a and b equally many times inside of γ.

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Proof:

Let $z_j(a)$ be the zeros of $f(z)=a$.

Let $z_j(b)$ be the zeros of $f(z)=b$

$$\eta(\Gamma, a) = \sum_j \eta(\gamma, z_j(a))$$

$$\eta(\Gamma, b) = \sum_j \eta(\gamma, z_j(b)).$$

(17)

But $\eta(\Gamma, a) = n(\Gamma, b)$

$$\sum_j \eta(\gamma, z_j(a)) = \sum_j \eta(\gamma, z_j(b))$$

$\therefore$ $f(z)$ takes the values of a and b equally many times inside of γ.

Local mapping Theorem (or) local correspondence

Theorem:

Statement:

Suppose that $f(z)$ is analytic at z_0, $f(z_0)=w_0$ and that $f(z)-w_0$ has a zero of order n at z_0. If $\varepsilon>0$ is sufficiently small, there exists a corresponding $\delta>0$ such that for all a with $|a-w_0|<\delta$. The equation $f(z)=a$ has exactly n roots in the disc $|z-z_0|<\varepsilon$.

Proof:

Given $f(z)$ is analytic function at z_0. $f(z)=w_0$ and $f(z)-w_0$ has zero of over n at z_0. Given $\varepsilon>0$ is sufficiently small.

Let $f(z)$ be defined and analytic for

$|z-z_0| \leq \epsilon$ and so that z_0 is the only zero of $f(z)-w_0$ in this disc.

Let γ be the circle $|z-z_0| = \epsilon$. where $\epsilon > 0$ and Γ is the image under the mapping $w = f(z)$

Since w_0 belongs to the complement of the closed set. If there exists a neighbourhood $(w-w_0)$.

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which does not intersect Γ.

Applying Rouche's theorem to the function $w_0 - a$ and $f(z)-w_0$

we have $f(z)-w_0$ and $f(z)-w_0+w_0-a = f(z)-a$ have the same number of zeros in

$|z-z_0| < \epsilon$.

But $f(z)-w_0$ has zero of order n to z_0. Since zero of multiplicity 'n' is counted n times.

The function $f(z)-w_0$ has n zeros in $|z-z_0| < \epsilon$. Hence the function $f(z)-a$ has n zeros in $|z-z_0| < \epsilon$. In otherwords $f(z)=a$ has exactly n roots in $|z-z_0| < \epsilon$

Hence proved.

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open set :

A set is open if it is a neighbourhood of each its element

$N_{\epsilon}(z_0)$ means neighbourhood at $z_0 \Rightarrow |z-z_0| < \epsilon$

$N_{\delta}(z_0)$ means neighbourhood at $z_0 \Rightarrow |z-z_0| < \delta$.

Open Mapping Theorem :

statement :

A non-constant analytic function maps open sets onto open sets.

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proof :

Let $f(z)$ be a non-constant analytic function in a region.

Let G be any open set in Ω

To prove :

$f(z)$ is open

Let $w_0 \in f(z)$ $\exists$ $z_0 \in G$ such that $w_0 = f(z_0)$ which is a non-constant analytic function by local mapping theorem, it can only have zero of finite order at z_0. we can find $\epsilon > 0$ such that $f(z) - w_0$ has z_0 which has only zero in $N_{\epsilon}(z_0)$ in G. where,

$$N_{\epsilon}(z_0) = \left\{ z / |z-z_0| < \epsilon \right\}$$

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By local correspondance, we can find $\delta > 0$ such that for all a with $|a - w_0| < \delta$, $f(z) - a$ has exactly n. zeros in $N_{\epsilon_1}(z_0)$.

If $a \in N_\delta(w_0)$ we can find $z_1 \in N_{\epsilon_1}$

$\Rightarrow f(z_1) - a = 0$

$\therefore a = f(z_1) \in f(N_{\epsilon_1}(z_0)) \in f(G)$

$\therefore N_\delta(w_0) \subset f(G)$

$\rightarrow f(G)$ is a neighbourhood of each of its point

$\rightarrow f(z)$ is open

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Hence proved.

The Maximal principle theorem :

If $f(z)$ is analytic and non-constant in a region Ω then its absolute value $|f(z)|$ has no maximum inside Ω.

Proof :

Let z_0 be any point in Ω

Take a circle C as follows $C : |z - z_0| = r$

$$z - z_0 = re^{i\theta} \qquad 0 \le \theta \le 2\pi$$

$$z = z_0 + re^{i\theta}$$

$$dz = re^{i\theta} \cdot i\,d\theta$$

By cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0}\,dz.$$

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$$= \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(z_0 + re^{i\theta})}{re^{i\theta}} \cdot ire^{i\theta} \, d\theta$$

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} f(z_0 + re^{i\theta}) \, d\theta \longrightarrow (1)$$

Suppose $f(z)$ has a maximum at z_0 in Ω

Then $|f(z_0 + re^{i\theta})| \leq |f(z_0)| \longrightarrow (2)$

Let $|f(z_0 + re^{i\theta})| < |f(z)|$

From (1)

(21)

$$|f(z)| = \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta}) \, d\theta|$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} |f(z_0 + re^{i\theta})| \, |d\theta|$$

$$< \frac{1}{2\pi} \int_0^{2\pi} |f(z_0)| \, |d\theta| \quad \text{(using (2))}$$

$$< \frac{1}{2\pi} |f(z_0)| \int_0^{2\pi} |d\theta|.$$

(i.e) $|f(z)| < \frac{1}{2\pi} |f(z_0)| \, (2\pi)$

$|f(z)| < |f(z_0)|$

This is not possible.

$\therefore$ $|f(z)|$ is equal to $|f(z_0)|$ throughout on all sufficiently small circle δ.

$|z - z_0| = r$ and hence is a neighbourhood of z_0.

i.e) $|f(z)| = |f(z_0)|$ (or) Ω

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$\Rightarrow f(z)$ is constant

This contradiction proves that $f(z)$ has no maximum inside Ω.

Theorem :

If $f(z)$ is defined and continuous on a closed bounded set E. and analytic on the interior of E. Then the max of $|f(z)|$ on E is assumed on the boundary of E.

(22)

Proof :

Let E be compact. Then $|f(z)|$ has a maximum on E. suppose this maximum is attained at z_0

If z_0 is on the boundary then the theorem is proved.

If z_0 is inside E, then $|f(z_0)|$ is the maximum of $|f(z)|$ in a disk $|z-z_0| < \delta$ which is contained in E

This is not possible unless $f(z)$ is constant is the component of interior of E, which contains z_0 from the property of continuity of $f(z)$, $|f(z)|$ is equal to its maximum on the whole boundary of that component.

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This boundary is not empty and it is contained in the boundary of E.

Then the maximum of $|f(z)|$ is obtained only on the boundary of E.

Schwarz Lemma :

If $f(z)$ is analytic for $|z|<1$ and satisfies the conditions $|f(z)| \leq 1$, $f(0)=0$ then $|f(z)| \leq |z|$. and $|f'(0)| \leq 1$. If $|f(z)| = |z|$ for some $z \neq 0$ (or) if $|f'(0)|=1$ then $f(z)=cz$ with a constant c of absolute value 1.

Proof : (23)

Define a function

$$f_1(z) = \begin{cases} \dfrac{f(z)}{z}, & z \neq 0 \\ f'(z), & z=0 \end{cases}$$

Take the circle $|z| = r < 1$

$$|f_1(z)| = \left| \frac{f(z)}{z} \right|$$

$$\leq \frac{|f(z)|}{|z|}$$

$$\leq \frac{1}{r}$$

As $r \to 1$, $|f_1(z)| \leq 1$

$$\frac{|f(z)|}{|z|} \leq 1$$

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$$|f(z)| \leq |z|$$

If $f_1(z) = 1$ at a single point then $|f_1(z)|$ attains its maximum inside

$$C : |z| = r$$

$\Rightarrow f_1(z)$ is a constant say C.

$$\therefore \frac{f(z)}{z} = c$$

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$$f(z) = cz$$

Here $|f(z)| = |c||z|$

$$|z| = |c||z|$$

$$1 = |c|$$

Hence the proof of the lemma.

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The General form of cauchy's Theorem :

Introduction :

we assume cauchy's theorem for a circular region. The generalised region in two ways.

i) To identity the region in which cauchy's theorem is valid universely.

ii) In the region to find the type of curves γ for which the assertion of cauchy's theorem is valid.

Definition : [chain]

A formal sum $a_1\gamma_1 + a_2\gamma_2 + \cdots + a_n\gamma_n$ of the arc $\gamma_1, \gamma_2 \ldots \gamma_n$ is called a chain

The following operations are valid in a chain.

i) Permutation of an arc

ii) subdivision of an arc

iii) Fusion of sub arcs to a single arc

iv) Reparametrization of an arc

v) cancellation of opposite arcs.

The chains are said to be "Identical" iff they valid the "same line Integral" for all functions $f(z)$.

A chain is called "zero chain" (or) "valid chain." if all $a_i = 0$.

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A chain is a cycle if it can be represented as a "sum of closed curves".

(2)

Note :

1) we have seen already that the integral of an exact differential for a closed curve $\gamma = 0$. The same is true for cycle also.

2) The index of a point with respect to a cycle is defined as in the case of a single closed curve.

3) If γ_1 and γ_2 are two cycles. Then

$$n(\gamma_1 + \gamma_2, a) = n(\gamma_1, a) + n(\gamma_2, a)$$

2m **Definition :**

A region is "simply connected" if its complement with respect to the extended plane is connected.

Example :

A circular disk a half plane and a parallel strip are simply connected.

Theorem :

A region Ω is simply connected iff $n(\gamma, a) = 0$ for all cycles γ in Ω and all points a which do not belong to Ω

Proof :

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Necessary Part:

Let γ be a cycle in Ω.

since Ω is simply connected. If the complement of Ω is connected

$a \notin \Omega \Rightarrow a \in \Omega^c$

since $a \in \Omega^c$, Ω^c is unbounded.

$\Rightarrow a \in$ the unbounded region determined by γ.

$$\therefore \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a} = 0.$$

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i.e) $n(\gamma, a) = 0$

Sufficient part:

Let $n(\gamma, a) = 0$ for every cycle γ in Ω. we have to prove that Ω is simply connected.

Assume that Ω is not simply connected.

then by definition Ω^c is not connected.

i.e) $\Omega^c = A \cup B$. where A and B are non-empty disjoint closed sets.

Since $\Omega^c = A \cup B$, either A [or] B should be unbounded.

Take B has unbounded and A is unbounded.

Take $a \in A$

Take $\delta = \min d(A, B)$, $\delta > 0$.

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cover the whole plane with a set of squares Q of side $< \delta / \sqrt{2}$.

choose $a \in A$ lies at the centre of a square.

consider the cycle $\gamma = \sum \delta Q_j$.

(4)

where,

i) δQ_j denote the boundary of the squares Q_j

ii) The sum ranges over all squares Q_j in the net which have a point in common with A.

Then, $n(\gamma, a) = \sum_j n(\delta Q_j, a) = 1$ [$\because$ a is the centre of h]

Further γ does not meet B, if the cancellation are carried out, then also it is clear that γ does not meet A.

$\therefore \gamma \subset \Omega$

Thus we have prove that if Ω is not simply connected there for a cycle $\gamma \subset \Omega$ and $a \subset \Omega$, $n(\gamma, a) \neq 0$.

Taking the (-ve) statement of the above (contra positive).

$\therefore$ The theorem is proved.

Definition :

A cycle γ is an open set Ω is said to be "homologus" to zero with respect to Ω if $n(\gamma, a) = 0$

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for all points a in the complement of Ω

$\gamma \sim 0 \pmod{\Omega}$ (or). $\gamma = 0 \pmod{\Omega}$

Note :

$\gamma_1 - \gamma_2 \Longleftrightarrow \gamma_2 - \gamma_1 \sim 0 \pmod{\sim}$. Homologus can be added and subtract. If $\Omega \subset \Omega'$ then $\gamma \sim 0 \pmod{(\Omega)} \Rightarrow \gamma \sim 0 \pmod{\Omega'}$

General form of cauchy's theorem :

(8) 10m If $f(z)$ is analytic in Ω then $\int_\gamma f(z)\,dz = 0$. for every cycle γ. which is homologus to zero in Ω

proof:

(5)

case (i) :

Let Ω be bounded.

Given, $\delta > 0$ cover the plane by a set of square of side δ.

Let Q_j, $j \in J$ be the closed squares that are contained in Ω.

Since, Ω is bounded, j is finite

Since, δ is small, J is non empty.

The union of square Q_j $j \in J$ consists of closed regions are oriented boundaries make up the cycle.

$$\Gamma_\delta = \sum_{j \in J} \partial Q_j$$

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Γ_δ is the sum of oriented line segments which are sides of exactly one Q_j.

Let $\Omega\delta$ = Interior of $\cup Q_j$

Let γ be a cycle which is homologus to zero in Ω.

choose δ. so small such that, γ is connected in $\Omega\delta$.

(6)

consider a point $\in \Omega$ $\Omega\delta$. It belongs to atleast one Q, which is not a Q_j.

There is a point $\zeta_0 \in Q$ which is not in Ω

Join and $\underline{\zeta_0}$ by a line segment which lies in Q and does not meet $\Omega\delta$.

i.e) $n(\gamma, \zeta) = n(\gamma, \zeta) = 0 \longrightarrow (1)$

$\Rightarrow n(\gamma, \ ) = 0$ for all points on Γ_δ

suppose that f is analytic in Ω

If z lies in the interior of Q_{j_0} for some j_0.

Then,
$$\frac{1}{2\pi i}\int_{\partial Q_j} \frac{f(\zeta)d}{\zeta - z} = \begin{cases} f(z) & \text{if } j = j_0 \\ 0 & \text{if } j \neq j_0 \end{cases}$$

Here,
$$f(z) = \frac{1}{2\pi i}\int_{\Gamma_\delta} \frac{f(\zeta)\,d\zeta}{\zeta - z}, \quad z \in \Omega\delta.$$

$$\int_\gamma f(z)dz = \int_\gamma \frac{1}{2\pi i}\int_{\Gamma_\delta} \frac{f(\zeta)\,d\zeta}{\zeta - z}\,dz$$

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$$= \int_{\Gamma_\gamma} \frac{1}{2\pi i}\left(\int_\gamma \frac{dz}{\zeta - z}\right) f(\zeta)\, d\zeta \rightarrow (2)$$

But, $\int_\gamma \frac{dz}{\zeta - z} = n(\gamma, \zeta) = 0$ [from (1)].

R.H.S of (2) is zero

$\therefore$ i.e) $\int_\gamma f(z)\,dz = 0$

(7)

case (ii) :

Let Ω be unbounded we replace it by its intersection Ω' with a disk $|z| < d$ which is large enough to contain any point a in the complement of Ω' is either in the complement of Ω or $|z| < \gamma$.

In either case $n(\gamma, a) = 0$. So that $\gamma \sim 0$ (mod Ω) is valid to "Ω" by case (i).

$\therefore$ Hence the theorem is proved for Ω in this case also.

Definition :

2m A function $f(z)$ which is analytic in a region Ω except for poles is said to be "meromorphic in Ω".

Note :

Let $f(z)$ be an analytic function in a region

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Ω except a point $a \in \Omega$ so that a is an isolated singularity of $f(z)$ consider the conditions.

1. $\lim_{z \to a} |z-a|^{\alpha} \cdot |f(z)| = 0$

2. $\lim_{z \to a} |z-a|^{\alpha} \cdot |f(z)| = \infty$ for all real values of α.

If (1) holds for a certain α then it holds for all larger α and hence for some integer m.

Then $(z-a)^m f(z)$ has a removable singularity at $z = a$.

(8)

Definition :

A differential $p\,dx + q\,dy$ is said to be "locally exact in Ω" if it is exact in some neighbourhood of each point in Ω.

Multiply connected Regions :

A region Ω is said to be multiply connected if it is not simply connected.

Ω is said to have the "finite connectivity n" if the complement of Ω has exactly n components and "infinite connectivity". if the complement has infinitely many components.

Note :

Ω have a finite connectivity n.

Let $A_1, A_2 \ldots\ldots A_n$ be the components of the

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complement of Ω.

Let $\infty \in A_n$

If γ is an arbitrary cycle in Ω, then $n(\gamma, a)$ is constant. when a varies over any one of the components A_i and that $n(\gamma, a)=0$ in A_n.

we can find cycles γ_i, $i = 1, 2 \ldots n-1$ such that $n(\gamma_1, a) = 1$ for all other points outside of Ω.

For a given cycle γ in Ω. Let c_i be the constant values of $n(\gamma, a)$ for $a \in A_i$. we find that any point outside of Ω has the index zero with respect to the cycle

(9)

$\gamma_1 - c_1\gamma_1 - c_2\gamma_2 \ldots c_{n-1}\gamma_{n-1}$

Hence $\gamma \sim c_1\gamma_1 + c_2\gamma_2 + \ldots + c_{n-1}\gamma_{n-1}$

These cycles $\gamma_1, \gamma_2 \ldots \gamma_{n-1}$ are unique and linearly independent. They form a homology basis for the region Ω.

Hence we can conclude that every region Ω which has a finite homology basis has finite connectivity and the number of basis element is one less than the connectivity.

Definition:

If $\int_\gamma f dz = c_1 \int_{\gamma_1} f dz + c_2 \int_{\gamma_2} f dz + \ldots + c_{n-1} \int_{\gamma_{n-1}} f dz$

where $\gamma \sim c_1\gamma_1 + c_2\gamma_2 + \ldots + c_{n-1}\gamma_{n-1}$

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Then the numbers $P_i \int_{\gamma_i} f\,dz$ depends only on the function and not on Ω.

They are called modulus of periodicity of the differential $f\,dz$.

The calculus of Residues:

The Residue of $f(z)$ at an isolated singularity a is the unique complex number R which makes $f(z) - \frac{R}{(z-a)}$. The derivative of a single valued analytic function in an annulus $0 < |z-a| < \delta$.

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Remark:

1) Since $f(z) - \frac{R}{(z-a)}$ is analytic in $0 < |z-a| < \delta$

we get $\int_c \left(f(z) - \frac{R}{z-a}\right) dz = 0$

where c is closed curve in the annulus

$$\int_c f(z)\,dz - R \int \frac{dz}{z-a} = 0$$

$$\Rightarrow \int_c f(z)\,dz = R \int \frac{dz}{z-a}$$

$$= R\,(2\pi i)$$

$$\Rightarrow R = \frac{1}{2\pi i} \int_c f(z)\,dz.$$

2) If $P = \int_c f(z)\,dz$ then $P = 2\pi i$ [Residues of $f(z)$]

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Residue Theorem :

Let $f(z)$ be analytic for isolated singularities a_j in a region Ω. Then $\frac{1}{2\pi i}\int_{\gamma} f(z)dz = \sum_j n(\gamma, a_j) \operatorname{Res}_{z=a_j} f(z)$ for any cycle γ which is homologus to zero in Ω and does not pass through any of the points a_j.

Proof:

Let $f(z)$ be analytic except for isolated singularities a_j in Ω

(11)

case (i) :

Let $a_1\, a_2\, a_3 \ldots . a_n$ be a finite number of singularities in Ω then $\Omega' = \Omega - \bigcup_{j=1}^{n} a_j$ then $f(z)$ is analytic in Ω'.

Let γ be a cycle homologus to zero in Ω such that γ does not pass through any a_j.

Then, $\operatorname{Res}_{z=a_j} (f(z)) = \frac{1}{2\pi i} \int_{c_j} f(z)dz$

Further γ satisfies the homology and $\gamma \sim \sum_j n(\gamma, a_j) c_j$. consider the cycle

$$\Gamma = \gamma - \sum_j n(\gamma, a_j) c_j .$$

Then

$$n(\Gamma, a_k) = n(\gamma, a_k) - n\left(\sum_j n(\gamma, a_j) c_j\, a_k\right)$$

$$= n(\gamma, a_k) - n(\gamma, a_k)\, n(c_j\, a_k)$$

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$= n(\gamma, a_k) - n(\gamma, a_k) - 1$

$= 0.$

If $a \notin \Omega$ then

$n(\Gamma, a) = n(\gamma, a) - \sum n(\gamma, a)\, n(c_j, a)$

$= 0 - 0 \qquad \gamma \sim 0 \pmod{\Omega}.$

$= 0$

The Γ is homologus to zero, which does not pass through any a_j

By general form of cauchy's theorem

$$\int_{\Gamma} f(z).dz = 0$$ (12)

$$\int f(z) dz = 0$$

$$\gamma - \sum_j n(\gamma, a_j) c_j$$

$$\int_{\gamma} f(z).dz = \int f(z).dz \cdot \sum_j n(\gamma, a_j) c_j.$$

$$= \sum_j n(\gamma, a_j) \int_{c_j} f(z).dz$$

$$\Rightarrow \frac{1}{2\pi i} \int_{\gamma} f(z).dz = \sum_j n(\gamma, a_j) \frac{1}{2\pi i} \int_{c_j} f(z).dz$$

$$= \sum_j n(\gamma, a_j) \operatorname{Res}_{z = a_j} f(z).$$

case (ii)

Let the singularities be infinitely many number in Ω. Now the set of all points 'a' with $n(\gamma, a) = 0$ is open and contain all points.

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outside of a large circle. The complement is a compact set hence it cannot contain more than a finite no. of the isolated points a_j.

$\therefore n(\gamma, a_j) \neq 0$ only for a finite no. of the singularities a_js.

Hence case (i) applies to these points.

Corollary :

If γ is a simple closed curve then $n(\gamma, a_j) = 1$ for all with in γ. Then the result of the above theorem reduces to $\frac{1}{2\pi i}\int_\gamma f(z)\,dz = \sum_j \operatorname{Res}_{z=a_j} f(z)$

Formula :

1) If $z=a$ is a simple pole for $f(z)$ then

$$\operatorname{Res}_{z=a} f(z) = \lim_{z\to a} (z-a) f(z)$$

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2) If $f(z) = \frac{h(z)}{g(z)}$ and $z=a$ is a simple pole then

$$\operatorname{Res}_{z=a} f(z) = \frac{h(a)}{g'(a)}$$

3) If $z=a$ is a pole of order m for $f(z)$ then the residue at $z=a$.

$$\operatorname{Res}_{z=a} f(z) = \frac{1}{\lfloor m-1} \lim_{z\to a} \left[\frac{d^{m-1}}{dz^{m-1}} (z-a)^m f(z) \right]$$

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Definition:

A cycle γ is said to "bound the region" Ω iff $n(\gamma, a)$ is defined and equal to 1 for all points $a \in \Omega$ and either undefined (or) equal to zero for all a not in Ω.

The argument principle theorem:

(14)

Statement:

If $f(z)$ is meromorphic in Ω with the zeros a_j and the poles b_K. Then

$$\frac{1}{2\pi i}\int_{\gamma} \frac{f'(z)}{f(z)}\, dz = \sum_j n(\gamma, a_j) - \sum_K n(\gamma, b_K).$$

For every cycle γ which is homologus to zero in Ω and does not pass through any of the zeros and poles.

Proof:

Let a_j be a zero of $f(z)$ with order m_j and b_K be a pole of $f(z)$ with order m_K. Enclose the zeros by circles c_j and poles by the circles c_K such that γ satisfies.

$$\gamma \sim \sum_j n(\gamma, a_j) + \sum_K n(\gamma, b_K)\, c_K.$$

Now a_j is a zero of order m_j inside c_j.

$$\therefore f(z) = (z - a_j)^{m_j} . g(z).$$

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where $g(a_j) \neq 0$.

logarithmic diff. given

$$\frac{f'(z)}{f(z)} = \frac{m_j}{z-a_j} + \frac{g'(z)}{g(z)}$$

This means a_j is a simple zero of $\frac{f'(z)}{f(z)}$ with residue m_j

b_K is the pole of order m_K for $f(z)$

$$\therefore f(z) = \frac{\phi(z)}{(z-b_K)^{m_K}}$$

(15)

log diff gives

$$\frac{f'(z)}{f(z)} = \frac{-m_K}{z-b_K} + \frac{\phi'(z)}{\phi(z)}$$

$\Rightarrow b_K$ is a simple pole of $\frac{f'(z)}{f(z)}$ with residue $-m_K$

Hence we can applied residue theorem for a_j and b_K.

Hence $\frac{1}{2\pi i}\int \frac{f'(z)}{f(z)}\,dz = \sum_j n(\gamma, a_j) - \sum_K n(\gamma, b_K)$.

corollary:

1) Argument principle for an analytic function

If $f(z)$ is analytic in Ω.

Then it has no poles

Hence the above result residues to

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_j n(\gamma, a_j)$$

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2) If N is the numbers of zeros of $f(z)$ inside γ. Then $N = \frac{1}{2\pi} \Delta_\gamma (\arg f(z))$ then $\Delta_\gamma[\arg f(z)]$ denotes the change of argument $f(z)$ as z varies over γ.

Proof :

we have $N = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz$

put $f(z) = Re^{i\theta}$

$|f(z)| = R$ and $\arg f(z) = \theta$

$$f'(z)dz = Re^{i\theta} . i d\theta + e^{i\theta} . dR$$ (16)

$$= e^{i\theta} [R i d\theta + dR]$$

$$= \frac{f(z)}{R} [R i d\theta + dR]$$

$$= f(z) \left[i d\theta + \frac{dR}{R} \right]$$

$$\frac{f'(z)}{f(z)} dz = i d\theta + \frac{dR}{R}$$

$$\therefore N = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} dz$$

$$= \frac{1}{2\pi i} \int_\gamma i d\theta + \frac{dR}{R}$$

$$= \frac{1}{2\pi i} \int_\gamma \frac{dR}{R} + \frac{1}{2\pi i} \int_\gamma i d\theta$$

$$= \frac{1}{2\pi i} (\log R)_\gamma + \frac{1}{2\pi} (\theta)_\gamma$$

$$= 0 + \frac{1}{2\pi} \Delta_\gamma (\arg f(z)).$$

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Rouche's theorem :

Statement :

Let γ be homologus to zero in Ω and such that $n(\gamma, z)$ is either 0 or 1 for any point z not on γ. Suppose that $f(z)$ and $g(z)$ are analytic in Ω and satisfy the inequality $|f(z) - g(z)| < |f(z)|$ on γ. Then $f(z)$ and $g(z)$ have the same number of zeros enclosed by γ

Proof :

By defn both $f(z)$ and $g(z)$ are zero for on γ.

since if $f(z) = 0$ on γ then

$|f(z) - g(z)| < |f(z)|$ on γ

(17)

$|0 - g(z)| < |0|$ on γ

$\Rightarrow |g(z)| < 0$ on γ

This is ab

Next suppose that $g(z) = 0$ on γ

Then $|f(z) = 0| < |f(z)|$ on γ

$|f(z)| < |f(z)|$

This is also

Further $|f(z) - g(z)| < |f(z)|$ on γ

$$\left| 1 - \frac{g(z)}{f(z)} \right| < 1 \text{ on } \gamma$$

$|f(z) - 1| < 1$ on γ.

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where $F(z) = \frac{g(z)}{f(z)}$

$\Rightarrow$ The values of $F(z)$ are continuous in the open disk with centre 1 and radius 1. Now $F(z)$ is meromorphic in Ω.

Then N = The number of zeros of $f(z)$ enclosed by γ

= The number of zeros of $f'(z)$ enclosed by γ

P = The number of poles of $f(z)$ enclosed by γ

= The number of poles of $f'(z)$ enclosed by γ.

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$$\therefore N-P = \frac{1}{2\pi i}\int_{\gamma} \frac{F'(z)}{F(z)}\,dz.$$

If Γ be the image of γ under F. Then $\Gamma = F(\gamma)$. Then $N-P = \frac{1}{2\pi i}\int_{\gamma} \frac{dw}{w}$ where $w = F(z)$

$= n(\Gamma, 0) \longrightarrow (1)$

But $|f(z)-1|<1$ and $\Gamma = F(\gamma)$

$\Rightarrow \Gamma$ in the unit disk with centre 1

$n(\Gamma, 0) = 1$

$N-P = 0$ [using (1)]

$N = P$

Number of zeros of $f(z)$ = Number of poles of $g(z)$ enclosed by γ.

Note :

1) The above Theorem $F(z)$ and $g(z)$ morinter. changable.

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2) we can apply this thm and prove Fundamental thrm of algebra.

Fundamental theorem of Algebra.

Statement:

If $p(z)$ is a polynomial of n^{th} polynomial degree with real or complex co-efficients then $p(z)=0$ has exactly n-roots

(19)

proof:

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n, \quad a_n \neq 0$$

Take $f(z) = a_n z^n$ and

$$g(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_{n-1} z^{n-1}$$

Let C be a circle $|z| = R$ $(R>1)$ on the circle C

$$\left|\frac{g(z)}{f(z)}\right| = \frac{|a_0 + a_1 z + \cdots a_{n-1}(z)^{n-1}|}{|a_n z^n|}.$$

$$\leq \frac{|a_0| + |a_1||z| + \cdots + |a_{n-1}||z^{n-1}|}{|a_n||z^n|}.$$

$$\leq \frac{|a_0| R^{n-1} + |a_1| R^{n-1} + \cdots + |a_{n-1}| R^{n-1}}{|a_n| R^n},$$

$$\leq \frac{|a_0| + |a_1| + \cdots + |a_{n-1}|}{|a_n| R}$$

we can choose R large enough such that

R.H.S < 1

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$\therefore \left|\frac{g(z)}{f(z)}\right| < 1$

$|g(z)| < |f(z)|$ on c

By Rouche's theorem $g(z)$, $f(z)+g(z)$ will have the same number of zeros inside c.

But $f(z) = a_n z^n$ has exactly n zeros inside c.

Hence $p(z) = f(z) + g(z)$ has exactly n zeros inside c.

i.e) $p(z) = 0$ has exactly n roots inside c

Hence the theorem.

Evaluation of integrals [contour integration]

TYPE-I :

Integration around the unit circle

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$$

where R is a rational function

put $z = e^{i\theta}$

$dz = i e^{i\theta} d\theta$, $d\theta = iz\, d\theta$

$d\theta = \frac{dz}{iz}$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}(z + 1/z) = \frac{z^2+1}{2z}.$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{1}{2i}\left[z - \frac{1}{z}\right] = \frac{z^2-1}{2iz}$$

$|z| = |e^{i\theta}| = 1$

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Take c as $|z| = 1$

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\,d\theta = \int_c R\left[\frac{z^2+1}{2z}, \frac{z^2-1}{2z}\right]\frac{dz}{iz}$$

$$= 2\pi i$$

Problem :

Evaluate $\displaystyle\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$.

Soln :

Let $I = \displaystyle\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$ (21)

Take $c : |z| = 1$, $z = e^{i\theta}$

$$dz = ie^{i\theta}\,d\theta$$

$$\frac{dz}{ie^{i\theta}} = d\theta \quad ; \quad d\theta = \frac{dz}{iz}$$

$$I = \int_c \frac{dz/iz}{a+\frac{z^2+1}{2z}} = \int_c \frac{2\,dz}{i(2az+z^2+1)}$$

$$= \frac{2}{i}\int_c \frac{dz}{z^2+2az+1}$$

$$= \frac{2}{i}\int_c f(z)\,dz$$

$\left[\text{where } f(z) = \dfrac{1}{z^2+2az+1}\right]$

$$f(z) = \frac{1}{(z-\alpha)(z-\beta)}$$

where $\alpha = \dfrac{-2a \pm \sqrt{4a^2-4}}{2}$

$\alpha = -a + \sqrt{a^2+1}$ [simple pole]

$\beta = -a - \sqrt{a^2-1}$

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Here $z=\alpha$ lies inside c (β lies outside c)

$$\underset{z=\infty}{\text{Resi}}\, f(z) = \lim_{z\to\alpha} (z-\alpha) f(z) = \lim_{z\to\alpha} \frac{(z-\alpha)}{(z-\alpha)(z-\beta)}$$

$$= \lim_{z\to\alpha} \frac{1}{(z-\beta)}$$

$z = \infty$

$$= \frac{1}{\alpha-\beta} = \frac{1}{2\sqrt{a^2-1}}$$

By cauchys Preside theorem

$$I = \frac{2}{i} \times 2\pi i \, (\Sigma \text{ Resi})$$

$$= 4\pi \left[\frac{1}{2\sqrt{a^2-1}}\right] = \frac{2\pi}{\sqrt{a^2-1}}$$

(22)

i.e) $$\int_0^{\pi} \frac{d\theta}{a+\cos\theta} = \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+\cos\theta}$$

$$= \frac{1}{2} \times \frac{2\pi}{\sqrt{x^2-1}}$$

$$= \frac{\pi}{\sqrt{x^2-1}}$$

Evaluate $\displaystyle\int_0^{\pi/2} \frac{dx}{a+\sin^2 x}$ $|a|>1$

soln :

$$I = \int_0^{2\pi} \frac{dx}{a+\sin^2 x} = \int_0^{\pi/2} \frac{dx}{a+\frac{1-\cos 2x}{2}}$$

$$= \int_0^{\pi/2} \frac{2\,dx}{2a+1-\cos 2x}$$

Put $2x=\theta$

$2dx = d\theta$

$x=0,\ \theta=0;\quad x=\pi/2,\ \theta=\pi$

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$$I = \int_0^{\pi} \frac{d\theta}{2a+1-\cos\theta}$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{2a+1-\cos\theta}$$

$$2I = \int_0^{2\pi} \frac{d\theta}{2a+1-\cos\theta}$$

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Take C,

$|z|=1$, $z = e^{i\theta}$

$$\therefore d\theta = \frac{dz}{iz}$$

$$\cos\theta = \frac{z^2+1}{2z}$$

$$= \int_C \frac{dz/iz}{2a+1-\frac{(z^2+1)}{2z}}$$

$$= \int_C \frac{dz/iz}{\frac{2(2a+1)z-z^2-1}{2z}} \Rightarrow \int_C \frac{2dz}{i(z^2-2)(2a+1)z+1}$$

$$= -\frac{2}{i}\int_C \frac{dz}{z^2-2(2a+1)z+1}$$

$$2I = -\frac{2}{i}\int_C f(z)\,dz.$$

$$I = -\frac{1}{c}\, 2\pi i\,[\Sigma \text{Res}] \longrightarrow (1)$$

$$f(z) = \frac{1}{z^2-2(2a+1)z+1} = \frac{1}{(z-\alpha)(z-\beta)}$$

$$\alpha = \frac{2(2a+1) \pm \sqrt{4(2a+1)^2-4}}{2} = \frac{2\left[(2a+1) \pm \sqrt{(2a+1)^2-1}\right]}{2}$$

$$= 2a+1 \pm \sqrt{(2a+1)^2-1}$$

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$= (2a+1) \pm \sqrt{4a^2+4a}$

$\alpha = 2a+1 + 2\sqrt{a^2+a}$

$\beta = 2a+1 - 2\sqrt{a^2+a}$

$\left[\because (2a+1)^2 - 1 \right.$

$4a^2+4a+1-1$

$\left. = 4a^2+4a \right]$

Here β lies inside c, α lies outside c

$$\underset{z \to \beta}{\text{Resi}} f(z) = \lim_{z \to \beta} (z-\beta) \frac{1}{(z-\alpha)(z-\beta)}$$

$z = \beta$ sub

sub in (1) $I = -\frac{1}{i} \cdot 2\pi i \left(\frac{1}{-4\sqrt{a^2+a}} \right)$

$$I = -2\pi \left[\frac{1}{-4\sqrt{a^2+a}} \right]$$

$$I = \frac{\pi}{2\sqrt{a^2+a}}$$

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Type-II

Integral of the type $\int_{-\infty}^{\infty} R(x)\,dx = \int_{-\infty}^{\infty} \frac{P(x)}{Q(x)}\,dx$.

where i) The degree of $Q(x)$ is greater than that of $P(x)$ by atleast 2.

$Q(x)$ has no zeros on real axis

[$R(x)$ has no poles on the real axis].

Take the contour c on the closed curve —

of i) The segment on the real —

ii) The upper half semicircle $\Gamma = |z| = R$

Make use of the result that.

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$$\int_{\Gamma} f(z)\,dz \to 0 \text{ as } R \to \infty$$

$$\text{if } \lim_{z \to \infty} z f(z) = 0.$$

problem :

1. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\,dx$

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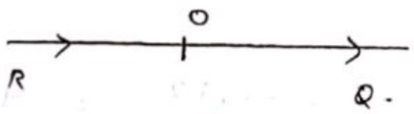
soln :

Take $f(z) = \dfrac{z^2 - z + 2}{(z^4 + 10z^2 + 9)} = \dfrac{z^2 - z + 2}{(z^2+9)(z^2+1)}$

consider $\int_C F(z)\,dz$ where C is closed contour consisting of

i) The segment on the real axis $[-R, R]$

The upper half semicircle $\Gamma : |z| = R$ $(\text{Im } z > 0)$



By Residue thrm $\int_C f(z)\,dz = 2\pi i$ [sum of the residues of $f(z)$ in the upper half plane].

$$= 2\pi i \left(\Sigma R^{+}\right)$$

$$\int_{-R}^{R} f(x)\,dx + \int_{\Gamma} f(z)\,dz = 2\pi i \left[\Sigma R^{+}\right]$$

As $R \to \infty$ if $\displaystyle\lim_{z \to \infty} z \cdot f(z) = 0$

$$\lim_{z \to \infty} \frac{z(z^2 - z + 2)}{z^2 + 10z^2 + 9} = 0$$

Then

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UNIT-V

HARMONIC FUNCTIONS

Definition :

A real valued function $u(z)$ (or) $u(x,y)$ defined and single valued in a region Ω is said to be harmonic in Ω (or) a partial function if it is continuous together with its partial derivatives of the first two orders and satisfies the Laplace's equation. (1)

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Note :

1) The sum of two harmonic function are harmonic
2) The constant multiple of a harmonic function is harmonic

(i-e) If u_1 & u_2 are harmonic then for any c_1 & c_2 are constants then $c_1u_1 + c_2u_2$ is harmonic.

Note :

Laplace's equation in polar co-ordinate (r,θ) as $r \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial \theta^2} = 0.$

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Example :

1) $u = ax+by$

$$\frac{\partial u}{\partial x} = a, \quad \frac{\partial u}{\partial y} = b$$

$$\frac{\partial^2 u}{\partial x^2} = 0, \quad \frac{\partial^2 u}{\partial y^2} = 0$$

2

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore$ The given function is harmonic.

2) $u = \log r$

2m

$$\frac{\partial u}{\partial r} = \frac{1}{r}, \quad \frac{\partial u}{\partial \theta} = 0$$

$$r \cdot \frac{\partial u}{\partial r} = 1, \quad \frac{\partial^2 u}{\partial \theta^2} = 0$$

$$r \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) = 0$$

$$\therefore r \frac{\partial}{\partial r}\left(r \frac{\partial u}{\partial r}\right) + \frac{\partial^2 u}{\partial \theta^2} = 0$$

$\therefore$ The given function is harmonic

NOTE :

1) The real and imaginary parts of an analytic function $f(z) = u+iv$ in Ω satisfies the C-R equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

conversely, if u & v satisfies the C-R equations

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in Ω and if the partial derivatives are continuous then $f(z) = u + iv$ is an analytic function

2) If $\Delta u = 0$ & $\Delta v = 0$ the real and imaginary parts of analytic function (or) conjugate harmonic function.

Transition form Harmonic function to Analytic function

If u is harmonic is a region Ω then

$$f(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$ is analytic in Ω

(3)

Proof :

For write, $u = \frac{\partial u}{\partial x}$ and $v = -\frac{\partial u}{\partial y}$

We have $f(z) = u + iv$

and $\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}$; $\frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial x \partial y}$

$$\frac{\partial v}{\partial x} = -\frac{\partial^2 u}{\partial x \partial y} \; ; \; \frac{\partial v}{\partial y} = -\frac{\partial^2 u}{\partial y^2}$$

Since u is harmonic it satisfies the C-R equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$u_x = v_y$$

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As u is continuous we have

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

$$\Rightarrow -\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} \qquad (4)$$

$$-v_x = u_y$$

Thus u & v satisfies the C-R equations.

Further u is harmonic implies its partial derivatives of 1st two orders exists and hence,

$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are all continuous.

$\therefore u+iv$ is analytic in Ω

$\Rightarrow f(z) = u+iv$ is analytic in Ω

$\therefore f(z)$ is analytic in Ω

$\therefore$ Hence the proof.

Remark :

If $u(x,y)$ is harmonic define du & $f du$

$$f(z)dz = \left[\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}\right] (dx+idy)$$

$$\Rightarrow f(z)dz = \left[\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy\right] + i\left[\frac{\partial u}{\partial x}dy - \frac{\partial u}{\partial y}dx\right] \quad (1)$$

Here the real part of $f(z)dz$ is,

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = du$$

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If u has a harmonic conjugate v then the imaginary part of $f(z)dz$ is,

$$du = \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx$$

(5)

Applying C-R equation

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy$$

In general there is no single valued conjugate function and in these circumstances we used the following notations

$$* du = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$$

"du" is called the conjugate differential of du".

$$(1) \Rightarrow f(z)dz = du + i * du \longrightarrow (2)$$

Next we find $\int_{\gamma} * du$ where γ is homologus to zero in Ω

"By cauchy's Theorem the integral of $f(z)dz$ vanishes along any cycle which is homologus to zero in Ω".

$$\text{i.e.)} \int_{\gamma} f(z)dz = 0 \longrightarrow (3)$$

On the other hand the integral of the exact.

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du vanishes along all cycles

i.e) $\int_{\gamma} du = 0. \longrightarrow (4)$

$(2) \Rightarrow \int_{\gamma} f(z)dz = \int_{\gamma} du + i\int_{\gamma} {}^*du$

(6)

$0 = 0 + i\int_{\gamma} {}^*du$

$\Rightarrow \int_{\gamma} {}^*du = 0$

Thus, $\int_{\gamma} {}^*du = \int_{\gamma} \left[-\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy\right] = 0$ for all cy

which are homologus to zero in Ω.

Theorem:

If u_1 & u_2 are harmonic in a region Ω

then $\int_{\gamma} u_1 {}^*du_2 - u_2 {}^*du_1 = 0$ for every cycle γ

which is homologus to zero in Ω.

Proof:

It is suffices to show that the integral vanishes over every rectangle contained in Ω.

Let $\gamma = \delta R$. where R is a rectangle contains because R is a rectangle it is simply connected and hence in R, both u_1 & u_2 have single valued harmonic conjugate say v_1 & v_2 respectively.

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$\therefore\ * du_1 = dv_1\ ,\ * du_2 = dv_2$

Now, $u_1 * du_2 - u_2 * du_1 = u_1 dv_2 - u_2 dv_1$

$= u_1 dv_2 + v_1 du_2 - du$

$[\because d(u_2 v_1) = v_1 du_2 + u_2 dv_1 - u_2 dv_1$

$= v_1 du_2]$

since, $d(u_2 v_1)$ is an exact differential [and R is closed] (7)

$$\int_{\delta R} d(u_2 v_1) = 0$$

$u_1 dv_2 + v_1 du_2$ is the imaginary part of

$(u_1 + iv_1)(du_2 + i dv_2)$

(i.e) $u_1 dv_2 + v_1 du_2 = Im[(u_1 + iv_1)(du_2 + i dv_2)]$

$= Im[(u_1 + iv_1) d(u_2 + iv_2)]$

$u_1 dv_2 + v_1 du_2 = Im[f_1 d f_2]$

[where $f_1 = u + iv$ & $f_2 = u_2 + iv_2$. Both f_1 & f_2 are analytic]

$$\int_{\delta R} f_1 d f_2 = 0$$

$$\int_{\delta R} Im(f_1 f_2) = 0$$

i.e) $\int_{\delta R} (u_1 dv_2 + v_2 dv_2) = 0$

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Hence $\int_{\partial R} u_1{}^* du_2 - u_2{}^* du_1 = \int_{\partial R} u_1 dv_2 + v_1 du_2 - \int_{\partial R} d(u_2 v)$

$= 0 - 0$

$= 0$

$$\int_{\gamma} u_1{}^* du_2 - u_2{}^* du_1 = 0$$

For every cycle γ which is homologues to zero in Ω.

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Interpretation of $*du$:

If γ be the regular curve with the equation $z = z(t)$. The direction of the tangent is determined by the angle

$$\alpha = \arg z'(t)$$

we can write $dx = |dz| \cos\alpha$

$dy = |dz| \sin\alpha$

The normal which point to the right of the tangent has the direction

$$\beta = \alpha - \pi/2$$

$$\Rightarrow \alpha = \beta + \pi/2$$

From this,

$\cos\alpha = \cos(\beta + \pi/2) = -\sin\beta$

$\sin\alpha = \sin(\beta + \pi/2) = \cos\beta$

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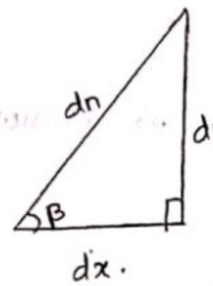
Thus, $* du = -\frac{\partial u}{\partial y} dx + \frac{\partial u}{\partial x} dy$

$$= -\frac{\partial u}{\partial y} [|dz| \cos\alpha] + \frac{\partial u}{\partial x} [|dz| \sin\alpha]$$

$$= |dz| \left[-\frac{\partial u}{\partial y} \cos\alpha + \frac{\partial u}{\partial x} \sin\alpha \right]$$



$$= |dz| -\frac{\partial u}{\partial y} [(-\sin\beta) + \frac{\partial u}{\partial x} \cos\beta]$$

$$= \left[\frac{\partial u}{\partial x} \cos\beta + \frac{\partial u}{\partial y} \sin\beta \right] |dz|$$

$$* du = \left[\frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial n} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial n} \right] |dz|$$

$* du = \frac{\partial u}{\partial n} |dz|$ where "$\frac{\partial u}{\partial n}$ is the normal derivatives of u".

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The Mean value property [or] Mean value Theorem For Harmonic function :

The arithmetic mean of a harmonic function over concentric circles $|z| = r$ is a linear functions of $\log r$, $\frac{1}{2\pi} \int_{|z|=r} u \, d\theta = \alpha \log r + \beta$

and if u is harmonic in a disck $\alpha = 0$ and arithmetic mean is constant

Proof :

Assume that $I = \int u_1 * du_2 - u_2 * du_1 = 0$

Let us take $u_1 = \log r$, $u_2 = u$ the function

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harmonic in $|z| < \rho$

for all region Ω we punctured disc $0<|z|<\rho$

and for the cycle γ we take $c_1 - c_2$

where c_i $(i = 1, 2 \ldots\ldots)$ is circle $|z| = r_i < \rho$

described in the +ve sense

on a circle $|z| = r$ we have the formula

$$* du = r \left(\frac{\partial u}{\partial r} \right) d\theta \qquad \left[\because * du = \frac{\partial n}{\partial x} |dz| \right]$$

$$\therefore I = \int_{\gamma} \log r \; * du - u^* \, d(\log r) = 0 \qquad (10)$$

$$\Rightarrow \int_{c_1 - c_2} \left[\log r \left(r \frac{\partial u}{\partial r} \right) d\theta - u \left(r \cdot \frac{1}{r} \right) d\theta \right] = 0$$

$$\Rightarrow \int_{c_1} \left[\log r \left(r \frac{\partial u}{\partial r} \right) d\theta - u d\theta \right] - \int_{c_2} \log r \left(r \cdot \frac{\partial u}{\partial r} \right) d\theta - u d\theta] = 0$$

$$\Rightarrow \int_{|z|=r_1} \log r_1 \left(r_1 \frac{\partial u}{\partial r_1} \right) d\theta - \int_{|z|=r_1} u d\theta - \int_{|z|=r_2} \log r_2 \left(r_2 \frac{\partial u}{\partial r_2} \right) d\theta + \int_{|z|=r_2} u d\theta = 0.$$

$$\Rightarrow \int_{|z|=r_1} \left[\log r_1 \left(r_1 \frac{\partial u}{\partial r_1} \right) d\theta - u d\theta \right] = \int_{|z|=r_2} \left[\log r_2 \left(r_2 \frac{\partial u}{\partial r_2} \right) d\theta - u d\theta \right]$$

for $r_2 < r < r_1 < \rho$

$$\int_{|z|=r} \left[\log r \left(r \frac{\partial u}{\partial r} \right) d\theta - u d\theta \right] = \text{constant}.$$

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[or]

$$\Rightarrow \int_{|z|=r} u\,d\theta - \int_{|z|=r} \log r \left(r \frac{\partial u}{\partial r} \right) d\theta = \text{constant} \rightarrow (1)$$

This is true even if u is only known to be harmonic in an annulus.

consider $\int_{|z|=r} \log r \cdot r \cdot \frac{\partial u}{\partial r}\, d\theta = \log r \int_{|z|=r} r \cdot \frac{\partial u}{\partial r}\, d\theta$

$$= \log r \int_{|z|=r} {}^{*}du$$

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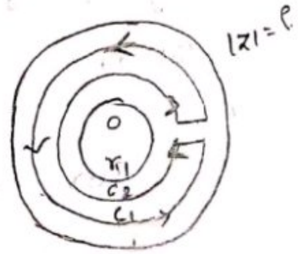
we see that the above integral is constant in the case of an annulus and zero if u is harmonic in the whole disc

$\therefore \log r \int_{|z|=r} r \cdot \frac{\partial u}{\partial r}\, d\theta = \log r \ \frac{2\pi\alpha}{2\pi\alpha}$ (or) $\log r \ \alpha \ u$

we have,

$$\int_{|z|=r} u . d\theta - [2\pi\alpha \log r \ [\text{or}] \ 0] = \text{constant} = \beta c \ [\text{say}]$$

$$\Rightarrow \int_{|z|=r} u\,d\theta - 2\pi\alpha \log r = \beta c$$



$$\Rightarrow \int_{|z|=r} u\,d\theta = 2\pi\alpha \log r + \beta c$$

[or] $\frac{1}{2\pi} \int_{|z|=r} u\,d\theta = \alpha \log r + \beta$

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Note:

In the above result that $\beta = u(0)$ and consider the case when $\alpha = 0$ we have,

$$\frac{1}{2\pi} \int_{|z|=r} u \, d\theta = (0) \log r + u(0)$$

changing the origin from 0 to new point z_0

we have, $\frac{1}{2\pi} \int_{|z|=r} u(z) \, d\theta = u(z_0)$ (12)

[or]

$$\Rightarrow u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) \, d\theta$$

where the circle is $|z - z_0| = r$ and $z = z_0 + re^{i\theta}$

$0 \leq \theta \leq 2\pi$

$\therefore$ The arithmetic mean is constant.

Maximum principle for Harmonic Functions:

A non constant harmonic function has neither maximum nor a minimum in its region of definition consequently, the maximum and the minimum on a closed bounded set E are taken on the boundary of E.

Proof:

If u is harmonic throughout the circular disc $|z| \leq r$ then by mean value property

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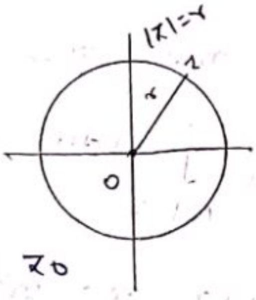
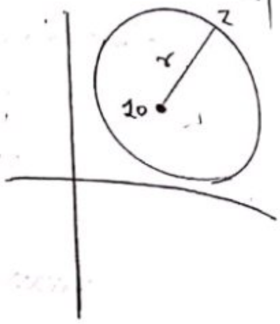
its arithmetic mean is a constant

$\therefore \frac{1}{2\pi} \int_{|z|=r} u\,d\theta = \beta$ [$\because$ by above note]





$\Rightarrow u(0) = \frac{1}{2\pi} \int_{|z|=r} u\,d\theta$

By change of origin to z_0

we have,

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta \longrightarrow (A)$$

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This shows that the value of $u(z)$ at the

center z_0.

i-e) $u(z_0)$ equals the arithmetic mean of the

value of $u(z)$ on the circumference

suppose, $u(z)$ has the max at z_0 in Ω

Then $|u(z_0 + re^{i\theta})| \leq |u(z_0)| \longrightarrow (1)$

$\therefore$ we have $|u(z_0)| = \left| \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta})\,d\theta \right|$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} |u(z_0 + re^{i\theta})|\,d\theta$$

$$\leq \frac{1}{2\pi} |u(z_0 + re^{i\theta})| \int_0^{2\pi} d\theta$$

$$\leq \frac{1}{2\pi} |u(z_0 + re^{i\theta})| (2\pi)$$

$$|u(z_0)| \leq |u(z_0 + re^{i\theta})|$$

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This inequality holds throughout

$$(A) \Rightarrow \frac{1}{2\pi}\int_0^{2\pi} (u(z_0+re^{i\theta})d\theta = u(z_0)$$

Now, $$\frac{1}{2\pi}\int_0^{2\pi} |u(z_0+re^{i\theta})|d\theta = \frac{1}{2\pi}\int_0^{2\pi} |u(z_0)|d\theta$$

$$\Rightarrow \int_0^{2\pi} |u(z_0+re^{i\theta})|\, d\theta = \int_0^{2\pi} |u(z_0)|d\theta$$

$$\Rightarrow \int_0^{2\pi} [|u(z_0+re^{i\theta})| - |u(z_0)|]d\theta = 0.$$

(14)

[or] $$\Rightarrow \int_0^{2\pi} [|u(z_0)| - |u(z_0+re^{i\theta})|]\, d\theta = 0.$$

since the integer and is non-negative

$$\therefore |u(z_0)| - |u(z_0+re^{i\theta})| = 0$$

$$\Rightarrow |u(z_0)| = |u(z_0+re^{i\theta})| \qquad 0 \le \theta \le 2\pi$$

This equality holds on all $|z-z_0| = r$ and therefore $|u(z)|$ is constant in any neighbourhood of z_0.

Hence $|u|$ is constant in Ω. This is a contradiction.

From this it follows that if u is a harmonic function defined on a closed bounded set E then the maximum is taken over on the boundary of E

Hence the theorem.

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Poisson's Formula for Harmonic Function :

Statement :

Suppose that $u(z)$ is harmonic for $|z| < R$ continuous for $|z| \le R$ then,

$$u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2} u(z) \, d\theta \quad \forall \; |a| < R$$

Proof

By hypothesis $u(z)$ is harmonic in an open disc $|z| < R$

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The linear transformation

$$z = S(\zeta) = \frac{R(R\zeta + a)}{R + \bar{a}\zeta} \longrightarrow (1)$$

maps the circle $|\zeta| \le 1$ onto the circle $|z| \le R$ with $\zeta = 0$ corresponding to the point $z = a$

Since the function $u(S(\zeta))$ is harmonic for $|\zeta| < 1$ we know that and continuous for $|\zeta| \le 1$

we know that,

$$u(z_0) = \frac{1}{2\pi} \int_0^{2\pi} u(z_0 + re^{i\theta}) \, d\theta$$

Now,

$$u(a) = \frac{1}{2\pi} \int_{|z|=R} u(S(\zeta)) \, d(\arg \zeta) \longrightarrow (2)$$

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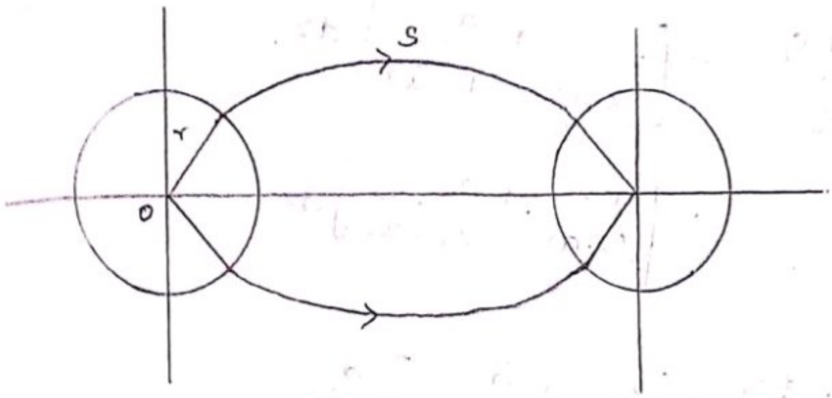

from (1),

$$z = \frac{R(R\zeta+a)}{R+\bar{a}\zeta}$$

(16)

$$z(R+\bar{a}\zeta) = R(R\zeta+a)$$

$$Rz + z\bar{a}\zeta = R^2\zeta + Ra$$

$$\Rightarrow Rz - Ra = R^2\zeta - z\bar{a}\zeta$$

$$R(z-a) = \zeta(R^2 - \bar{a}z)$$

$$\zeta = \frac{R(z-a)}{R^2-\bar{a}z}$$

Taking log on both sides

$$\log\zeta = \log[R(z-a)] - \log[R^2-\bar{a}z]$$

$$\log\zeta = \log R + \log(z-a) - \log(R^2-\bar{a}z)$$

Diff on both sides we get,

$$\frac{d\zeta}{\zeta} = 0 + \left[\frac{dz}{(z-a)} + \frac{\bar{a}\,dz}{R^2-\bar{a}z}\right]. (-dz)\bar{a}$$

$$\frac{d\zeta}{\zeta} = \frac{dz}{z-a} + \frac{\bar{a}\,dz}{R^2-\bar{a}z}$$

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$$\frac{dg}{g} = \left[\frac{1}{(z-a)} + \frac{\bar{a}}{R^2 - \bar{a}z}\right] dz$$

$$= \left[\frac{1}{(z-a)} + \frac{\bar{a}}{z\bar{z} - \bar{a}z}\right] dz$$

$$\left\{ \therefore \frac{dg}{g} = \left[\frac{z}{z-a} + \frac{\bar{a}}{(\bar{z}-\bar{a})}\right] \frac{dz}{z} \right.$$

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multiply $(-i)$ we get

$$-i\frac{dg}{g} = \left[\frac{z}{(z-a)} + \frac{\bar{a}}{(\bar{z}-\bar{a})}\right] -i\frac{dz}{z}$$

$$-i\frac{dg}{g} = \left[\frac{z}{(z-a)} + \frac{\bar{a}}{(\bar{z}-\bar{a})}\right] d\theta \Big\}$$

$$= i\left[\frac{z}{(z-a)} + \frac{\bar{a}}{(\bar{z}-\bar{a})}\right] d\theta \longrightarrow (3)$$

$$-i\frac{dg}{g} = \left[\frac{z}{(z-a)} + \frac{\bar{a}}{(\bar{z}-\bar{a})}\right] d\theta \longrightarrow (4)$$

$$\Rightarrow d(\arg g) = \left[\frac{z(\bar{z}-\bar{a}) + \bar{a}(z-a)}{(z-a)(\bar{z}-\bar{a})}\right] d\theta$$

$$d(\arg g) = \left[\frac{z\bar{z} - z\bar{a} + \bar{a}z - \bar{a}a}{(z-a)(\bar{z}-\bar{a})}\right] d\theta$$

$$d(\arg g) = \frac{|z|^2 - |a|^2}{|z-a|^2}\, d\theta \qquad [\because |z| = R]$$

$$d(\arg g) = \frac{R^2 - |a|^2}{|z-a|^2}\, d\theta \longrightarrow (5)$$

But, $u[s(g)] = u(z) \longrightarrow (6)$

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Sub (5) & (6) in (2)

$$(2) \Rightarrow u(a) = \frac{1}{2\pi} \int_{|z|=R} u(z) \, \frac{R^2 - |a|^2}{|z-a|^2} \, d\theta$$

$$u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2} \, u(z) \, d\theta \qquad |a| < R$$

This is called the poissons formula.

polar form of poisson's formula:

Take, $z = Re^{i\theta}$, $a = re^{i\phi}$ (18)

$a = r$

$|a|^2 = r^2$

$|z-a|^2 = |Re^{i\theta} - re^{i\phi}|^2$

$= (Re^{i\theta} - re^{i\phi})(Re^{-i\theta} - re^{-i\phi})$

$= R^2 - Rre^{i(\theta-\phi)} - Rre^{-i(\theta-\phi)} + r^2$

$= R^2 - Rr(e^{i(\theta-\phi)} + e^{-i(\theta-\phi)}) + r^2$

$|z-a|^2 = R^2 - Rr\,2\cos(\theta-\phi) + r^2$

$$\Rightarrow u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2} \, u(z) \, d\theta \qquad \left[\because \frac{e^{i\theta} + e^{-i\theta}}{2} = \cos\theta\right]$$

$$u(re^{i\phi}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta-\phi) + r^2} \, u(Re^{i\theta}) \, d\theta$$

This is called the "polar form of poisson's formula.

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Note :

The poissions formula is

$$u(a) = \frac{1}{2\pi} \int_{|z|=R} \frac{R^2 - |a|^2}{|z-a|^2} u(z)\, d\theta$$

Now, $\frac{1}{2}\left[\frac{z+a}{z-a} + \frac{\bar{z}+\bar{a}}{\bar{z}-\bar{a}}\right] = \operatorname{Re}\left[\frac{z+a}{z-a}\right]$, $|z|=R$, $|a|<R$

$$\frac{1}{2}\left[\frac{(z+a)(\bar{z}-\bar{a})+(\bar{z}+\bar{a})(z-a)}{(z-a)(\bar{z}-\bar{a})}\right] = \operatorname{Re}\left[\frac{z+a}{z-a}\right]$$

$\therefore$ The poission's formula takes the form

as, $u(a) = \frac{1}{2\pi} \int_{|z|=R} \operatorname{Re}\left(\frac{z+a}{z-a}\right) u(z)\, d\theta$, $|a|<R$.

(19)

Definition :

Poission integral

consider a disc of radius $R=1$ and let $u(\theta)$ be a piecewise continuous function defined on $0 \le \theta \le 2\pi$

$$P_u(z) = \frac{1}{2\pi} \int_0 \operatorname{Re}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) u(\theta)\, d\theta$$

This is called the poission integral of u.

$P_u(z)$ is not only a function of z but also a function of the function 'θ' as such that it is called a functional

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The linear functional is in as much as

$$P_{u+v}(z) = \frac{1}{2\pi}\int_0^{2\pi} Re\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)[u(\theta+v\theta)]\,d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} Re\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)u(\theta)\,d\theta + \frac{1}{2\pi}\int_0^{2\pi} Re\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)v(\theta)\,d\theta.$$

$$P_{u+v} = P_u + P_v$$

III'ly

$$P_{cu} = cP_u \quad \text{where } c \text{ is constant.}$$

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Schwarz's Theorem :

Statement :

The function $P_u(z)$ is harmonic for $|z|<1$ and

then $\lim\limits_{z \to e^{i\theta}} P_v(z) = u(\theta)$.

Proof :

we have, Given

$$P_v(z) = \frac{1}{2\pi}\int_0^{2\pi} Re\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)u(\theta)\,d\theta$$

$$P_u(z) = Re\left[\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)u(\theta)\,d\theta\right]$$

$$= Re\,[g(z)]$$

where,

$$g(z) = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{i\theta}+z}{e^{i\theta}-z}\,u(\theta)\,d\theta$$

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