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Unit - 1

Definition: Let F be a field.

Let
$$A_{11}x_1 + A_{12}x_2 + \dots + A_{1n}x_n = y_1 \longrightarrow (1)$$
$$A_{21}x_1 + A_{22}x_2 + \dots + A_{2n}x_n = y_2 \longrightarrow (2)$$
$$\vdots$$
$$A_{m1}x_1 + A_{m2}x_2 + \dots + A_{mn}x_n = y_m \longrightarrow (m)$$

be a system of m linear equations in 'n' unknowns $x_1, \dots, x_n$.

$A_{ij}, y_i \in F$ are known.

Any n tuples $(x_1, \dots, x_n)$ which satisfies each equation in the system is called a solution of the system.

We write the system of equation in matrix form as

$$AX = Y.$$

where A is called coefficient matrix

X is an $n \times 1$ matrix

Y is an $m \times 1$ matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & A_{22} & \dots & A_{2n} \\ \vdots & & & \\ A_{m1} & A_{m2} & \dots & A_{mn} \end{bmatrix}; \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

If $y_1 = y_2 = \dots = y_m = 0$, then the system is called homogeneous.

Linear combination: Let $c_1, c_2, \dots, c_m \in F$

$$(1) \times c_1 + (2) \times c_2 + \dots + (m) \times c_m \Rightarrow$$

$$[c_1A_{11} + c_2A_{21} + \dots + c_mA_{m1}]x_1 + [c_1A_{12} + c_2A_{22} + \dots + c_mA_{m2}]x_2 + \dots$$
$$+ [c_1A_{1n} + c_2A_{2n} + \dots + c_mA_{mn}] = 0$$

is called a linear combination of m equation in the system.

Clearly any solution of the system of equation is also a solution of above linear combination.

Equivalent of the linear equation:-

If each equation in one system in a linear combination of equation in the II system. Then the system are said to be equivalent

Theorem:-

Prove the equivalent system of linear combination equations have exactly the same solution.

Proof: Let $A_{11}x_1 + A_{12}x_2 + \dots + A_{1n}x_n = y_1$

$A_{21}x_1 + A_{22}x_2 + \dots + A_{2n}x_n = y_1$

$\vdots$

$A_{m1}x + A_{m2}x_2 + \dots + A_{mn}x_n = y_m$ be a one system of linear equations.

Let $B_{11}x_1 + B_{12}x_2 + \dots + B_{1n}x_n = z_1$

$B_{21}x_1 + B_{22}x_2 + \dots + B_{2n}x_n = z_2$

$\vdots$

$B_{k1}x_1 + B_{k2}x_2 + \dots + B_{kn}x_n = z_k$ be a solution of k linear equations in which each equation is a linear combination of the equations in II system then every solution I system in a solution of II system.

It may happen that some solution of II system may not be a solution of I system.

This will not happen if each equation in II system is a linear combination of equation in I system.

(ii) if the 2 system are equivalent they have the same solution.

Definition of an $m \times n$ matrix :-

An $m \times n$ matrix over the field F is a function (Rule) from the set of Pare of integers (i, j) ; $1 \leq i \leq m$ into the field F.

Definition :- An ERO [Elementary Row operation] is a function (or) a rule e which associates with each $m \times n$ matrix A an $m \times n$ matrix e(A)

They are three ERO

(i) Multiplying any Row by a non-zero Scalar

(ii) replacement of r^{th} row by row r plus c times row s ($r \neq s$)

(iii) Interchange of two rows of A.

Definition :- Row Equivalent

If A and B are $m \times n$ matrix over the field F, then B is said to be row equivalent to A. If B can be obtained from A by a finite sequence of EROS.

Definition :- Row Reduced matrix

An $m \times n$ matrix R is called a row reduced matrix if

1. First non-zero entry in each non-zero row is 1
2. each column of R which has the leading non-zero entry of some row has all its open entries zero.

Example $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an Row-reduced matrix

Definition:- Row-reduced Echelon matrix

An $m \times n$ matrix R is said to be an row-reduced echelon matrix if

(i) R is called row-reduced matrix

(ii) every zero row of R occurs below every row which has a non-zero entry.

(iii) If rows $1, 2, \ldots, r$ all the non-zero row of R and if the leading non-zero entry of row i occurs in column k_i, $i = 1, 2 \ldots r$ then $k_1 < k_2 < \ldots < k_n$.

Example:-

$$\begin{bmatrix} 0 & 1 & -3 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Theorem:-

If A is an $m \times n$ matrix and $m < n$, then the homogeneous system of equation $Ax = 0$ has a non-trivial solution.

Proof:- Let A be an $m \times n$ matrix

Let $m < n \longrightarrow (1)$

Let R be the row reduced echelon matrix which is row equivalent to A.

Then $Ax = 0$ and $Rx = 0$ have the same solution $\rightarrow (2)$

Let r be the number of non-zero rows

Then $r \leq m$

But by (1) $m < n$

$\therefore r < n$

$\therefore$ $Rx = 0$ has a non-trivial solution

by ② $\therefore$ $Ax = 0$ has a non-trivial solution.

Theorem: if A, B, C are matrices over F and the Products BC and $A(BC)$ are defined then so are AB and $(AB)C$ and that $A(BC) = (AB)C$

Proof: Suppose B is an $m \times p$ matrix

BC is defined

$\therefore$ C has p rows and BC has m rows

$A(BC)$ is defined

$\therefore$ A must have n column.

$\therefore$ we take A as an $m \times n$ matrix

A is of type $m \times n$

B is of type $n \times p$

$\therefore$ (AB) is defined and is of type $m \times p$

C has p rows

$\therefore$ $(AB)C$ is defined

- Next we Prove:- $A(BC) = (AB)C$ for that it's enough if we Prove

$$[A(BC)]_{ij} = [(AB)C]_{ij}$$

$$\therefore [A(BC)]_{ij} = \sum_r A_{ir} (BC)_{rj}$$

$$= \sum_r A_{ir} (BC)_{ij}$$

$$= \sum_r A_{ir} \sum_s B_{rs} C_{sj}$$

$$= \sum_r \sum_s A_{ir} B_{rs} C_{sj} = \sum_s \left[\sum_r A_{ir} B_{rs} \right] C_{sj}$$

$$= \sum_s (AB)_{is} C_{sj} = (AB)C_{ij}$$

$\therefore A(BC) = (AB)C$.

Definition :- Elementary matrix

An $m \times n$ matrix A is said to be elementary matrix if it can be obtained from a $m \times m$ identity matrix by a single elementary row operation.

Definition :- Invertible matrix

Let A be an $n \times n$ matrix. If $\exists$ an $n \times n$ matrix B such that $BA = I$ then B is called a left inverse of A. If $\exists$ an $m \times n$ matrix B, such that $AB = I$, then B is called a right inverse of A.

If $AB = BA = I$, then B is called a 2-side inverse or inverse of A.

Lemma :- If A has a left inverse of B and a right inverse C, then Prove that $B = C$

Proof :- B is a left of A

$\therefore BA = I \longrightarrow (1)$

C is a right inverse of A

$\therefore AC = I \longrightarrow (2)$

To Prove : $B = C$

Consider $B = BI$

$= B(AC) = (BA)C$

$= IC$

$\therefore B = C$

Vector space

vector addition :-

$(V,+)$

1. $a+b \in V$ closure Property
2. $a+(b+c) = (a+b)+c$ Associative.
3. $a+0 = 0+a = a$
4. $a+(-a) = (-a)+a = 0$
5. $a+b = b+a$

scalar multiplication :-

1. $1.a = a$ $1 \in F$
2. $c_1(c_2 a) = (c_1 c_2)a$
3. $c_1(a+b) = c_1 a + c_2 b$
4. $(c_1+c_2)a = c_1 a + c_2 a$

Definition :- Dimensional

The number of elements in a basis of a vector space v is called the dimension of v.

if the dimension is finite, v is said to be finite dimensional.

Theorem :-

If v is finite dimension vector space, then Prove that any two basis of v have the same number of elements.

Proof :

v is finite dimensional

$\therefore$ It has a finite basis

let $\{\beta_1, \beta_2 \ldots, \beta_m\}$ be a basis of v.

let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a another bases of v.

$\{\beta_1, \beta_2 \ldots \beta_m\}$ spans v and $\{\alpha_1, \alpha_2 \ldots, \alpha_n\}$ is linearly independent

Then $n \le m \longrightarrow (1)$

By the same argument $m \le n \longrightarrow (2)$

from (1) & (2)

$\therefore m = n$

Theorem :

If w_1, w_2 are finite dimension sub space of vector space V. Then $w_1 + w_2$ is finite dimension and $\dim w_1 + \dim w_2 = \dim(w_1 \cap w_2) + \dim(w_1 + w_2)$.

Proof :

w_1, w_2 are sub space of V.

$\therefore w_1 \cap w_2$ is also sub space of V.

let $S_1 = \{\alpha_1, \alpha_2, \dots \alpha_k\}$ be a basis of $w_1 \cap w_2$

$\therefore \dim(w_1 \cap w_2) = k$

Thus S_1 is a Part of a basis $S_2 = \{\alpha_1, \alpha_2 \dots \alpha_k, \beta_1, \dots \beta_m\}$ of w_1

Also S_1 is a Part of a basis $S_3 = \{\alpha_1 \dots \alpha_k, \gamma_1, \gamma_2 \dots \gamma_n\}$ of w_2

The sub space $w_1 + w_2$ is spanned by

$S = \{\alpha_1, \alpha_2 \dots \alpha_k, \beta_1, \beta_2 \dots \beta_m, \gamma_1, \gamma_2 \dots, \gamma_n\} \longrightarrow (1)$

next we Prove S is linearly independent

we consider $\sum x_i \alpha_i + \sum y_j \beta_j + \sum z_r \gamma_r = 0$

$\Rightarrow \sum z_r \gamma_r = -\left[\sum x_i \alpha_i + \sum y_j \beta_j\right]$

$\Rightarrow \sum z_r \gamma_r \in w_2$

$\sum x_i \alpha_i + \sum y_j \beta_j \in w_1$

These 2 are equal there are in $w_1 \cap w_2$

$\sum z_r \gamma_r$ is a linearly combination of $\alpha_1, \alpha_2 \dots, \alpha_k$

i.e. $\sum z_r \gamma_r = \sum c_i \alpha_i$

$\Rightarrow \left.\begin{array}{l} c_i = 0 \ \forall i \\ z_r = 0, \forall r \end{array}\right\}$ because S_3 is a basis of w_2

Put $\gamma_r = 0$, $\forall r$ in (2)

Then (2) $\Rightarrow$ $\sum x_i \alpha_i + \sum y_j \beta_j = 0$

$\Rightarrow$ $x_i = 0$, $\forall i$

$y_j = 0$ $\forall j$ because S_3 is a basis of W_1

$\therefore$ $x_i = 0$

$y_j = 0$

$\gamma_r = 0$

$\Rightarrow$ S is a linearly independent of set

$\therefore$ $\dim(W_1 + W_2) = k+m+n \longrightarrow (3)$

Consider $\dim W_1 + \dim W_2 = k+m + k+n$

$= k + (m+k+n)$

$= \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$

$\therefore$ $\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$.

Ordered Basis :-

Let V be a finite dimension vector space. An ordered basis of V is a finite sequence of vector which is linearly independent and spans V.

Spanned :-

Let S be a set of vectors in V. Then the sub space spanned by S is defined to be the intersection W of all subspace of V which contains S

Let $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ then W is called sub space spanned by S.

Basis and Dimension

Definition :- Let V be a vector space over F, Let $S = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be a sub space of V. Then S is said to be a linearly dependent set of vectors if $\exists$ scalar $c_1, c_2, \dots, c_n$ in F not all zero such that

$$c_1x_1 + c_2x_2 + \dots + c_nx_n = 0 \longrightarrow (1)$$

If S is not linearly independent then it is said to be linearly independent

In that case $(1) \Rightarrow c_1 = c_2 = c_3 = \dots = c_n = 0$

Definition :- Basis

Let V be a vector space over F and let S be a subset of V. If

1. S is linearly independent
2. S spans V then S is said to be a basis of V

V is said to be finite dimensional if it has a finite basis

Theorem :

Prove that row equivalent matrices have the same row space.

Proof :

Let A be $m \times n$ matrix over F.

Let $\alpha_1, \alpha_2, \dots, \alpha_m$ be the m row vectors of A

These m vectors span a subspace called row space of A.

If B is the row-equivalent matrix to A

Then $\exists$ an $m \times m$ matrix P such that $B = PA \longrightarrow (1)$

Let $\beta_1, \beta_2, \dots, \beta_m$ be the m row vector of B

Then from (1) we see that each β_i is a linear combination of $\alpha_1, \alpha_2, \dots, \alpha_m$ and

$$\beta_i = P_{i1}\alpha_1 + P_{i2}\alpha_2 + \dots + P_{im}\alpha_m$$

$\Rightarrow \beta_i$ is in the subspace spanned by $\alpha_1, \alpha_2, \dots, \alpha_m$

$\Rightarrow$ Row space of B is a subspace of row space of A $\rightarrow$ (2)

$$B = PA$$

$\Rightarrow$ $A = P^{-1}B$

$\Rightarrow$ row space of A is a subspace of row space of B

By (2) & (3)

row space of A = row space of B.

Linear transformation:

Let V & W be two vector space over the same field F. Then a function $T: V \to W$ is called a linear transformation if $T(c\alpha + \beta) = cT\alpha + T\beta$ $\forall\ \alpha, \beta \in V$ and $c \in F$

Theorem:

Let V be a vector space over the field F and $\beta = \{\alpha_1 \dots \alpha_n\}$ be an ordered basis of V. Let W be a vector space over the same field F & let $\beta_1, \beta_2, \dots, \beta_n$ be any n vectors in W.

Then Prove that $\exists$ Precisely one linear transformation from V into W defined by $T\alpha_j = \beta_j$, $j = 1, 2, \dots, n$.

Proof:

Let $\alpha \in V$

Since $\beta = \{\alpha_1, \alpha_2 \dots, \alpha_n\}$ is an ordered Basis of vectors. Then α be expressed as a unique linear combination of $\alpha_1, \alpha_2, \dots, \alpha_n$

Let $\alpha = x_1\alpha_1 + \dots + x_n\alpha_n$ $\longrightarrow (1)$

Define $T: V \to W$ such that

$T\alpha = x_1\beta_1 + \dots + x_n\beta_n$ $\longrightarrow (2)$

from (1) & (2), we see that

$T\alpha_j = \beta_j$ $j = 1, 2, \dots, n$.

is a linear transformation from V into W.

To Prove :- T is a linear transformation from V into W.

Let $\beta \in V$.

Let $\beta = y_1\alpha_1 + y_2\alpha_2 + \dots + y_n\alpha_n$ $\longrightarrow (3)$

Let $c \in F$

Consider $c\alpha + \beta = c(x_1\alpha_1 + \dots + x_n\alpha_n) + (y_1\alpha_1 + \dots + y_n\alpha_n)$

$$= (cx_1 + y_1)\alpha_1 + \dots + (cx_n + y_n)\alpha_n \longrightarrow (4)$$

Consider $cT\alpha + T\beta = c(x_1\beta_1 + \dots + x_n\beta_n) + (y_1\beta_1 + \dots + y_n\beta_n)$

$$= (cx_1 + y_1)\beta_1 + \dots + (cx_n + y_n)\beta_n \longrightarrow (5)$$

By (4) & (5) T is a linear transformation

T is unique:

Let $U : V \longrightarrow W$ be another linear transformation such that

$$U\alpha_j = \beta_j, \quad j = 1, 2, \dots n$$

Let $\alpha \in V$

Consider $U\alpha = U(x_1\alpha_1 + \dots + x_n\alpha_n)$

$$= U\left(\sum_{i=1}^{n} x_i\alpha_i\right)$$

$$= \sum_{i=1}^{n} x_i U\alpha_i = \sum_{i=1}^{n} x_i\beta_i$$

$\therefore U\alpha = T\alpha \Rightarrow U = T$

$\therefore T$ is unique.

Null space & Range space

Let V and W be two vector space over the same field F. If T is a linear transformation from V into W, then the set $N = \{\alpha \in V \mid T\alpha = 0\}$ is a subspace of V called the null space of T.

Let V and W be two vector space over the same field F. If T is a linear transformation from V into W, then the set $R_T = \{\beta \in W \mid \beta = T\alpha, \alpha \in V\}$ is a subspace of W called the Range of T.

Theorem: Let V & W be two vector space over F. If T is a linear transformation from V into W, then $\text{rank}(T) + \text{Nullity}(T) = \dim V$.

Proof:

Let $\dim V = n$

If $N = \{\alpha \in V \mid T\alpha = 0\}$. N is null space of T.

Let $R_T = \{\beta \mid \beta = T\alpha, \alpha \in V\}$ be the range of T

Let $S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_k\}$ be a basis of N

Then nullity of $T = k$.

S_1 can be completed to a basis of V as

$S_2 = \{\alpha_1, \alpha_2, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_n\}$

Let $\alpha \in V$. Then $\alpha = c_1\alpha_1 + c_2\alpha_2 + \ldots c_k\alpha_k + c_{k+1}\alpha_{k+1} + \ldots + c_n\alpha_n$.

$T\alpha = c_1T\alpha_1 + c_2T\alpha_2 + \ldots + c_kT\alpha_k + c_{k+1}T\alpha_{k+1} + \ldots + c_nT\alpha_n$

$= 0 + 0 + \ldots + 0 + c_{k+1}T\alpha_{k+1} + \ldots + c_nT\alpha_n$.

$\therefore T\alpha = c_{k+1}T\alpha_{k+1} + \ldots + c_nT\alpha_n$

$\Rightarrow S_3 = \{T\alpha_{k+1}, T\alpha_{k+2}, \ldots, T\alpha_n\}$ spans R_T

To Prove: S_3 is linearly independent

Consider $c_{k+1}T\alpha_{k+1} + \ldots + c_nT\alpha_n = 0 \longrightarrow (1)$

$$\Rightarrow \sum_{i=k+1}^{n} c_iT\alpha_i = 0$$

$$\Rightarrow T\left(\sum_{i=k+1}^{n} c_i\alpha_i\right) = 0$$

$$\Rightarrow \sum_{i=k+1}^{n} c_i\alpha_i \in N$$

$S_1 = \{\alpha_1, \alpha_2, \ldots, \alpha_K\}$ is a basis of N.

Then $\sum_{i=K+1}^{n} C_i \alpha_i$ can be expressed as a linear combination of

$\alpha_1, \alpha_2, \ldots, \alpha_K$

(ii) $$\sum_{i=K+1}^{n} C_i \alpha_i = d_1\alpha_1 + d_2\alpha_2 + \cdots + d_K\alpha_K$$

$$= \sum_{i=1}^{K} d_i \alpha_i$$

$$\therefore \sum_{i=1}^{K} d_i \alpha_i - \sum_{i=K+1}^{n} C_i \alpha_i = 0$$

$$\Rightarrow d_1 = d_2 = \cdots = d_K = 0$$

$$C_{K+1} = C_{K+2} = \cdots = C_n = 0$$ because

$\{\alpha_1, \alpha_2, \ldots, \alpha_K, \alpha_{K+1}, \ldots, \alpha_n\}$ is a basis of V.

In equation (1), $C_{K+1} = C_{K+2} = \cdots = C_n = 0$

$\Rightarrow \{T\alpha_{K+1}, T\alpha_{K+2}, \ldots, T\alpha_n\}$ is linearly independent.

Hence S_3 is a basis for R_T

$\therefore$ dimension of $R_T = n-K$.

(ie) $\text{Rank}(T) = n-K$.

$$\therefore \text{Rank } T + \text{nullity } T = (n-K) + K$$

$$= n$$

$$= \dim V.$$

$\therefore$ Rank T + Nullity T = $\dim V$.

Let $U \in L(V,W)$ and $U = \sum_p \sum_q A_{pq} E^{pq}$

Consider $U\alpha_j = \sum_p \sum_q A_{pq} E^{pq}(\alpha_j)$

$= \sum_p \sum_q A_{pq} \delta_{jq} \beta_p$

$= \sum_p A_{pj} \beta_p$

ie $U\alpha_j = T\alpha_j \Rightarrow U = T$

T is a linear combination mn linear transformation in $L(V,W)$

mn linear transformation spans $L(V,W)$

To Prove: mn linear transformation are linearly independent

Consider $\sum_p \sum_q A_{pq} E^{pq} = 0$

$\Rightarrow \sum_p \sum_q A_{pq} E^{pq}(\alpha_j)$

$\Rightarrow A_{pj} = 0, \forall p \,\&\, j$

since $\{\beta_1, \beta_2, \ldots, \beta_m\}$ is a basis for W

Hence mn linear transformation independent. $\longrightarrow$ (3)

from (2) & (3)

These mn linear transformation form a basis for $L(V,W)$

$\dim L(V,W) = mn$.

$L(V,W)$ is finite dimensional & its dim is mn

Hence the Proof.

Theorem: let V be n-dimensional vector space over F & W be m dimensional vector space over F and then $L(V, W)$ is finite dimensional and its dimensional is $m \times n$.

Proof:- $L(V, W)$ is the space of all linear transformation from V into W

Let $\beta = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of V and

$\beta' = \{\beta_1, \beta_2, \ldots, \beta_m\}$ be a basis of W

For each pair of integers (p, q) define a linear transformations

$$E^{p,q}(\alpha_i) = \begin{cases} 0 & \text{if } i \neq q \\ \beta_p & \text{if } i = q \end{cases}$$

$$= \delta_{iq} \beta_p \longrightarrow (1)$$

Then there is a unique linear transformation.

From V into W satisfying these conditions

We claim that these mn linear transformations from a basis to $L(V, W)$

First we prove mn linear transformations spans $L(V, W)$

for that we have show that if $T \in L(V, W)$.

$$\text{Then } T = \sum_{p=1}^{m} \sum_{q=1}^{n} A_{pq} E^{pq}$$

if $\alpha_j \in V$, then $T\alpha_j \in W$

$\{\beta_1, \beta_2, \ldots, \beta_m\}$ is a basis of W.

Then $T\alpha_j$ is unique linear transformation of $\beta_1, \beta_2 \ldots \beta_m$

$$\text{as } T\alpha_j = \sum_{p=1}^{m} A_{pj} \beta_p \longrightarrow (2)$$

Definition :- Linear operator.

Let V be a vector space over the field F. Then T is called a linear operator on V if T is a linear Transformation from V into V.

Definition: Invertible Transformation

A function $T: V \to W$ is said to be invertible if $\exists$ $U: W \to V$ such that UT is a the identity function on V & TU is the identity function on W

(ie) if $TT^{-1} = T^{-1}T = I$, then T is said to be invertible.

Theorem : If T is invertible, then (i) T is 1-1. (ii) T is onto

Proof: (i) Let $\alpha, \beta \in V$

consider $T\alpha = T\beta$

$$T\alpha - T\beta = 0$$

$$T(\alpha - \beta) = 0$$

$$T^{-1}T(\alpha - \beta) = T^{-1}(0)$$

$$I(\alpha - \beta) = 0$$

$$\alpha = \beta$$

Hence T is one to one

(ii) Let $N = \{\alpha, / T\alpha = 0\}$ and $R_T = \{\beta \in W / \beta = T\alpha, \alpha \in V\}$

Let $\beta \in W$, then

$$\beta = c_1 T\alpha_1 + c_2 T\alpha_2 + \dots + c_n T\alpha_n.$$

where $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a basis for V

Let $\alpha \in N$

$\Rightarrow T\alpha = 0$

$\Rightarrow \alpha = 0$

we know that Rank T + Nullity T = dim V.

Rank T + 0 = dim V

$R(T) = W$

Theorem:- Let $T: V \rightarrow W$ be a linear transformation. Then T is non singular iff T carries each linearly independent subset of V into a linearly independent subset of W.

Proof: Let T be non-singular, let $\{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a linearly independent subset of V.

To Prove: $\{T\alpha_1, T\alpha_2, \ldots, T\alpha_K\}$ is linearly independent in W.

Consider $c_1 T\alpha_1 + \ldots + c_K T\alpha_K = 0$

$\Rightarrow T(c_1\alpha_1 + \ldots + c_K\alpha_K) = 0$

$\Rightarrow c_1\alpha_1 + \ldots + c_K\alpha_K = 0$ [$\because T$ is invertible]

$\Rightarrow c_1 = c_2 = 0 \ldots = c_K = 0$

$\therefore \{T\alpha_1, T\alpha_2, \ldots, T\alpha_K\}$ is linearly independent in W.

Conversely, Let T carry linearly independent set into linearly independent set

Let $\alpha \neq 0 \in V$.

Then $\{\alpha\}$ is linearly independent

$\therefore \{T\alpha\}$ is linearly independent

$\therefore T\alpha \neq 0$

$\therefore$ null space of $T = \{0\}$

$\Rightarrow$ T is non-singular.

Isomorphism

Definition:- Let $T: V \to W$ be a linear transformation from a vector space V into W over the same field F. If T is 1-1 & onto, then T is said to be an isomorphism of V onto W.

Theorem: Every n-dimension vector space is isomorphic to the space F^n.

Proof: Let V be a vector space F & let $\dim V = n$.

Let $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ be an ordered basis of V.

Let $\alpha \in V$, then α is linear combination of $\alpha_1, \alpha_2, \dots, \alpha_n$ as

$$\alpha = x_1\alpha_1 + \dots + x_n\alpha_n, \quad x_i \in F$$

Define $T: V \to F^n$ such that $T\alpha = (x_1, \dots, x_n)$

To Prove: T is linear transformation, T is 1-1 & T is onto

Let $\alpha, \beta \in F$ & $c \in F$

Let $\alpha = x_1\alpha_1 + \dots + x_n\alpha_n$, $\beta = y_1\alpha_1 + \dots + y_n\alpha_n$

Consider $c\alpha + \beta = (cx_1 + y_1)\alpha_1 + \dots + (cx_n + y_n)\alpha_n$

$\therefore T(c\alpha + \beta) = cT\alpha + \beta$

$\therefore T$ is a linear transformation.

T is 1-1 : consider $T\alpha = T\beta$

$\Rightarrow (x_1, x_2, \dots, x_n) = (y_1, y_2, \dots, y_n)$

$\Rightarrow \alpha = \beta$

$\therefore T$ is 1-1.

T is onto : If $(x_1, x_2, \dots, x_n) \in F^n$, then $\exists\ \alpha \in V$ such that

$T\alpha = (x_1, x_2, \dots, x_n)$

T is onto

Then V is isomorphic to F^n.

Matrix of a linear operators

Definition: Let $T: V \to V$ be a linear operator on V, let B be an ordered basis of V. Then the matrix of T relative to the basis B is an $m\times n$ matrix A whose obtains A_{ij} all given by

$$T\alpha_j = \sum_{i=1}^{n} A_{ij}\,\alpha_{ij}, \quad j = 1, 2, \dots n$$

We denotes by $[T]_B$

Linear function

Definition: Let V be a vector space over the field F. If f is a function from V into F such that $f(c\alpha+\beta) = cf(\alpha) + f(\beta)$, $\forall\ \alpha, \beta \in V$ & $c \in F$, then f is called a linear functional on V.

Dual Basis

Definition: Let V be a vector space over F, & let $\dim V = n$

Let V^* be a dual space of V.

If $B = \{\alpha_1, \alpha_2, \dots, \alpha_n\}$ is a basis of V, then $\exists$ a unique linear functional f_i on V such that $f_i(\alpha_j) = \delta_{ij}$

Then n linear functions $\{f_1, f_2, \dots, f_n\}$ from a basis of V^* called dual bases of B.

Definition : Annihilator

If V is a vector space over F and S is a subset of V, the annihilator of S is the set S^0 of linear functionals f on V such that $f(\alpha) = 0$ for every α in S.

Theorem: Let V be a finite dimension vector space over F and let W be a sub space of V then $\dim W + \dim W^0 = \dim V$.

Proof :- Let $\dim W = K$ and let $\{\alpha_1, \alpha_2 \dots, \alpha_n\}$ be a basis of W

choose vector $\alpha_{K+1}, \dots, \alpha_n$ such that

$S_1 = \{\alpha_1, \dots, \alpha_n\}$ is a basis for V.

Let $\{f_1, f_2 \dots f_n\}$ be the dual basis of S_1 in V^*

To Prove : $\{f_{K+1}, \dots, f_n\}$ is a basis for the annihilator W^0

Then $f_i(\alpha_j) = \delta_{ij}$ and $\delta_{ij} = 0$ for $i \geq K+1$ & $j \leq K$ $\longrightarrow$ (1)

To Prove : $f_{K+1}, \dots, f_n \in W^0$

Let $\alpha \in W$, then α is a linear combination of $\alpha_1, \alpha_2 \dots, \alpha_K$

$\Rightarrow \alpha = c_1\alpha_1 + \dots + c_K\alpha_K$

$\Rightarrow f_i(\alpha) = c_1 f_i(\alpha_1) + \dots + c_K f_i(\alpha_K)$

$= 0$ for $r \geq K+1$.

$\therefore f_i(\alpha) = 0$, $\forall$ $i \geq K+1$ & $\alpha \in W$

$\therefore S_2 = \{f_{K+1}, \dots, f_n\} \in W^0$ and it is linearly independent set.

Next we Prove S_2 spans W^0

Let $f \in V^*$, then $f = \sum_{i=1}^{n} f(\alpha_i) f_i \longrightarrow (2)$

If $f \in W^0$, then $f(\alpha_i) = 0$ for $i \leq k$.

$\therefore$ (2) becomes $f = \sum_{i=k+1}^{n} f(\alpha_i) f_i$

$\therefore$ S_2 spans W^0

$\therefore \dim W^0 = n-k$

$\therefore \dim W + \dim W^0 = k+n-k = n$

$\therefore \dim W + \dim W^0 = \dim V$.

The transpose of a linear transformation

Definition:- Let V & W be two vector space over F. For each linear transformation T from V into W, there is a unique linear transformation T^t from W^* into V^* such that $(T^t g)(\alpha) = g(T\alpha)$ $\forall$ $g \in W^*$, $\alpha \in V$.

Unit - 3

Polynomial

Definition: Linear Algebra

Let F be a field. A linear Algebra over the field F is a vector space over F with an additional operation called multiplication of vectors which associates with each pair of vectors α, β in a vector $\alpha\beta$ in a called the Product of α and β in such a way that

(i) Multiplication is associative

$$\alpha(\beta\gamma) = (\alpha\beta)\gamma.$$

(ii) Multiplication is distributive w.r.t addition

$$\alpha(\beta+\gamma) = \alpha\beta + \alpha\gamma \quad \text{and}$$

$$(\alpha+\beta)\gamma = \alpha\gamma + \beta\gamma$$

(iii) For each scalar $c \in F$, $c(\alpha\beta) = \alpha(c\beta) = (c\alpha)\beta$

If there is an element $1 \in a$ such that $1\alpha = \alpha 1 = \alpha$, $\forall \alpha \in a$

Then $\mathcal{L}$ is called linear Algebra with identity over F.

The algebra is called commutative if $\alpha\beta = \beta\alpha$, $\alpha, \beta \in a$.

Theorem: Prove that F^{∞} is a linear Algebra identity.

Proof: A vector in F^{∞} is an infinite sequence of scalars in F

Let $f, g \in F^{\infty}$, $a, b \in F$

Let $f = (f_0, f_1, \ldots)$ & $g = (g_0, g_1, \ldots)$

Then $af + bg = (af_0 + bg_0, af_1 + bg_1, \ldots)$

Which define addition in F^{∞}

Product is defined by $(fg)_p = \sum_{i=0}^{n} f_i g_{n-i}$, $n = 0,1,2 \dots$

$\Rightarrow fg = (f_0g_0 + f_1g_0,\ f_0g_2 + 2f_1g_1 + f_2g_0 \dots)$

Consider $(gf_n) = \sum_{i=1}^{n} g_i f_{ni} = \sum f_i g_{n-i} = (fg)_n$

$\therefore fg = gf$

$\therefore$ Multiplication is commutative

Let $h = (h_0, h_1, \dots) \in F^{\infty}$

Consider $[(fg)h]_n = \sum_{i=0}^{n} (fg)_i h_{n-i}$

$= \sum_{i=1}^{n} \left(\sum_{j=0}^{i} f_i g_{i-j} \right) h_{n-i}$

$= \sum_{j=0}^{n} f_i [gh]_{n-j} = [f(gh)_n]$, $n = 0,1 \dots$

$\Rightarrow (fg)h = f(gh)$

$\therefore$ Multiplication is commutative

Multiplication is distributive:

$[f(g+h)]_n = \sum_{i=0}^{n} f_i (g+h)_{n-i}$

$= (fg)_n + (fh)_n$, $n = 0,1,2 \dots$

$\Rightarrow f(g+h) = fg + fh$

Let $\alpha, \beta \in F$ & $f \in F^{\infty}$

Taking $a = \alpha$, $b = \beta$ and $g = f$ in ①

We see that $a(\alpha + \beta) f \in F^{\infty}$

and $(\alpha + \beta) f = \alpha f + \beta f$

clearly $a(fg) = (\alpha f) g = f(\alpha g)$

$i = (1, 0, \dots) \in F^{\infty}$ is the identity elements of F^{∞}

$\therefore F^{\infty}$ is a commutative linear Algebra with identity over F

Definition: Algebra of formal Power Series

Let F be a field and S be the set of non-negative integers. Then the set of all function from S into F is a vector space over F denoted by F^{∞}

F^{∞} is a linear Algebra with identity and it is called the algebra of formal Power series over F

Polynomial.

Definition! Let $F[x]$ be the sub space of F^{∞} spanned by the vectors $1, x, x^2, \ldots$ An element of $F(x)$ is called a Polynomial over F.

$F(x)$ consists of all finite linear combination of x and its Power.

Definition: Degree of Polynomial.

A non-zero vector $f \in F^{\infty}$ is a Polynomial if and only if $\exists$ an integer $n \geq 0$ such that

i. $f_n \neq 0$

ii. $f_K = 0$, $\forall$ integer $K > n$

The integer n is unique and is called the degree of the Polynomial. It is denoted by degree of Polynomial.

Definition: Let L be a linear Algebra over F. Let $f = \sum_{i=0}^{n} f_i x^i \in L$ If $\alpha \in L$, then $f(\alpha) = \sum_{i=0}^{n} f_i \alpha^i$.

Theorem :- Let F is a field and L be a linear Algebra with identity over F. Suppose f & g are polynomials over F.

Let $\alpha \in L$ and $c \in F$. Then Prove that

1. $(cf+g)(\alpha) = cf(\alpha)+g(\alpha)$
2. $(fg)(\alpha) = f(\alpha)g(\alpha)$

Proof :

Let $f = \sum_{i=0}^{m} f_i x^i$ & $g = \sum_{j=0}^{n} g_j x^j$

(i) $(cf+g) = \sum_i (cf_i+g_i)\alpha^i$

$= c\sum_i f_i\alpha^i + \sum_i g_i\alpha^i$

$= cf(\alpha)+g(\alpha)$

(ii) $fg = \sum_{ij} f_i g_j x^{i+j}$

Then $(fg)(\alpha) = \sum_{i,j} f_i g_j \alpha^{i+j} = \left(\sum_{i=0}^{m} f_i\alpha^i\right)\left(\sum_{j=0}^{n} g_j\alpha^j\right)$

$= f(\alpha)g(\alpha)$.

Definition : Isomorphism

Let F be a field and L and $L^{\sim}$ be a linear Algebra over

L and $L^{\sim}$ are said to be isomorphic

If $\exists$ a 1-1 mapping $\alpha \to \alpha^{\sim}$ from L onto $L^{\sim}$ such that

1. $c(c\alpha+d\beta)^{\sim} = c\alpha^{\sim}+d\beta^{\sim}$
2. $(\alpha\beta)^{N} = \alpha^{N}\beta^{N}$, $\forall \alpha,\beta \in V$ & $c,d \in F$

The mapping is called an isomorphism of L onto $L^{\sim}$

Polynomial Ideals

Lemma : Suppose f and d are non-zero Polynomial over a field F such that $\deg d \leq \deg f$. Then Prove that $\exists$ a Polynomial $g \in F[x]$ such that either $f - \deg g = 0$ & $\deg(f-d) < \deg f$.

Proof:- Let $f = a_0 + a_1 x + \dots + a_{m-1} x^{m-1} + a_m x^m$; $a_m \neq 0$

and $d = b_0 + b_1 x + \dots + b_{n-1} x^{n-1} + b_n x^n$; $b_n \neq 0$

(ie) $f = a_m x^m + \sum_{i=0}^{m-1} a_i x^i \longrightarrow (1)$

$d = b_n x^n + \sum_{i=0}^{n-1} a_i x^i \longrightarrow (2)$

Let $\deg d \leq \deg f$

$m \geq n$

Multiplying (2) by $\left(\frac{a_m}{b_n}\right) x^{m-n}$

we get $d\left(\frac{a_m}{b_n}\right) x^{m-n} = a_m x^n + \left(\frac{a_m}{b_n}\right) x^{m-n} \sum_{i=0}^{n-1} b_i x^i \longrightarrow$

from (1) - (3) we see that either

$f - d\left(\frac{a_m}{b_n}\right) x^{m-n} = 0$ or

$\deg\left[f - d\left(\frac{a_m}{b_n}\right) x^{m-n} \right] < \deg f$

Then $\exists$ a Polynomial $g(x) = \frac{a_m}{b_n} x^{m-n} \in F[x]$

satisfying the given condition.

Definition :- Let d be a non-zero Polynomial over the field F. If $f \in F[$ then $\exists$ at most one Polynomial $q \in F[x]$ such that $f = dq$

we say that d divides f and f is divisible by d. It is denoted by $q = f|d$

q is called the quotient of f and d.

Corollary :

Let $f \in F[x]$ and $c \in F$. $f(x)$ is divisible by $x-c$ if and only if $f(c) = 0$

Definition :- Let F be a field and $f \in F[x]$. If $f(c) = 0$, then c in F is called a root or a zero of Polynomial f.

Corollary :- Prove that a Polynomial f of degree n over a field F has at most n roots in F

Proof :- The result is true for Polynomial of degree 0 and 1

Suppose that it is true for Polynomials of degree $(n-1)$

If a is a root of f, then

$$f = (x-a)q \quad \text{where } q \text{ is of degree } (x-1)$$

Since $f(b) = 0$ if $a = b$ or $q(b) = 0$

q has at most $(n-1)$ roots

$\therefore$ By induction f has at most n roots.

Definition :- If $f = c_0 + c_1 x + c_2 x^2 + \dots + c_n x^n \in F(x)$, then the dirivative of their Polynomial is defined by

$$Df = f' = c_1 + 2c_2 x + \dots + n c_n x^{n-1}$$

$$f'' = D^2 f$$

$$f''' = D^3 f \quad \& \text{ so on.}$$

Theorem : Taylor's formula

Let F be a field of characteristic, let $c \in F$ and n be a +ve integer, let f be a polynomial over F and let $\deg \leq n$, Then

$$f = \sum_{k=1}^{n} \frac{(D^k f)(c)}{\lfloor k} (x-c)^k.$$

Proof :- From binomial theorem, we have

$$(a+b)^m = \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k \text{ where}$$

$$\binom{m}{k} = \frac{\lfloor m}{\lfloor k \lfloor m-k} = \frac{m(m-1)\ldots(m-k+1)}{1 \cdot 2 \ldots k}.$$

By the binomial theorem

$$x^n = [c + (x-c)]^n$$

$$= \sum_{k=0}^{m} \binom{m}{k} c^{m-k} (x-c)^k$$

$$= c^m + mc^{m-1}(x-c) + \ldots + (x-c)^m \longrightarrow (1)$$

If we take $f = x^m$, then $f(c) = c^m$.

Then $(D^0 f)(c) = D^0 c^m = c^m$

$$(D^1 f)(c) = Dc^m = mc^{m-1}$$

$$(D^m f)(c) = D^m c^m = m(m-1)\ldots 1 = \lfloor m$$

$$(1) \Rightarrow x^m = D^0 c^m + \frac{Dc^m}{1!}(x-c) + \ldots + \frac{D^m c^m}{m!}(x-c)^m$$

$$= \sum_{k=0}^{m} \frac{(D^k f)(c)}{k!} (x-c)^k \longrightarrow (2)$$

If $f = \sum_{m=0}^{n} a_m x^m \longrightarrow$ (3)

Then $D^k f(c) = D^k \left(\sum_{m=1}^{n} a_m x^m \right)(c)$

$= \sum a_m (D^k x^m)(c)$

$\therefore \sum_{k=1}^{n} \frac{D^k f(c)}{k!} (x-c)^k = \sum_k \sum_m a_m \left(\frac{D^k x^m}{k!} \right)(c)\,(x-c)^k$

$= \sum_m a_m x^m$ [using (2)]

$= f$

Definition :- Let c be a root of the polynomial f then the largest +ve integer r such that $(x-c)^r$ divides f is called the multiplicity of c.

Definition : Ideal

Let F be a field let m be a subspace of $F[x]$, m is said to be an ideal of $F[x]$ if $f \in F[x]$ and $g \in m \Rightarrow fg \in m$.

Definition : Principal ideal.

Let m be an ideal of $F[x]$. If every element in M is a said to be an ideal of $F[x]$. if $f \in F[x]$ and $g \in m \Rightarrow fg \in M$

Definition :- if $P_1, \ldots, P_n$ are polynomial over a field F, not all of which are 0, the monic generator d of the ideal $P_1F[x] + \ldots + P_nF[x]$ is called the ~~generalist~~ greatest common divisor (gcd) of $P_1, P_2, \ldots, P_n$.

Relative Prime :- The polynomials are said to be relatively prime if their gcd is 1 if ideal generated by primes $F[x]$.

The Prime factorization of a Polynomial

Definition :- Let F be a field and f be a Polynomial in $F[x]$, f is Said to be reducible over F, if $\exists$ Polynomial $g, h \in F[x]$ such that $f = gh$

Theorem: If F is a field a non-scalar monic Polynomial in $F[x]$ can be factored as a Product of monic Primes in $F[x]$ in one and except for order only one way ?

Proof :- Suppose that f is a non-scalar monic polynomial over F

Polynomial of degree one are irreducible

$\therefore$ if $\deg f = 1$, then there is nothing to Prove

$\therefore$ $\deg f = n$ and $n > 1$ by induction assume that theorem is true for non-scalar monic Polynomial of degree less then r.

If f is irreducible, there is nothing to Prove

Let $f = gh$ where g and h are non-scalar monic Polynomials of degree less than n.

Then by assumption g and h can be factored as a Product of monic Primes in $F[x]$

Suppose that $f = P_1 \dots . P_m = q_1 \dots q_n \longrightarrow (1)$

where $P_1, P_2 \dots, P_m$ and $q_1, q_2 \dots q_n$ are monic Primes in $F[x]$

$\therefore$ consider $P_1 \dots P_m = q_1 \dots q_n$

clearly P_m divides LHS

$\therefore$ P_m divides $q_1 q_2 \dots q_n$

$\Rightarrow P_m$ divides q_i

But P_m and q_i are monic Primes

$\therefore q_i = P_m \longrightarrow$ (2)

if $m=1$ or $n=1$.

then $f = P_1 = q_1$ and there is nothing to Prove

Suppose that $m>1$ and $n>1$.

By rearranging q's, we can assume $P_m = q_n$

$\therefore$ (1) becomes $P_1 \ldots P_{m-1}, P_m = q_1 \ldots q_{n-1}, q_n$

$\Rightarrow P_1 \ldots P_{m-1} = q_1 . q_2 \ldots q_{n-1}$

Now the Polynomials $P_1 \ldots P_{m-1}$ has degree less than n

$\therefore$ By induction hypothesis we see that $q_1 q_2 \ldots q_{n-1}$ is atmost a rearrangement of $P_1 \ldots P_{m-1}$

Hence the theorem.

Determinants

Definition of a n-linear

Let K be a commutative ring with identity 1. Let n be a +ve integer.

Let $D: A \rightarrow D(A)$ be a function.

which gives to each $n \times n$ matrix A a scalar $D(A)$

D is said to be n-linear if for each i $(1 \leq i \leq n)$

D is a linear function of the i^{th} row when the other $(n-1)$ rows are held fixed.

Alternating function :-

Let D be a n-linear function. D is said to be alternating

if 1. $D(A) = 0$ when 2 rows of A are equal

ii. $D(A') = -D(A)$ where A' obtained from A by interchanging any two rows of A.

Determinant function :-

Let K be a commutative ring with identity 1. Let n be a +ve integer. Let $D: K^{n \times n} \to K$ be a function from $n \times n$ matrices into K

1. D is n-linear
2. D is alternating
3. $D(I) = 1$; I - identity matrix.

Lemma :- Let D be a two linear function on $n \times n$ matrices over K. Suppose D has the property that $D(A) = 0$ where two adjoint rows are equal. Prove that D is alternating.

Proof :- we have to prove

(i) $D(A) = 0$ when any two row are equal

(ii) $D(A') = -D(A)$ where A' is obtained from A by interchange two rows

when A' is obtained from A by interchanging two adjacent rows of A

then $D(A') = -D(A)$

Let B be the matrix obtained from A by interchanging rows i & j where $i < j$

First we interchange row i with row $(i+1)$ and continue till we get the rows in the order.

$\alpha_1 \ldots \alpha_{i-1}, \alpha_{i+1} \ldots \alpha_j, \alpha_i, \alpha_{j+1} \ldots \alpha_n$

The number of interchange in these case $= j-i$

Next we prove α_j to the jth position.

The number of interchange of rows $= j-1-1$

Thus B is obtained from A by $(j-i)+[(j-i)-1]$ interchanges

$2j-2i-1$ interchanges

$$D(B) = (-1)^{2j-2i-1} D(A)$$

$$= -D(A)$$

Condition ② is true.

Next we prove $D(A)=0$ when any 2 rows are equal

Given $D(A)=0$ when 2 adjacent rows are equal

If A, two adjacent rows are equal, then $D(A)=0$

if not, we can make two any equal rows adjacent by row interchanges as we did before.

$\therefore$ In that case also $D(A)=0$.

Unit - 4

Elementary Canonical Forms

Characteristic values

Let V be a vector space over the field F and let T be a linear operator on V. A characteristic value of T is a scalar c in F such that there is a non-zero vector α in V with $T\alpha = c\alpha$. If c is a characteristic value of T, then

(a) any α such that $T\alpha = c\alpha$ is called a characteristic vector of T associated with the characteristic value c.

(b) the collection of all α such that $T\alpha = c\alpha$ is called the characteristic space associated with c.

Result: Let T be a linear operator on a finite-dimensional space V and let c be a scalar. The following are equivalent.

1. c is a characteristic value of T
2. The operator $(T - cI)$ is singular (not invertible)
3. $\det(T - cI) = 0$

Definition: If A is an $n \times n$ matrix over the field F, a characteristic value of A in F is a scalar c in F such that the matrix $(A - cI)$ is singular

Lemma: Similar matrices have the same characteristic polynomial.

Proof: If $B = P^{-1}AP$, then $\det(xI - B) = \det(xI - P^{-1}AP)$.

$$= \det(P^{-1}(xI - A)P)$$

$$= \det P^{-1} \det(xI - A) \det P$$

$$= \det(xI - A)$$

Definition :- Let T be a linear operator on the finite dimensional space V. We say that T is diagonalizable if there is a basis for V each vector of which is a characteristic vector of T.

Lemma :- Suppose that $T\alpha = c\alpha$. If f is any Polynomial, then $f(T)\alpha = f(c)\alpha$

Proof :- Given $T\alpha = c\alpha$

(ie) c is a characteristic value of T and α is the characteristic vector such that $\alpha \neq 0$

Now $T\alpha = c\alpha \Rightarrow T\alpha - c\alpha = 0$

$\Rightarrow (T - cI)\alpha = 0$

$\Rightarrow f(T - cI)\alpha = f(0)$

$\Rightarrow f(T)\alpha - f(c)\alpha = 0$

$\Rightarrow f(T)\alpha = f(c)\alpha$

Annihilating Polynomial

Definition :- Minimal Polynomial

Let T be a linear operator on a finite dimensional vector space V over the field F. The minimal polynomial for T is the unique monic generator of the ideal of polynomials over F which annihilate T.

Theorem :- Let T be a linear operator on an n-dimensional vector space V. The characteristic and minimal polynomial for T have same roots except for multiplicities.

Proof :-

Let P be the minimal Polynomial for T. Let c be a Scalar.

To Prove: $P(c) = 0$ iff c is a characteristic value for T.

Suppose $P(c) = 0$, then $P = (x-c)q$, where q is a polynomial

Since $\deg q < \deg P$

Then by define of minimal Polynomial P tells us that $q(T) \neq 0$

choose a vector β such that

$$q(T)\beta \neq 0$$

Let $\alpha = q(T)\beta$, then $0 = P(T)\beta$

$$0 = (x-c)\,q(T)\beta$$

$$0 = (x - cI)\alpha$$

Thus c is a characteristic value of T

Conversely suppose c is a characteristic value of T

Then $T\alpha = c\alpha$, with $\alpha \neq 0$

$$P(T)\alpha = P(c)\alpha$$

Since $P(T) = 0$ and $\alpha \neq 0$

$$\Rightarrow P(c) = 0$$

Invariant subspace.

Definition :-

Let V be a vector space and T be a linear operator on V if W is a subspace of V we say that W invariant under T if for each vector α in W the vector $T\alpha$ is in W (ie) if $T(W)$ is contained in W.

(ie) $T(W) \subseteq W$

Lemma:-

Let W be an invariant subspace for T. The characteristic Polynomial for the restriction operator T_W divides the characteristic Polynomial for T. The minimal Polynomial for T_W divides the minimal Polynomial for T.

Proof:-

Let the characteristic Polynomial for T_W divides the characteristic Polynomial for T.

$$A = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix} \quad \text{where } A = [T]_B \ \& \ B = [T]_B$$

$$\det(xI - A) = \det \begin{bmatrix} x - A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & x - A_{22} & \cdots & A_{2n} \\ \vdots & & & \\ A_{n1} & A_{n2} & \cdots & x - A_{nn} \end{bmatrix}$$

Since $A_{ij} = 0$ if $j \leq r$, $i > r$.

$$\det(xI - A) = \det[xI - B] \det(xI - D)$$

$\det(xI - A)$ is characteristic Polynomial for T

$\det(xI - A)$ is characteristic Polynomial for T_W.

$$\therefore A^k = \begin{bmatrix} B^k & C^k \\ 0 & D^k \end{bmatrix} \quad \text{where } C^k \text{ is some } r \times (n-r) \text{ matrix}$$

$\therefore$ Any Polynomial which annihilates A

$$f(A) = 0$$

$$f(A) = \det(xI - A).$$

$$0 = \det[xI - B] \det(xI - D).$$

Also annihilates B

(ie) Any Polynomial which annihilates A also annihilates B.

$\therefore$ The minimal Polynomial for B divides the minimal Polynomial for A.

Definition :-

Let W be an invariant subspace for T and let α be a vector in V. The T- conductor of α into W is the set $S_T(\alpha; W)$ which consists of all Polynomials g such that $g(T)\alpha$ is in W.

Notes :-

If the special case $W = \{0\}$ the conductor is called the T- conductor of α.

Direct sum decomposition

Definition :-

Let $W_1, W_2, \ldots, W_k$ be subspaces of the vector V, we say that $W_1, \ldots, W_k$ are independent if $\alpha_1 + \alpha_2 + \ldots + \alpha_k = 0$, α_i in W_i implies that each α_i is 0.

Definition :- If V is a vector space, a Projection of V is a linear operator E on V such that $E^2 = E$

Theorem :-

If $V = W_1 \oplus \ldots \oplus W_k$, then there exists k linear operators $E_1, \ldots, E_k$ on V such that

(i) each E_i is a Projection $(E_i^2 = E_i)$

(ii) $E_iE_j = 0$, if $i \neq j$

(iii) $I = E_1 + \cdots + E_k$, (iv) the range of E_i is W_i

Conversely if $E_1, \ldots, E_K$ are K linear operators on V which satisfy conditions (i), (ii), and (iii) and if we let W_i be the range of E_i, then $V = W_1 \oplus W_2 \oplus \ldots \oplus W_K$

Proof:

We have to proof only the converse statement suppose $E_1, \cdots, E_K$ are linearly operators on V which satisfy the first three conditions and let w_i be the range of E. Then $V = W_1 + W_2 + \cdots + W_K$.

For by condition (iii)

$$\alpha = E_1\alpha + \cdots + E_K\alpha$$

For each $\alpha \in V$, and $E_i\alpha \in W_i$

This expression for α's unique, because $\alpha = \alpha_1 + \cdots + \alpha_K$ with $\alpha_i \in W_i$ say $\alpha_i = E_i\beta_i$

Then using (i) and (ii) we have

$$E_j\alpha = \sum_{i=1}^{K} E_j E_i$$

$$= \sum_{i=1}^{K} E_j E_i \beta_i$$

$$= E_j^2 \beta_j = E_j\beta_j = \alpha_j$$

This show that V is direct sum of the W_i's

Hence the Proof.

Unit - 5

Invariant Direct Sum

Definition :-

If α is a vector in V, then $\exists$ unique vector $\alpha_1, \alpha_2, \ldots, \alpha_k$ with $\alpha_i \in W_i$ such that $\alpha = \alpha_1 + \alpha_2 + \cdots + \alpha_k$ and then $T\alpha = T_1\alpha_1 + \cdots + T_k\alpha_k$ we say that T is direct sum of the operators $T_1, T_2, \ldots, T_k$.

Theorem :-

Let T be a linear operator on the space V and let $W_1, \ldots, W_k$ and $E_1, \ldots, E_k$ be on the subspace of V and Projection on V. Then a necessary and sufficient condition that each subspace W_i be invariant under T is that T commutative with each of the Projections E_i that is $TE_i = E_iT$, $i = 1, 2, \ldots, k$.

Proof :-

Suppose that T commutes with each E_i

Let α be in W_j

Then $E_j\alpha = \alpha$ and $T\alpha = T(E_j\alpha)$

$= E_j(T\alpha)$

which show that $T\alpha$ is in the range of E_j

i.e., W_j is invarient under T

Conversely, now Assume that each W_i is invariant under T

To show that $TE_j = E_jT$

Let α be any vector in V, Then $\alpha = \alpha_1 + \cdots + \alpha_k$

$\alpha = E_1\alpha + \cdots + E_k\alpha$ [$\because W_i$ is invariant under T $\therefore \alpha_j = E_j\alpha$]

$$T\alpha = TE_1\alpha + \dots\dots + TE_k\alpha$$

Since each $E_i\alpha$ is in W_i which is invariant under T

we have $T(E_i\alpha) = E_i\beta_i$ for some vector β_i

Then $E_j T E_i\alpha = E_j E_i \beta_i$

$$= \begin{cases} 0 & \text{if } i \neq j \\ E_j\beta_j & \text{if } i = j \end{cases}$$

Thus $E_j T\alpha = E_j TE_1\alpha + \dots\dots + E_j TE_k\alpha$

$$= E_j\beta_j$$

$$= TE_j\alpha$$

$$\therefore \; E_j T\alpha = TE_j\alpha$$

$$E_j T = TE_j$$

This holds for each $\alpha \in V$

Hence $E_j T = TE_j$

i.e. T commutes with each of the projections E_i, $i = 1, \dots, k$.

Primary Decomposition theorem.

Theorem: Let T be a linear operator on the finite dimensional vector space V over the field F. Let P be the minimal polynomial for T, $P = p_1^{r_1} \dots p_k^{r_k}$ where the p_i are distinct irreducible monic polynomials over F and the r_i are positive integers. Let W_i be the null space of $p_i(T)^{r_i}$, $i = 1, \dots, k$, then

(i) $V = W_1 \oplus W_2 \oplus \dots \oplus W_k$

(ii) each W_i is invariant under T

(iii) If T_i is the operator for induced on W_i by T, then the minimal polynomial for T_i is $p_i^{r_i}$

Proof:-

If the direct sum decomposition (1) is valid.

Our aim is to get a projections $E_1, E_2, \dots, E_k$ associated with the decomposition.

The projection E_i will be the identity on W_i and zero on the other W_j

$\therefore$ we shall find a polynomial h_i such that $h_i(T)$ is the identity on W_i and zero on the other W_j

$$\therefore h_1(T) + \dots + h_k(T) = I$$

For each i, let $f_i = \dfrac{P}{P_i^{r_i}} = \prod_{j \neq i} P_j^{r_j}$

Since $P_1 \dots P_k$ are distinct prime polynomials

The polynomials $f_1, f_2, \dots, f_k$ are relatively prime

$$\therefore (f_i, f_j) = 1$$

Thus there are polynomials $g_1, g_2, \dots, g_k$ such that $\sum_{j=1}^{n} f_i g_i = 1$

Also note that if $i \neq j$ then $f_i f_j$ is divisible by the polynomial P because $f_i f_j$ contains each $P_m^{r_m}$ as a factor

we shall show that the polynomials $h_i = f_i g_i$

let $E_i = h_i(T)$

$= f_i(T)\, g_i(T)$

Since $h_1 + \cdots + h_K = 1$ and P divides $f_i f_j$ for $i \neq j$

$$E_i E_j = 0 \quad \text{if} \quad i \neq j$$

Thus the E_i are Projections which correspond to some direct sum decomposition of the space V.

The range of E_i is exactly the subspace of W_i

$\Rightarrow$ Each vector in the range of E_i is in W_i

If α is the range of E_i, then $\alpha = E_i\alpha$

$$P_i(T)^{r_i}\alpha = P_i(T)^{r_i} E_i\alpha$$

$$= P_i(T)^{r_i} (T)\, g_i(T)^{\alpha}$$

$$= 0$$

because $p^{r_i} f_i g_i$ is divisible by the minimal Polynomial P.

Conversely

let g be any Polynomial such that

$$g(T_i) = 0$$

Then $g(T) f_i(T) = 0$

Thus $g f_i$ is divisible by the minimal Polynomial P of T

(ie) $P_i^{r_i} f_i$ divides $g f_i$

$\Rightarrow$ $P_i^{r_i}$ divides g

Hence the minimal Polynomial for T_i is $P_i^{r_i}$

Theorem:- Let T be a linear operator on finite dimensional space V. If T is diagonalizable and if $c_1, c_2, \ldots, c_K$ are distinct characteristic value T, then there exists linear operators

$E_1, E_2, \ldots, E_k$ on V such that

(i) $T = c_1E_1 + c_2E_2 + \cdots + c_kE_k$

(ii) $I = E_1 + E_2 + \cdots + E_k$

(iii) $E_iE_j = 0 \quad i \neq j$

(iv) $E_i^2 = E_i$ (E_i is a Projection).

(v) The range of E_i is the characteristic space for T with c_i

Conversely, if there exists k distint scalar $c_1, \ldots, c_k$ and k non zero linear operators $E_1, \ldots, E_k$ which satisfy conditions (i), (ii), and (iii) then T is diagonalizable $c_1, \ldots, c_k$ are distint characteristic value of T and conditions (iv) and (v) are also satisfied.

Proof:- Suppose that T is diagonalizable, with distinct characteristic value $c_1, \ldots, c_k$

Let W_i be the space of characteristic vector associated with the characteristic value c_i

$$\therefore V = W_1 \oplus \cdots \oplus W_k$$

Let $E_1, E_2, \ldots, E_k$ be the Projections associated with this decomposition

Then (i), (ii), (iii), (iv) and (v) are satisfied

To verify (i) Proceed as follows

For each α in V, $\alpha = E_1\alpha + E_2\alpha + \cdots + E_k\alpha$.

$$T\alpha = TE_1\alpha + \cdots + TE_k\alpha.$$

$\therefore T\alpha = (c_1E_1 + \dots + c_KE_K)\alpha$

$\therefore T = c_1E_1 + \dots + c_KE_K.$

Now suppose that we are given a linear operator T along with the distinct scalar c_i and non-zero operator E_i which satisfy (i), (ii), and (iii)

Since $E_iE_j = 0$ when $i \neq j$

we multiply both side of $E = E_1 + \dots + E_K$ by E_i

$\Rightarrow E_i = E_iE_i + \dots + E_iE_i + \dots + E_KE_i$

$\Rightarrow E_i = E_i^2$

Also multiplying $T = c_1E_1 + \dots + c_KE_K$ by E_i

$\Rightarrow TE_i = c_iE_i^2$

Since each E_i is a Projection.

$E_i^2 = E_i$

$TE_i = c_iE_i$

$TE_i - c_iE_i = 0$

$(T - c_i)E_i = 0$

which show that any vector in the range of E_i is in the null space of $(T - c_iT)$

Assume that $E_i \neq 0$

$T - c_iI = 0$

Hence c_i is the characteristic values of T

If c is any scalar then $T - cI = (c_1 - c)E_1 + \dots + (c_K - c)E_K$

$$(T-cI)\alpha = (c_1-c)E_1\alpha + \dots + (c_k-c)E_k\alpha$$

If $(T-cI)\alpha = 0$, then $(c_i-c)E_i\alpha = 0$

If $\alpha \neq 0$, then $E_i\alpha \neq 0$ for some i

$\therefore c_i - c = 0 \quad [\because E_i\alpha \neq 0]$

Since T is diagonalizable

Every non zero vector in the range of E_i is a characteristic value of T

And $I = E_1 + E_2 + \dots + E_k$ these characteristic vectors spans V.

The null space $(T - c_i I)$ is exactly the range of E_i

If $T\alpha = c_i\alpha$, then

$$\sum_{j=1}^{k} (c_j - c_i) E_i\alpha = 0$$

Hence $(c_j - c_i)E_i\alpha = 0$ for each j

and then $E_i\alpha = 0$ for $j \neq i$

Since $\alpha = E_1\alpha + \dots + E_k\alpha$ and $E_i\alpha = 0$ for $j \neq i$

$\Rightarrow \alpha = E_i\alpha$ which prove that α is in the range of E_i

Definition: let N be a linear operator on the vector space V. we say that N is nilpotent if there is some positive integer r such that $N^r = 0$

Cyclic subspace and Annihilators

Definition:- If α is any vector in V, the T-cyclic subspace generated by α is the subspace $Z(\alpha ; T)$ of all vectors of the form $g(T)\alpha$, g in $F[x]$. If $Z(\alpha ; T) = V$, then α is called a cyclic vector for T.

Definition:- If α is any vector in V, the T-annihilator of α is the ideal $M(\alpha ; T)$ in $F[x]$ consisting of all polynomials 'g' over F such that $g(T)\alpha = 0$. The unique monic polynomial P_α which generates this ideal will also be called the T-annihilator of α.

Definition:- Let T be a linear operator on a vector space V and let W be a subspace of V. We say that W is T-admissible if

(i) W is invariant under T

(ii) if $f(T)\beta$ is in W, there exists a vector γ in W such that $f(T)\beta = f(T)\gamma$.