CORE COURSE VIII

PARTIAL DIFFERENTIAL EQUATIONS

Objectives

1. To give an in-depth knowledge of solving partial differential equations and apply them in scientific and engineering problems.
2. To study the other aspects of PDE

UNIT I

Partial differential equations- origins of first order Partial differential equations- Cauchy's problem for first order equations- Linear equations of the first order- Integral surfaces Passing through a Given curve- surfaces Orthogonal to a given system of surfaces -Non linear Partial differential equations of the first order.

UNIT II

Cauchy's method of characteristics- compatible systems of first order equations- Charpits method- Special types of first order equations- Solutions satisfying given conditions- Jacobi's method.

UNIT III

Partial differential equations of the second order : The origin of second order equations –second order equations in Physics – Higher order equations in Physics - Linear partial differential equations with constant co-efficient- Equations with variable coefficients- Characteristic curves of second order equations

UNIT IV

Characteristics of equations in three variables- The solution of Linear Hyperbolic equations-Separation of variables. The method of Integral Transforms – Non Linear equations of the second order.

Unit V

Laplace equation : Elementary solutions of Laplace's equations-Families of equipotential Surfaces- Boundary value problems-Separation of variables –Problems with Axial Symmetry.

TEXT BOOK

Ian N. Sneddon,Elements of Partial differential equations**,** Dover Publication –INC, New York, 2006.
UNIT I Chapter II Sections 1 to 7
UNIT II Chapter II Sections 8 to 13
UNIT III Chapter III Sections 1 to 6
UNIT IV Chapter III Sections 7 to 11
UNIT V Chapter IV Sections 2 to 6

REFERENCES

1. **M.D.Raisinghania**, Advanced Differential Equations , S.Chand and company Ltd., New Delhi,2001.
2. **E.T.Copson,** Partial Differential Equations, Cambridge University Press

*****

UNIT – V

Elementary soln of laplaces equations – Families of Equipotential surfaces – Boundary value problems – separation of variables – problems with Axial symmetry.

TEXT BOOK:

Ian. N. Sneddon, Elements of partial Differential equation Dover publications – INC. Newyork – 2006.

UNIT – I → ch – II → sec 1 – 7

UNIT – II → ch – II → sec 8 – 13

UNIT – III → ch III → sec 1 – 6.

UNIT – IV → ch – III → sec 7 – 11

UNIT – V → ch – IV → sec 2 – 6

REFERENCES:

1. M. D. Raisinghanier, Advanced D.E. S. chand & company, New Delhi – 2001.
2. E. T. Copson P.D.E. Cambridge university press.

UNIT - I

partial differential Equation :

An equation involving a function and / or if partial derivatives $\frac{\partial \theta}{\partial x}$, $\frac{\partial \theta}{\partial y}$, $\frac{\partial \theta}{\partial z}$, $\frac{\partial \theta}{\partial t}$ will be non - zero.

Higher derivateives of The types,

$\frac{\partial^2 \theta}{\partial x^2}$, $\frac{\partial^2 \theta}{\partial y^2 \partial x \partial t}$, $\frac{\partial^3 \theta}{\partial x^2 \partial t}$ etc....

A relation between The derivatives of a kind

$$F\left(\frac{\partial^2 \theta}{\partial x}, \frac{\partial^2 \theta}{\partial x^2}, \frac{\partial^2 \theta}{\partial x \partial t} \cdots\cdots\right) = 0$$

Such an equation relating partial derivatives is called "partial differential equation".

First order equation in two variables,

The equation, $\left(\frac{\partial \theta}{\partial x}\right)^3 + \frac{\partial \theta}{\partial t} = 0$

First order equation in Three Variables,

The equation, $x \cdot \frac{\partial \theta}{\partial x} + y \cdot \frac{\partial \theta}{\partial y} + z \frac{\partial \theta}{\partial z} = 0$

Second order equation in two Variables,

The equation, $\frac{\partial^2 \theta}{\partial x^2} = \frac{\partial x}{\partial t}$

consider partial differential equations of The first order, (i,e) The equation of the type

$$F\left(\theta, \frac{\partial \theta}{\partial x} \cdots\cdots\right) = 0$$

Scanned with CamScanner

Suppose That There are two independent Variable x, y

and that The dependent variable is denoted by z.

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}$$ (2)

Such an equation can be written in The form

$$F(x, y, z, p, q) = 0$$

Origins of first order PDE:

1. Eliminate The orbitary constants for This equation $x^2+y^2+(z-c)^2 = a^2$. where The constant a and c are orbitary.

Solu:

Given equation :- $x^2+y^2+(z-c)^2 = a^2$ —— (1)

Diff equ (1) partially w.r. to "x".

$$2x + 2(z-c) \cdot \frac{\partial z}{\partial x} = 0$$

$$2x + 2(z-c)p = 0$$

$$x + (z-c)p = 0$$

$$(z-c)p = -x$$

$$z - c = \frac{-x}{p} \quad \text{—— (2)}$$

Diff equ (1) partially w.r. to "y".

$$2y + 2(z-c)\frac{\partial z}{\partial y} = 0$$

Scanned with CamScanner

$$2y + 2(z-c)q = 0$$

$$y + (z-c)q = 0$$

$$(z-c)q = -y$$

$$z - c = \frac{-y}{q} \quad \text{——} (3)$$

Equating equation (2), (3)

$$\frac{-x}{p} = \frac{-y}{q}$$

$$xq = yp$$

$$xq - yp = 0$$

3

2. Eliminate the constant from the given equation $x^2 + y^2 = (z-c)^2 \tan^2\alpha$ where the constants c and α are orbitary.

Solu:

Given equ is $x^2 + y^2 = (z-c)^2 \tan^2\alpha$ —— (1)

Diff equ (1) partially w.r.to "x"

$$2x = 2(z-c) \cdot \frac{\partial z}{\partial x} \tan^2\alpha$$

$$x = (z-c)\, p \tan^2\alpha$$

$$\frac{x}{p} = (z-c)\tan^2\alpha \quad \text{——} (2)$$

Diff equ (1) partially w.r.to "y"

$$2y = 2(z-c) \cdot \frac{\partial z}{\partial y} \tan^2\alpha$$

$$y = (z-c) \cdot q \tan^2\alpha$$

$$\frac{y}{q} = (z-c)\tan^2\alpha \quad \text{——} (3)$$

Equating (2), (3)

$$\frac{x}{p} = \frac{y}{q}$$

$$xq = yp$$

$$xq - yp = 0$$

3. Eliminate the constant from the equation $(x-a)^2 + (y-b)^2 + z^2 = 1$ where a, b are orbitary constant

$(x-a)^2 + (y-b)^2 + z^2 = 1$ —— (1)

Given equ is $(x-a)^2 + (y-b)^2 + z^2 = 1$

Diff. (1) partially w.r.to "x"

$$2(x-a)(1) + 2z \cdot \frac{\partial z}{\partial x} = 0$$

$$2(x-a) = -2zp$$

$$x-a = -zp \Rightarrow x-a/p = -z \quad \text{—— (2)}$$

Diff (1) partially w.r.to "y"

$$2(y-b) + 2z \cdot \frac{\partial z}{\partial y} = 0$$

$$y-b = -zq$$

$$y-b/q = -z \quad \text{—— (3)}$$

Equating (2), (3) $\Rightarrow \dfrac{x-a}{p} = \dfrac{y-b}{q}$

$$q(x-a) = p(y-b)$$

$$q(x-a) - p(y-b) = 0$$

$$\left.\begin{aligned}(2) &\Rightarrow -zp = x-a\\ (3) &\Rightarrow -zq = y-b\end{aligned}\right\} \text{ in } (1)$$

$$(-zp)^2 + (-zq)^2 + z^2 = 1$$

$$z^2p^2 + z^2q^2 + z^2 = 1 \Rightarrow z^2(p^2+q^2+1) - 1 = 0$$

7. Eliminate the constant from the equation $z = (x+a)(y+b)$ where a, b are constants

5

Solu:

Given equ is $z = (x+a)(y+b)$ —— (1)

Diff with (1) partially with r.to "x"

$$\frac{\partial z}{\partial x} = (1)(y+b)$$

$$\frac{\partial z}{\partial x} = y+b$$

$$p = y+b \text{ —— (2)}$$

$$p - y = b$$

$$b = p - y$$

Diff (1) partially w.r.to "y"

$$\frac{\partial z}{\partial y} = (x+a)(1+0)$$

$$\frac{\partial z}{\partial y} = x+a$$

$$q = x+a \text{ —— (3)}$$

Sub (2), (3) in equ (1) we get

$$z = (p)(q)$$

$$z = pq$$

$$\boxed{z - pq = 0}$$

Scanned with CamScanner

Eliminate the constant from the equation

$2z = (ax+y)^2+b$

6

Soln:

Given equation $2z = (ax+y)^2+b$ —— (1)

Diff (1) partially with r. to "x"

$$2\frac{\partial z}{\partial x} = 2(ax+y)a$$

$$p = (ax+y)a$$

$$\frac{p}{a} = ax+y \quad \text{—— (2)}$$

Diff (1) partially with r. to "y"

$$2.\frac{\partial z}{\partial y} = 2(ax+y)(1)$$

$$\frac{\partial z}{\partial y} = (ax+y)$$

$$q = ax+y \quad \text{—— (3)}$$

Equating (2), (3) we get

$$\frac{p}{a} = q \Rightarrow p/q = a \quad \text{—— (4)}$$

$p = qa$

$p - aq = 0$

$\boxed{p - qa = 0}$

Sub equ (4) in (3) we get

$q = (p/q)x + y$

$q = \frac{px+qy}{q} \Rightarrow q^2 = px+qy$

6. Eliminate the constant from the equation

$ax^2+by^2+z^2=1$

Given equ is $ax^2 + by^2 + z^2 = 1$ ——(1)

Diff (1) partially with r. to "x".

$$2ax + 0 + 2z \cdot \frac{\partial z}{\partial x} = 0$$

$$2ax + 2z \cdot \frac{\partial z}{\partial x} = 0$$

$$2z \cdot p = -2ax$$

$$zp = -ax \quad \text{——(2)} \Rightarrow a = -\frac{zx}{p} \Rightarrow a = -\frac{zp}{px}$$

Diff (1) partially with r to "y"

$$0 + 2by + 2z\frac{\partial z}{\partial y} = 0$$

$$2by = -2z\frac{\partial z}{\partial y}$$

$$2by = -2qz$$

$$by = -qz \Rightarrow b = \frac{-zq}{y} \quad \text{——(3)}$$

$$\frac{-by}{q} = z \quad \text{——(3)}$$

Equating (2), (3) we get

$$\frac{ax}{p} = \frac{by}{q}$$

$$axq = byp$$

$$axq - byp = 0$$

$$\left(\frac{-zp}{x}\right)x^2 + \left(\frac{-zq}{y}\right)y^2 + z^2 = 1$$

$$-zpx - zqy + z^2 = 1$$

$$-zpx - zqy = -z^2 + 1$$

$$-z(px + qy) = -z^2 + 1$$

$$z(px + qx) = z^2 - 1$$

1. Eliminate the arbitary function f from the equation

$z = f(x^2+y^2)$

Given equ is $z = f(x^2+y^2)$ —— ①

diff (1) partially w.r.to "x"

$$\frac{\partial z}{\partial x} = f'(x^2+y^2) \cdot 2x$$

$$\frac{p}{2x} = f'(x^2+y^2) \text{ —— } (2)$$

Diff (1) partially w.r.to "y"

$$\frac{\partial z}{\partial y} = f'(x^2+y^2) \cdot 2y$$

$$\frac{q}{2y} = f'(x^2+y^2) \text{ —— } (3)$$

Equating (2), (3) $\Rightarrow$ $p/2x = q/2y \Rightarrow p/x = q/y$

$py = qx \Rightarrow \boxed{py - qx = 0}$

2. Eliminate the arbitary function from the equation

$z = xy + f(x^2+y^2)$.

Solu:

Given equation $z = xy + f(x^2+y^2)$ —— ①

diff (1) partially w.r. to "x"

$$\frac{\partial z}{\partial x} = (1)y + f'(x^2+y^2) \cdot 2x$$

$$\frac{\partial z}{\partial x} = y + f'(x^2+y^2) \cdot 2x$$

$$\frac{p-y}{2x} = f'(x^2+y^2) \text{ —— } (2)$$

Scanned with CamScanner

diff (1) partially w.r. to "y"

$$\frac{\partial z}{\partial y} = x(1) + f^1(x^2+y^2).2y$$

$$\frac{q-x}{2y} = f^1(x^2+y^2) \quad \text{——(3)}$$

9

Equating (2), (3) $\Rightarrow$ $\frac{p-y}{\cancel{2}x} = \frac{q-x}{\cancel{2}y}$

$$\frac{p-y}{x} = \frac{q-x}{y}$$

$$y(p-y) = x(q-x)$$

$$py - y^2 = xq - x^2$$

$$py - xq - y^2 + x^2 = 0$$

$$x^2 - y^2 - xq + py = 0$$

3. Eliminate the arbitary function from the equation

$z = x+y+f(xy)$

Solu:

Given equ $z = x+y+f(xy)$ —— (1)

diff (1) partially w.r. to x.

$$\frac{\partial z}{\partial x} = 1+0+f^1(xy)(y)$$

$$\frac{\partial z}{\partial x} = 1+f^1(xy)y$$

$$\frac{p-1}{y} = f^1(xy)$$

$$\frac{p-1}{y} = f^1(xy) \quad \text{——(2)}$$

diff (1) partially w.r. to "y".

$$\frac{\partial z}{\partial y} = 1+0+f^1(xy)(x) \Rightarrow \frac{q-1}{x} = f^1(xy) \quad \text{——(3)}$$

Equating (2), (3) $\Rightarrow \frac{p-1}{y} = \frac{q-1}{x}$

10

$$x(p-1) = y(q-1)$$

$$px - x = qy - y$$

$$-x + y - yq + xp$$

$$px - qy - x + y = 0$$

7. Eliminate The arbitary function from the equation

$$z = f\left(\frac{xy}{z}\right)$$

Given That $z = f(xy/z)$ —— (1)

Diff (1) partially w. r. to "x"

$$\frac{\partial z}{\partial x} = f'(xy/z)\left[\frac{zy - x \cdot y\, \partial z/\partial x}{z^2}\right]$$

$$p = f'(xy/z)\left[\frac{zy - xyp}{z^2}\right]$$

$$\frac{pz^2}{zy - xyp} = f'(xy/z) \text{ —— } (2)$$

Diff (1) partially w. r. to "y"

$$\frac{\partial z}{\partial y} = f'(xy/z)\left[\frac{zx - xy \cdot \partial z/\partial y}{z^2}\right]$$

$$q = f'(xy/z)\left[\frac{zx - xyq}{z^2}\right]$$

$$\frac{qz^2}{zx - xyq} = f'(xy/z) \quad \text{——— (3)}$$

Equating (2), (3) we get

$$\frac{pz^2}{zy - xyp} = \frac{qz^2}{zx - xyq}$$

$$p(zx - xyq) = q(zy - xyp)$$

$$pzx - pxyq = qzy - xyqp$$

$$xpz - pxyq - qzy + xypq = 0$$

$$xpz - yqz = 0$$

$$z(xp - yq) = 0$$

$$xp - yq = 0$$

5. Eliminate the arbitary function from the equation

$$z = f(x-y)$$

Given equ $z = f(x-y)$ ——— (1)

Diff (1) partially w.r.to "x".

$$\frac{\partial z}{\partial x} = f'(x-y)(1)$$

$$p = f'\,\cancel{(x)}\cdot(x-y)(1)$$

$$p = f'(x-y) \quad \text{——— (2)}$$

Diff (1) partially w.r.to "y"

$$\frac{\partial z}{\partial y} = f'(x-y)(-1)$$

$$-q = f'(x-y) \quad \text{——— (3)}$$

Equating (2), (3) ...

Linear Equation of 1st order P.D.E :-

12

Lagrange's linear Equation :-

The equation of the form $Pp + Qq = R \rightarrow (1)$ is known as lagrange's Equation, where P, Q, R are functions of x, y and z

To solve the equation it is enough to solve the subsidary equations.

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \rightarrow (2)$$

If the soln of the subsidary equation is of the form $u(x,y) = c_1$ and $v(x,y) = c_2$ ∴ Then the soln of the gn lagrange's equation is $\phi(u,v) = 0$.

To solve the subsidary equation we have two methods.

i) Method of grouping

ii) Method of multipliers.

(i) Method of grouping:-

Consider the subsidary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Take any two members say first two members (or) last two members are first and last member.

Now, consider the first two members.

$\frac{dx}{P} = \frac{dy}{Q}$. If P and Q contains z (other than x & y)

try to eliminate it.

Now direct integration gives $u(x,y) = c_1$. 13

Similarly, take another two members say $\frac{dy}{Q} = \frac{dz}{R}$. If Q and R contains x [(i.e) other than y & z] try to eliminate it

Now, direct integration gives $v(y,z) = c_2$

$\therefore$ The soln of the given Lagrange's equation is $\phi(u,v) = 0$

(ii) Method of Multipliers:-

Choose any three multipliers l, m, n be constants (or functions of x, y and z. Such that in,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

The expression $lP + mQ + nR = 0$.

Hence, $l\,dx + m\,dy + n\,dz = 0$.

[Each of the above ratios is equal to a constants]

(i.e) $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$

(i.e) $l\,dx + m\,dy + n\,dz = k\,(lP + mQ + nR)$

If $lP + mQ + nR = 0$.

The $l\,dx + m\,dy + n\,dz = 0$]

Now direct integration gives $u(x,y,z) = c_1$

Similarly,

Choose another set of multipliers l', m', n' which that in $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l'dx + m'dy + n'dz}{l'P + m'Q + n'R}$

The expression $lP+mQ+nR=0$, $l'dx+m'dy+n'dz=0$ [as explained Earlier]

14

Direct integration gives $v(x,y,z)=c_2$

∴ The soln of the lagrange's equ is $\phi(u,v)=0$.

1. Find the general lagrange's soln of $y^2p+xyq=x(z-2y)$

Gn equ is $y^2p+xyq=x(z-2y)$.

Lagrange's equ. $P.p+Qq=R$

$P=y^2$, $Q=-xy$, $R=x(z-2y)$.

The subsidary equ, $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

$$\frac{dx}{y^2}=\frac{dy}{-xy}=\frac{dz}{x(z-2y)}$$

Consider 1st & 2nd ratio,

$$\frac{dx}{y^2}=\frac{dy}{-xy}$$

$$-x\,dx=y\,dy$$

$\int$ing on b.s,

$$-\int x\,dx=\int y\,dy$$

$$\frac{-x^2}{2}=\frac{y^2}{2}+c$$

$$-\frac{x^2}{2}-\frac{y^2}{2}=c$$

$$\frac{-x^2-y^2}{2}=c$$

$$\frac{x^2+y^2}{2}=c$$

$x^2+y^2=2c$

$x^2+y^2=c_1$

15

Consider 2nd & 3rd ratio:

$$\frac{dy}{-xy} = \frac{dz}{x(z-2y)}$$

$$\frac{dy}{-y} = \frac{dz}{(z-2y)}$$

$dy\,(z-2y) = dz\,(-y)$

$z\,dy - 2y\,dy = -y\,dz$

$z\,dy + y\,dz = 2y\,dy$

$d(yz) = 2y\,dy$

$\int$ ing on b.s,

$\int d(yz) = \int 2y\,dy$

$yz = \frac{2y^2}{2} + c_2$

$yz - y^2 = c_2$

The solution is $\phi(c_1, c_2) = 0$

$\phi(x^2+y^2,\ yz-y^2) = 0$

2. Find the g.s of the equ $x^2p+y^2q=(x+y)z$ (or)

$x^2\frac{\partial z}{\partial x} + y^2\frac{\partial z}{\partial y} = (x+y)z$.

Gen $x^2p+y^2q=(x+y)z$ is a Lagrange's equation.

Lagrange's equ is $Pp+Qq=R$

where $P = x^2$, $Q = y^2$, $R = (x+y)z$.

The subsidary equation $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ (16)

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{(x+y).z}$$

1. Consider 1st & 2nd ratios,

$$\frac{dx}{x^2} = \frac{dy}{y^2}$$

$$x^{-2}dx = y^{-2}dy$$

$\int$ing on b.s we get,

$$\int x^{-2}dx = \int y^{-2}dy$$

$$\frac{x^{-1}}{-1} = \frac{y^{-1}}{-1}$$

$$-x^{-1} = -y^{-1}$$

$$-\frac{1}{x} = -\frac{1}{y}$$

$$\frac{-1}{x} + \frac{1}{y} = 0$$

$$\frac{1}{y} - \frac{1}{x} = 0$$

choose $(1, -1, 0)$ as lagrange's multipliers,

$$\frac{dx - dy}{x^2 - y^2} = \frac{dz}{(x+y)z}$$

$$\frac{dx - dy}{(x+y)(x-y)} = \frac{dz}{(x+y)z}$$

$$\frac{dx-dy}{(x-y)} = \frac{dz}{z}$$

$\int$ing on b.s

$\log(x-y) = \log z + \log c$

$\log(x-y) - \log z = \log c$

$\log\left(\frac{x-y}{z}\right) = \log c$

$$\frac{x-y}{z} = c_2$$

The soln is $\phi(c_1, c_2) = 0$

$$\phi\left(\frac{1}{y} - \frac{1}{x}, \frac{x-y}{z}\right) = 0$$

3. Find the g.s of $z(xp - yq) = y^2 - x^2$

Gen equ is $z(xp - yq) = y^2 - x^2$ is Lagrange's equ

Lagrange's equ, $xzp - yzq = y^2 - x^2$.

$$Pp + Qq = R$$

where $P = xz$, $Q = -yz$, $R = y^2 - x^2$.

The subsidary equation is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{xz} = \frac{dy}{-yz} = \frac{dz}{y^2 - x^2}$$

Consider 1st & 2nd ratio,

$$\frac{dx}{xz} = \frac{dy}{-yz}$$

Scanned with CamScanner

$$\frac{dx}{x} = \frac{dy}{-y}$$

∫ing on b.s we get

(18)

$\log x = -\log y + \log c$

$\log x + \log y = \log c$

$\log (xy) = \log c$

$xy = c_1$

~~Consider~~ choose $(1, -1, 0)$ as lagrange's multepliers,

$$\frac{dx - dy}{xz + yz} = \frac{dz}{y^2 - x^2}$$

$$\frac{dx - dy}{xz + yz} = \frac{dz}{-(x^2 - y^2)}$$

$$\frac{dx - dy}{z(x+y)} = \frac{dz}{-(x+y)(x-y)}$$

$(x-y)(dx - dy) = -z\,dz$

∫ing on b.s, $\frac{(x-y)^2}{2} = -\frac{z^2}{2} + c$

$$\frac{(x-y)^2}{2} + \frac{z^2}{2} = c$$

$(x-y)^2 + z^2 = c_2$

The soln is $\phi(c_1, c_2) = 0$

$\phi(xy, (x-y)^2 + z^2) = 0$

Scanned with CamScanner

4. Find the g.s. of $(y+zx)p - (x+yz)q = x^2 - y^2$.

Gen equ $(y+zx)p - (x+yz)q = x^2 - y^2$.

Lagrange's general equ is $Pp + Qq = R$

The auxillary equ, $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$P = y + zx$, $Q = -(x+yz)$, $R = x^2 - y^2$

$$\frac{dx}{y+zx} = \frac{dy}{-(x+yz)} = \frac{dz}{x^2-y^2}$$

Choose the $(x, y, -z)$ as Lagrange's multipliers

$$\frac{dx}{y+zx} = \frac{dy}{-(x+yz)} = \frac{dz}{x^2-y^2} \Rightarrow \frac{x\,dx + y\,dy - z\,dz}{x(y+zx) - y(x+yz) - z(x^2-y^2)}$$

$$= \frac{x\,dx + y\,dy - z\,dz}{xy + zx^2 - xy - y^2z - zx^2 + zy^2}$$

$$= \frac{x\,dx + y\,dy - z\,dz}{0}$$

$$x\,dx + y\,dy - z\,dz = 0$$

$\int$ing on b.s

$$\int x\,dx + \int y\,dy - \int z\,dz = 0$$

$$x^2/_2 + y^2/_2 - z^2/_2 = 0$$

$$x^2 + y^2 - z^2 = 2c$$

$$x^2 + y^2 - z^2 = c_1$$

Choose $(y, x, 1)$ as Lagrange's multipliers,

$$\frac{dx}{y+zx} = \frac{dy}{-(x+yz)} = \frac{dz}{x^2-y^2} \Rightarrow \frac{y\,dx + x\,dy + dz}{y^2 + zxy - x^2 - xyz + x^2 - y^2}$$

$$\Rightarrow \frac{y\,dx + x\,dy + dz}{y^2 + zxy - x^2 - xyz + x^2 - y^2} \Rightarrow \frac{y\,dx + x\,dy + dz}{0}$$

$y\,dx + x\,dy + dz = 0$

$\int$ ing on b.s we get,

$\int d(xy) + \int dz = 0$

$xy + z = c_2$

The Solution is $\phi(c_1, c_2) = 0$

$$\phi\left(x^2+y^2-z^2, xy+z\right) = 0$$

5. Find The g.s of $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$

Given equ is $x(y^2+z)p - y(x^2+z)q = z(x^2-y^2)$

Lagrange's g. equation is $Pp + Qq = R$

$P = x(y^2+z)$; $Q = -y(x^2+z)$; $R = z(x^2-y^2)$

The Subsidary equ is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

Choose $(x, y, -1)$ as Lagrange's multipliers,

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} \Rightarrow \frac{x\,dx + y\,dy + dz}{x^2(y^2+z) - y^2(x^2+z) + (x^2-y^2)(-z)}$$

$$\Rightarrow \frac{x\,dx + y\,dy + dz}{x^2y^2 + zx^2 - x^2y^2 - zy^2 + \cancel{x^2} - \cancel{y^2} - zx^2 + zy^2}$$

$$\Rightarrow \frac{x\,dx + y\,dy + dz}{0}$$

$x\,dx + y\,dy + dz = 0$

$\int$ing on b.s,

$\int x\,dx + \int y\,dy + \int dz = 0$

21

$x^2/_2 + y^2/_2 + z = 0$

$x^2/_2 + y^2/_2 - z = 0 + c$

$$\frac{x^2+y^2-2z}{2} = c$$

$x^2+y^2-2z = 2c$

$x^2+y^2-2z = c_1$

choose $(1/x, 1/y, 1/z)$ as lagrange's multipliers,

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} \Rightarrow \frac{1/x\,dx + 1/y\,dy + 1/z\,dz}{y^2+z-x^2-z+x^2-y^2}$$

$$\Rightarrow \frac{1/x\,dx + 1/y\,dy + 1/z\,dz}{0}$$

$1/x\,dx + 1/y\,dy + 1/z\,dz = 0$

$\int$ing on b.s,

$\int dx/x + dy/y + dz/z = 0$

$\log x + \log y + \log z = \log c$

$\log(xyz) = \log c$

$xyz = c_2$

The solution is $\phi(c_1, c_2) = 0$

$\phi(x^2+y^2-2z,\ xyz) = 0$

6. Find The g.s of $px(z-2y^2) = (z-qy)(z-y^2-2x^3)$

Gn' $px(z-2y^2) = (z-qy)(z-y^2-2x^3)$

$px(z-2y^2) = z(z-y^2-2x^3) - qy(z-y^2-2x^3)$ 22

$px(z-2y^2) + qy(z-y^2-2x^3) = z(z-y^2-2x^3)$

The auxillary equ is, $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\frac{dx}{x(z^2-2y^2)} = \frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)}$$

Consider 2nd & 3rd ratio we get

$$\frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)}$$

$$\frac{dy}{y} = \frac{dz}{z}$$

$\int$ing on b.s we get

$$\int dy/y - \int dz/z = c$$

$$\log y - \log z = \log c$$

$$\log(y/z) = \log c$$

$$y/z = c_1$$

choose $(0, +1, -1)$ as Lagrange's multipliers.

$$\frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)}$$

Scanned with CamScanner

$$= \frac{dy - dz}{y(z-y^2-2x^3) - z(z-y^2-2x^3)}$$ (23)

$$= \frac{dy - dz}{(y-z)(z-y^2-2x^3)}$$

Consider 2nd & 4th ratio

$$\frac{dy}{y(z-y^2-2x^3)} = \frac{dy-dz}{(y-z)(z-y^2-2x^3)}$$

$$\frac{dy}{y} = \frac{dy-dz}{(y-z)}$$

$\int$ing on b.s

$$\int dy/y = \int \frac{dy-dz}{y-z}$$

$$\log y = \log(y-z) + \log c$$

$$\log y - \log(y-z) = \log c$$

$$\log\left(y/y-z\right) = \log c$$

$$\frac{y}{y-z} = c_2$$

The solution is $\phi(c_1, c_2) = 0$

$$\phi\left(y/z,\ y/y-z\right) = 0$$

Integral surfaces passing through a given curve:-

The general solution may be used to determ[ine] the integral surface which passes through the gi[ven] curve. Suppose that we have found two solutions $u(x,y,z)=c_1;\ v(x,y,z)=c_2$ } — ① Any soln of the corresponding equ is of the form $F(u,v)=0$ — ② arising from a relation $F(c_1,c_2)=0$ — ③ b/w the constants c_1 and c_2

Suppose we wish to find the integral surface which passes through the curve c. whose parametric equations are $x=x(t),\ y=y(t),\ z=z(t)$ where t is a parameter. Then the particular solution must be such that

$$\left.\begin{aligned} u\{x(t),y(t),z(t)\} &= c_1 \\ v\{x(t),y(t),z(t)\} &= c_2 \end{aligned}\right\} \longrightarrow (4)$$

Next we eliminate the single parameter "t" from (4) and relation c_1 & c_2.

Finally we replace c_1 and c_2 with the help of equation (1) obtain the required integral surface.

1. Find the integral surface of the linear partial diff. equ $x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$ which contains the straight line $x+y=0$, $z=1$

10m Ex.

25

Gen equ is $x(y^2+z) - y(x^2+z)q = z(x^2-y^2)$

Lagrange's equ is $Pp + Qq = R$.

$P = x(y^2+z)$; $Q = -y(x^2+z)$; $R = z(x^2-y^2)$

The subsidary equ is

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)}$$

choose $(x, y, -1)$ as Lagrange's multipliers,

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} \Rightarrow \frac{x\,dx + y\,dy - dz}{x^2(y^2+z) - y^2(x^2+z) + (-z)(x^2-y^2)}$$

$$\Rightarrow \frac{x\,dx + y\,dy - dz}{x^2y^2 + zx^2 - x^2y^2 - zy^2 - zx^2 + zy^2}$$

$$\Rightarrow \frac{x\,dx + y\,dy - dz}{0}$$

$$x\,dx + y\,dy - dz = 0$$

$\int$ing on b.s,

$$\int x\,dx + \int y\,dy - \int dz = 0 + c$$

$$x^2/_2 + y^2/_2 - z = 0 + c$$

$$\frac{x^2+y^2+2z}{2} = c$$

$$x^2+y^2+2z = 2c \Rightarrow x^2+y^2+2z = c_1 \quad \text{---} (1)$$

Scanned with CamScanner

Choose $\left(\frac{1}{x}, \frac{1}{y}, \frac{1}{z}\right)$

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{z(x^2-y^2)} = \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2+z-x^2-z+x^2-y^2}$$

$$\Rightarrow \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$$

$$\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$\int$ing on b.s,

$$\int dx/x + \int dy/y + \int dz/z = 0 + c$$

$$\log x + \log y + \log z = \log c$$

$$\log(xyz) = \log c$$

$$xyz = c_2 \quad \text{—— (2)}$$

$$\left.\begin{aligned} x^2+y^2+2z &= c_1 \\ xyz &= c_2 \end{aligned}\right\} \text{—— (3)}$$

$$x+y=0,\ z=1 \longrightarrow (4)$$

Assume That $x=t$.

Sub $x=t$ in equ (4) we get,

$$t+y=0,\quad z=1$$

$$y=-t,\quad z=1$$

Sub $x=t,\ y=-t,\ z=1$ in (3)

$t^2+t^2+2(1)=c_1$	$t(-t)(1)=c_2$
$2t^2-2=c_1$ — (5)	$-t^2=c_2$ — (6)

Scanned with CamScanner

Solve equ (5) and (6) $\Rightarrow$ (5) $\Rightarrow 2t^2 - 2 = c_1$

$(6) \times 2 \Rightarrow -2t^2 = 2c_2$

$-2 = c_1 + 2c_2$

$$c_1 + 2c_2 + 2 = 0$$ (7)

Sub equ (3) in (7)

$$x^2 + y^2 - 2z + 2(xyz) + 2 = 0$$

$$x^2 + y^2 + 2xyz - 2z + 2 = 0$$

which is the requuied integral surface.

2. Find the equation of the integral surface of the I.E $2y(z-3)p + (2x-z)q = y(2x-3)$ which passes through the circle $z=0$, $x^2+y^2 = 2x$

Con equ is $2y(z-3)p + (2x-z)q = y(2x-3)$

Lagrange's general equ is $Pp + Qq = R$

$P = 2y(z-3)$; $Q = (2x-z)$; $R = y(2x-3)$

The subsidary equ is $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

Choose $\frac{dx}{2y(z-3)} = \frac{dy}{(2x-z)} = \frac{dz}{y(2x-3)}$

Consider 1st and 3rd ratio,

$$\frac{dx}{2y(z-3)} = \frac{dz}{y(2x-3)}$$

$$\frac{dx}{2(z-3)} = \frac{dz}{(2x-3)}$$

Scanned with CamScanner

$(2x-3)\,dx = 2(z-3)\,dz$

$2x \cdot dx - 3 \cdot dx = 2z \cdot dz - 6 \cdot dz$

$\int$ing on b.s,

$\int 2x\,dx - 3\int dx = 2\int z\,dz - 6\int dz$

$2\,x^2/_2 - 3x = 2\,z^2/_2 - 6z + c$

$x^2 - 3x = z^2 - 6z + c$

$x^2 - z^2 + 6z - 3x = c_1 \longrightarrow (1)$

choose $(x, 3y, -z)$ as L.multipliers,

$$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)} \Rightarrow \frac{x\,dx + 3y\,dy - z\,dz}{2xy(z-3) + 3y(2x-z) - zy(2x-3)}$$

$$\frac{x\,dx + 3y\,dy - z\,dz}{2xyz - 6xy + 6xy - 3yz - 2xyz + 3zy}$$

$$\Rightarrow \frac{x\,dx + 3y\,dy - z\,dz}{0}$$

$\Rightarrow x\,dx + 3y\,dy - z\,dz = 0$

$\int$ing on b.s,

$\int x\,dx + 3\int y\,dy - \int z\,dz = 0 + c$

$x^2/_2 + 3y^2/_2 - z^2/_2 = c$

$$\frac{x^2 + 3y^2 - z^2}{2} = c$$

$$x^2+3y^2-z^2=2c$$

$$x^2+3y^2-z^2=c_2 \longrightarrow (2)$$

$$\left.\begin{aligned} x^2-z^2-3x+6z &= c_1 \\ x^2+3y^2-z^2 &= c_2 \end{aligned}\right\} \longrightarrow (3)$$

29

$$z=0,\ x^2+y^2=2x \longrightarrow (4)$$

Assume That $x=t$,

Sub $x=t$ in equ. (4) we get,

$$t^2+y^2=2x,\quad z=0$$

$$\cancel{-t^2=y^2,\ z=0}\qquad t^2+y^2=2t,\ z=0$$

$$y^2=2t-t^2,\ z=0$$

Sub $x=t$, $y^2=2t-t^2$, $z=0$ in (3) we get

$$x^2-z^2-3x+6z=c_1$$

$$(t)^2-(0)^2-3(t)+6(0)=c_1$$

$$t^2-3t=c_1 \text{ —— } (6)$$

$$x^2+3y^2-z^2=c_2$$

$$(t)^2+3(2t-t^2)-(0)=c_2$$

$$t^2+6t-3t^2=c_2$$

$$6t-2t^2=c_2 \text{ —— } (7)$$

Solving equation (6) & (7).

$$(6)\times 2 \Rightarrow 2t^2-6t=2c_1$$

$$(7) \Rightarrow -2t^2+6t=c_2$$

$$\Rightarrow 2c_1+c_2=0 \text{ —— } (8)$$

Sub (3) in (8) $\Rightarrow$ $2c_1+c_2=0$

$$2(x^2-z^2-3x+6z)+(x^2+3y^2-z^2)=0$$

$$2x^2-2z^2-6x+12z+x^2+3y^2-z^2=0$$

$$3x^2+3y^2-3z^2-6x+12z=0$$

3. Find the integral surface of the equation $(x-y)y^2p + (y-x)x^2q = (x^2+y^2)z$ through the curve $xz = a^3,\ y=0$

30

Gn equ is $(x-y)y^2p + (y-x)x^2q = (x^2+y^2)z$

Lagrange's g. equ is $Pp + Qq = R$

$P = (x-y)y^2\ ;\ Q = (y-x)x^2\ ;\ R = (x^2+y^2)z$.

The Subsidiary equ is $\dfrac{dx}{P} = \dfrac{dy}{Q} = \dfrac{dz}{R}$

$$\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} = \frac{dz}{(x^2+y^2)z}$$

Consider 1st and 2nd ratio,

$$\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} \Rightarrow \frac{dx}{(x-y)y^2} = \frac{dy}{-(x-y)x^2}$$

$$\cancel{x^2(y-x)\,dx = y^2(x-y)\,dy} \Rightarrow \frac{dx}{y^2} = \frac{dy}{-x^2}$$

$$-x^2\,dx = y^2\,dy$$

$\int$ing on b.s,

$$-\int x^2\,dx = \int y^2\,dy + c$$

$$-x^3/3 = y^3/3 + c$$

$$-x^3/3 - y^3/3 = c$$

$$-(x^3+y^3) = 3c \qquad \boxed{x^3+y^3 = c_1} \longrightarrow (1)$$

$$x^3+y^3 = -3c$$

Scanned with CamScanner

choose $(1,-1,0)$ as lagrage's multipliers,

$$\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} = \frac{dz}{(x^2+y^2)z} \Rightarrow \frac{dx-dy+0}{(x-y)y^2-x^2(y-x)}$$

$$\frac{dx-\cancel{dy}}{xy^2-\cancel{y^3}-x^2y-x^3} \qquad \frac{dx-dy}{(x-y)y^2+x^2(x-y)}$$

31

$$\Rightarrow \frac{dx-dy}{(x-y)(x^2+y^2)}$$

consider 3rd and 4th ratio,

$$\frac{dz}{(x^2+y^2)z} = \frac{dx-dy}{(x-y)(x^2+y^2)}$$

$$\frac{dz}{z} = \frac{dx-dy}{x-y}$$

Sing on b.s,

$$\int dz/z = \int \frac{dx-dy}{x-y}$$

$$\log z = \log(x-y) + \log c$$

$$\log z - \log(x-y) = \log c$$

$$\log\left(z/x-y\right) = \log c$$

$$z/x-y = c_2 \longrightarrow (2)$$

$$\left.\begin{aligned} x^3+y^3 &= c_1 \\ z/x-y &= c_2 \end{aligned}\right\} \longrightarrow (3)$$

$xz = a^3, y = 0 \rightarrow$

Assume That $x = t$

$(t) z = a^3$

$tz = a^3$

$z = a^3/t$

put $x = t, \ y = 0, \ z = a^3/t$ in (3) we get

$x^3 + y^3 = c_1$

$(t)^3 + (0)^3 = c_1$

$t^3 = c_1$

$t^2 \cdot t = c_1$

$t^2 = c_1/t \rightarrow (6)$

$z/x-y = c_2$

$\frac{(a^3/t)}{t-(0)} = c_2$

$a^3/t/t = c_2$

$a^3/t^2 = c_2 \rightarrow (7)$

$a^3/c_1/t = c_2$

$a^3 t/c_1 = c_2$

$t = \frac{c_1 c_2}{a^3}$

$\left(\frac{c_1 c_2}{a^3}\right)^3 = c_1$

$\frac{c_1^3 c_2^3}{(a^3)^3} = c_1$

$\frac{c_1^2 c_2^3}{a^9} = 1$

$c_1^2 c_2^3 - a^9 = 0 \rightarrow (8)$

Sub equ (3) in (8) we get,

$(x^3+y^3)^2 (z/(x-y))^3 - a^9 = 0$

which is the required equation.

17.12.19

Compatible system of first order PDE's :-

Definition:

If $f(x,y,z,p,q)=0$ and $g(x,y,z,p,q)=0$

are compatible on a domain D if.

$$J = \frac{\partial(f,g)}{\partial(p,q)} = \begin{vmatrix} \frac{\partial f}{\partial p} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial p} & \frac{\partial g}{\partial q} \end{vmatrix} \neq 0 \quad \text{—(2)}$$

$p=\phi(x,y,z)$ and $q=\psi(x,y,z)$ obtained by solving equ (1) and (2) gives,

$dz = \phi(x,y,z) + \psi(x,y,z)$ is integrable.

Theorem:

The necessary and sufficient condition for the gn two PDE's $f(x,y,z,p,q)=0$ and $g(x,y,z,p,q)=0$ to be compatible is that,

$$[f,g] = \frac{\partial(f,g)}{\partial(x,p)} + \frac{\partial(f,g)}{\partial(y,q)} + p\frac{\partial(f,g)}{\partial(z,p)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0$$

Proof:

Gn two PDE's are,

$f(x,y,z,p,q)=0$ —(1)

$g(x,y,z,p,q)=0$ —(2)

Let equations (1) & (2) be compatible then

$J = \frac{\partial(f,g)}{\partial(p,q)} \neq 0$ and we can find

$p = \phi(x,y,z)$ and $q = \psi(x,y,z)$

2

$dz = \phi\,dx + \psi\,dy$ is integrable

$\phi\,dx + \psi\,dy - dz = 0$ is integrable.

$\bar{x} \cdot (\text{curl})\,\bar{x} = 0$

where $\bar{x} = \phi\vec{i} + \psi\vec{j} - \vec{k}$

$\text{curl}\,\bar{x} = \nabla \times \bar{x}$

$$\nabla = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}$$

$$\text{curl}\,\bar{x} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \phi & \psi & -1 \end{vmatrix}$$

$$= \left[\vec{i}\left(\partial/\partial y\,(-1) - \partial/\partial z\,(\psi)\right)\right] - \vec{j}\left[\partial/\partial x\,(-1) - \partial/\partial z\,(\phi)\right]$$

$$+ \vec{k}\left[\partial/\partial x\,(\psi) - \partial/\partial y\,(\phi)\right]$$

$$= \vec{i}\left[0 - \partial\psi/\partial z\right] - \vec{j}\left[0 - \partial/\partial z\,(\phi)\right] + \vec{k}\left[\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right]$$

$$= -\frac{\partial\psi}{\partial z}\vec{i} + \frac{\partial\phi}{\partial z}\vec{j} + \vec{k}\left[\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right]$$

$$\bar{x}\cdot\text{curl}\,\bar{x} = \left[(\phi\vec{i} + \psi\vec{j} - \vec{k})\cdot\left(-\frac{\partial\psi}{\partial z}\vec{i} + \frac{\partial\phi}{\partial z}\vec{j} + \vec{k}\,\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right)\right]$$

$$0 = \phi\left(-\frac{\partial\psi}{\partial z}\right) + \psi\left(\frac{\partial\phi}{\partial z}\right) - (1)\left(\frac{\partial\psi}{\partial x} - \frac{\partial\phi}{\partial y}\right)$$

$$= -\phi\left(\frac{\partial\psi}{\partial z}\right) + \psi\left(\frac{\partial\phi}{\partial z}\right) \quad \frac{\partial}{\partial x} \quad \frac{\partial}{\partial y}$$

$$\phi \cdot \frac{\partial\psi}{\partial z} + \frac{\partial\psi}{\partial x} = \psi \frac{\partial\phi}{\partial z} + \frac{\partial\phi}{\partial y} \quad \text{---}(3)$$ 3

Equ (1) gives $f(x, y, z, p, q) = 0$

Sub $p = \phi$, $q = \psi$ values in above equ we get,

$$f(x, y, z, \phi, \psi) = 0 \quad \text{---}(4)$$

Diff equ (4) w.r. to x we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial \psi} \cdot \frac{\partial \phi}{\partial x} + \frac{\partial f}{\partial \psi} \cdot \frac{\partial \psi}{\partial x} = 0.$$

$$f_x + f_\phi \cdot \phi_x + f_\psi \cdot \psi_x = 0 \quad \text{---}(5)$$

Diff equ (4) w.r. to z we get

$$\frac{\partial f}{\partial z} + \frac{\partial f}{\partial \phi} \cdot \frac{\partial \phi}{\partial z} + \frac{\partial f}{\partial \psi} \cdot \frac{\partial \psi}{\partial z} = 0$$

$$f_z + f_\phi \, \phi_z + f_\psi \, \psi_z = 0 \quad \text{---}(6)$$

Multiply ϕ on b.s on equ (6)

$$\phi f_z + \phi f_\phi \cdot \psi_z + \phi f_\psi \cdot \psi_z = 0 \quad \text{---}(7)$$

Adding (5) & (7) we get.

$$f_x + f_\phi \, \phi_x + f_\psi \, \psi_x + \phi f_z + \phi f_\phi \, \phi_z + \phi \cdot f_\psi \, \psi_z = 0$$

$$f_x + \phi f_z + f_\phi (\phi_x \cdot \phi \cdot \phi_z) + f_\psi (\psi_x + \phi \psi_z) = 0 \quad \text{---}(8)$$

From equ (2) $\Rightarrow g(x, y, z, p, q) = 0$

Sub $p = \phi$ and $q = \psi$ in the above equ

$g(x, y, z, \phi, \psi) = 0$ — (9)

Diff equ (9) partially w. r. to x

4

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial \phi} \cdot \frac{\partial \phi}{\partial x} + \frac{\partial g}{\partial \psi} \cdot \frac{\partial \psi}{\partial x} = 0$$

$g_x + g_\phi \cdot \phi_x + g_\psi \, \psi_x = 0$ — (10)

Diff (9) partially w. r. to z

$$\frac{\partial g}{\partial z} + \frac{\partial g}{\partial \phi} \cdot \frac{\partial \phi}{\partial z} + \frac{\partial g}{\partial \psi} \cdot \frac{\partial \psi}{\partial z} = 0$$

$g_z + g_\phi \cdot \phi_z + g_\psi \cdot \psi_z = 0$ — (11)

Multiply ϕ in equ (11) we get

$\phi g_z + \phi \, g_\phi \cdot \phi_z = \phi \, g_\psi \cdot \psi_z = 0$ — (12)

Adding equ (10) & (12)

$g_x + g_\phi \cdot \phi_x + g_\psi \cdot \psi_x + \phi g_z + \phi g_\phi \cdot \phi_z + \phi g_\psi \cdot \psi_z = 0$

$g_x + \phi g_z + g_\phi (\phi_x + \phi \phi_z) + g_\psi (\psi_x + \phi \psi_z) = 0$ — (13)

Equ (8) $\times g_\phi$ − Equ (13) $\times f_\phi$

$g_\phi f_x + g_\phi \cdot \phi f_z + g_\phi f_\phi (\phi_x + \phi \cdot \phi_z) + g_\phi \cdot f_\psi (\psi_x + \phi \psi_z)$

$- g_x f_\phi - \phi g_z f_\phi - g_\phi f_\phi (\phi_x + \phi \cdot \phi_z) - f_\phi g_\psi (\psi_x + \phi \cdot \psi_z) = 0$

$\Rightarrow g_\phi f_x + g_\phi \cdot \phi f_z + g_\phi f_\psi (\psi_x + \phi \cdot \psi_z) - f_\phi g_x -$

$f_\phi \cdot \phi g_z - f_\phi \cdot g_\psi (\psi_x + \phi \cdot \psi_z) = 0$

$g_\phi f_x - f_\phi g_x + \phi (g_\phi f_z - f_\phi g_z) + (\psi_x + \phi \cdot \psi_z) \cdot (g_\phi f_\psi - f_\phi g_\psi) = 0$

$$\frac{\partial(f,g)}{\partial(x,\phi)} + \phi\frac{\partial(f,g)}{\partial(z,\phi)} + (\psi_x + \phi.\psi_z)\left[-\frac{\partial(f,g)}{\partial(\phi,\psi)}\right] =$$

$$\frac{\partial(f,g)}{\partial(x,\phi)} + \phi\frac{\partial(f,g)}{\partial(\phi,z)} + (\psi_x + \phi\psi_z)(-J) = 0$$

5

$$\frac{\partial(f,g)}{\partial(x,\phi)} + \phi\frac{\partial(f,g)}{\partial(\phi,z)} = J(\psi_x + \phi\psi_z)$$

$$\frac{1}{J}\left[\frac{\partial(f,g)}{\partial(x,\phi)} + \phi\frac{\partial(f,g)}{\partial(\phi,z)}\right] = \psi_x + \phi\psi_z$$

changing from x into y and ϕ into ψ.

$$-\frac{1}{J}\left[\frac{\partial(f,g)}{\partial(y,\psi)} + \psi\frac{\partial(f,g)}{\partial(z,\psi)}\right] = \phi_y + \psi\phi_z$$

from equ (3) gives

$$\psi_x + \phi\psi_z = \psi\phi_z + \phi_y$$

$$\frac{1}{J}\left[\frac{\partial(f,g)}{\partial(x,\phi)} + \phi\frac{\partial(f,g)}{\partial(z,\phi)}\right] = \frac{-1}{J}\left[\frac{\partial(f,g)}{\partial(y,\psi)} + \psi\frac{\partial(f,g)}{\partial(z,\psi)}\right]$$

$$\frac{\partial(f,g)}{\partial(x,\phi)} + \phi\frac{\partial(f,g)}{\partial(z,\phi)} + \frac{\partial(f,g)}{\partial(y,\psi)} + \psi\frac{\partial(f,g)}{\partial(z,\psi)} = 0$$

Replace p and q in place of ϕ & ψ.

$$\frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0$$

1. Show that the equation $xp = yq$ and $z(xp+yq) = 2xy$ are compatible and solve it.

6

$$f(x, y, z, p, q) = xp - yq \quad \text{—— (1)}$$

$$g(x, y, z, p, q) = xzp + yzp - 2xy \quad \text{—— (2)}$$

$$\frac{\partial(f,g)}{\partial(x,p)} + p \cdot \frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0 \quad \text{—— (3)}$$

$\frac{\partial f}{\partial x}$ $f_x = p$ $\qquad g_x = zp - 2y$

$\frac{\partial f}{\partial y}$ $f_y = -q$ $\qquad g_y = zq - 2x$

$\frac{\partial f}{\partial z}$ $f_z = 0$ $\qquad g_z = xp + yq$

$\frac{\partial f}{\partial p}$ $f_p = x$ $\qquad g_p = xz$

$\frac{\partial f}{\partial q}$ $f_q = -y$ $\qquad g_q = yz$

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \partial f/\partial x & \frac{\partial f}{\partial p} \\ \partial g/\partial x & \partial g/\partial p \end{vmatrix} = \begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix}.$$

$$= \begin{vmatrix} p & x \\ zp-2y & xz \end{vmatrix}$$

$$= xzp - xzp + 2xy$$

$$\frac{\partial(f,g)}{\partial(x,p)} = 2xy$$

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} f_z & f_p \\ g_z & g_p \end{vmatrix} \Rightarrow \begin{vmatrix} 0 & x \\ xp+yq & xz \end{vmatrix}$$

$$= 0 - x(xp+yq)$$

Scanned with CamScanner

$$\frac{\partial(f,g)}{\partial(z,p)} = -x^2p - xyq$$

(7)

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial q} \end{vmatrix} \Rightarrow \begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix}$$

$$= \begin{vmatrix} -q & -y \\ zq-2x & yz \end{vmatrix} \Rightarrow -qyz + y(zq-2x)$$

$$= -y\cancel{q}z + y\cancel{q}z - 2xy$$

$$\frac{\partial(f,g)}{\partial(y,q)} = -2xy.$$

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \frac{\partial f}{\partial z} & \frac{\partial f}{\partial q} \\ \frac{\partial g}{\partial z} & \frac{\partial g}{\partial q} \end{vmatrix} \Rightarrow \begin{vmatrix} f_z & f_q \\ g_z & g_q \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -y \\ xp+yq & yz \end{vmatrix} \Rightarrow 0 + y(xp+yq) \Rightarrow xpy + y^2q$$

$$\frac{\partial(f,g)}{\partial(z,q)} = xpy + y^2q.$$

Equ (3) $\Rightarrow$ $\frac{\partial(f,g)}{\partial(x,p)} + p.\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 0$

$= \cancel{xzp} = 2xy + p(-x^2p - xyq) - 2xy + q(xpy + y^2q) = 0$

$= -x^2p^2 - xypq + xypq + y^2q^2 = 0$

$$-x^2p^2 + y^2q^2 = 0$$

$$-x^2p^2 + x^2p^2 = 0 \qquad [\text{Given that } xp = yq]$$

$$0 = 0$$

(8)

$\therefore$ The two given PDE is compatible.

$(1) \Rightarrow xp - yq = 0$

$(2) \Rightarrow xzp + yzq - 2xy = 0$

$(1) \times z \Rightarrow xzp - yzq = 0$

$(2) \Rightarrow xzp + yzq = 2xy$

$$2xzp = 2xy$$

$$\boxed{p = y/z}$$

Sub $p = y/z$ in (1)

$$xp - yq = 0$$

$$x(y/z) - yq = 0$$

$$xy/z = yq$$

$$\boxed{x/z = q}$$

The solution is $dz = p\,dx + q\,dy$

$$dz = y/z\,dx + x/z\,dy$$

$$dz = \frac{y\,dx + x\,dy}{z}$$

$$z\,dz = y\,dx + x\,dy$$

$$z\,dz = d(xy)$$

Scanned with CamScanner

$\int$ing on both sides,

$$z^2/2 = xy + c$$

$$z^2 = 2xy + 2c$$

$$z^2 - 2xy = c$$

Which is the required common solution.

Note:-

(1) If the gn two P.D.E's are compatible Then They will posses a common Solution.

(2) The system of equations $p = P(x,y)$ $q = Q(x,y)$ are compatible iff $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

(3) If $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$. Then the gn two PDE are not compatible and the gn PDE's posses not solutions.

1. Show That the PDE's $p = 5x - 7y$; $q = 6x + 8y$ are not compatible.

Given, $p = 5x - 7y$

$$q = 6x + 8y$$

Not compatible $\Rightarrow \frac{\partial p}{\partial y} \neq \frac{\partial q}{\partial x}$

$$\frac{\partial p}{\partial y} = -7 \; ; \; \frac{\partial q}{\partial x} = 6$$

$$\frac{\partial p}{\partial y} \neq \frac{\partial q}{\partial x} \qquad -7 \neq 6$$

2. S.T The PDE's $p = x^2 - ay$, $q = y^2 - ax$ are compatible. Find their common solution.

$f(x, y, z, p, q) = p - x^2 + ay$ (10)

Gn $p = x^2 - ay$; $q = y^2 - ax$.

$\frac{\partial p}{\partial y} = -a$; $\frac{\partial q}{\partial x} = -a$

$$\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$$

$-a = -a$

The system of the eqns p and q are compatible

The Solution is $dz = p\,dx + q\,dy$

$$dz = (x^2 - ay)dx + (y^2 - ax)dy$$

$$dz = x^2 dx - ay\,dx + y^2 dy - ax\,dy$$

$\int$ ing on b.s

$$\int dz = \int x^2 dx - \int ay\,dx + \int y^2 dy - \int ax\,dy$$

$$\cancel{\int dz} = \cancel{\frac{x^3}{3}} - a\cancel{\frac{y^2}{2}}$$

$$\int dz = \int x^2 dx + \int y^2 dy - a\int (y\,dx + x\,dy)$$

$$\int dz = \int x^2 dx + \int y^2 dy - a\int d(xy)$$

$$z = \frac{x^3}{3} + \frac{y^3}{3} - a(xy) + c$$

$$z = \frac{x^3 + y^3 - 3a(xy) + 3c}{3}$$

$$3z = x^3 + y^3 - 3a(xy) + 3c$$

$$3c = x^3 + y^3 - 3a(xy) - 3z$$

$$c = x^3 + y^3 - 3a(xy) - 3z$$

Which is the required common solution.

2. Show that the equations $xp - yq = x$; $x^2p + q = xz$ are compatible. and find the solution.

$$f(x,y,z,p,q) = xp - yq - x \quad \text{——(1)}$$

$$g(x,y,z,p,q) = x^2p + q - xz \quad \text{——(2)}$$

$$\frac{\partial(f,g)}{\partial(x,p)} + p.\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q.\frac{\partial(f,g)}{\partial(z,q)} = 0 \quad \text{—(3)}$$

$f_x = p-1$ $\qquad g_x = 2xp - z$

$f_y = -q$ $\qquad g_y = 0$

$f_z = 0$ $\qquad g_z = -x$

$f_p = x$ $\qquad g_p = x^2$

$f_q = -y$ $\qquad g_q = 1$

$$\frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \partial f/\partial x & \partial f/\partial p \\ \partial g/\partial x & \partial g/\partial p \end{vmatrix} \Rightarrow \begin{vmatrix} f_x & f_p \\ g_x & g_p \end{vmatrix}$$

$$= \begin{vmatrix} p-1 & x \\ 2xp-z & x^2 \end{vmatrix} \Rightarrow x^2(p-1) - x(2xp-z)$$

$$\Rightarrow x^2p - x^2 - 2x^2p + xz$$

$$= -x^2p - x^2 + xz$$

$$= xz - x^2p - x^2$$

$$\frac{\partial(f,g)}{\partial(x,p)} = xz - x^2p - x^2$$

Scanned with CamScanner

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} f_z & f_p \\ g_z & g_p \end{vmatrix} \Rightarrow \begin{vmatrix} 0 & x \\ -x & x^2 \end{vmatrix}$$

12

$$= 0 + x^2$$

$$\frac{\partial(f,g)}{\partial(z,p)} = x^2$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} f_y & f_q \\ g_y & g_q \end{vmatrix} \Rightarrow \begin{vmatrix} -q & -y \\ 0 & 1 \end{vmatrix}$$

$$= -q + 0$$

$$\frac{\partial(f,g)}{\partial(y,q)} = -q$$

$$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} f_z & f_q \\ g_z & g_q \end{vmatrix} \Rightarrow \begin{vmatrix} 0 & -y \\ -x & 1 \end{vmatrix}$$

$$= 0 + xy$$

$$\frac{\partial(f,g)}{\partial(z,q)} = -xy$$

Equ (3) $\Rightarrow \left(xz - x^2p - x^2\right) + p\,(x^2) - q + q\,(-xy) = 0$

$\Rightarrow xz - \cancel{x^2p} - x^2 + \cancel{px^2} - q - qxy = 0$

$\Rightarrow xz - x^2 - q - qxy = 0$

2 The given eq

lc $xp - yq = x$

$P = x$; $Q = -y$; $R = x$

The subsidary equations,

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{x}$$

Consider 1st and 2nd ratio,

$$\frac{dx}{x} = \frac{dy}{-y}$$

$\int$ing on b.s

$$\int \frac{dx}{x} = \int dy/-y$$

$$\log x = -\log y + \log c$$

$$\log x + \log y = \log c$$

$$\log(xy) = \log c$$

$$xy = c$$

Which is the common required solution.

(13)

Special types of first order Equations :- [Type I]

i] Equations involving only p and q : (14)

$$F(p,q)=0$$

Let $z = ax+by+c$ —— (1) be the solution of $F(p,q)=0$

Diff equ (1) partially w.r.to 'x' & 'y'

$$p = \frac{\partial z}{\partial x} \Rightarrow a$$

$$q = \frac{\partial z}{\partial y} \Rightarrow b$$

Replace $p=a$ & $q=b$ in equ (1) we get

$$(1) \; z = ax+by+c$$

To find complete integral :-

Solving for b from $F(a,b)=0$ we get,

$$b = \phi(a)$$

$$\therefore z = ax + \phi(a)y + c$$

Define complete Integral: (or) Complete solution :-

A solution in which the number of arbitary constant is equal to the number of independence variable is called complete integral (or) complete solution

particular integral :-

A complete integral if we give particular values to the arbitary constants we get particular integral

Singular integral:

Let $F(x,y,z,p,q)=0$ be a partial diff. equ whose complete integral is $\phi(x,y,z,p,q)=0$ —— (1)

Scanned with CamScanner

Diff equ (1) partially w.r. to a and b we get

$$\frac{\partial \phi}{\partial a} = 0 \quad \text{and} \quad \frac{\partial \phi}{\partial b}$$ (15)

Eliminate ~~equ (a) & (3)~~ (a) and b by using (1), (2) & (3)

The elimination a and b is called singular integral.

1. Find the complete integral of the equation

$pq = 1$

Sdu:

Gn equ $pq = 1$ —— (*)

Let ~~us~~ assume that $z = ax + by + c$ —— (1)

Diff equ (1) partially w.r. to x & y

$$p = \frac{\partial z}{\partial x} = a$$

$$q = \frac{\partial z}{\partial y} = b$$

Sub p, q in (*) we get

$$ab = 1$$

$$\boxed{b = \frac{1}{a}}$$

Sub $b = 1/a$ in equ (1)

$$z = ax + by + c$$

$$z = ax + \frac{1}{a}y + c$$

$$z = \frac{a^2x + y + ac}{a}$$

$$za = a^2x + y + ac$$

$ax + y - za + ac = 0$

which is the required complete integral.

2\. Find the complete integral of the equ $\sqrt{p} + \sqrt{q} = 1$

Solu:
Gn equ $\sqrt{p} + \sqrt{q} = 1$ ——— $(*)$

Let us assume that $z = ax + by + c$ ——— (1)

Diff equ (1) partially w. r. to x & y

$$p = \frac{\partial z}{\partial x} = a$$

$$q = \frac{\partial z}{\partial y} = b$$

Sub p, q in $(*)$ we get

$\cancel{\sqrt{a}\sqrt{b}=1}$ $\quad \sqrt{a} + \sqrt{b} = 1$

$$\sqrt{b} = 1 - \sqrt{a}$$

$$b = (1-\sqrt{a})^2$$

Sub $b = (1-\sqrt{a})^2$ in equ (1)

$$z = ax + by + c$$

$$z = ax + (1-\sqrt{a})^2 y + c$$

$$ax + (1-\sqrt{a})^2 y + c - z = 0$$

which is the required complete integral.

3\. Find the complete integral of the equ $p+q = pq$

Solu:
Gn equ $p + q = pq$ ——— $(*)$

Let us assume that $z = ax + by + c$ ——— (1)

Diff (1) partially w. r. to x & y

$$p = \frac{\partial z}{\partial x} = a$$

$$q = \frac{\partial z}{\partial y} = b$$

Sub p, q in $(*)$ $\Rightarrow$ $a+b = ab$

$b = ab - a$

$b = a(b-1)$

$a+b = ab$

$a = ab - b$

$a = b(a-1)$

$\frac{a}{a-1} = b$

Sub b value in (1) $\Rightarrow$ $z = ax + by + c$

$$z = ax + \left(\frac{a}{a-1}\right)y + c$$

$$z = (a-1)ax + ay + (a-1)c$$

$$z = a^2x - ax + ay + ac - c$$

$$a^2x - ax + ay + ac - z - c = 0$$

which is the required complete integral.

4. Find the complete integral of the equ $p^2 + q^2 = 4$

Soln:

Gn equ $p^2 + q^2 = 4$ —— $(*)$

Let us assume that $z = ax + by + c$ —— (1)

Diff equ (1) partially w.r.to x & y

$$p = \frac{\partial z}{\partial x} \Rightarrow a$$

$$q = \frac{\partial z}{\partial y} \Rightarrow b$$

Sub p, q in $(*)$ $\rightarrow$ $a^2 + b^2 = 4$

$$b^2 = 4 - a^2$$

$$b = \pm\sqrt{4 - a^2}$$

Sub b value in (1) $\Rightarrow$ $z = ax + \sqrt{4-a^2}\, y + c$

$$ax + \sqrt{4-a^2}\, y - z + c = 0 .$$

which is the required complete integral.

Type - 2

To solve this type of equation.

$F_1(x, p) = F_2(y, q) = a$ (say constant)

$$\left.\begin{aligned} F_1(x, p) &= a \\ F_2(y, q) &= a \end{aligned}\right\} \text{—(1)}$$

From equ (1) we get

$$\left.\begin{aligned} p &= F_1(x, a) \\ q &= F_2(y, a) \end{aligned}\right\} \text{—(2)}$$

We know that, $dz = p\,dx + q\,dy$ —(3)

sub (2) in (3) we get

$$dz = F_1(x, a)\,dx + F_2(y, a)\,dy .$$

$\int$ing on b.s,

$$\int dz = \int F_1(x, a)\,dx + \int F_2(y, a)\,dy$$

$$z = \int F_1(x, a)\,dx + \int F_2(y, a)\,dy$$

which gives the required complete integral of the equation.

1. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}(x/a)$

2. $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1}(x/a)$

3. $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}(x/a)$

4. $\int \sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}(x/a)$

5. $\int \sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\cosh^{-1}(x/a)$

6. $\int \sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{a^2+x^2} + \frac{a^2}{2}\sin^{-1}(x/2)$

We now have five equs involving the four arbitary quantities f', f'', g', g'' If we eliminate these four quantities from the five equs, we obtain the relation.

$$\begin{vmatrix} p-w_x & u_x & v_x & 0 & 0 \\ q-w_y & u_y & v_y & 0 & 0 \\ r-w_{xx} & u_{xx} & v_{xx} & u^2_x & v^2_x \\ s-w_{xy} & u_{xy} & v_{xy} & u_x u_y & v_x v_y \\ t-w_{yy} & u_{yy} & v_{yy} & u^2_y & v^2_y \end{vmatrix} = 0 \quad (2)$$

which involves only the derivatives p, q, r, s, t and known functions of x & y.

1. If $z = f(x+ay) + g(x-ay)$ where f & g are aribitary functions and a is constant then P.T $t = a^2 r$

(i.e) $\dfrac{\partial^2 z}{\partial y^2} = a^2 \cdot \dfrac{\partial^2 z}{\partial x^2}$.

Soln:

Gn' $z = f(x+ay) + g(x-ay)$ —— (1)

Diff equ (1) w.r.to 'x'

$$\frac{\partial z}{\partial x} = f'(x+ay) + g'(x-ay)$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + g''(x-ay) \text{ —— (2)}$$

Diff. equ (1) w.r.to y

$$\frac{\partial z}{\partial y} = f'(x+ay)\,a + g'(x-ay)(-a)$$

Scanned with CamScanner

Again diff. w.r. to y

$$\frac{\partial^2 z}{\partial y^2} = f''(x+ay)a^2 + g''(x-ay)(-a)^2$$

$$= a^2\left[f''(x+ay) + g''(x-ay)\right]$$

③

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2} \quad (\text{by equ } (2))$$

$$t = a^2 r.$$

2. Verify The PDE $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \frac{2z}{x}$ is satisfied by

$z = \frac{1}{x}\phi(y-x) + \phi'(y-x)$

Soln: Gn $z = \frac{1}{x}\phi(y-x) + \phi'(y-x)$ —— (1)

Diff w.r. to 'x'.

$$\frac{\partial z}{\partial x} = \frac{1}{x}\phi'(y-x)(-1) + \phi(y-x)\left(\frac{-1}{x^2}\right) + \phi''(y-x)(-1)$$

$$= \frac{-1}{x}\phi'(y-x) - \frac{1}{x^2}\phi(y-x) - \phi''(y-x).$$

$$\frac{\partial^2 z}{\partial x^2} = -\left[\frac{1}{x}\phi''(y-x)(-1) + \phi'(y-x)\left(-1/x^2\right)\right] -$$

$$\left[\frac{1}{x^2}\phi'(y-x)(-1) + \phi(y-x)\left(-2/x^3\right)\right] - \phi'''(y-x)(-1)$$

$$= \frac{1}{x}\phi''(y-x) + \frac{1}{x^2}\phi'(y-x) + \frac{1}{x^2}\phi'(y-x) + \frac{2}{x^3}\phi(y-x) + \phi'''(y-x).$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{x}\phi''(y-x) + \frac{2}{x^2}\phi'(y-x) + \frac{2}{x^3}\phi(y-x) + \phi'''(y-x) \text{ —— (2)}$$

Diff equ (1) w.r. to 'y'.

$$\frac{\partial z}{\partial y} = \frac{1}{x} \cdot \phi'(y-x) + \phi''(y-x)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{1}{x} \phi''(y-x) + \phi'''(y-x) \text{ --- } (3)$$

(4)

$(2) - (3) \Rightarrow$

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \frac{1}{x} \phi''(y-x) + \frac{2}{x^2} \phi'(y-x) + \frac{2}{x^3} \phi(y-x) + \phi'''(y-x) - \frac{1}{x}\phi''(y-x) - \phi'''(y-x)$$

$$= \frac{2}{x^2} \phi'(y-x) + \frac{2}{x^3} \phi(y-x)$$

$$= \frac{2}{x^2} \left[\phi'(y-x) + \frac{1}{x} (y-x) \right]$$

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \frac{2z}{x^2}$$

3. If $u = f(x+iy) + g(x-iy)$ where the function $f \& g$ are arbitary show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Solu:

Gen' $u = f(x+iy) + g(x-iy)$ --- (1)

Diff (1) w.r.to x

$$\frac{\partial u}{\partial x} = f'(x+iy) + g'(x-iy)$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x+iy) + g''(x-iy) \text{ --- } (2)$$

Diff (1) w.r. to "y".

$$\frac{\partial u}{\partial y} = f'(x+iy)i + g'(x-iy)(-i)$$

$u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y)$ is a soln of equ

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \cdot \frac{\partial^2 u}{\partial t^2}$$ provided that $\alpha^2 = 1 - v^2/c^2$

(b)

Soln:

Gn'

$u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y)$ —(1)

Diff (1) w.r.to 'x'

$$\frac{\partial u}{\partial x} = f'(x - vt + i\alpha y) + g'(x - vt - i\alpha y).$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y) \quad \text{—(2)}$$

Diff (1) w.r.to "y"

$$\frac{\partial u}{\partial y} = f'(x - vt + i\alpha y)(i\alpha)^2 + g''(x - vt - i\alpha y)(-i\alpha)^2.$$

$$\frac{\partial^2 u}{\partial y^2} = f''(x - vt + i\alpha y)(i\alpha)^2 + g''(x - vt - i\alpha y)(-i\alpha)^2$$

$$\frac{\partial^2 u}{\partial y^2} = -\alpha^2 f''(x - vt + i\alpha y) - \alpha^2 g''(x - vt - i\alpha y).$$

$$\frac{\partial^2 u}{\partial y^2} = -\alpha^2 \left[f''(x - vt + i\alpha y) - g''(x - vt - i\alpha y)\right] \quad \text{—(2)}$$

Diff (1) w.r.to "t"

$$\frac{\partial u}{\partial t} = f'(x - vt + i\alpha y)(-v) + g'(x - vt - i\alpha y)(-v).$$

$$\frac{\partial^2 u}{\partial t^2} = f''(x - vt + i\alpha y)v^2 + g''(x - vt - i\alpha y)v^2.$$

$(2) + (3) \Rightarrow$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y) - \alpha^2 \left[f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y)\right]$$

Gn $\alpha^2 = 1 - v^2/c^2$

$$= f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y) - (1 - v^2/c^2)\left[f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y)\right]$$

$$\frac{\partial^2 u}{\partial y^2} = f''(x+iy)(i)^2 + g''(x-iy)(-i)^2$$

$$\frac{\partial^2 u}{\partial u^2} = -f''(x+iy) - g''(x-iy) \quad \text{---} (3)$$

equ $(2)+(3) \Rightarrow$

$$f''(x+iy) + g''(x-iy) - f''(x+iy) - g''(u-iy) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial u^2}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

4. If $z = f(x^2-y) + g(x^2+y)$ where the fn f & g are arbitary P.T $\frac{\partial^2 z}{\partial x^2} - \frac{1}{x} \cdot \frac{\partial z}{\partial x} = 4x^2 \cdot \frac{\partial^2 z}{\partial y^2}$

Gn $z = f(x^2-y) + g(x^2+y)$ --- (1)

Diff (1) w.r.to x.

$$\frac{\partial z}{\partial x} = f'(x^2-y) \cdot 2x + g'(x^2+y) 2x$$

$$= 2x \left[f'(x^2-y) + g'(x^2+y) \right]$$

$$\frac{\partial^2 z}{\partial x^2} = \left[f''(x^2-y) + g''(x^2+y) \right] 2x + 2\left[f'(x^2-y) + g'(x^2+y) \right]$$

$$\frac{\partial^2 z}{\partial x^2} = 4x^2 \left[f''(x^2-y) + g''(x^2+y) \right] \quad \text{---} (2)$$

Diff. (1) w.r.to "y".

$$\frac{\partial z}{\partial y} = f'(x^2-y)(-1) + g'(x^2+y)$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x^2-y) + g''(x^2+y) \quad \text{---} (3)$$

Sub (3) in (2) $\Rightarrow$

$$\frac{\partial^2 z}{\partial x^2} = 4x^2 \cdot \frac{\partial^2 z}{\partial y^2}$$

5. S.T if f & g are arbitary of a single variable. Then $u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y)$ is a soln of equ

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \cdot \frac{\partial^2 u}{\partial t^2}$ · provided That $\alpha^2 = 1 - v^2/c^2$

Soln:

Gn'

$$u = f(x - vt + i\alpha y) + g(x - vt - i\alpha y) \text{ --- (1)}$$

Diff (1) w.r.to 'x'

$$\frac{\partial u}{\partial x} = f'(x - vt + i\alpha y) + g'(x - vt - i\alpha y).$$

$$\frac{\partial^2 u}{\partial x^2} = f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y) \text{ --- (2)}$$

Diff (1) w.r.to "y"

$$\frac{\partial u}{\partial y} = f'(x - vt + i\alpha y)(i\alpha)^2 + g''(x - vt - i\alpha y)(-i\alpha)^2.$$

$$\frac{\partial^2 u}{\partial y^2} = f''(x - vt + i\alpha y)(i\alpha)^2 + g''(x - vt - i\alpha y)(-i\alpha)^2$$

$$\frac{\partial^2 u}{\partial y^2} = -\alpha^2 f''(x - vt + i\alpha y) - \alpha^2 g''(x - vt - i\alpha y).$$

$$\frac{\partial^2 u}{\partial y^2} = -\alpha^2 \left[f''(x - vt + i\alpha y) - g''(x - vt - i\alpha y) \right] \text{ --- (2)}$$

Diff (1) w.r.to "t"

$$\frac{\partial u}{\partial t} = f'(x - vt + i\alpha y)(-v) + g'(x - vt - i\alpha y)(-v).$$

$$\frac{\partial^2 u}{\partial t^2} = f''(x - vt + i\alpha y)v^2 + g''(x - vt - i\alpha y)v^2.$$

(2) + (3) ⇒

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y) - \alpha^2 \left[f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y) \right]$$

Gn $\alpha^2 = 1 - v^2/c^2$

$$= f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y) - (1 - v^2/c^2) \left[f''(x - vt + i\alpha y) + g''(x - vt - i\alpha y) \right]$$

1) show ...

$= (4RT - S^2)(\xi_x \eta_y - \xi_y \eta_x)^{2/4}$ —— (1)

The problem now is to choose ξ and η so that equation $A(\xi_x, \xi_y)u_{\xi\xi} + 2B(\xi_x \xi_y ; \eta_x \eta_y)u_{\xi\eta} + A(\eta_x \eta_y)u_{\eta\eta} = G(\xi, \eta, u, u_\xi, u_\eta)$ —— (*) takes a simple form.

(8)

Solu:

Case (i):- $S^2 - 4RT > 0$.

Then we shall show that ξ and η can be so choosen that the co-efficients of $u_{\xi\xi}$ and $u_{\eta\eta}$ in equ (1) vanish,

Consider $R\alpha^2 + S\alpha + T = 0$

This equ has two real distinct roots $\lambda_1(x,y)$ and $\lambda_2(x,y)$ due to the condition

$S^2 - 4RT > 0$

we choose ξ and η such that

$$\frac{\partial \xi}{\partial x} = \lambda_1 \frac{\partial \xi}{\partial y}$$

$$\frac{\partial \eta}{\partial x} = \lambda_2 \frac{\partial \eta}{\partial y}.$$

These are first order partial diff. equation for ξ and η

If $f_1(x,y) = c_1$ and

$f_2(x,y) = c_2$ are the solu of the ordinary diff. equation. $dy/dx + \lambda_1(x,y) = 0$

$dy/dx + \lambda_2(x,y) = 0$ respectivel

1) show that $A(\xi_x, \xi_y) A(\eta_x, \eta_y) - B^{\cdot}(\xi_x, \xi_y; \eta_x \eta_y)$

$= (ART - s^2)(\xi_x \eta_y - \xi_y \eta_x)^{2/4}$ —— (1)

The problem now is to choose ξ and η so that

equation $A(\xi_x, \xi_y) u_{\xi\xi} + 2B(\xi_x \xi_y; \eta_x \eta_y) u_{\xi\eta} +$

$A(\eta_x \eta_y) u_{\eta\eta} = G(\xi, \eta u, u_\xi, u_\eta)$ —— (*) takes a

simple form.

Solu:

Case (i):- $S^2 - 4RT > 0$.

Then we shall show that ξ and η can be so choosen

that the co-efficients of $u_{\xi\xi}$ and $u_{\eta\eta}$ in equ (1) vanish.

Consider $R\alpha^2 + S\alpha + T = 0$

This equ has two real distinct roots $\lambda_1(x,y)$ and

$\lambda_2(x,y)$ due to the condition

$S^2 - ART > 0$

we choose ξ and η such that

$$\frac{\partial \xi_1}{\partial x} = \lambda_1 \frac{\partial \xi_1}{\partial y}$$

$$\frac{\partial \eta}{\partial x} = \lambda_2 \frac{\partial \eta}{\partial y}.$$

These are first order partial diff. equation for ξ and η

If $f_1(x,y) = c_1$ and

... the solu of the ordin...

1) Show that $A(\xi_x, \xi_y) A(\eta_x, \eta_y) - B^2(\xi_x, \xi_y; \eta_x \eta_y)$
$= (ART - s^2)(\xi_x \eta_y - \xi_y \eta_x)^{2/4}$ —(1)

The problem now is to choose ξ and η so that equation $A(\xi_x, \xi_y) u_{\xi\xi} + 2B(\xi_x \xi_y; \eta_x \eta_y) u_{\xi\eta} + A(\eta_x \eta_y) u_{\eta\eta} = G(\xi, \eta, u, u_\xi, u_\eta)$ —(*) takes a simple form.

8

Solu:

Case (i): - $S^2 - ART > 0$.

Then we shall show that ξ and η can be so choosen that the co-efficients of $u_{\xi\xi}$ and $u_{\eta\eta}$ in equ (1) vanish.

Consider $R\alpha^2 + S\alpha + T = 0$

This equ has two real distinct roots $\lambda_1(x,y)$ and $\lambda_2(x,y)$ due to the condition

$S^2 - ART > 0$

we choose ξ and η such that

$$\frac{\partial \xi}{\partial x} = \lambda_1 \frac{\partial \xi}{\partial y}$$

$$\frac{\partial \eta}{\partial x} = \lambda_2 \frac{\partial \eta}{\partial y}.$$

These are first order partial diff. equation for ξ and η

If $f_1(x,y) = c_1$ and

$f_2(x,y) = c_2$ are the solu of the ordinary diff. equation. $dy/dx + \lambda_1(x,y) = 0$

$dy/dx + \lambda_2(x,y) = 0$ respectively

Scanned with CamScanner

show That $A(\xi_x, \xi_y) A(\eta_x, \eta_y) - B^2(\xi_x, \xi_y; \eta_x \eta_y)$

$= (ART - S^2)(\xi_x \eta_y - \xi_y \eta_x)^{2/4}$ —— (1)

The problem now is to choose ξ and η so that

quation $A(\xi_x, \xi_y) u_{\xi\xi} + 2B(\xi_x \xi_y; \eta_x \eta_y) u_{\xi\eta} +$

$A(\eta_x \eta_y) u_{\eta\eta} = G(\xi, \eta, u, u_\xi, u_\eta)$ —— (*) takes a

simple form.

Solu:

Case (i):- $S^2 - ART > 0$.

Then we shall show That ξ and η can be so choosen

That The co-efficients of $u_{\xi\xi}$ and $u_{\eta\eta}$ in equ (1) vanish.

Consider $R\alpha^2 + S\alpha + T = 0$

This equ has two real distinct roots $\lambda_1(x,y)$ and

$\lambda_2(x,y)$ due to the condition

$S^2 - ART > 0$

we choose ξ and η such That

$$\frac{\partial \xi}{\partial x} = \lambda_1 \frac{\partial \xi}{\partial y}$$

$$\frac{\partial \eta}{\partial x} = \lambda_2 \frac{\partial \eta}{\partial y}.$$

These are first order partial diff. equation for ξ and η

If $f_1(x,y) = c_1$ and

$f_2(x,y) = c_2$ are the solu of the ordinary

diff. equation. $dy/dx + \lambda_1(x,y) = 0$

$dy/dx + \lambda_2(x,y) = 0$ respectively

Scanned with CamScanner

$= (4RT - S^2)(\xi_x \eta_y - \xi_y \eta_x)^{2/4}$ — (1)

The problem now is to choose ξ and η so that equation $A(\xi_x, \xi_y)u_{\xi\xi} + 2B(\xi_x \xi_y ; \eta_x \eta_y)u_{\xi\eta} + A(\eta_x \eta_y)u_{\eta\eta} = G(\xi, \eta u, u_\xi, u_\eta)$ — (*) takes a simple form.

Solu:

Case (i):- $S^2 - 4RT > 0$.

Then we shall show that ξ and η can be so choosen that the co-efficients of $u_{\xi\xi}$ and $u_{\eta\eta}$ in equ (1) vanish,

Consider $R\alpha^2 + S\alpha + T = 0$

This equ has two real distinct roots $\lambda_1(x,y)$ and $\lambda_2(x,y)$ due to the condition

$$S^2 - 4RT > 0$$

we choose ξ and η such that

$$\frac{\partial \xi}{\partial x} = \lambda_1 \frac{\partial \xi}{\partial y}$$

$$\frac{\partial \eta}{\partial x} = \lambda_2 \frac{\partial \eta}{\partial y}.$$

These are first order partial diff. equation for ξ and η

If $f_1(x,y) = c_1$ and

$f_2(x,y) = c_2$ are the solu of the ordinary diff. equation. $dy/dx + \lambda_1(x,y) = 0$

$dy/dx + \lambda_2(x,y) = 0$ respectively

Scanned with CamScanner

$$= \left[f''(x-vt+i\alpha y) + g''(x-vt-i\alpha y)\right]\left(-1-1+v^2/c^2\right)$$

$$= v^2 \left[f''(x-vt+i\alpha y)+g''(x-vt-i\alpha y)\right]. \frac{1}{c^2}$$

$$= \frac{\partial^2 y}{\partial t^2} \cdot \frac{1}{c^2} \qquad 7$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2} \cdot \frac{\partial^2 u}{\partial t^2} \quad //.$$

Equations with Variable co-efficients :-

Consider the equations of the type.

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \quad \text{---}(1)$$

which may be written in the form,

$$L(z) + f(x, y, z, p, q) = 0 \quad \text{---}(2)$$

where L is the differential operaton defined by the equation,

$$L = R. \frac{\partial^2 z}{\partial x^2} + S \frac{\partial^2 z}{\partial x \partial y} + T \frac{\partial^2 z}{\partial y^2} \quad \text{---}(3)$$

In which R, S, T are continuous functions of x and y possesing continuous partial derivatives of as high an order as necessary By a suitable change of the Independent variables. we shall show that any equation of the type (2) can be reduced to one of three canonical forms.

Suppose we change the independent variables form x, y to ξ, η where $\xi = \xi(x, y)$ $\eta = \eta(x, y)$

Scanned with CamScanner

1) show That $A(\xi_x, \xi_y) A(\eta_x, \eta_y) - B^2(\xi_x, \xi_y; \eta_x \eta_y)$

$= (4RT - S^2)(\xi_x \eta_y - \xi_y \eta_x)^{2/4}$ —— (1)

The problem now is to choose ξ and η so that equation $A(\xi_x, \xi_y) u_{\xi\xi} + 2B(\xi_x \xi_y; \eta_x \eta_y) u_{\xi\eta} +$

$A(\eta_x \eta_y) u_{\eta\eta} = G(\xi, \eta u, u_\xi, u_\eta)$ —— (*) takes a Simple form.

(8)

Solu:

Case (i):- $S^2 - 4RT > 0$.

Then we shall show that ξ and η can be so choosen that The co-efficients of $u_{\xi\xi}$ and $u_{\eta\eta}$ in equ (1) vanish,

Consider $R\alpha^2 + S\alpha + T = 0$

This equ has two real ditinct roots $\lambda_1(x,y)$ and $\lambda_2(x,y)$ due to the condition

$S^2 - 4RT > 0$

we choose ξ and η such that

$$\frac{\partial \xi}{\partial x} = \lambda_1 \frac{\partial \xi}{\partial y}$$

$$\frac{\partial \eta}{\partial x} = \lambda_2 \frac{\partial \eta}{\partial y}.$$

These are first order partial diff. equation for ξ and η

If $f_1(x,y) = c_1$ and

$f_2(x,y) = c_2$ are the solu of the ordinary diff. equation. $dy/dx + \lambda_1(x,y) = 0$

$dy/dx + \lambda_2(x,y) = 0$ respectively.

Then $\xi = f_1(x,y)$

$\eta = f_2(x,y)$ will be the suitables choice.

This choice of ξ and η makes

$$A(\xi_x, \xi_y) = A(\eta_x, \eta_y) = 0 \qquad (a)$$

From equ (*) in This case we have

$$B^2 > 0.$$

Hence equ (1) reduces to.

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = \phi(\xi, \eta, u, u_\xi, u_\eta) \text{——} (2)$$

The curves $\xi(x,y) = \text{constant}$ and $\eta(x,y) = \text{Constant}$ are called the characteristic curves of equation.

$$Lu + g(x,y,u,u_x,u_y) = 0.$$

Case (ii) :-

$$S^2 - 4RT = 0.$$

In This case The roots of The equation.

$R\alpha^2 + S\alpha + T = 0$ canonical (say $(\lambda(x,y))$).

Define $\xi = f(x,y)$

where $f(x,y) = c_1$ is The solu of $dy/dx + \lambda(x,y) = 0$

Take η as any orbitary function of x and y

Independent of ξ and form equ (*) observe That $B = 0$

Since $A(\xi_x, \xi_y) = 0$.

Since η is Independent of ξ.

$A(\eta_x, \eta_y) \neq 0$.

$\therefore$ Equations equ (1) reduces to

$$\frac{\partial^2 u}{\partial \eta^2} = \phi(\xi, \eta, u, u_\xi, u_\eta) \quad \text{--- (3)}$$

Case (iii) :-

$S^2 - 4RT < 0$.

This is the same as case (i).

However here the roots are complex.

proceeding as in case (i) we find that equ (1) reduces to the form equ (2)

But the variables ξ and η are not real and infact complex conjugates.

Therefore there are no real characteristic curves in this case.

We make the further transformation.

$$\alpha = \frac{1}{2}(\xi + \eta)$$

$$\beta = i/2\,(\eta - \xi)$$

Equ (1) becomes,

$$\frac{\partial^2 u}{\partial \alpha^2} + \frac{\partial^2 u}{\partial \beta^2} = \phi(\alpha, \beta, u, u_\alpha, u_\beta) \longrightarrow (4).$$

problem :-

1. Reduce the equ $\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$ (or)

U.Q

Sm $u_{xx} - x^2 u_{yy} = 0$ to a canonical form

$y = \frac{x^2}{2} + c_2$

$c_2 = y - \frac{x^2}{2}$

$\eta = y - \frac{x^2}{2}$ —(2)

$\eta_x = -x$ $\quad\quad \eta_y = 1$

$\eta_{xx} = -1$ $\quad\quad \eta_{yy} = 0$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial \xi^2} \xi_x^2 + \frac{\partial^2 z}{\partial \xi \partial \eta} \xi_x \eta_x + \frac{\partial z}{\partial \xi_1} \cdot \xi_{xx} + \frac{\partial^2 z}{\partial \eta^2} \eta^2 x +$$

$$\frac{\partial^2 z}{\partial \eta \partial \xi_1} \eta_x \xi_x + \frac{\partial z}{\partial \eta} \cdot \eta_{xx}.$$

$$= \frac{\partial^2 z}{\partial \xi_1^2} x^2 + \frac{\partial^2 z}{\partial \xi \partial \eta} (x)(-x) + \frac{\partial z}{\partial \xi_1}(1) + \frac{\partial^2 z}{\partial \eta^2}(-x)^2 +$$

$$\frac{\partial^2 z}{\partial \eta \partial \xi_1}(-x)(x) + \frac{\partial z}{\partial \eta}(-1).$$

$$\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial \xi_1^2} - 2x^2 \frac{\partial^2 z}{\partial \xi_1 \partial \eta} + x^2 \frac{\partial^2 z}{\partial \eta^2} + \frac{\partial z}{\partial \xi_1} - \frac{\partial z}{\partial \eta} \quad \text{—(2)}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial \xi^2} \xi^2 y^2 + \frac{\partial^2 z}{\partial \xi_1 \partial \eta} \xi_y \eta_y + \frac{\partial z}{\partial \xi_1} \xi_{yy} + \frac{\partial^2 z}{\partial \eta^2} \eta^2 y^2 +$$

$$\frac{\partial^2 z}{\partial \eta \partial \xi_1} \eta_y \xi_y + \frac{\partial z}{\partial \eta} \eta_{yy}$$

$$= \frac{\partial^2 z}{\partial \xi^2}(1)^2 + \frac{\partial^2 z}{\partial \xi_1 \partial \eta}(1)(1) + \frac{\partial z}{\partial \xi_1}(0) + \frac{\partial^2 z}{\partial \eta^2}(1)^2 +$$

$$\frac{\partial^2 z}{\partial \eta \partial \xi_1}(1)(1) + \frac{\partial z}{\partial \eta}(0)$$

Solu:-

Con $u_{xx} - x^2 u_{yy} = 0$

The second order P.D.E of the form

$R \frac{\partial^2 z}{\partial x^2} + S \frac{\partial^2 z}{\partial x \partial y} + T \frac{\partial^2 z}{\partial y^2}$.

$R = 1, S = 0, T = -x^2$.

$S^2 - 4RT = 0 - 4(1)(-x^2)$

$= 4x^2 > 0$. (greater type 1

$R\alpha^2 + S\alpha + T = 0$

$\alpha^2 - x^2 = 0$

$\alpha^2 = x^2$

$\alpha = \pm x$

$\alpha = x$; $\alpha = -x$.

$\frac{dy}{dx} + x = 0 \Rightarrow \frac{dy}{dx} = -x$.

$dy = -x\,dx$

$y = \frac{-x^2}{2} + c_1$

$y + \frac{x^2}{c} = c_1$

$\xi = y + \frac{x^2}{2}$ —(1)

$\xi_x = x$ $\xi_y = 1$

$\xi_{xx} = 1$ $\xi_{yy} = 0$

$\frac{dy}{dx} - x = 0$

$\frac{dy}{dx} = x$

$dy = x\,dx$

Scanned with CamScanner

$$y = \frac{x^2}{2} + c_2$$

$$c_2 = y - \frac{x^2}{2}$$

(12)

$$\eta = y - \frac{x^2}{2} \quad \text{---} (2)$$

$$\eta_x = -x \qquad \eta_y = 1$$

$$\eta_{xx} = -1 \qquad \eta_{yy} = 0$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial \xi^2} \xi_x^2 + \frac{\partial^2 z}{\partial \xi \partial \eta} \xi_x \eta_x + \frac{\partial z}{\partial \xi_1} \cdot \xi_{1xx} + \frac{\partial^2 z}{\partial \eta^2} \cdot \eta^2_x +$$

$$\frac{\partial^2 z}{\partial \eta \partial \xi_1} \eta_x \xi_x + \frac{\partial z}{\partial \eta} \cdot \eta_{xx} .$$

$$= \frac{\partial^2 z}{\partial \xi_1^2} x^2 + \frac{\partial^2 z}{\partial \xi \partial \eta} (x)(-x) + \frac{\partial z}{\partial \xi_1} (1) + \frac{\partial^2 z}{\partial \eta^2} (-x)^2 +$$

$$\frac{\partial^2 z}{\partial \eta \partial \xi_1} (-x)(x) + \frac{\partial z}{\partial \eta} (-1) .$$

$$\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial \xi_1^2} - 2x^2 \frac{\partial^2 z}{\partial \xi_1 \partial \eta} + x^2 \frac{\partial^2 z}{\partial \eta^2} + \frac{\partial z}{\partial \xi_1} - \frac{\partial z}{\partial \eta} \quad \text{---} (2)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial \xi^2} \xi^2 y^2 + \frac{\partial^2 z}{\partial \xi_1 \partial \eta} \xi_y \eta_y + \frac{\partial z}{\partial \xi_1} \xi_{yy} + \frac{\partial^2 z}{\partial \eta^2} \eta^2 y^2 +$$

$$\frac{\partial^2 z}{\partial \eta \partial \xi_1} \eta_y \xi_{1y} + \frac{\partial z}{\partial \eta} \eta_{yy}$$

$$= \frac{\partial^2 z}{\partial \xi^2} (1)^2 + \frac{\partial^2 z}{\partial \xi_1 \partial \eta} (1)(1) + \frac{\partial z}{\partial \xi_1} (0) + \frac{\partial^2 z}{\partial \eta^2} (1)^2 +$$

$$\frac{\partial^2 z}{\partial \eta \partial \eta \xi_1} (1)(1) + \frac{\partial z}{\partial \eta} (0)$$

$R.\frac{\partial^2 z}{\partial x^2} + S.\ \frac{\quad}{\partial x \partial y}$

Compare eqn (1) & (2)

$R = 1,\ S = 0,\ T = x^2.$

$S^2 - 4RT = (0)^2 - 4(1)(x^2)$

$= -4x^2 < 0$

$R\alpha^2 + S\alpha + T = 0$

$\alpha^2 + x^2 = 0$

$\alpha^2 = -x^2$

$\boxed{\alpha = \pm ix}$

$\frac{dy}{dx} \pm ix = 0$

$\frac{dy}{dx} + ix = 0 \Rightarrow \frac{dy}{dx} = -ix$

$dy = -ix\,dx$

$ix\,dx + dy = 0 \quad (\div) i$

$x\,dx + \frac{1}{i}\,dy = 0$

$x\,dx + \frac{i}{i^2}\,dy = 0$ (Taking conjugate)

$x\,dx - i\,dy = 0$

$\int$ing on b.s

$\frac{x^2}{2} - iy = c_1$

The

$$R \cdot \frac{\partial^2 z}{\partial x^2} + S \cdot \frac{\partial^2 z}{\partial x \partial y} + T \cdot \frac{\partial^2 z}{\partial y^2} = 0 \quad \text{--- (2)}$$

compare equ (1) & (2)

$R = 1,\ S = 0,\ T = x^2$

$S^2 - 4RT = (0)^2 - 4(1)(x^2)$

$= -4x^2 < 0$

$R\alpha^2 + S\alpha + T = 0$

$\alpha^2 + x^2 = 0$

$\alpha^2 = -x^2$

$$\boxed{\alpha = \pm ix}$$

$\frac{dy}{dx} \pm ix = 0$

$\frac{dy}{dx} + ix = 0 \Rightarrow \frac{dy}{dx} = -ix$

$dy = -ix\,dx$

$ix\,dx + dy = 0 \quad (\div)\ i$

$x\,dx + \frac{1}{i}\,dy = 0$

$x\,dx + \frac{i}{i^2}\,dy = 0$ (Taking Conjugate)

$x\,dx - i\,dy = 0$

$\int$ing on b.s

$\frac{x^2}{2} - iy = c_1$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial \xi^2} + 2\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2} \quad \text{---}(3)$$

$$\frac{\partial^2 z}{\partial x^2} - x^2 \frac{\partial^2 z}{\partial y^2} = 0$$

13

$$\Rightarrow x^2 \frac{\partial^2 z}{\partial \xi^2} - 2x^2 \frac{\partial^2 z}{\partial \xi \partial \eta} + x^2 \frac{\partial^2 z}{\partial \eta^2} + \frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta}$$

$$- x^2 \frac{\partial^2 z}{\partial \xi^2} - 2x^2 \frac{\partial^2 z}{\partial \xi \partial \eta} - x^2 \frac{\partial^2 z}{\partial \eta^2} = 0$$

$$-4x^2 \frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta} = 0$$

$$-4x^2 \frac{\partial^2 z}{\partial \xi \partial \eta} = \frac{-\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}$$

$$\frac{\partial^2 z}{\partial \xi \partial \eta} = \frac{-1}{4x^2}\left(-\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}\right)$$

$$= \frac{1}{4x^2}\left(\frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta}\right) \quad \text{---}(4)$$

$$\xi = y + \frac{x^2}{2}, \quad \eta = y - x^2/2$$

$$\xi - \eta = y + \frac{x^2}{2} - y + \frac{x^2}{2} = \frac{2x^2}{2} = x^2$$

Sub $x^2 = \xi - \eta$ in (4)

$$\frac{\partial^2 z}{\partial \xi \partial \eta} = \frac{1}{4(\xi - \eta)}\left(\frac{\partial z}{\partial \xi} - \frac{\partial z}{\partial \eta}\right) //$$

Reduce the equ $\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0$

$$\frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0 \quad \text{---}(1)$$

The general equ

$$R \cdot \frac{\partial^2 z}{\partial x^2} + S \cdot \frac{\partial^2 z}{\partial x \partial y} + T \cdot \frac{\partial^2 z}{\partial y^2} = 0 \quad \text{---} (2)$$

Compare equ (1) & (2)

14

$R = 1,\ S = 0,\ T = x^2$.

$S^2 - 4RT = (0)^2 - 4(1)(x^2)$

$= -4x^2 < 0$

$R\alpha^2 + S\alpha + T = 0$

$\alpha^2 + x^2 = 0$

$\alpha^2 = -x^2$

$$\boxed{\alpha = \pm ix}$$

$\frac{dy}{dx} \pm ix = 0$

$\frac{dy}{dx} + ix = 0 \Rightarrow \frac{dy}{dx} = -ix$

$dy = -ix\,dx$

$ix\,dx + dy = 0 \quad (\div)\, i$

$x\,dx + \frac{1}{i}\,dy = 0$

$x\,dx + \frac{i}{i^2}\,dy = 0$ (Taking Conjugate)

$x\,dx - i\,dy = 0$

$\int$ ing on b.s

$\frac{x^2}{2} - iy = c_1$

$$= \frac{\partial}{\partial \alpha} \qquad \partial \beta$$

(16)

$$\frac{\partial z}{\partial x} = x \frac{\partial z}{\partial \alpha}.$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left[x \cdot \frac{\partial z}{\partial \alpha}\right]$$

$$= x \cdot \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \alpha}\right) + \frac{\partial z}{\partial \alpha}(1)$$

$$= x \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \alpha}\right) + \frac{\partial z}{\partial \alpha}$$

$$= x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right) \cdot \frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \beta}{\partial x}\right] + \frac{\partial z}{\partial \alpha}$$

$$= x\left[\frac{\partial^2 z}{\partial \alpha^2}(x) + \frac{\partial^2 z}{\partial \alpha \partial \beta}(0)\right] + \frac{\partial z}{\partial \alpha}.$$

$$\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial z}{\partial \alpha} \text{ ——— } (3)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta} \cdot \frac{\partial \beta}{\partial y}.$$

$$= \frac{\partial z}{\partial \alpha}(0) + \frac{\partial z}{\partial \beta}(1)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \beta}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \beta}\right)$$

$$= \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial y} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right) \cdot \frac{\partial \beta}{\partial y}$$

$$\frac{x^2}{2} - iy = \eta .$$

$$\frac{dy}{dx} - ix = 0 \Rightarrow \frac{dy}{dx} = ix .$$

$$dy = ix\,dx \Rightarrow ix\,dx - dy = 0$$

$$x\,dx - \frac{1}{i}\,dy = 0$$

$$x\,dx + i\,dy = 0 \qquad \text{(Taking conjugate)}$$

$\int$ing on b.s.

$$x^2/2 + iy = c_2$$

$$x^2/2 + iy = \xi$$

$$\alpha = \frac{1}{2}(\xi + \eta)$$

$$\alpha = \frac{1}{2}\left(\frac{x^2}{2} + iy + \frac{x^2}{2} - iy\right)$$

$$= \frac{1}{2}\left(2x^2/2\right)$$

$$\alpha = x^2/2 \qquad \left[\therefore \frac{\partial\alpha}{\partial x} = x,\ \frac{\partial\alpha}{\partial y} = 0\right]$$

$$\beta = i/2\,(\eta - \xi)$$

$$= i/2\left(x^2/2 - iy - x^2/2 - iy\right)$$

$$= i/2\,(-2iy) = -i^2 y$$

$$\beta = y \qquad \left[\therefore \frac{\partial\beta}{\partial x} = 0,\ \frac{\partial\beta}{\partial y} = 1\right]$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial\alpha}\cdot\frac{\partial\alpha}{\partial x} + \frac{\partial z}{\partial\beta}\cdot\frac{\partial\beta}{\partial x} .$$

13

$$= \frac{\partial z}{\partial \alpha}(x) + \frac{\partial z}{\partial \beta}(0).$$

(16)

$$\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial \alpha}.$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left[x \cdot \frac{\partial z}{\partial \alpha}\right]$$

$$= x \cdot \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \alpha}\right) + \frac{\partial z}{\partial \alpha}(1)$$

$$= x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \alpha}\right) + \frac{\partial z}{\partial \alpha}$$

$$= x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\cdot\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \beta}{\partial x}\right] + \frac{\partial z}{\partial \alpha}$$

$$= x\left[\frac{\partial^2 z}{\partial \alpha^2}(x) + \frac{\partial^2 z}{\partial \alpha \partial \beta}(0)\right] + \frac{\partial z}{\partial \alpha}.$$

$$\frac{\partial^2 z}{\partial x^2} = x^2\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial z}{\partial \alpha} \quad \text{---} (3)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial y}.$$

$$= \frac{\partial z}{\partial \alpha}(0) + \frac{\partial z}{\partial \beta}(1)$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \beta}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \beta}\right)$$

$$= \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial y} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\cdot\frac{\partial \beta}{\partial y}$$

$$= 0 + \frac{\partial^2 z}{\partial \beta^2} (1)$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial \beta^2} \quad \text{---} (4)$$

17

$$(1) \Rightarrow \frac{\partial^2 z}{\partial x^2} + x^2 \frac{\partial^2 z}{\partial y^2} = 0$$

$$x^2 \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial z}{\partial \alpha} + x^2 . \frac{\partial^2 z}{\partial \beta^2} = 0$$

$$x^2 \left[\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} \right] = -\frac{\partial z}{\partial \alpha}$$

$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \frac{-1}{x^2} . \frac{\partial z}{\partial \alpha} \qquad \left[\because \alpha = x^2/2 \quad 2\alpha = x^2 \right]$$

$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \frac{-1}{2\alpha} . \frac{\partial z}{\partial \alpha}$$

3- Reduce the equ $\frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ to canonical form and hence solve it.

Solu: Gn $\frac{\partial^2 z}{\partial x^2} + 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ —— (1)

$$R . \frac{\partial^2 z}{\partial x^2} + S . \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0 \quad \text{---} (2)$$

Comparing (1) & (2)

$R = 1$, $S = 2$, $T = 1$

$$S^2 - 4RT = (2)^2 - 4(1)(1) = 4 - 4 = 0$$

The roots are equal

$R\alpha^2 + S\alpha + T = 0$

$\alpha^2 + 2\alpha + 1 = 0$

(18)

$(\alpha+1)(\alpha+1) = 0$

$\alpha = -1, -1$

$$\frac{dy}{dx} - 1 = 0$$

$$\frac{dy}{dx} = 1$$

$$dy = dx$$

$$dx - dy = 0$$

$$x - y = c_1$$

$$x - y = \xi_1$$

we take η as arbitary function of x and y independent of ξ_1.

So $\eta = x + y$.

$\therefore \frac{\partial \xi_1}{\partial x} = 1$

$\frac{\partial \xi_1}{\partial y} = -1$

$$\frac{\partial \eta}{\partial x} = 1 \; ; \; \frac{\partial \eta}{\partial y} = 1$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi_1} \cdot \frac{\partial \xi_1}{\partial x} + \frac{\partial z}{\partial \eta} \cdot \frac{\partial \eta}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial \xi_1} + \frac{\partial z}{\partial \eta}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left[\frac{\partial z}{\partial \xi_1} + \frac{\partial z}{\partial \eta}\right]$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial \xi_1}\left[\frac{\partial z}{\partial \xi_1} + \frac{\partial z}{\partial \eta}\right] \cdot \frac{\partial \xi_1}{\partial x} + \frac{\partial}{\partial \eta}\left[\frac{\partial z}{\partial \xi_1} + \frac{\partial z}{\partial \eta}\right]\frac{\partial \eta}{\partial x}$$

Scanned with CamScanner

$$= \frac{\partial^2 z}{\partial \xi^2} - \frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta \partial \xi} + \frac{\partial^2 z}{\partial \eta^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = -\frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \eta^2}$$

20

$$\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

$$\frac{\partial^2 z}{\partial \xi^2} + 2\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2} + 2\left[-\frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \eta^2}\right] + \frac{\partial^2 z}{\partial \xi^2} - \frac{2\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2} = 0$$

$$\frac{\partial^2 z}{\partial \xi^2} + 2\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2} - 2\frac{\partial^2 z}{\partial \xi^2} + 2\frac{\partial^2 z}{\partial \eta^2} + \frac{\partial^2 z}{\partial \xi^2} - 2\frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2} = 0$$

$$2\frac{\partial z}{\partial \xi^2} + 4\frac{\partial^2 z}{\partial \eta^2} - 2\frac{\partial^2 z}{\partial \xi^2} = 0$$

$$4\frac{\partial^2 z}{\partial \eta^2} = 0$$

$$\frac{\partial^2 z}{\partial \eta^2} = 0 \quad //$$

Linear partial differential equation with constant co-efficients:

Consider the solution of linear P.D.E that with constant co-eff. Such an equ can be written in the form

$$F(D, D')z = f(x, y) \quad \text{---}(1)$$

where $F(D, D')$ denotes a diff. operator of the type

$$F(D, D') = \sum_r \sum_s c_{rs} D^r D'^s \quad \text{---}(2)$$

$$= \frac{\partial}{\partial \xi} \left[\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta} \right] + \frac{\partial}{\partial \eta} \left[\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta} \right]$$

$$= \frac{\partial^2 z}{\partial \xi^2} + \frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2}$$

$$= \frac{\partial^2 z}{\partial \xi^2} + 2 \frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial \xi} \cdot \frac{\partial \xi}{\partial y} + \frac{\partial z}{\partial \eta} \cdot \frac{\partial \eta}{\partial y}$$

$$\frac{\partial z}{\partial y} = -\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y} \left[-\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta} \right]$$

$$= \frac{\partial}{\partial \xi} \left[-\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta} \right] \cdot \frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta} \left[-\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta} \right] \cdot \frac{\partial \eta}{\partial y}$$

$$= \frac{\partial}{\partial \xi} \left[-\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta} \right] (-1) + \frac{\partial}{\partial \eta} \left[-\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta} \right]$$

$$= \frac{\partial^2 z}{\partial \xi^2} - \frac{\partial^2 z}{\partial x \partial \eta} - \frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial \xi^2} - 2 \frac{\partial^2 z}{\partial \xi \partial \eta} + \frac{\partial^2 z}{\partial \eta^2}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y} \left[\frac{\partial z}{\partial x} \right]$$

$$= \frac{\partial}{\partial y} \left[\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta} \right].$$

$$= \frac{\partial}{\partial \xi} \left[\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta} \right] \cdot \frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta} \left[\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta} \right] \cdot \frac{\partial \eta}{\partial y}$$

$$= \frac{\partial}{\partial \xi} \left[\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta} \right] (-1) + \frac{\partial}{\partial \eta} \left[\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta} \right] (1)$$

$$= \frac{\partial}{\partial \xi} \left[\frac{\partial z}{\partial \xi} + \frac{\partial z}{\partial \eta} \right]$$

$$\left.\begin{aligned} \bar{u} &= F(T_1, T_2) \\ \overline{\frac{\partial u}{\partial n}} &= G(T_1, T_2) \end{aligned}\right\} \rightarrow (4)$$ 2

$$\frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \frac{\partial n}{\partial x_3} dx_3 = 0 \rightarrow (*)$$

The bar denoting that these are the Values assumed by the relevant quantity on the surface S.

$$(2) \Rightarrow \boxed{f = 0} \Rightarrow \boxed{\partial f = 0}$$

(ie) $\frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 + \frac{\partial f}{\partial x_3} dx_3 = 0$

where $dx = \frac{\partial x_i}{\partial T_1} dT_1 + \frac{\partial x_i}{\partial T_2} dT_2$,

$i = 1, 2, 3 \ldots$

The above eqn becomes.

$$\Rightarrow \frac{\partial f}{\partial x_1}\left[\frac{\partial x_1}{\partial T_1} dT_1 + \frac{\partial x_1}{\partial T_2} dT_2\right] + \frac{\partial f}{\partial x_2}\left[\frac{\partial x_2}{\partial T_1} dT_1 + \frac{\partial x_2}{\partial T_2} dT_2\right] + \frac{\partial f}{\partial x_3}\left[\frac{\partial x_3}{\partial T_1} dT_1 + \frac{\partial x_3}{\partial T_2} dT_2\right] = 0$$

ie) $\sum_{i=1}^{3} \frac{\partial f}{\partial x_i} \left[\frac{\partial x_i}{\partial T_1} dT_1 + \frac{\partial x_i}{\partial T_2} dT_2 \right] = 0 \quad \to (5)$

Equating to zero the coeff dT_1, dT_2.

we have,

$$\sum_{i=1}^{3} \delta_i P_{ij} = 0 \quad , \quad j = 1, 2 \ldots$$

where $\boxed{\delta_i = \frac{\partial f}{\partial x_i}}$, $P_{ij} = \frac{\partial x_i}{\partial T_j}$

solving these eqn we find that,

$$\frac{\delta_1}{\Delta_1} = \frac{\delta_2}{\Delta_2} = \frac{\delta_3}{\Delta_3} = \rho \text{ (say)} \quad \to (6)$$

where Δ_1 denotes the Jacobian,

$$\boxed{\frac{\partial(x_2, x_3)}{\partial(T_1, T_2)} = \Delta_1} \quad , \quad \boxed{\frac{\partial(x_1, x_3)}{\partial(T_1, T_2)} = \Delta_2}$$

$$\frac{\partial(x_1, x_2)}{\partial(T_1, T_2)} = \Delta_3$$

From (*) taking the total derivatives of $\bar{u}$.

$$\partial \bar{u} = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \frac{\partial u}{\partial x_3} dx_3$$

$$d\bar{u} = \sum_{i=1}^{3} \sum_{j=1}^{2} P_i P_{ij} \, dT_j$$

where $P_i = \frac{\partial u}{\partial x_i}$, $\bar{u} = F(T_1, T_2)$

$$d\bar{u} = \frac{\partial F}{\partial T_1} dT_1 + \frac{\partial F}{\partial T_2} dT_2 \rightarrow (7)$$

$$\frac{\partial F}{\partial T_1} = \frac{\partial u}{\partial x_1} \cdot \frac{\partial x_1}{\partial T_1} + \frac{\partial u}{\partial x_2} \cdot \frac{\partial x_2}{\partial T_1} + \frac{\partial u}{\partial x_3} \cdot \frac{\partial x_3}{\partial T_1}$$

$$\therefore \frac{\partial F}{\partial T_i} = \sum_{i=1}^{3} \frac{\partial u}{\partial x_i} \cdot \frac{\partial x_i}{\partial T_j}$$

$$\frac{\partial F}{\partial T_j} = \sum_{i=1}^{3} P_i P_{ij} \quad , \quad j = 1, 2 \rightarrow (8)$$

now, $q = \frac{\partial \bar{u}}{\partial n} = \nabla u \cdot \hat{n}$

where,

$$\hat{n} = \frac{\frac{\partial F}{\partial x_1} \vec{i} + \frac{\partial F}{\partial x_2} \vec{j} + \frac{\partial F}{\partial x_3} \vec{k}}{\sqrt{\left(\frac{\partial F}{\partial x_1}\right)^2 + \left(\frac{\partial F}{\partial x_2}\right)^2 + \left(\frac{\partial F}{\partial x_3}\right)^2}}$$

$$\nabla \cdot u \cdot \hat{n} = \frac{\left(\frac{\partial u}{\partial x_1} \vec{i} + \frac{\partial u}{\partial x_2} \vec{j} + \frac{\partial u}{\partial x_3} \vec{k}\right)\left(\frac{\partial f}{\partial x_1} \vec{i} + \frac{\partial f}{\partial x_2} \vec{j} + \frac{\partial f}{\partial x_3} \vec{k}\right)}{\sqrt{\left(\frac{\partial f}{\partial x_1}\right)^2 + \left(\frac{\partial f}{\partial x_2}\right)^2 + \left(\frac{\partial f}{\partial x_3}\right)^2}}$$

where $P_i = \frac{\partial u}{\partial x_i}$, $\delta_i = \frac{\delta F}{\partial x_i}$

$$q = \nabla \cdot u \cdot \hat{n} = \sum_{i=1}^{3} \frac{P_i \delta_i}{\sqrt{\delta_1^2 + \delta_2^2 + \delta_3^2}}$$

$$q = \sum_{i=1}^{n} \frac{P_i \delta_i}{\sqrt{\delta_1^2+\delta_2^2+\delta_3^2}}$$

$$\sum_{i=1}^{n} P_i \delta_i = q \left(\delta_1^2 + \delta_2^2 + \delta_3^2 \right)^{1/2} \rightarrow (9)$$

From (8) & (9) are sufficient for the determinant of P_1, P_2, P_3 at all pts of surface.

now, consider,

$$d\left(\frac{\partial u}{\partial x_1}\right) = \frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_i}\right)dx_1 + \frac{\partial}{\partial x_2}\left(\frac{\partial u}{\partial x_i}\right)dx_2 + \frac{\partial}{\partial x_3}\left(\frac{\partial u}{\partial x_i}\right)dx_3$$

$$dx = \frac{\partial x_i}{\partial T_1} dT_1 + \frac{\partial x_i}{\partial T_2} dT_2$$

$$= \frac{\partial}{\partial x_1}\left(\frac{\partial u}{\partial x_i}\right)\frac{\partial x_1}{\partial T_1} dT_1 + \frac{\partial}{\partial x_2}\left(\frac{\partial u}{\partial x_i}\right)\frac{\partial x_2}{\partial T_2} dT_2 + \frac{\partial}{\partial x_3}\left(\frac{\partial u}{\partial x_i}\right)\frac{\partial x_3}{\partial T_3} dT_3$$

$$\frac{\partial \bar{P_i}}{\partial T_j} = \sum_{r=1}^{3} P_{rj} P_{ir} \quad , j = 1, 2 \rightarrow (10)$$

For each value of i

This pair of eqn is not sufficient of the soln of P_{i1}, P_{i2}, P_{i3} so that add eqn.

$$\sum_{i=1}^{3} \alpha_r P_{ir} = \lambda_i \longrightarrow (11)$$

where λ_i is a parameter interms of all the P_{ir} are expressed linearly and the α's are numerial constants chosen in such a way as to ensure that the determinant.

$$\Delta = \begin{vmatrix} P_{11} & P_{21} & P_{31} \\ P_{12} & P_{22} & P_{32} \\ \alpha_1 & \alpha_2 & \alpha_3 \end{vmatrix} \text{ is non-zero}$$

Suppose now that the quantities P_{ir} Constitute a set of Solution of the eqn (10), then

$$\sum_{r=1}^{3} P_{rj} \left(P_{ir} - P_{ir}' \right) = 0, \quad j = 1, 2.$$

for each $j=1$, $i=1$

$$P_{11}\left(P_{11} - P_{11}'\right) + P_{21}\left(P_{12} - P_{12}'\right) + P_{31}$$
$$\left(P_{13} - P_{13}'\right) = 0$$

which ie),

$$\frac{P_{ij} - P_{ij}'}{\Delta_j} = \delta_j = \frac{P_{i1} - P_{i1}'}{\Delta_1} = \frac{P_{i2} - P_{i2}'}{\Delta_2} = \delta_j$$

$$P_{ij} - P_{ij}' = \delta_j \Delta_j \Rightarrow P_{ij}' + \delta_j \Delta_j$$

$$\Rightarrow P_{ij} = \quad \Rightarrow P_{ij}' + P_i \Delta_j$$

$$(\because P_i = \delta_j \text{ (or) } P_j = \delta_j)$$

which can be written in the form of

$$\boxed{P_{ij} = P_{ij}' + P_i \Delta_j} \rightarrow (13)$$

where, P_i are constant

now $\boxed{P_{ij} = P_{ji}}$ and $\boxed{P_{ij}' = P_{ji}'}$, so that we must have

$$P_i \Delta_j = P_j \Delta_i \rightarrow (14) \quad \frac{P_i}{P_j} = \frac{\Delta_i}{\Delta_j}$$

But $P_i / P_j = \frac{\Delta_i}{\Delta_j} = \frac{\delta_i}{\delta_j} = \mu$, so that $P_i = \mu \delta_i$,

where μ is a constant, and from (6)

$$\Delta_j = \mu \delta_j / P_i \quad \Rightarrow P_i \Delta_j = \lambda \delta_i \delta_j \quad \Rightarrow P_i \Delta_j = \underline{\mu} \delta_j$$

$$\boxed{P_i \Delta_j = \lambda \delta_i \delta_j}$$

where, $\boxed{\lambda = \mu / P}$ is a constant

14) $\Rightarrow P_{ij} = P_{ij}' + \lambda \delta_i \delta_j \rightarrow$ (15)

Sub 15) in (1) we get

$$\lambda \sum_{i,j=1}^{3} a_{ij} \delta_i \delta_j + \sum_{i,j=1}^{3} a_{ij} P_{ij}' + \sum_{i=1}^{3} b_i P_i + cu = 0 \rightarrow (16)$$

This equ has a solu for λ unless the characteristic function

$$\phi = \sum_{i,j}^{3} a_{ij} \delta_i \delta_j \rightarrow (17)$$

Vanishes ie) Unless f is such that

$$\sum_{i,j}^{3} a_{ij} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} = 0 \rightarrow (18)$$

when $\phi \neq 0$, we can solve equ (16) for λ, so that there all the second derivatives can be found and the procedure repeated for higher derivatives of u on s. The complete solution can then be found by a Taylor expansion.

The equ 18) ie) $\phi = 0$ defines the charac-teratic surfaces. If $f(x_1, x_2, x_3)$ is a soln of (18) then the direction raties $(\delta_1, \delta_2, \delta_3)$ of the normal at any point of the surface satisfy

$$\sum_{i,j} a_{ij} \delta_i \delta_j = 0 \longrightarrow (19)$$

Which is the equ of a cone

Therefore at any point in space the

normals to all possible characteristic
Surfaces through the point lie on a cone.

The planes perpendicular to these normals therefore also envelop a cone. This cone is called the characteristic cone through the point the characteristic cone at a point therefore touches all the characteristic surfaces at the point

now according to equs (8) the cauchy characteristics of the first Order equation (18) are defined by the eqns

$$\frac{dx_i}{\partial \phi / \partial \xi_i} = \frac{-d\xi_i}{\partial \phi / \partial x_i} \quad i = 1, 2, 3 \ldots$$

The integrals of these eqns satisfying the correct initial conditions at a given point represent lines which are called the bicharacteristics of the equ (1). These lines in turn generate a Surface called a conoid, which reduces, in the case of constant a_{ij}'s to the characteristic, cone.

we may use the quadratic form (17) to classify Second Order eqns in three independent Variables.

a) If ϕ is positive definite in the ξ's at the point $p(x_1^\circ, x_2^\circ, x_3^\circ)$, The charateristic cones is conoid are imaginary and we say that the eqn is elliptic at p.

b) If ϕ is indefinite, The characteristic cones are real, and we say that the eqn is Hyperbolic at the point

c) If the determinant $\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix}$ of the form ϕ Vanishes, we say that the equ is parabolic.

Example:-

classify the eqn,

a) $u_{xx} + u_{yy} = u_z$

Here $a_{11} = 1$, $a_{22} = 1$, $a_{33} = 0$.

$$\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$\therefore$ This equ is parabolic

$$\left\{ \therefore \begin{vmatrix} u_{xx} & u_{xy} & u_{xz} \\ u_{yx} & u_{yy} & u_{yz} \\ u_{zx} & u_{zy} & u_{zz} \end{vmatrix} \right.$$

b) $u_{xx} + u_{yy} = u_{xx}$.

Here $a_{11} = 1$, $a_{22} = 1$, $a_{33} = -1$

$$\begin{vmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = -1 < 0$$

The equ is hyperbolic.

$u_{xx} + u_{yy} + u_{zz} = 0$

Here $a_{11} = 0$, $a_{22} = 0$, $a_{33} = 0$

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 > 0$$

$\therefore$ The eqn is elliptic

$u_{xx} + 2u_{yy} - 2u_{xy} = 2u_{yz}$

$$u_{xx} + 2u_{yy} + u_{zz} - 2u_{xy} - 2u_{yz} = 0$$

$a_{11} = 0$, $a_{22} = 2$, $a_{33} = 1$, $a_{12} = -2$, $a_{23} = 0$

$$\begin{vmatrix} 1 & -2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{vmatrix} = 1(2) = 2 > 0$$

$\therefore$ The eqn is elliptic.

The soln of linear hyperbolic eqns:-

we shall briefly sketch the existence theorems for two types of initial conditions on the equs

$$\frac{\partial^2 z}{\partial x \, \partial y} = f(x, y, z, z_x, z_y)$$

For both kinds of initial condition it is assumed that the function $f(x, y, z, p, q)$ is continuous at all points of a region R defined by $\alpha < x < \beta$, $\gamma < y < \delta$ for all values of x, y, z, p, q concerned and that it satisfies a lipschitz condition.

$$|F(x,y,z_2,p_2,q_2) - f(x,y,z_1,p_1,q_1)|$$
$$\leq M\{|z_2-z_1| + |p_2-p_1| + |q_2-q_1|\}.$$

in all bounded subrectangles $\mathfrak{R}$ of R. now State (without proof) two existence theorem.

Theorem 1:-

Initial Conditions of the first kind:-

If $F(x)$ & $G(x)$ are defined in the open intervals (α,β), (γ,δ) respectively & have continuous first derivatives & if (ϵ,η) is a point inside R such that $F(\epsilon) = G(\eta)$ Then the gn' diff equ has atleast one integral $z = \phi(x,y)$ in R such that

$$\phi(x,y) = \begin{cases} F(x) & \text{when } y=\eta \\ G(x) & \text{when } x = \epsilon \end{cases}$$

Theorem 2:-

Initial Conditions of the Second kind:.

let c be a space curve defined by $x = x(\lambda)$, $y=y(\lambda)$, $z = z(\lambda)$ interms of a single parameter λ & also let C_0 be the projection of C_1 on the xy plane. If we are given (x,y,z,p,q) along a strip C_1 then the gn' equ has a integral which takes on the given values of z,p,q along the curve C_0. This integral exists at every point of the region R, which is defined as the smallest rectangle completely enclosing the curve C_0.

Riemann method of soln of general linear hyperbolic equ of the second order.

Assume that the equ has already been reduced to canonical form,

$$L(z) = f(x,y) \rightarrow ①$$

where L denotes the linear operator.

$$L = \frac{\partial^2}{\partial x \partial y} + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c \rightarrow ②$$

where a, b, c are functions of x & y only.

let w be another function with continuous derivatives of the first order.

Again, let M be another operator defined by the relation.

$$Mw = \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial(aw)}{\partial x} - \frac{\partial(bw)}{\partial y} + cw \rightarrow ③$$

The operator M defined by (4) is called the adjaint operator to the operator L.

$$\therefore wLz - zMw$$

$$= w\left(\frac{\partial^2 z}{\partial x \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + c\right) -$$

$$z\left(\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial(aw)}{\partial x} - \frac{\partial(bw)}{\partial y} + cw\right)$$

$$= \left(w \frac{\partial^2 z}{\partial x \partial y} - z \frac{\partial^2 w}{\partial x \partial y} \right) + \left(wa \frac{\partial z}{\partial x} + z \frac{\partial (aw)}{\partial x} \right)$$

$$\left(wb \frac{\partial z}{\partial y} + z \frac{\partial (bw)}{\partial y} \right)$$

$$= \frac{\partial}{\partial y}\left(w \frac{\partial z}{\partial x} \right) - \frac{\partial}{\partial x}\left(z \frac{\partial w}{\partial y} \right) + \frac{\partial (awz)}{\partial x} + \frac{\partial (bwz)}{\partial y}$$

$$= \frac{\partial}{\partial x}\left(awz - z \frac{\partial w}{\partial y} \right) + \frac{\partial}{\partial y}\left(bwz + w \frac{\partial z}{\partial x} \right)$$

$$= \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} \rightarrow (4)$$

where
$$\left. \begin{aligned} U &= awz - z \frac{\partial w}{\partial y} \\ V &= bwz + w \frac{\partial z}{\partial x} \end{aligned} \right\} \rightarrow (5)$$

now if 'c' is a closed curve enclosing an area S.

Then,

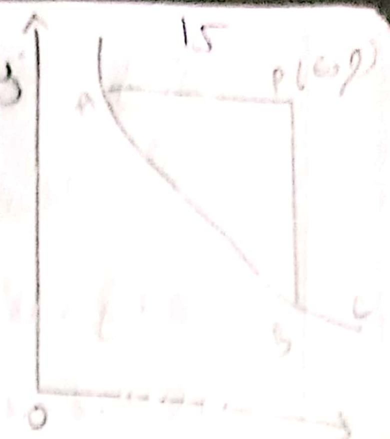
$$\iint_S (wLz - zMw)\,dx\,dy = \iint_S \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) dx\,dy \quad (\text{by } (4))$$

$$= \int_C (U\,dy - V\,dx) \rightarrow (6)$$

by green's Theorem

Assume that the Values of z and $\frac{\partial z}{\partial x}$ or $\frac{\partial z}{\partial y}$ are Prescribed along a curve c in the xy plane. and futher assume that we are required to determine the soln of (1) at the point $P(\xi, \eta)$ agreeing

With these boundary conditions, draw PA, PB ∥ to x-axis & y-axis and cutting the curve c in the points A & B respectively. The closed circuit PABP can be taken as the closed curve c'.



Then (5) reduces to

$$\iint_S (wLz - zMw)\,dx\,dy = \int_{AB} (Udy - Vdx) + \int_{BP} (Udy - Vdx) + \int_{PA} (Udy - Vdx)$$

$$= \int_{AB} (Udy - Vdx) + \int_{BP} Udy - \int_{PA} Vdx \rightarrow (7)$$

where we have used the facts that along BP

x = constant so that $dx = 0$ & along PA.

y = constant so that $dy = 0$

now,

$$\int_{PA} Vdx = \int_{PA} \left(bwz + w\frac{\partial z}{\partial x}\right) dx \quad (\text{by } 5)$$

$$= \int_{PA} bwz\,dz + \int_{PA} w\frac{\partial z}{\partial x}\,dx$$

$$= \int_{PA} bwz\,dz + [wz]_P^A - \int_{PA} z\frac{\partial w}{\partial x}\,dx.$$

Scanned with CamScanner

$\int$ing by parts.

$$= [wz]_A - [wz]_P + \int_{PA} z\left(bw - \frac{\partial w}{\partial x}\right) dx \rightarrow (8)$$

Using (6) & (8), (7) becomes

$$\iint_S (wLz - zMw)\,dx\,dy = \int_{AB} (u\,dy - v\,dx) +$$

$$\int_{Bp} \left(awz - z\frac{\partial w}{\partial y}\right) dy - [wz]_A + [wz]_P - \int_{PA} z\left(bw - \frac{\partial w}{\partial x}\right) dx$$

$$\therefore [wz]_P = [wz]_A + \int_{PA} z\left(bw - \frac{\partial w}{\partial x}\right) dx - \int_{Bp} z\left(aw - \frac{\partial w}{\partial y}\right)$$

$$- \int_{AB} (u\,dy - v\,dx) + \iint_S (wLz - zMw)\,dx\,dy \rightarrow (9)$$

So far the function w has been arbitrary. Suppose now that we choose a function $w(x, y; \epsilon, \eta)$

which has the properties.

i) $Mw = 0$

ii) $\frac{\partial w}{\partial x} = b(x, y)w$ when $y = \eta$

iii) $\frac{\partial w}{\partial y} = a(x, y)w$ when $x = \epsilon$

iv) $w = 1$ when $x = \epsilon$, $y = \eta$

Such a function is called a Green's function for the problem or sometimes a Riemann-Green function. Since also $Lz = f$, we find that.

$$[z]_P = [wz]_A - \int_{AB} (u\,dy - v\,dx) + \iint_S wLz\,dx\,dy.$$

$$= [wz]_A - \iint_{AB} \left(awz - z\frac{\partial w}{\partial y}\right)dy + \int_{AB} \left(bwz + \frac{\partial w}{\partial x}\right)dx$$

$$+ \iint_S (wf)\,dx\,dy \quad [\text{using } (1)\ \&\ (5)]$$

$$= [wz]_A - \int_{AB} wz(a\,dy - b\,dx) + \int_{AB} \left(z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx\right)$$

$$+ \iint_S wf\,dx\,dy \rightarrow (10).$$

Equ(10) may be used to determined the value of z at the point P. when $\frac{\partial z}{\partial x}$ is prescribed along the curve C.

Suppose in plane of prescribed value of $\frac{\partial z}{\partial x}$ we are now gives a prescribed value of $\frac{\partial z}{\partial y}$.

Then we make use of the following relation.

$$\int_{AB} d(wz) = \int_{AB} \left(\frac{\partial(wz)}{\partial x} dx + \frac{\partial(wz)}{\partial y} dy \right).$$

$$\Rightarrow 0 = [wz]_B - [wz]_A - \int_{AB} \left(\frac{\partial(wz)}{\partial x} dx + \frac{\partial(wz)}{\partial y} dy \right) \rightarrow (11)$$

Adding the corresponding sides of (10) & (11) we get,

$$[z]_P = [wz]_B - \int_{AB} wz\,(a\,dy - b\,dx) + \int_{AB} \left(z\frac{\partial w}{\partial y} dy + w\frac{\partial z}{\partial y} \right.$$

$$\left. - \int_{AB} \left(\frac{\partial(wz)}{\partial x} dx + \frac{\partial(wz)}{\partial y} dy \right) + \iint_S (wF)\,dx\,dy \right.$$

$$= [wz]_B - \int_{AB} wz\,(a\,dy - b\,dx) - \iint_{AB} \left(z\frac{\partial w}{\partial x} dx + w\frac{\partial z}{\partial y} d \right.$$

$$+ \iint_S (wf)\,dx\,dy \rightarrow (12)$$

Eqn (12) may be used to determine by z at the Point P when $\frac{\partial z}{\partial y}$ is prescribed along the curve c.

Finally by adding (10) & (12) we get the following Symmetrical result which can be used. To find Value of z at the point P when both $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ are prescribed along the curve c.

$$[z]_P = \frac{1}{2}\left\{ [wz]_A + [wz]_B \right\} - \int_{AB} wz\,(a\,dy - b\,dx):$$

$$\iint_S (wf)\,dx\,dy - \frac{1}{2}\int_{AB} w\left(\frac{\partial z}{\partial y}dy - \frac{\partial z}{\partial x}dx\right)$$

$$-\frac{1}{2}\int_{AB} z\left(\frac{\partial w}{\partial x}dx - \frac{\partial w}{\partial y}dy\right)$$

$\to$ (13)

By means of whichever of the formulas (10), (12) & (13) is suitable, we may determine the soln of (1) at any point in terms of the prescribed values of z, $\frac{\partial z}{\partial x}$ or l and $\frac{\partial z}{\partial y}$ along a given curve C.

Note:-

$$x\left(\frac{1}{x+\eta} - \frac{\partial x}{1+x^2} + \frac{1}{x}\right) = \frac{x}{x+\eta} - \frac{\partial x^2}{1+x^2} + 1$$

$$= \frac{(x+\eta-x)}{x+\eta} - \frac{\partial(1+x^2)}{1+x^2} + 1$$

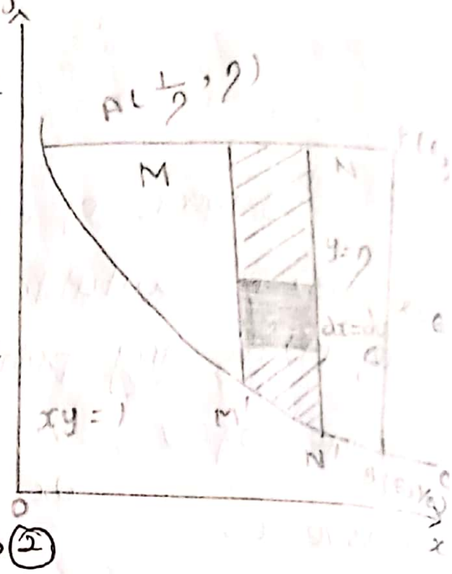
$$= \frac{-\eta}{x+\eta} + \frac{2}{1+x^2}$$

Find the sol valid when $x, y > 0$, $xy > 1$ of the equ $\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{x+y}$ such that $z = 0$, $P = \frac{\partial y}{x+y}$ on the hyperbola $xy = 1$.

Soln :-

Comparing the gn' equ with $L(z) = f(x,y)$ we have, $a = b = c = 0$ & $f(x,y) = 1/_{x+y}$.

Hence the adjoint Operator M of the Operator L is the given by $M = \frac{\partial^2}{\partial x \partial y}$



So Green's function can be taken as $w = 1 \rightarrow (1)$

In the present problem, the values of z & $\frac{\partial z}{\partial x} = p$ are gn' by

$$z = 0, \quad \frac{\partial z}{\partial x} = (Py) / (x+y) \rightarrow (2)$$

along the curve c, which is hyperbola

$$xy = 1 \rightarrow (3)$$

Then we wish to finish the soln of gn' equations at the point $P(\xi, \eta)$ agreeing with those boundaring conditions. Through P we draw PA $\parallel^{le}$ to the x-axis & cutting $xy=1$ in the point A & PB $\parallel^{le}$ to the y-axis & cutting $xy=1$ in B. Then region enclosed by $xy=1$, $x = \xi$, $y = \eta$ is denoted by S.

now, we know that (refer equ (1))

$$[z]_P = [wz]_A - \int_{AB} wz(a\,dy - b\,dx) +$$

$$\int_{AB} \left(z \frac{\partial w}{\partial y} dy + w \frac{\partial w}{\partial x} dx\right) + \iint_C w f \, dx\, dy$$

(or)

$$[z]_P = \int_{AB} \frac{\partial y}{x+y} dx = 2 \int_A^B \frac{xy}{x^2 + xy} dx .$$

$$= 2 \int_{1/\eta}^{\epsilon} \frac{1}{1+x^2}\, dx$$

$$= 2 \left\{ \tan^{-1}\epsilon - \tan^{-1}(1/\eta) \right\}$$

and $$\iint_{e} \frac{1}{1+x}\, dx\, dy = \int_{x=1/\eta}^{\epsilon} \left\{ \int_{y=\frac{1}{x}}^{\eta} \frac{1}{x+y}\, dy \right\} dx \quad \text{(6)}$$

Since to integrate over area bounded by PABP. we first integrate along the strip MNN'M' by fixing x & varying y from $y = 1/x$ at M' to $y = \eta$ at M & then integrate from A to P (keeping y fixed) by varying x from $x = 1/\eta$ to $x = \epsilon$.

Evaluating the double integral on R.H.S of (6) by the usual Rule, we have.

$$\iint_{S} \frac{1}{x+y}\, dx\, dy = \int_{1/\eta}^{\epsilon} \Big[\log(x+y) \Big]_{1/x}^{\eta} dx$$

$$= \int_{1/\eta}^{\epsilon} \left[\log(x+\eta) - \log(x + 1/x) \right] dx$$

$$= \int_{1/\eta}^{\epsilon} \left\{ \log(x+\eta) - \log(1+x^2) + \log x \right\} dx$$

$$= \Big[\left\{ \log(x+\eta) - \log(1+x^2) + \log x \right\} x \Big]_{1/\eta}^{\epsilon}$$

$$-\int_{1/\eta}^{\epsilon} x\left(\frac{1}{x+\eta} - \frac{2x}{1+x^2} + \frac{1}{x}\right)dx.$$

$$= \epsilon\{\log(\epsilon+\eta) - \log(1+\epsilon^2) + \log\epsilon\} - 1/\eta\{\log(1/\eta+\eta)$$

$$-\log(1+1/\eta^2) + \log 1/\eta\} - \int_{1/\eta}^{\epsilon}\left(\frac{2}{1+x^2} - \frac{\eta}{x+\eta}\right)dx.$$

$$= \epsilon\log\frac{\epsilon(\epsilon+\eta)}{1+\epsilon^2} - 2\left[\tan^{-1}x - \eta\log(x+\eta)\right]_{1/\eta}^{\epsilon}$$

$$= \epsilon\log\frac{\epsilon(\epsilon+\eta)}{1+\epsilon^2} - 2\left[\tan^{-1}\epsilon - \tan^{-1}1/\eta\right] + \eta\log\frac{\eta(\epsilon+\eta)}{1+\eta^2} \rightarrow (7)$$

Using (5) & (7), (4) reduces to

$$[z]_P = \epsilon\log\frac{\epsilon(\epsilon+\eta)}{1+\epsilon^2} + \eta\log\frac{\eta(\epsilon+\eta)}{1+\eta^2} \rightarrow (8)$$

Replacing ϵ & η by x & y respectively in (8) The value of z, soln of the gn equ at any point (x,y) is gn' by

$$z = x\log\frac{x(x+y)}{1+x^2} + y\log\frac{y(x+y)}{1+y^2}.$$

2. P.T for the equ $(\partial^2 z/\partial x\partial y) + (z/4) = 0$ The green's function is $w(x,y;\epsilon,\eta) = J_0\sqrt{(x-\epsilon)(y-\eta)}$ whose $J_0(z)$ denotes Bessel's functions of the first kind of Order zero.

Soln :

Here $L(Z) = \left(\partial^2 z / \partial x \partial y \right) + (z/4) = 0 \rightarrow (1)$

$$L = \frac{\partial^2}{\partial x \partial y} + a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c$$

$$= \frac{\partial^2}{\partial x \partial y} + \frac{z}{4}$$

$\Rightarrow a = 0 \; , \; b = 0 \; , \; c = z/4 \rightarrow (2)$

So the adjacent Operator M to the Operator L is gn' by,

$$M = \frac{\partial^2}{\partial x \partial y} + \frac{1}{4} \rightarrow (3)$$

gn'

$$\omega = J_0 \sqrt{(x-\epsilon)(y-\eta)} \rightarrow (4)$$

$$(4) \Rightarrow \frac{\partial \omega}{\partial x} = \frac{\sqrt{(y-\eta)}}{2\sqrt{(x-\epsilon)}} J_0' \rightarrow (5)$$

$$\frac{\partial^2 \omega}{\partial y \partial x} = \frac{1}{4} \frac{1}{\sqrt{(x-\epsilon)(y-\eta)}} J_0' + \frac{\sqrt{(y-\eta)}}{2\sqrt{(x-\epsilon)}} \frac{\sqrt{(x-\epsilon)}}{2\sqrt{(y-\eta)}} J_0''$$

$$= \frac{1}{4} \left\{ J_0'' + \frac{1}{(x-\epsilon)(y-\eta)} J_0' \right\} \rightarrow (6)$$

So (3) & (6)

$$\Rightarrow MW = \frac{1}{4} \left\{ J_0'' + \frac{1}{\sqrt{(x-\epsilon)(y-\eta)}} J_0' + J_0 \right\} \rightarrow (7)$$

now, Bessel equ of Order zero is gn' by

$x^2 y'' + x y' + x^2 y = 0$ (or) $y'' + (1/x) y' + y = 0 \rightarrow (8)$

Since, $y = J_0 \left\{ \sqrt{(x-\epsilon)(y-\eta)} \right\}$ is a soln of (8)

we get,

$$J_0'' + \frac{1}{\sqrt{(x-\epsilon)(y-\eta)}} J_0' + J_0 = 0 \quad (or) \quad M(\omega) = 0$$

(by (7))

$\hookrightarrow$ (9)

Again (5) $\Rightarrow$ $\left(\frac{\partial \omega}{\partial x}\right) = 0 = b\omega$ when $y = \eta$ $(\because b = 0)$

$\hookrightarrow$ (10)

$||^{ly}$

$\left(\frac{\partial \omega}{\partial y}\right) = 0 = a\omega$ when $x = \epsilon$ $(\because a = 0) \rightarrow$ (11)

finally, when $x = \epsilon$, $y = \eta$, $\omega = J_0(0) = 1 \rightarrow$ (12)

Since ω satisfies four properties. (9), (10), (11), & (12) of a green's fnc it follows that ω must be a green's fnc of the gn equ (1).

3. P.T for the eqn $\frac{\partial^2 z}{\partial x \partial y} + \frac{2}{x+y}\left(\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\right) = 0$

The green's function is

$$\omega(x, y; \epsilon, \eta) = \frac{(x+y)\left\{2xy + (\epsilon - \eta)(x+y) + 2\epsilon\eta\right\}}{(\epsilon+\eta)^3}$$

Hence find the soln of the diff eqn which satisfies the conditions $z = 0$, $\frac{\partial z}{\partial x} = 3x^2$ on $y = x$.

Soln:-

Compute the gn' equ with $L(z) = f(x, y)$

where,

$$L = \frac{\partial^2}{\partial x \partial y} + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c$$

we find,

$a = 2/(x+y)$, $b = 2/(x+y)$, $c = 0$, $f(x, y) = 0$

$\hookrightarrow$ (1)

So the adjacent Operator M to the Operator L is gn' by,

$$Mw = \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial}{\partial x}\left(\frac{2}{x+y}\, w\right) - \frac{\partial}{\partial y}\left(\frac{2}{x+y}\, w\right) \to (2)$$

gn'

$$w(x, y; \epsilon, \eta) = \frac{(x+y)\{2xy + (\epsilon-\eta)(x-y) + 2\epsilon\eta\}}{(\epsilon+\eta)^3} \to (3)$$

$$(3) \Rightarrow \frac{\partial w}{\partial x} = \frac{2xy + (\epsilon-\eta)(x-y) + 2\epsilon\eta + (x+y)(2y+\epsilon-\eta)}{(\epsilon+\eta)^3} \to (4)$$

$$(3) \Rightarrow \frac{\partial w}{\partial y} = \frac{2xy + (\epsilon-\eta)(x-y) + 2\epsilon\eta + (x+y)(2y-\epsilon+\eta)}{(\epsilon+\eta)^3} \to (5)$$

$$(5) \Rightarrow \frac{\partial^2 w}{\partial x \partial y} = \frac{2y + \epsilon - \eta + 2x - \epsilon + \eta + 2(x+y)}{(\epsilon+\eta)^2}$$

$$= \frac{4(x+y)}{(\epsilon+\eta)^3} \to (6)$$

Using (6) & (3), (2) reduces to

$$Mw = \frac{4(x+y)}{(\epsilon+\eta)^3} - 2\frac{\partial}{\partial x}\left\{\frac{2xy + (\epsilon-\eta)(x-y) + 2\epsilon\eta}{(\epsilon+\eta)^3}\right\}$$

$$-2\frac{\partial}{\partial y}\left\{\frac{2xy + (\epsilon-\eta)(x-y) + 2\epsilon\eta}{(\epsilon+\eta)^3}\right\}$$

$$= \frac{4(x+y)}{(\xi+\eta)^3} - \frac{2(2y+\xi-\eta)+2(2x-\xi+\eta)}{(\xi+\eta)^3} = 0 \rightarrow (7)$$

At $y=\eta$,

$$\frac{\partial w}{\partial x} = \frac{2x\eta+(\xi-\eta)(x-\eta)+2\xi\eta+(\xi+\eta)(2\eta+\xi-\eta)}{(\xi+\eta)^3}$$

(by (4))

$$= \frac{2\{x(\xi+\eta)+\eta^2+\xi\eta\}}{(\xi+\eta)^3} \rightarrow (8)$$

From (1) & (3).

$$bw = \frac{2\{2xy+(\xi-\eta)(x-y)+2\xi\eta\}}{(\xi+\eta)^3} \rightarrow (9)$$

$$= \frac{2\{x(\xi+\eta)+\eta^2+\xi\eta\}}{(\xi+\eta)^3} \rightarrow (10)$$

from (8) & (10), $\frac{\partial w}{\partial x} = bw$ when $y=\eta$ $\rightarrow (11)$

$||^{ly}$ $\frac{\partial w}{\partial y} = aw$, when $x=\xi$ $\rightarrow (12)$

from (3), when $x=\xi$, $y=\eta$

$$w = \frac{(\xi+\eta)\{2\xi\eta+(\xi-\eta)^2+2\eta\}}{(\xi+\eta)^3} - 1 \rightarrow (13)$$

Since, w satisfies four properties (7), (11), (12), & (13) of a greens function it follows that w must be a greens fnc of the gn' equ.

To find the soln of the eqn;

In the present problem the values of z & $\frac{\partial z}{\partial x} = p$ are gn' by

$$z = 0, \quad \frac{\partial z}{\partial x} = 3x^2 \rightarrow (14)$$

along the line AB is $y = x \rightarrow (15)$

Then we wish to find the soln of gn' equ at the point $P(\epsilon, \eta)$ agreeing with these boundary conditions. Through P we draw PA $\parallel^{le}$ to the x-axis & cutting $y = x$ in the point A & PB $\parallel^{le}$ to the y-axis and cutting $y = x$ in B. Then Δ^{le} region enclosed by straight lines $y = x$, $y = \eta$ & $x = \epsilon$ is denoted by S.

Then we know that (refer 10))

$$[z]_P = [wz]_A - \int_{AB} wz\,(a\,dy - b\,dx) + \int_{AB} \left(z \frac{\partial w}{\partial y} dy + w \frac{\partial z}{\partial x} dx\right) + \iint_S wf\,dx\,dy \rightarrow (16)$$

now on line AB,

$$\text{From (3)} \Rightarrow w = \frac{4x(x^2 + \epsilon\eta)}{(\epsilon + \eta)^2} \quad \text{as } y = x \rightarrow (17)$$

using (1), (14), (17) reduces to

$$[z]_P = \int_A^B \frac{4x(x^2 + \epsilon\eta)}{(\epsilon + \eta)^3} 3x^2\,dx$$

$$= \frac{12}{(\epsilon + \eta)^3} \int_\eta^\epsilon (x^5 + \epsilon\eta x^3)\,dx.$$

$$= \frac{12}{(\epsilon + \eta)^3} \left[\frac{x^6}{6} + \epsilon\eta \frac{x^4}{4}\right]_\eta^\epsilon$$

$$= \frac{12}{(\epsilon+\eta)^3}\left[\frac{\epsilon^6-\eta^6}{6}+\frac{\epsilon\eta}{4}(\epsilon^4-\eta^4)\right]$$

$$= (\epsilon+\eta)^{-3}\left\{2(\epsilon^3+\eta^3)(\epsilon^3-\eta^3)+3\epsilon\eta(\epsilon^2-\eta^2)(\epsilon^2+\eta^2)\right\}$$

$$= (\epsilon+\eta)^{-3}\left\{2(\epsilon^3+\eta^3)(\epsilon-\eta)(\epsilon^2+\eta^2+\epsilon\eta)+3\epsilon\eta(\epsilon-\eta)(\epsilon-\eta)(\epsilon^2+\eta^2)\right\}$$

$$= (\epsilon+\eta)^{-3}(\epsilon-\eta)\left\{2(\epsilon^3+\eta^3)(\epsilon^2+\eta^2)+2\epsilon\eta(\epsilon^3+\eta^3)-3\epsilon\eta(\epsilon+\eta)(\epsilon^2+\eta^2)\right\}$$

$$= (\epsilon+\eta)^{-3}(\epsilon-\eta)\left\{2(\epsilon^3+\eta^3)(\epsilon^2+\eta^2)+6\epsilon\eta(\epsilon+\eta)(\epsilon^2+\eta^2)-3\epsilon\eta(\epsilon+\eta)(\epsilon^2+\eta^2)+2\epsilon\eta(\epsilon^3+\eta^3)\right\}.$$

$$= (\epsilon+\eta)^{-3}(\epsilon-\eta)\left[2(\epsilon^2+\eta^2)\left\{\epsilon^3+\eta^3+3\epsilon\eta(\epsilon+\eta)\right\}-\epsilon\eta\left\{3(\epsilon+\eta)(\epsilon^2+\eta^2)-2(\epsilon^3+\eta^3)\right\}\right]$$

$$= (\epsilon+\eta)^{-3}(\epsilon-\eta)\left\{2(\epsilon^2+\eta^2)(\epsilon+\eta)^3-\epsilon\eta(\epsilon+\eta)^3\right\}$$

$$= (\epsilon-\eta)(2\epsilon^2+2\eta^2-\epsilon\eta).$$

(or) $[z]_P = 2\epsilon^3+3\epsilon\eta^2-3\epsilon^2\eta-2\eta^3 \rightarrow (19)$

Replacing ϵ & η by x respectively in (18) the value of z.

(e) Soln of the gn' equ at any point (x,y) is gn' by,

$$z = 2x^3+3xy^2-3x^2y-2y^3.$$

Vibrations of a String of finite length (Method of Seperation of Variables):.

let us consider the following problem

$$y_{tt} - c^2 y_{xx} = 0, \quad 0 < x < L, \quad t > 0 \rightarrow (1)$$

$$y(x,0) = f(x) \quad , 0 \leq x \leq L \rightarrow (2)$$

$$y_t(x,0) = g(x) \quad , 0 \leq x \leq L \rightarrow (3)$$

$$y(0,t) = y(L,t) = 0, \quad t > 0 \rightarrow (4)$$

So, f & g are the initial displacement & velocity respectively. let us assume the soln. of equ (1) in the form.

$$y(x,t) = X(x) T(t)$$

Then,
$$\frac{X''}{X} = \frac{T''}{c^2 T}$$

Observe that the right hand side is a function of t alone while the left hand side is fnc of x above, Hence each of them must be constant & equal to say λ.

Therefore,

$$X'' - \lambda X = 0$$

$$T'' - c^2 \lambda T = 0$$

From (4) we have,

$$y(0,t) = X(0) T(t) = 0 \quad \forall \; t \geq 0.$$

Since,

$T(t) \neq 0$ we get $X(0) = 0$

11^{ly}

$y(L,t)=0 \Rightarrow X(L)=0$

Therefore we have, $X'' = \lambda X$

$X(0)=0 \Rightarrow X(l)$

which is an eigen value problem.

Case i):-

$\lambda>0$ the soln of the above eigen value Problem is

$$X(x) = Ae^{-\sqrt{\lambda}x} + Be^{\sqrt{\lambda}x}$$

where A & B are arbitrary constants To satisfy the boundary conditions.

$$A+B = Ae^{-\sqrt{\lambda}L} + Be^{\sqrt{\lambda}L} = 0.$$

The only possibility is $A=B=0$. Hence there is no eigen value $\lambda>0$

Case ii):-

$\lambda=0$. In this case the soln of the eigen value problem is of the form

$$X(x) = A+Bx$$

The boundary conditions imply that $A=0$ & $A+Bl=0$

$\therefore A=B=0$. Hence $\lambda=0$ is not a eigen value.

Case iii):-

$\lambda<0$, The soln in this case is of the form

$$X(x) = A\cos\sqrt{-\lambda}x + B\sin\sqrt{-\lambda}x$$

The condition $X(0)=0$ implies that $A=0$ & $X(l)=0$ implies that $B\sin\sqrt{-\lambda}l = 0$.

As $B=0$ gives only a trivial soln, we must have $\sin\sqrt{-\lambda}\,l = 0$ for a non-trivial soln.

i.e) $\sqrt{-\lambda}\,l = n\pi$, $n = 1, 2, 3 \ldots$

$$-\lambda_n = \left(\frac{n\pi}{l}\right)^2$$

These λ_n are called eigen values and the fnc $\sin(n\pi x/l)$ are the corresponding eigen fnc.

$$\therefore \quad X_n = B_n \sin(n\pi x/l).$$

For each, λ_n we have,

$$T_n(t) = C_n \cos\left(\frac{n\pi c t}{l}\right) + D_n \sin\left(\frac{n\pi c t}{l}\right)$$

where C_n & D_n are arbitary constants

Hence,

$$y_n(x,t) = \left(a_n \cos\left(\frac{n\pi c t}{l}\right) + b\sin\left(\frac{n\pi c t}{l}\right)\right)$$

$$\sin\left(\frac{n\pi t}{l}\right)$$

is a soln of equ ① & satisfies the boundary conditions (4).

If y_1 & y_2 are two soln of a linear homogeneous equ satisfying linear homogeneous boundary conditions, Then $y_1 + y_2$ is also a solu of that equ & satisfies the same boundary conditions.

This is called the principle of super position observe that eqn ①, and the boundary conditions (4) are linear & homogeneous.

∴ ~~This~~ The principle of superposition the series $y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) \rightarrow (5)$

If it converges is also a soln of equ (1) satisfying the boundary condition (4). In fact we assume that term by term diff is possible. and that the derived series is also convergent. now, a_n & b_n must be choosen such that y as gn' equ (5) satisfies the initial conditions (2) & (3).

The initial conditions $y(x,0) = f(x)$. gives,

$$f(x) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{l}\right), \quad 0 < x < l \rightarrow (6)$$

11^{ly}, The initial conditions $y_t(x,0) = g(x)$ gives

$$g(x) = \sum_{n=1}^{\infty} b_n \left(\frac{n\pi c}{l}\right) \sin\left(\frac{n\pi x}{l}\right), \quad 0 < x < l \rightarrow (7)$$

Hence, a_n & b_n are gn' by the fourier co.eff of the half range sine series of $f(x)$ & $g(x)$ respectively,

$$\therefore a_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx \rightarrow (8)$$

and,

$$b_n = \frac{2}{n\pi c} \int_0^l g(x) \sin\left(\frac{n\pi x}{l}\right) dx \text{ --- } (9)$$

we have derived the same result earlier in d' Alembert's soln.

The method of integral transforms:-

Suppose we have to determine a fnc 'u' which depends on the independent Variables $x_1, x_2 \ldots x_n$ and whose behaviour is determined by the linear PDE.

$$a(x_1)\frac{\partial^2 u}{\partial x_1^2} + b(x_1)\frac{\partial u}{\partial x_1} + c(x_1)u + Lu = f(x_1, x_2, \ldots x_n) \to ①$$

in which L is a Linear differential Operator in the Variables $x_2 \ldots x_n$ and the range of Variation of x_1 is $\alpha \leq x_1 \leq \beta$.

$$\bar{u}(\epsilon, x_2, \ldots x_n) = \int_\alpha^\beta u(x_1, x_2, \ldots x_n)\, k(\epsilon, x_1)\,dx_1 \to ②$$

Then an integration by parts shows that

$$\int_\alpha^\beta \left\{ a(x_1)\frac{\partial^2 u}{\partial x_1^2} + b(x_1)\frac{\partial u}{\partial x_1} + c(x_1)u \right\} k(\epsilon, x_1)\,dx_1$$

$$= g(\epsilon, x_2 \ldots x_n) + \int_\alpha^\beta u\left\{ \frac{\partial^2}{\partial x_1^2}(ak) - \frac{\partial}{\partial x_1}(bk) + ck \right\} dx_1$$

where,

$$g(\epsilon, x_2 \ldots x_n) = \left[a\frac{\partial u}{\partial x_1} k(\epsilon, x_1) + u\left\{ bk - \frac{\partial}{\partial x_1}(ak) \right\} \right]_\alpha^\beta$$

let $\bar{U} = \{\epsilon, x_2 \ldots x_n\} = \int_\alpha u(x_1, x_2 \ldots x_n) k(\epsilon, x_1)\,dx_1$

for that consider,

Elementary Solu. of Laplace Equation :-

Let ψ be the functions given by,

$\psi = q / (\bar{r} - \bar{r}') = q / \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$

$$\psi = q \{ (x-x')^2 + (y-y')^2 + (z-z')^2 \}^{-1/2} \longrightarrow ①$$

where q is a constant & (x', y', z') are the co-ordinates of the fixed points.

Diff !- equ ① w.r. to x,

$$\frac{\partial \psi}{\partial x} = (-1/2)\, q \{ (x-x')^2 + (y-y')^2 + (z-z')^2 \}^{-3/2} (2(x-x'))$$

$$= \frac{-q(x-x')}{\{[(x-x')^2 + (y-y')^2 + (z-z')^2]^{1/2}\}^3}$$

$$\frac{\partial \psi}{\partial x} = \frac{-q(x-x')}{|(\bar{r} - \bar{r}')|^3} \longrightarrow ②$$

$$\frac{\partial \psi}{\partial y} = \frac{-q(y-y')}{|(\bar{r} - \bar{r}')|^3} \longrightarrow ③$$

$$\frac{\partial \psi}{\partial z} = \frac{-q(z-z')}{|(\bar{r} - \bar{r}')|^3} \longrightarrow ④$$

Again Diff w.r. to x,

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{-q}{|\bar{r} - \bar{r}'|^3 (1-0) - q(x-x') \frac{\partial}{\partial x} \left[\frac{1}{|\bar{r} - \bar{r}'|^3} \right]}$$

$$= \frac{-q}{|\bar{r} - \bar{r}'|^3} - q(x-x') \frac{\partial}{\partial x} \left[(x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{-3/2}$$

$$= \frac{-q}{|\bar{r} - \bar{r}'|^3} - q(x-x')(-3/\cancel{2}) \left[(x-x')^2 + (y-y')^2 + (z-z')^2 \right]^{-5/2} \cancel{2}(x-x')$$

$$= \frac{-q}{|\bar{r}-\bar{r}'|^3} + \frac{3q(x-x')^2}{\{[(x-x')^2+(y-y')^2+(z-z')^2]^{1/2}\}^5}$$

$$\frac{\partial^2\varphi}{\partial z^2} = \frac{-q}{|\bar{r}-\bar{r}'|^3} + \frac{3q(z-z')^2}{|\bar{r}-\bar{r}'|^5} \longrightarrow (5)$$

(4)+(3)+(5) ⇒

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2} = \frac{-3q}{|\bar{r}-\bar{r}'|^3} + \frac{3q}{|\bar{r}-\bar{r}'|^5}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]$$

$$= \frac{-3q}{|\bar{r}-\bar{r}'|^3} + \frac{3q}{|\bar{r}-\bar{r}'|^5}|\bar{r}-\bar{r}'|^2$$

$$= \frac{-3q}{|\bar{r}-\bar{r}'|^3} + \frac{3q}{|\bar{r}-\bar{r}'|^3} = 0.$$

$$\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}+\frac{\partial^2\varphi}{\partial z^2} = 0$$

$\Rightarrow \nabla^2\varphi = 0$

$\Rightarrow \varphi$ is a solution.

Now, $\varphi = q/|\bar{r}-\bar{r}'|$ is a solu. & a true form for electrostatic potential corresponding to a space which apart from the isolated point (x', y', z') is empty of electric charge.

Let S be any sphere with centre (x', y', z') then $\int_S \frac{\partial\varphi}{\partial x}\, dS = -4\pi q \longrightarrow (*)$

$\varphi = q/|\bar{r}-\bar{r}'|$ is a soln of Laplace equation correspond to an electric charge $+q$. (By Gauss theorm, & eq

Let $q_1, q_2, \dots q_i, q_{i+1}, \dots q_n$ electric charges with position vector $\bar{r}_i$ $(i = 1, 2, \dots n)$ then,

$\psi = \sum_{i=1}^{n} q_i / |\bar{r} - \bar{r}_i|$ is a solu. of laplace equation

Assume that two electric charges $+q$ and $-q$ are located every closely to each other at point $\bar{r}'$ and $\bar{r}' + \delta\bar{r}'$ where $\delta\bar{r}' = (l, m, n)$ a for this distribution, laplace equation have a sol. of the form.

$$\psi = \frac{-q}{|\bar{r} - \bar{r}'|} + \frac{q}{|\bar{r} - \bar{r}' + \delta\bar{r}'|} \longrightarrow (6)$$

Consider,

$$\frac{1}{|\bar{r} + (-\bar{r}') + \delta\bar{r}'|} = \frac{1}{|\bar{r} - \bar{r}'|} + \frac{l(x-x') + m(y-y') + n(z-z')}{|\bar{r} - \bar{r}'|^3} a + o(a^2) \rightarrow (7)$$

Let (7) in (6) we get,

$$\psi = \cancel{\frac{-q}{|\bar{r} - \bar{r}'|}} + \cancel{\frac{q}{|\bar{r} - \bar{r}'|}} + q\frac{[l(x-x') + m(y-y') + n(z-z')]a}{|\bar{r} - \bar{r}'|^3}$$

$$\psi = q\frac{[l(x-x') + m(y-y') + n(z-z')]a}{|\bar{r} - \bar{r}'|^3} \longrightarrow (8)$$

If $a \to a$, $q \to \infty$ in such a way $q \cdot a \to \mu$ is an electric dipole is formed.

$$a = \mu / q \Rightarrow q = \mu / a \longrightarrow (9)$$

let (9) in (8) we get,

$$\psi = \mu/a \left[\frac{l(x-x') + m(y-y') + n(z-z')}{|\bar{r} - \bar{r}'|^3} \right] a$$

$$\mu = \frac{\mu}{|\bar{r} - \bar{r}'|^3} [l(x-x') + m(y-y') + n(z-z')] \longrightarrow (10)$$

let $\bar{m} = \mu(l, m, n)$

$= \mu l\vec{i} + \mu m\vec{j} + \mu n\vec{k}$

$m(\bar{r} - \bar{r}') = [\mu l\vec{i} + \mu m\vec{j} + \mu n\vec{k}]$

$$= \left[(x-x')\vec{i} + (y-y')\vec{j} + (z-z')\vec{k}\right]$$

$$= \mu\left[l(x-x') + m(y-y') + n(z-z')\right]$$

$\therefore$ (10) takes the form.

$$\varphi = \frac{\bar{m}(\bar{r}-\bar{r}')}{|\bar{r}-\bar{r}'|^3} \longrightarrow (11)$$

Consider

$$\frac{\partial}{\partial x'}\frac{1}{|\bar{r}-\bar{r}'|} = \frac{\partial}{\partial x'}\left[\frac{1}{\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\right]$$

$$= \frac{\partial}{\partial x'}\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-1/2}$$

$$= -1/2\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{-3/2} \cdot 2\left[(x-x')(\right.$$

$$= (x-x')/\left\{\left[(x-x')^2+(y-y')^2+(z-z')^2\right]^{1/2}\right\}^3$$

$$\left.\begin{aligned}
\frac{\partial}{\partial x'}\frac{1}{|\bar{r}-\bar{r}'|} &= \frac{(x-x')}{|\bar{r}-\bar{r}'|^3} \\
\text{Mly},\quad \frac{\partial}{\partial y'}\frac{1}{|\bar{r}-\bar{r}'|} &= \frac{(y-y')}{|\bar{r}-\bar{r}'|^3} \\
\frac{\partial}{\partial z'}\frac{1}{|\bar{r}-\bar{r}'|} &= \frac{(z-z')}{|\bar{r}-\bar{r}'|^3}
\end{aligned}\right\} \longrightarrow (12)$$

we have

$$\varphi = \mu\left[\frac{l(x-x')+m(y-y')+n(z-z')}{|\bar{r}-\bar{r}'|^3}\right]$$

$$= \mu\left[\frac{l(x-x')}{|\bar{r}-\bar{r}'|^3} + \frac{m(y-y')}{|\bar{r}-\bar{r}'|^3} + \frac{n(z-z')}{|\bar{r}-\bar{r}'|^3}\right]$$

$$= \mu\left[l\frac{\partial}{\partial x'}\frac{1}{|\bar{r}-\bar{r}'|^3} + m\frac{\partial}{\partial y'}\frac{1}{|\bar{r}-\bar{r}'|^3} + n\frac{\partial}{\partial z}\frac{1}{|\bar{r}-\bar{r}'|}\right.$$

$$= \mu\left[l\frac{\partial}{\partial x'} + m\frac{\partial}{\partial y'} + n\frac{\partial}{\partial z'}\right]\frac{1}{|\bar{r}-\bar{r}'|^3}$$

Suppose we have if continuous distribution of charges fills a region V of space then φ takes the form.

$$\varphi = \int_{V} \frac{dq}{|\bar{r}-\bar{r}'|}$$, where q is the "Steilges measure".

the charge at the point $\bar{r}'$

If ρ is the charge density,

$$\varphi(\bar{r}) = \int_{V} \frac{\rho(\bar{r})\,d\tau'}{|\bar{r}-\bar{r}'|}$$

If S is a surface that cause a electric charge of density σ; then

$$\varphi(\bar{r}) = \int_{S} \frac{\sigma(r')\,ds'}{|\bar{r}-\bar{r}'|}$$

Eg:-

If $\rho > 0$ & $\varphi(\bar{r})$ is given by,

$$\varphi(\bar{r}) = \int_{V} \frac{\rho(\bar{r}')\,d\tau'}{|\bar{r}-\bar{r}'|} \longrightarrow ①$$

where the volume V is bounded, prove that

$\lim_{r\to\infty} r\,\varphi(\bar{r}) = M$, where $M = \int_V \rho(\bar{r}')\,d\tau'$.

Solu:-

Let r_1, r_2 be the maximum & minimum values of distance $\frac{1}{|\bar{r}-\bar{r}'|}$ from the point $\bar{r}$ to the integration point $\bar{r}'$ of the bounded volume V.

Then by a theorem of elementary calculus.

$$M/r' < \int_{V} \frac{\rho(\bar{r}')(d\tau')}{|\bar{r}-\bar{r}'|} < M/r^2 \longrightarrow ②$$

Let ① in ②,

$\mu/r_1 < \varphi(r) < \mu/r_2$

Multiply the above inequality by r, we have

$$\mu/r_1 \cdot r < r\cdot\varphi(\bar{r}) < \mu/r_2 \cdot r \longrightarrow (3)$$

As $r \to \infty$, r/r_1 & $r/r_2 \to 1$

$$\therefore \lim_{r\to\infty} \mu r/r_1 < \lim_{r\to\infty} r\cdot\varphi(\bar{r}) < \lim_{r\to\infty} \mu r/r_2.$$

$$\mu \lim_{r\to\infty} r/r_1 < \lim_{r\to\infty} r\,\varphi(\bar{r}) < \mu \lim_{r\to\infty} r/r_2.$$

From (3),

$$\Rightarrow \mu < \lim_{r\to\infty} r\,\varphi(\bar{r}) < \mu.$$

$$\lim_{r\to\infty} r(\varphi(\bar{r})) = \mu.$$

Families of equipotential surfaces:-

If the function $\varphi(x, y, z)$ is a sol. of laplace's equation, then one parameter system of surfaces

$$\varphi(x, y, z) = c \longrightarrow (1)$$

is called equipotential surfaces.

Necessary condition for any one parameter family of surfaces:-

$f(x, y, z)$ is a family of equipotential if the potential function of φ is constant whenever $f(x, y, z)$ is constant

$$\text{let } \varphi = \left[F\{f(x, y, z)\}\right] \longrightarrow (2)$$

Diff (2) w.r.to x,

$$\frac{\partial\varphi}{\partial x} = \frac{\partial}{\partial x}\left[F\{f(x, y, z)\}\right]$$

$$\frac{\partial \psi}{\partial x} = \frac{\partial F}{\partial f} \cdot \frac{\partial f}{\partial x} \longrightarrow (3)$$

$$\left.\begin{aligned}
\frac{\partial^2 \psi}{\partial x^2} &= \frac{\partial^2 F}{\partial f^2}\left(\frac{\partial f}{\partial x}\right)^2 + \frac{\partial F}{\partial f} \cdot \frac{\partial^2 f}{\partial x^2} \\
\text{|||}^{ly},\quad \frac{\partial^2 \psi}{\partial y^2} &= \frac{\partial^2 F}{\partial f^2}\left(\frac{\partial f}{\partial y}\right)^2 + \frac{\partial F}{\partial f} \cdot \frac{\partial^2 f}{\partial y^2} \\
\frac{\partial^2 \psi}{\partial z^2} &= \frac{\partial^2 F}{\partial f^2}\left(\frac{\partial f}{\partial z}\right)^2 + \frac{\partial F}{\partial f} \cdot \frac{\partial^2 f}{\partial z^2}
\end{aligned}\right\} \longrightarrow (4)$$

Consider,

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2}$$

$$= \frac{\partial^2 F}{\partial f^2}\left(\frac{\partial f}{\partial x}\right)^2 + \frac{\partial F}{\partial f}\left(\frac{\partial^2 f}{\partial x^2}\right) + \frac{\partial^2 F}{\partial f^2}\left(\frac{\partial f}{\partial y}\right)^2 + \frac{\partial F}{\partial f}\left(\frac{\partial^2 f}{\partial y^2}\right) + \frac{\partial^2 F}{\partial f^2}\left(\frac{\partial f}{\partial z}\right)^2 + \frac{\partial F}{\partial f}\left(\frac{\partial^2 f}{\partial z^2}\right).$$

$$= \frac{\partial^2 F}{\partial f^2}\left\{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right\} + \frac{\partial F}{\partial f}\left\{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right\}$$

$$= F''(f)\left\{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2\right\} + F'(f)\left\{\left(\frac{\partial^2 f}{\partial x^2}\right) + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right\}$$

$$= F''\{f\}\,(\text{grad } f^2) + F'\{f\}\,\nabla^2 f \longrightarrow (5)$$

$$(\text{grad } f)^2 = \left\{\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right\}\left\{\frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}\right\}$$

$$= \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + \left(\frac{\partial f}{\partial z}\right)^2$$

But $\nabla^2 \psi = 0$

$$\Rightarrow F''\{f\}\,(\text{grad } f)^2 + F'\{f\}\,\nabla f = 0.$$

$$\Rightarrow F''\{f\}\,(\text{grad} f)^2 = -F'\{f\}\,\nabla^2 f.$$

$$\Rightarrow \frac{-F''(f)}{F'(f)} = \chi(f)$$

$\Rightarrow -F''(f) + \chi(f)\, F'(f) = 0$

(i.e.), $\frac{d^2F}{\partial f^2} + \chi(f) \cdot \frac{dF}{df} = 0.$

$\Rightarrow \frac{dF}{df} = A e^{-\int \chi(f)\, df}$ where A is constant

we have, $X = F\{f(x, y, z)\}$

$X = A \int e^{-\int \chi(f) df} \cdot df + B$

where B is constant.

Eg:-

S.T. the surfaces $x^2 + y^2 + z^2 = cx^{2/3}$ can form a family of equipotential surfaces & find the general form of the corresponding potential func.

Sol:-

Given that, $x^2 + y^2 + z^2 = cx^{2/3}$

$c = x^{-2/3}(x^2 + y^2 + z^2)$

$= x^{2(-2/3)} + x^{-2/3}(y^2) + x^{-2/3}(z^2)$

$= x^{-4/3} + x^{-2/3} y^2 + x^{-2/3} z^2$

Let $f = c, = x^{-4/3} + x^{-2/3} y^2 + x^{-2/3} z^2$

$= x^{-2/3}(x^2 + y^2 + z^2)$

$\text{grad } f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$

$\frac{\partial f}{\partial x} = (4/3) x^{4/3 - 1} + (-2/3) x^{-2/3 - 1} (y^2 + z^2)$

$\frac{\partial f}{\partial x} = 4/3\, x^{1/3} - 2/3\, x^{-5/3} (y^2 + z^2)$

$\frac{\partial f}{\partial y} = 2x^{-2/3} y$, $\frac{\partial f}{\partial z} = 2x^{-2/3} z$.

$\text{grad } f = \left[4/3\, x^{1/3} - 2/3\, x^{-5/3}(y^2 + z^2)\right]\vec{i} + 2x^{-2/3} y\vec{j} + 2x^{-2/3} z\vec{k}$

$\therefore \text{grad } f = 2/3\, x^{-5/3} \left\{ \left[2x^2 - (y^2 + z^2)\right]\vec{i} + 3xy\vec{j} + 3xz\vec{k} \right\}$

$$|\text{grad } f|^2 = \sqrt{(2/3)^2 (x^{-5/3})^2 \left\{(2x^2-y^2)^2 + (3xy)^2 + (3xz)^2\right\}}$$

$$= 4/9\, x^{-10/3} \left\{(2x^2-y^2-z^2) + 9x^2y^2 + 9x^2z^2\right\}$$

$$= 4/9\, x^{-10/3} \left\{4x^4+y^4+z^4 - 4x^2y^2 + 2y^2z^2 - 4x^2z^2 + 9x^2y^2 + 9x^2z^2\right\}$$

$$= 4/9\, x^{-10/3} \left\{4x^4+y^4+z^4+5x^2y^2+2y^2z^2+5x^4z^2\right\}$$

$$= 4/9\, x^{-10/3} \left\{(4x^2+y^2)+z^2\}\,(x^2+y^2+z^2)\right\}$$

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$\frac{\partial^2 f}{\partial x^2} = (4/3)(1/3)\, x^{1/3-1} - (2/3)\left\{-5/3\, x^{-5/3-1}(y^2+z^2)\right\}$$

$$= 4/9\, x^{-2/3} + 10/9\, x^{-8/3}(y^2+z^2)$$

$$\frac{\partial^2 f}{\partial y^2} = 2x^{-2/3}$$

$$\frac{\partial^2 f}{\partial z^2} = 2x^{-2/3}$$

$$\nabla^2 f = 4/9\, x^{-2/3} + 10/9\, x^{-8/3}(y^2+z^2) + 2x^{-2/3} + 2x^{-2/3}$$

$$= \frac{4}{9} x^{-2/3} + \frac{10}{9} x^{-8/3}(y^2+z^2)$$

$$= \frac{10}{9} x^{-8/3}\left[4x^{-2/3+8/3} + y^2+z^2\right] = \frac{10}{9} x^{-8/3}\left[4x^2+y^2+z^2\right]$$

Consider,

$$\frac{\nabla^2 f}{|\text{grad } f|^2} = \frac{10/9\, x^{-8/3}\left[4x^2+y^2+z^2\right]}{4/9\, x^{-10/3}\left\{(4x^2+y^2+z^2)(x^2+y^2+z^2)\right\}}$$

$$= 5/2\, x^{2/3} \cdot 1/x^2+y^2+z^2$$

$$= 5/2\, \frac{1}{x^{-2/3}(x^2+y^2+z^2)} = \frac{5/2}{f} \quad (\because c = r)$$

$$= 5/2 f = \alpha(f) \quad \cdots \left[\because \alpha(f) = \nabla^2 f / |\text{grad} f|^2\right]$$

where,

$\alpha(f) = 5/2 f$.

∴ The given set of surfaces forms a family of equipotential surfaces.

$$\varphi = A\int e^{-\int x(f)df} . df + B$$

$$= A\int e^{-\int 5/2 f df} . df + B$$

$$= A\int e^{-5/2 \log f} . df + B$$

$$= A\int e^{\log f^{(-5/2)}} . df + B$$

$$= \frac{A f^{-3/2}}{-3/2} + B = -2/3 A f^{-3/2} + B$$

$$= -2/3 A \left[x^{-2/3}(x^2+y^2+z^2)\right]^{-3/2} + B$$

The required potential function is,

$$\varphi = \left[-2/3 A x (x^2+y^2+z^2)\right]^{3/2} + B$$

where A & B are constants.

Boundary value problem:-

Defn:

In the discussion of certain physical problems of function φ whose analytic form satisfying laplace eqn. within a certain region of space V also satisfying conditions on the boundary S of the region.

Any problem in which we are required to find such a function φ is called boundary value problem for laplace's equation.

Dirichlet's Boundary value problem:-

i) Interior Dirichlet boundary value problem:-

If f is a continuous function defined on the boundary S of some finite region V, determine a function $\varphi(x, y, z)$ such that $\nabla^2 \varphi = 0$ within V & $\varphi = F$ on S.

ii) Exterior Dirichlet boundary value problem:-

If f is a continuous function defined on the boundary S of a finite simply connected region V, determine a function $\varphi(x, y, z)$ such that $\Delta^2 \varphi = 0$ outside V & is such that $\varphi = f$ on S.

For instance, the problem of finding the distribution of temperature within a body in the steady state when each point of its surface is kept at a prescribed steady temperature in a interior dirichlet problem while that of determining the distribution of potential outside a body whose surface potential is prescribed in an exterior dirichlet problem.

The exterior of the solutions of a dirichlet problem under very general condition can be established.

1) To prove that the existence of the soln. of an exterior dirichlet problem is unique.

Proof:-

Assume that φ_1 & φ_2 are soln. of the interior dirichlet problem.

let $\varphi = \varphi_1 - \varphi_2$

we have $\nabla^2 \varphi = 0$ within V cannot exceed its maximum on S (or) less than its minimum on S.

Scanned with CamScanner

$\Rightarrow \varphi = 0$ within V (i.e), $\varphi_1 - \varphi_2 = 0$ within V

$\Rightarrow \varphi_1 = \varphi_2'$ within V.

2) The sol. of the exterior dirichlet problem is not Unique unless some restriction is placed on the behaviour of $\varphi(x, y, z)$ as $r \to \infty$.

Eg:-

In three dimensional cases, the solu. of the exterior dirichlet problem is unique provided $\varphi(x, y, z) < C/r$ as $r \to \infty$ where c is a constant. In the two dimensional case, we require the function, φ to be bounded at infinity.

Thrm:-

Deduction of exterior dirichlet problem from interior problem.

Proof:-

Within the region V, Choose a Spherical surface C with centre O & radius a.

Invert the space outside the region V with respect to the sphere C. (i.e), map a point outside V into a point $\bar{\pi}$ inside the sphere C such that $OP \cdot O\bar{\pi} = (t^2) \to (1)$

The region exterior to the surface S mapped into a region V_k^* lying entirely within the sphere C.

If $f^*(\bar{\pi}) = \frac{a}{2R} f(R)$ & if $\varphi^*(\bar{\pi})$ is the solu. of the Interior Dirichlet problem.

$$\nabla^2 \psi^* = 0 \quad \text{within } V^* \longrightarrow (2)$$

$$\psi^* = f^*(\bar{\pi}) \quad \text{for } \pi \in S^*$$

let $\psi(P) = a/op \, \psi^*(\bar{\pi})$

Consider r,

$$\nabla^2 \psi(P) = \nabla^2 a/op \, \psi^*(\bar{\pi})$$

$$= a/op \, \nabla^2 \psi^*(\bar{\pi})$$

$$= a/op \cdot 0 \quad (\text{by } (2))$$

To prove that :-

$$\psi(P) = f(P) \quad \text{for } P \in S$$

Consider,

$$\psi(P) = a/op \; f^*(\bar{\pi}) = a/op \cdot a/o\pi \cdot f(P)$$

$$= a^2/a^2 \cdot f(P) = f(P) \quad \text{for } P \in S.$$

Also, $\nabla^2 \psi = 0$ outside V.

$\therefore \psi(P) = a/op \; \psi^*(\bar{\pi})$ is the solu. of the exterior dirichlet problem $\nabla^2 \psi = 0$ outside V & $\psi = f(P)$ for $P \in S$.

Neumann problem :-

i) Interior Neumann problem :-

If f is a continuous function which is defined uniquely at each point of the boundary S of a finite region V determine a function $\psi(x, y, z)$ such that $\nabla^2 \psi = 0$ within V and its normal derivative $\frac{\partial \psi}{\partial n} = f$ at every point of S.

ii) Exterior Neumann problem :-

If f is a continuous function

prescribed at each point of the (smooth) boundary S of boundary simple connected region v, find a function $\varphi(x, y, z)$ satisfying $\nabla^2\varphi = 0$ outside & $\frac{\partial \varphi}{\partial n} = f$ on S.

Thrm:-

Statement:

The necessary condition for the existence 'q' the function of the interior Neumann problem is $\int_C f(p)\,ds = 0$ that the integral of f over the boundary S show vanish.

Proof:

Let $\bar{a} = \text{grad}\,\varphi$ is Gauss theorem.

$$\Rightarrow \int_S a_n\,ds = \int_\gamma \text{div}\,\bar{a}\,d\bar{\imath}$$

(ie), $$\int_\gamma \nabla^2\varphi\,dz = \int_S \frac{\partial\varphi}{\partial n}\,ds \longrightarrow (1)$$

But $$\frac{\partial\varphi}{\partial n} = f(p) \quad ; (p \in S) \longrightarrow (2)$$

Let (2) in (1) : $$\int_V \nabla^2\varphi\,dz = \int_S f(p)\,ds.$$

But $\nabla^2\varphi = 0 \Rightarrow 0 = \int_S f(p)\,ds$

(ie), $$\int_S f(p)\,ds = 0.$$

Thrm:-

Statement:-

Reduce the Neumann problem to the Dirichlet equation.

Proof:

Assume that ψ is a solu. of Neumann problem.

$\nabla^2\psi = 0$ within S & $\frac{\partial \psi}{\partial n} = f(P)$ for $r, P \in C$ exists.

Assume that $\psi, \frac{\partial \psi}{\partial z}, \frac{\partial \psi}{\partial y}$ are continuous on C, of S choose ϕ within S & on C.

such that, $\frac{\partial \psi}{\partial x} = \frac{\partial \phi}{\partial y}$; $\frac{\partial \psi}{\partial y} = -\frac{\partial \phi}{\partial x}$

$\Rightarrow \psi + i\phi$ is an analytic function.

also, $\frac{\partial \phi}{\partial s} = \frac{\partial \psi}{\partial n}$

Let P, Q be two points on C.

then, $\phi(Q) - \phi(P) = \int_P^Q \frac{\partial \phi}{\partial s}\, ds = \int_P^Q f(s)\, ds$

But, $\int_C f(s)\, ds = 0 \Rightarrow \phi(Q) = \phi(P)$

$\Rightarrow \phi$ is a single valued function and ϕ is continuous, Also if ψ is harmonic, then ϕ is also harmonic.

we can determine the function ψ within S interior Churchill problem. If f is a continuous function prescribed on the boundary S of finite region V, determine $\psi(x, y, z) \ni \nabla^2\psi = 0$ within V & $\frac{\partial \psi}{\partial n} + (k+1)\psi = f$ at every point of S.

Exterior Churchill problem:-

If f is a continuous function prescribed.

on the boundary S of a finite region V, determined $\varphi(x,y,z)$ such that $\nabla^2\varphi = 0$ outside V & $\frac{\partial \varphi}{\partial n} + (k+1)\varphi = f$ at every point of S.

Separation of Variables:-

In this solu. we have dealing with laplace equ by finding solu to it by assigning method of separation of variables.

In spherical polar co-ordinates r, θ, ϕ. Laplace equ takes the form.

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{2}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2} + \frac{\cot\theta}{r^2}\cdot\frac{\partial \psi}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 \psi}{\partial \phi^2} = 0 \rightarrow ①$$

This equ has solu of the form

$$\left\{ A_n r^n + B_n / r^{n+1} \right\} \rightarrow ⑪ \qquad \cos\theta\, e^{Im\phi} \rightarrow ⑫$$

where A_n, B_n are constant

To prove that equ ② is the solu. of equ ①

let the solu be, $\psi(r,\theta,\phi) = R\,⑪\,\phi$.

where R is a function of r alone.

⑪ is a function of θ alone.

ϕ is a function of ϕ alone.

$$\left.\begin{aligned}
\partial\psi/\partial r &= R'\,⑪\,\Phi \;:\; \frac{\partial^2\psi}{\partial r^2} = R''\,⑪\,\Phi \\
\partial\psi/\partial\theta &= R\,⑪'\cdot\Phi \;:\; \frac{\partial^2\psi}{\partial\theta^2} = R\,⑪'\,\Phi \\
\partial\psi/\partial\phi &= R\,⑪\,\Phi' \;:\; \partial^2\psi/\partial\theta^2 = R\,⑪\,\Phi''
\end{aligned}\right\} \rightarrow ③$$

Let ③ in ① we have

$$R''\Theta\Phi + \frac{2}{r}R'\Phi \ \frac{1}{r^2}R\Theta''\Phi + \frac{\cot\theta}{r^2}R\Theta'\Phi + \frac{1}{r^2}\sin^2\theta\, R^2\Theta\Phi'' = 0$$

Dividing by $R\Theta\Phi$ & multiply by $r^2\sin^2\theta$ then

we have,

$$r^2\sin^2\theta\frac{R''}{R} + 2r\sin\theta\frac{R'}{R} + \sin^2\theta\frac{\Theta'}{\Theta} + \sin^2\theta\cot\theta\frac{\Theta}{\Theta} + \Phi''/\Phi = 0.$$

$$\Rightarrow \frac{\sin^2\theta}{R}\{r^2R'' + 2rR'\} + \sin^2\theta\frac{\Theta'}{\Theta} + \sin^2\theta\cot\theta\frac{\Theta''}{\Theta} = \frac{-\Phi''}{\Phi} \longrightarrow (4)$$

Let $\frac{\Phi''}{\Phi} = -m^2 \longrightarrow (5)$

R.H.S of (4) is purely a function of ϕ alone.

$$\therefore (4) \Rightarrow 1/R\{r^2R'' + 2rR'\} + \frac{1}{\Theta}\{\Theta' + \cot\theta\,\Theta'\} = \frac{m^2}{\sin^2\theta}$$

$$\Rightarrow 1/R\{r^2R'' + 2rR'\} = -\left[\frac{1}{\Theta}\{\Theta'' + \cot\theta\,\Theta'\} - \frac{m^2}{\sin^2\theta}\right]$$

L.H.S is a function of 'r' alone & R.H.S. is a function of θ alone.

$$\Rightarrow -1/R\{r^2R'' + 2rR'\} = -n(n+1)$$

$$\Rightarrow r^2R'' + 2rR' - n(n+1)R = 0.$$

Let $u = \log r$

$$D(D-1)R = r^2\frac{\partial^2R}{\partial r^2} \ \& \ D = r\frac{\partial R}{\partial V}$$

$$\Rightarrow [D(D-1) + 2D - n(n+1)]R = 0$$

$$\Rightarrow [D^2 + D - (n^2+n)]R = 0$$

The auxillary eqn is $m^2 + m - (n^2+n) = 0$ &

the Solu is, $R = Ae^{nu} + Be^{-(n+1)u}$

$$= Ae^{n\log r} + Be^{-(n+1)\log r}$$

$$= Ar^n + B/r^{n+1}$$

$$\therefore R = Ar^n + B/r^{n+1}$$

W.K.T, $\Phi'' - m^2 \Phi = 0$

The solu. of $\Phi = Ae^{\pm im\Phi}$.

Now,

$$\frac{1}{\Theta}\left[\Theta'' + \cot\theta\, \Theta'\right] - \frac{m^2}{\sin^2\theta} = -n(n+1)$$

$$\Rightarrow \Theta'' + \cot\theta\, \Theta' - m^2/\sin^2\theta + n(n+1)\Theta = 0.$$

$$\Rightarrow \Theta' + \cot\theta\, \Theta' - m^2/\sin^2\theta\, \Theta + n(n+1)\Theta = 0$$

$$\Rightarrow \Theta' + \cot\theta\, \Theta' - \left[n(n+1) - m^2/\sin^2\theta\right]\Theta = 0 \rightarrow (5)$$

$$\left[\sin^2\theta = 1 - \cos^2\theta\right] = 1 - \mu$$

$$\sin\theta = \sqrt{1-\mu^2}$$

put, $\cos\theta = \mu \Rightarrow \sin\theta = \sqrt{1-\mu^2}$

$$\frac{\partial\Theta}{\partial\mu} = \frac{\partial/\partial\theta\ \Theta}{\partial'/\partial\theta\,(\mu)}$$

$$\frac{\partial\Theta}{\partial\theta}\cdot\frac{\partial\phi}{\partial\mu} = \frac{\partial\Theta/\partial\theta}{\partial\theta/\partial\mu} = \frac{\Theta}{\partial\theta/\partial\mu}.$$

we have,

$$\frac{\partial\mu}{\partial\theta} = -\sin\theta \Rightarrow \frac{\Theta'}{-\sin\theta} \Rightarrow \frac{\partial\Theta}{\partial\mu} = \frac{\Theta'}{-\sin\theta}$$

$$\Rightarrow \Theta' = -\sin\theta\ \partial\Theta/\partial\mu.$$

$$\frac{\partial^2\Theta}{\partial\mu^2} = \frac{\partial}{\partial\mu}\left(\frac{\partial\Theta}{\partial\mu}\right) = \frac{\partial}{\partial\theta}\left(\frac{\partial\Theta}{\partial\mu}\right)\frac{\partial\theta}{\partial\mu}.$$

$$= \frac{\partial/\partial\theta\left[-\Theta'/\sin\theta\right]}{\partial/\partial\theta\,(\mu)} = \frac{\partial/\partial\theta\left[-\Theta'/\sin\theta\right]}{\partial\mu/\partial\theta}$$

$$= -\frac{\Theta''\sin\theta - \Theta'\cos\theta}{-\sin\theta} = \frac{-\Theta''/\sin\theta + \Theta''\operatorname{cosec}\theta\,(\cot\theta)}{-\sin\theta}$$

$$\frac{\partial^2\Theta}{\partial\mu^2} = \frac{1}{\sin^2\theta}\left[\Theta'' - \cot\theta\,\Theta'\right]$$

$$\sin^2\theta\,\frac{\partial^2\Theta}{\partial\mu^2} = \Theta'' - \cot\theta\left[-\sin\theta\cdot\partial\Theta/\partial\mu\right]$$

$$(11)'' = \sin^2\theta \frac{\partial^2 (11)}{\partial\mu^2} - \cot\theta \cdot \sin\theta \frac{\partial (11)}{\partial\mu}$$

$$\Rightarrow (11)'' = \sin^2\theta \left\{ \frac{\partial^2 (11)}{\partial\mu^2} - \frac{\cot\theta}{\sin\theta} \frac{\partial (11)}{\partial\mu} \right\}$$

$\therefore$ (5) becomes,

$$\Rightarrow \sin^2\theta \cdot \frac{\partial^2 (11)}{\partial\mu^2} - (\cot\theta \sin\theta + \sin\theta \cot\theta) \frac{\partial (11)}{\partial\mu} + \left[n(n+1) - \frac{m^2}{\sin^2\theta} \right] - (11) = 0.$$

$$\Rightarrow (1-\mu^2)\left\{ \left(\frac{\partial^2 (11)}{\partial\mu^2} \right) - 2\cos\theta \frac{\partial (11)}{\partial\mu} + n(n+1) - \frac{m^2}{\sin\theta} \right\} (11)$$

The solu. of the equ is,

$$(11) \cos\theta = E_{m,n} \cdot P_n^m(\cos\theta) + F_{m,n} Q_n^m(\cos\theta)$$

$\therefore$ The solu. of the equ (1), is given by.

$$\psi = \{ A_n r^n + B^n / r^{n+1} \} (11) \cos\theta \, e^{I m\phi}.$$

where, A_n, B_m, m are constant & $(11)\mu$ satisfies legendre's associated equation.

$$(1-\mu^2) \frac{\partial^2 (11)}{\partial\mu^2} - 2\mu \frac{\partial (11)}{\partial\mu} \left\{ n(n+1) - \frac{m^2}{1-\mu^2} \right\} = 0 \rightarrow (5)$$

Case-i):-

when $m=0$ the legendre associated equation becomes legendre's equ.

$$(1-\mu^2) \frac{\partial^2 (11)}{\partial\mu^2} - 2\mu \frac{\partial (11)}{\partial\mu} + n(n+1)(11) = 0 \rightarrow (*)$$

If 'n' is positive integer the equ has two independent solu's given by.

$$P_n(\mu) = 1/2^n \cdot n! \; \partial^n/\partial\mu^n \; (\mu^2-1)^n \rightarrow (6)$$

$$Q_n(\mu) = 1/2\, P_n(\mu) \log \frac{\mu+1}{\mu-1} - \sum_{s=0}^{\infty} \frac{2^{2} 4s-1}{(2s+1)(n-s)} P_{n-2s}(\mu).$$

where, $p = 1/2\,(n-1)$.

According as 'n' is odd (or) even, then general solu. of ⊛ is.

(11) $= C_n P_n(\mu) + D_n Q_n(\mu)$

where C_n & D_n are constants.

Now, when $\theta = 0$, $\mu = 1$, $Q_n(\mu)$ is infinite So, for is to remain finite on the polar axis take constant D_n to the identically zero.

∴ The solu. of laplace equn (1) becomes.

$$\varphi = \sum_n \left(A_n r^n + B_n/r^{n+1}\right) P_n \cos\theta \longrightarrow (7)$$

Case - ii) :-

when $m \neq 0$ & $0 \leq m \leq n$ the solu. of (5) is,

$$P_n^{(m)}(\mu) = (-\mu^2-1)^{\frac{1}{2}m} \frac{d^m (P_n(\mu))}{d\mu^m}.$$

$$Q_n^{(m)}(\mu) = (\mu^2-1)^{1/2\, m} \frac{d^m (Q_n(\mu))}{d\mu^m}.$$

Now, when $\mu = \pm 1$, $Q_n^m(\mu)$ is infinite, So taking D_n to be identically zero.

∴ The solu is given by.

(11) $\cos\theta = P_n^m(\cos\theta)$

The solu. of laplace's equation.

$$\varphi = \sum_{n=0}^{\infty} \sum_{m<n}^{n} \left(A_{mn} r^n + B_{mn} r^{(n+1)}\right) P_n^{(m)}(\cos\theta).$$

e^{time}

which may be written as.

$$\varphi = \sum_{n=0}^{\infty} (r/a)^n \Big[A_n P_n(\cos\theta) + \sum_{m=1}^{n} (A_{mn} \cos m\phi + B_{mn} \sin m(\phi)\, P_n^m(\cos\theta)\Big].$$

Example-3:-

A rigid sphere of radius 'a' is placed in a stream of fluid whose velocity in the undisturbed state is V, determine the velocity of the fluid at any point of the disturbed stream.

Soln:-

Let us take the polar axis OZ to be in the direction of the given velocity & let the polar co-ordinates be (r, θ, ϕ) with origin as centre of the fixed sphere.

The velocity of the fluid is given by the vector $q = -\text{grad}\,\varphi$

where, 1) $\nabla^2 \varphi = 0$

2) $\partial \varphi / \partial r = 0.$

3) $\varphi \sim -Vr \cos\theta = -Vr P_1 \cos\theta$

The auxillary asymmetrical function

$$\varphi = \sum_{n=0}^{\infty} \left(A_n r^n + B_n / r^{n+1}\right) P_n \cos\theta$$

In spherical polar co-ordinates (r, θ, ϕ) laplace equ. takes the form,

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{2}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} + \frac{\cot\theta}{r^2}\cdot\frac{\partial \varphi}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\cdot\frac{\partial^2 \varphi}{\partial \phi} = 0 \quad \to (1)$$

Equ (1) is seperable with soln of the form.

$$\left\{A_n r^n + B_n / r^{n+1}\right\} (\text{II}) \cos\theta\, e^{I m \phi}$$

where A_n, B_n, m are constant

Here (1) satisfies legendres associated equ is

$$(1-\mu^2) d^2 (\text{II}) / d\mu^2 - 2\mu\, d(\text{II}) / d\mu + \left\{ n(n+1) - m^2 / 1-\mu^2 \right\} (\text{II})' = 0 \quad \to (2)$$