UNIT –I

UNIT - I

Functional Analysis :

Functional analysis is the branch of mathematics and specifically of analysis, concerned with the Study of functional Spaces, in particular transformation of functions such as the Fourier transform as well as in the Study of differential and Integral equation. This usuage of the word goes back to the calculus of variation, implying a function whose argument is a function. It's use in general has been attributed to mathematician and physicist vito volterra and its founding is largely attributed to mathematician Stephen Banach.

Topological Vector Space :

Let x be a vector space over the field F. x is said to be a topological vector space if x is itself is a topological Space and both addition and scalar multiplication are joinetly continuous.

Compact Set : 1 2m

A Set S in a metric space is said to be compact iff every open covering on S contains a finite subcover that is every open cover contains a finite subcollection which also covers S.

Complete Metric Space :

A metric space (S, d) is called a complete metric space if every cauchy sequence in S

Converges in S.

Contraction Mapping:

A Mapping F of a metric space X into itself is said to be contractive if $d(Fx, Fy) < d(x,y)$, $x, y \in X$ $(x \neq y)$ and is said to be ϵ-contractive if $0 < d(x,y) < \epsilon \Rightarrow d(Fx, Fy) < d(x,y)$.

Banach Space:

Let X, Y be metric spaces and F be a mapping from X into Y, F is said to be Lipschitz, if K is the real numbers, $K>0$ such that for all $x, y \in X$, we have. $d(Fx, Fy) < K \cdot d(x,y)$

F is said to be contraction if $K<1$ and non-expensive if $K=1$, F is said to be contractive if for all x, y in X and $x \neq y$ we have,

$$d(Fx, Fy) < d(x,y) \quad x \neq y.$$

Normed Linear Space:

A normed Linear Space is a Linear Space on which there is defined a non-negative function which assigned to each elements x in the Space, a real number $\|x\|$ in such a manner that

i) $\|x\| \geq 0$ and $\|x\| = 0 \Leftrightarrow x = 0$

ii) $\|x+y\| \leq \|x\| + \|y\|$

iii) $\|\alpha x\| = |\alpha| \, \|x\|$.

Examples:

In R, let $\|x\| = |x|$ where $\forall x \in R$ ($\|\cdot\| : R \to R$)

To prove: Normed linear space. (3)

$\therefore$ i) Let $|x| \geq 0$

$$\Rightarrow \|x\| \geq 0$$

$$\|x\| = 0$$

$$\Leftrightarrow |x| = 0$$

$$\Leftrightarrow x = 0$$

ii) $\|x+y\| \leq |x+y|$

$$\leq |x| + |y|$$

$$\leq \|x\| + \|y\|$$

$$\therefore \|x+y\| \leq \|x\| + \|y\|$$

iii) $\|\alpha x\| = |\alpha x|$

$$= |\alpha| \, |x|$$

$$= |\alpha| \, \|x\|$$

$\therefore$ The space R is normed Linear Space.

Note:

A complete normed Linear Space is called a Banach Space.

Lipschitz continuous:

Given two metric spaces (X, d_X) and (Y, d_Y) where d_X and d_Y denotes the metric on the set X and Y. (For example Y might be the set of real numbers R with the metric $d_Y(x, y) = |x - y|$ and X might be the subset of R)

A Function $F : X \to Y$ is called Lipschitz continuous if there exist a real constant $K \geq 0$

Such that for all x_1, x_2 in X then,

$$d_Y(f(x_1), f(x_2)) \leq K\, d_X(x_1, x_2).$$

Examples:

R^n and C^n are Banach Spaces with respect to the norm defined by $\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$ where $x = (x_1, x_2 \ldots x_n)$ (Set of n-tuples)

Proof:

To proof: the ~~norm~~ Space is normed linear

i) $|x_i| \geq 0$

$$|x_i|^2 \geq 0$$

$$\sum_{i=1}^{n} |x_i|^2 \geq 0$$

$$\left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} \geq 0$$

$$\therefore \|x\| \geq 0$$

ii) $\|x\| = 0$

$$\Leftrightarrow \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} = 0$$

$$\Leftrightarrow \sum_{i=1}^{n} |x_i|^2 = 0$$

$$\Leftrightarrow |x_i|^2 = 0$$

$$\Leftrightarrow |x_i| = 0$$

$$\Leftrightarrow x_i = 0$$

$$\Leftrightarrow x = 0.$$

iii) To prove:

$$\|x + y\| \leq \|x\| + \|y\|$$

In L_p^n $\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$

$$\|x+y\| = \left(\sum_{i=1}^{n} |x_i + y_i|^2\right)^{1/2}$$

By Minkowski inequality, (5)

$$\|x+y\|_p \le \|x\|_p + \|y\|_p$$

$$\|x+y\|_2 \le \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2} + \left(\sum_{i=1}^{n} |y_i|^2\right)^{1/2}$$

$$\le \|x\|_2 + \|y\|_2$$

$$\|x+y\|_2 \le \|x\|_2 + \|y\|_2$$

iv) $\|\alpha x\| = \left(\sum_{i=1}^{n} |\alpha x_i|^2\right)^{1/2}$

$$= |\alpha| \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

$$= |\alpha| \, \|x\|$$

Note:

Let $\{x_m\}$ be cauchy sequence in R^n is given $\epsilon > 0$, there exists a positive integer n_0 such that, $\|x_m - x_n\| < \epsilon \ \forall \ m, n \ge n_0$

To prove,

R^n and C^n are Banach spaces

Let $\{x_m\}$ be a cauchy sequence in R^n and $\epsilon > 0$, $\exists$ a +ve integer n_0 such that,

$$\|x_m - x_p\| < \epsilon \ \forall \ m, p \ge n_0$$

$x_m = (x_1^m, x_2^m \dots x_n^m)$ where x_i^m is the i^{th} co-ordinate ~~[illegible]~~ of x_m

$$x_m - x_p = (x_1^m - x_1^p, x_2^m - x_2^p, \dots \dots x_n^m - x_n^p)$$

$$\|x_m - x_p\| = \left(\sum_{i=1}^{n} |x_i^m - x_i^p|^2\right)^{1/2}$$

$$\left(\sum_{i=1}^{n} |x_i^m - x_i^p|^2\right)^{1/2} < \epsilon \quad \forall\, m, p \geq n_0 \rightarrow (1)$$

$$\sum_{i=1}^{n} |x_i^m - x_i^p|^2 < \epsilon^2$$

(6)

$$\therefore |x_i^m - x_i^p|^2 < \epsilon^2$$

$$|x_i^m - x_i^p| < \epsilon \quad i = 1, \dots n \;\forall\, m, p \geq n_0$$

$\therefore \{x_i^j\}_{j=1}^{\infty}$ is a cauchy sequence for $i = 1, 2, \dots n$ in R

Hence $\therefore \{x_i^j\}_{j=1}^{\infty}$ is convergence.

Let it convergence to z_i that is

$$\lim_{j \to \infty} x_i^j = z_i$$

Let $z = (z_1, z_2 \dots z_n)$

$\therefore z \in l_2^n$ in eqn. (1) Let $p \to \infty$

$$\left(\sum |x_i^m - z_i|^2\right)^{1/2} < \epsilon \;,\; m \geq n_0$$

$$\therefore \|x_m - z\| < \epsilon$$

$\therefore \{x_m\}$ converges

In R^n every cauchy Sequence converges

$\therefore R^n$ is a Banach space.

2) Let p be any real number such that $1 \leq p \leq \infty$. then the Space is l_p^n of all n-tuples $x = (x_1, x_2 \dots x_n)$ of scalars is Banach spaces under the norm denoted by $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$

Soln:

i) $\|x\|_p \geq 0$ $(\because |x_i| \geq 0, \forall i)$

$\|x\|_p = 0$

$\Leftrightarrow \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} = 0$ ⑦

$\Leftrightarrow \sum_{i=1}^{n} |x_i|^p = 0$

$\Leftrightarrow |x_i|^p = 0$

$\Leftrightarrow |x_i| = 0 \ \forall\ i = 1 \text{ to } n$

$\Leftrightarrow |x| = 0$

$\Leftrightarrow x = 0$

ii) By Minkowshi inequality.

$\|x+y\|_p \leq \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} + \left(\sum_{i=1}^{n} |y_i|^p\right)^{1/p}$

$\leq \|x\|_p + \|y\|_p$

$\|\alpha x\|_p = \left(\sum_{i=1}^{n} |\alpha x_i|^p\right)^{1/p}$

$= |\alpha| \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$

$= |\alpha| \, \|x\|_p$

$\therefore l_p^n$ is a normed linear space

Let sequence $\{x_m\}$ be a cauchy sequence in l_p^n where $x_m = (x_1^m, x_2^m \ldots x_n^m)$, x_i^m denote the ith co-ordinate of x_m for $i = 1, 2 \ldots m$

Let $\epsilon > 0$ then $\exists$ a +ve integer n_0 such that

$\|x_m - x_l\| < \epsilon \ \forall\ m, l \geq n_0$

$\left(\sum_{i=1}^{n} |x_i^m - x_i^l|^p\right)^{1/p} < \epsilon \rightarrow$ Ⓐ

$\Rightarrow \sum_{i=1}^{n} |x_i^m - x_i^l|^p < \epsilon^p$

$\Rightarrow |x_i^m - x_i^l|^p < \epsilon^p$ i = 1 to n.

$\Rightarrow |x_i^m - x_i^l| < \epsilon$ i = 1 to n (8)

$\therefore \{x_i^m\}_{m=1}^{\infty}$ is a cauchy sequence in $\mathbb{C}$

(or) $\mathbb{C}$ for i = 1 to n

Hence $\{x_i^m\}$ converges for i = 1, ... to n

$z = (z_1, z_2 \dots z_n)$

$\therefore z \in l_p^n$

$\left(\sum_{i=1}^{n} |x_i^m - z_i|^p\right)^{1/p} < \epsilon \quad \forall\ m \geq n_0$

$\| x_i - z_i \|_p < \epsilon$

$\therefore \{x_m\}$ converges to ~~[illegible]~~

$z \in l_p^n$

$\therefore l_p^n$ is a banach spaces

2) The Space l_∞^n of all n tuples $x = (x_1, x_2 \dots x_n)$ of scalar in a Banach Space under the norm defined by $\|x\|_\infty = \max\{|x_1| |x_2| \dots |x_n|\}$

Proof:

$|x_i| \geq 0 \quad \forall\ i = 1$ to n

$\Rightarrow \|x\|_\infty \geq 0$

$\|x\|_\infty = 0 \Leftrightarrow \max\{|x_1| |x_2| \dots |x_n|\}$

$\Leftrightarrow |x_i| = 0$ for i = 1 to n

$\Leftrightarrow x_i = 0 \quad \forall\ i = 1$ to n

$\Leftrightarrow x = 0$

$\|\alpha x\| = \max\{|\alpha x_1|, |\alpha x_2| \dots |\alpha x_n|\}$

$= \max\{|\alpha||x_1|, |\alpha||x_2| \dots |\alpha||x_n|\}$

$= |\alpha| \max\{|x_1|, |x_2|, \dots |x_n|\}$ 9

$= |\alpha| \, \|x\|_\infty$

$\|x+y\|_\infty = \max\{|x_1+y_1|, |x_2+y_2|, \dots |x_n+y_n|\}$

$\leq \max\{|x_1|+|y_1|, |x_2|+|y_2|, \dots |x_n|+|y_n|\}$

$\leq \max\{|x_1|, |x_2| \dots |x_n|\} + \max\{|y_1|, |y_2| \dots |y_n|\}$

$= \|x\|_\infty + \|y\|_\infty$

Hence L^n_∞ is a normal linear space

Let $\{x_m\}$ be a cauchy sequence in l^n_∞.

where $x_m = (x_1^m, x_2^m \dots x_n^m)$

Let $\epsilon > 0$ then there exist a +ve integer no.

Such that $\|x_m - x_p\| < \epsilon$ for all $m, p \geq n_0$.

$\max\{|x_1^m - x_1^p|, |x_2^m - x_2^p| \dots |x_n^m - x_n^p|\} < \epsilon$ (*)

$\Rightarrow |x_i^m - x_i^p| < \epsilon, \; i = 1, 2 \dots n$

$\{x_i^m\}_{m=1}^{\infty}$ cauchy sequence in R (or) C

Hence it converges for $i = 1, 2 \dots n$

let, $\lim_{m \to \infty} x_i^m = z_i, \quad i = 1, 2 \dots n$

let $z = (z_1, z_2 \dots z_n)$

In (*), let $p \to \infty$ then

$\max\{|x_1^m - z_1|, |x_2^m - z_2| \dots |x_n^m - z_n|\} < \epsilon$

$\forall n, m \geq n_0$

$|x_i^m - z_i| < \epsilon$

$\Rightarrow \|x_m - z_i\|_\infty < \epsilon \; \forall m \geq n_0$

$\therefore \{x_m\}$ converges to $z \in L^n_\infty$

$\therefore L^n_\infty$ is a Banach space.

3) The Space $C(X)$ defined as the space of all bounded continuous scalar valued functions defined on X and it is a Banach Space with the norm of $f \in C(X)$ defined by, $\|f\| = \operatorname{Sup}\{|f(x)| \; x \in X\}$

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Soln :

i) $|f(x)| \geq 0$

$\operatorname{Sup}\{|f(x)| \geq 0\} \; \forall x \in X$

$\|f\| \geq 0$

ii) $\|f\| = 0$

$\Leftrightarrow \operatorname{Sup}\{|f(x)|\} = 0 \; \forall x \in X$

$\Leftrightarrow |f(x)| = 0$

$\Leftrightarrow f(x) = 0$

iii) $\|\alpha f\| = \operatorname{Sup}\{|\alpha f(x)|\}$

$= |\alpha| \operatorname{Sup}\{|f(x)|\}$

$\|\alpha f\| = |\alpha| \; \|f\|$

iv) $\|f(x) + g(x)\| = \operatorname{Sup}\{|f(x) + g(x)|\}$

$= \operatorname{Sup}\{|f(x)| + \operatorname{Sup}\{|g(x)|\}$

$\|f(x) + g(x)\| = \operatorname{Sup}\{|f(x) + g(x)|\}$

$= \operatorname{Sup}\{|f(x)|\} + \operatorname{Sup}\{|g(x)|\}$

$\|f(x) + g(x)\| = \|f(x)\| + \|g(x)\|$

$\|f + g\| \leq \|f\| + \|g\|$

Hence the Space $C(X)$ is a normed linear space

$\|f_m - f_n\| < \epsilon.$

$\sup \{ |f(x_m) - f(x_n)|, x_n, x_m \in X \} < \epsilon$ (11)

$|f(x_m) - f(x_n)| < \epsilon$.

$\{f(x_n)\}_{n=1}^{\infty}$ is a cauchy's sequence

$\Rightarrow \{f_n\}_{n=1}^{\infty}$ is convergent

$\Rightarrow \lim_{n\to\infty} f(x_n) = f(x) \Rightarrow f(x_n) \to f(x)$

$\therefore \lim_{n\to\infty} \| f(x_m) - f(x_n) \| < \epsilon, \ m, n \geq n_0$

$|f(x_m) - f(x)| < \epsilon, \ m > n.$

$\Rightarrow \sup \{ |f(x_m) - f(x)|, x_m \in X \} < \epsilon, \ m \geq n.$

$\therefore f(x_m) \to f(x)$:

Theorem: (X) 5M

(X) Let M be a closed linear subspace of a normed linear space N for each coset $x+M$ is the quotient space N/M, we define

$\| x+M \| = \inf \{ \|x+m\|, m \in M \}$ Then N/M is a normed linear space further if N is a banach space then N/M is also a banach's space.

Proof:

i) $\forall \ m \in M, \|x+m\| \geq 0.$

$\Rightarrow \inf \{ \|x+m\| ; m \in M \} \geq 0$

$\Rightarrow \|x+M\| \geq 0$

ii) To prove, $\|x+M\| = 0$

$\Leftrightarrow x+M$ is the zero element of N/M

$\Leftrightarrow x+M = M$

ie) To prove

$\|x+M\| = 0 \Leftrightarrow x \in M$

$\|x+M\| = 0 \Leftrightarrow \inf\{\|x+m\|, m\in M\} = 0$

$\Leftrightarrow$ There exist a sequence $\{m_n\}$ in M such that

$m_n + x \to 0$

ie) $\|m_n + x\| \to 0$ as $n \to \infty$

ie) $\lim\limits_{n\to\infty} m_n = -x$

$\Leftrightarrow -x \in M$ $\{\because M$ is linear$\}$

iii) $\|\alpha(x+M)\| = \inf\{\|\alpha(x+m)\| ; m\in M\}$

$= \inf\{|\alpha| \|x+m\| ; m\in M\}$

$= |\alpha| \inf\{\|x+m\|; m\in M\}$

$= |\alpha| \|x+M\|$

iv) $\|(x+M)+(y+M)\| = \inf\{\|x+m_1+y+m_2\| ; m_1, m_2 \in M\}$

$\leq \inf\{\|x+m_1\| + \|y+m_2\| ; m_1, m_2 \in M\}$

$\leq \inf\{\|x+m_1\|; m_1 \in M\} + \inf\{\|y+m_2\| ; m_2 \in M\}$

$= \|x+M\| + \|y+M\|.$

$\therefore \|(x+M)+(y+M)\| \leq \|x+M\| + \|y+M\|$

$\therefore N/M$ is a normed linear space.

Suppose, $\{s_N + M\}$ is a cauchy sequence in N/M

we can find the subsequence of $\{s_N+M\}$

$x_1, x_2 \ldots x_n$ such that

$\|(x_1+M) - (x_2+M)\| < 1/2 \to (1)$

$\|(x_2+M) - (x_3+M)\| < (1/2)^2 \to (2)$

$\|(x_n+M)-(x_{n+1}+M)\| < (1/2)^n \rightarrow (n)$

Let us choose the vector $y_1 \in x_1+M$ (13)

$\ni, \; y_1 = x_1+m_1 ; \; m_1 \in M$

Then,

from (1), we have

$\|(x_1+M)-(x_2+M)\| < 1/2$

$\Rightarrow \|(x_1-x_2)+M\| < 1/2$

$\inf \{\|(x_1+m)-(x_2+m)\| ; m \in M\} < 1/2.$

$\Rightarrow \inf \{\|(x_1-x_2)+m\| ; m \in M\} < 1/2.$

$\Rightarrow \exists \; m_0 \ni \|x_1-x_2+m_0\| < 1/2.$

$\Rightarrow \|x_1-m_1+m_1-x_2+m_0\| < 1/2.$

$\Rightarrow \|(x_1+m_1)-(x_2+m_1-m_0)\| < 1/2$

$\Rightarrow \|y_1-(x_2+m_1-m_0)\| < 1/2.$

$\Rightarrow \|y_1-y_2\| < 1/2.$

where,

$y_2 = x_2+m_1-m_0$

$= x_2+m_2$

$y_2 \in x_2+M$

Now,

we can choose $y_3 \ni$

$y_3 = x_3+m_3 \; , m_3 \in M$

$\|(y_2-y_3)\| < (1/2)^n$

continuing this process, we get,

$\|y_n-y_{n+1}\| < (1/2)^n.$

Let us consider,

$\| \;\; \| \;\; \| \;\; y_{m+1} + y_{m+1} - y_{m+1} + \cdots$

$\leq \|y_m - y_{m+1}\| + \|y_{m+1} - y_{m+2}\| + \dots + \|y_{n+1} - y_n\|$

$< (1/2)^m + (1/2)^{m+1} + \dots + (1/2)^{n-1}$ (14)

$= \sum_{k=m}^{n-1} (1/2)^k$

$= \|y_m - y_n\| < \sum_{k=m}^{\infty} (1/2)^k$

now, $\sum_{k=m}^{\infty} (1/2)^k = (1/2)^m + (1/2)^{m+1} + \dots$ is a geometric progression with common ratio and whose sum is

G.P (formula) $= \frac{a(1-r^n)}{1-r}$

$$\frac{(1/2)^m \left[1 - (1/2)^n\right]}{1 - 1/2} = \frac{1/2^m - 1/2^{m+n}}{1/2}$$

$$= \frac{1}{2^{m-1}} - \frac{1}{2^{m+n-1}}$$

$$< \frac{1}{2^{m-1}}$$

Let us choose, $\epsilon > \frac{1}{2^{m_0 - 1}}$ for a positive integer m_0

For $n, m \geq m_0$

$$\|y_m - y_n\| < \sum_{m}^{\infty} (1/2)^k < \frac{1}{2^{m-1}} = \epsilon$$

$\|y_m - y_n\| < \epsilon$

$\therefore \{y_n\}$ is a cauchy sequence in N

But N is complete.

$\therefore \{y_n\}$ converges in N

Let. $\lim(y_n) = y$ where $y \in N$

Let us consider,

$$\|(x_n+m) - (y+M)\| = \inf\{\|x_n - y + M\|, m \in M\}$$

$$\leq \|x_n - y + M\|$$

$$= \|(x_n+m) - y\|$$

$$= \|y_n - y\|$$

$$\|(x_n+m) - (y_n+m)\| \leq \|y_n - y\|$$

Since $y_n \to y$

$$\|y_n - y\| \to 0 \text{ as } n \to \infty$$

$$\therefore \|(x_n+y) - (y+M)\| \to 0 \text{ as } n \to \infty$$

$\{s_n + M\}$ is a cauchy sequence in N/M and

$\{x_n + M\}$ is a convergent sequence in N/M.

We know that,

Every cauchy sequence has a convergent

Subsequence.

Here, $\{x_n + M\}$ is a convergent sequence of

$\{s_n + M\}$

$\therefore$ $\{s_N + M\}$ converges in N/M

$\therefore$ N/M is a Banach space.

Theorem:

Let N and N' be the normed linear Space and T be a linear transformation of N into N'. Then the following condition equivalent one another.

i) T is continuous

ii) T is continuous at the origin.

ie) $x_n \to 0 \Rightarrow T(x_n) \to 0$.

iii) There exist a real number $k \geq 0$ such that

$\|T(x)\| \leq k\|x\| \ \forall x \in N$

iv) If $S = \{x : \|x\| < 1\}$ is a closed unit sphere in N then its image T(S) is bounded in N'.

Proof :

(i) $\Leftrightarrow$ (ii)

T is cts.

$x_n \to x \Rightarrow T(x_n) \to T(x)$

Suppose, $x_n \to 0$

$T(x_n) \to T(0)$

$\Rightarrow T(x_n) \to 0$

$\Rightarrow T(x_n) \to 0$

$\Rightarrow$ T is cts at the origion

conversely,

Let T be cts at the origin

$x_n \to 0 \Rightarrow T(x_n) = 0$

To prove : T is cts

Let $x_n \to x$

$x_n - x \to 0 \Rightarrow T(x_n - x) \to T(0) = 0$

$\Rightarrow T(x_n - x) \to 0$

$\Rightarrow T(x_n) - T(x) \to 0$

$\Rightarrow T(x_n) \to T(x)$

$\Rightarrow T$ is cts ($\because T$ is a linear transformation)

(ii) $\Rightarrow$ (iii)

Let T is cts at the origin

To prove there exist $k \geq 0$, such that

$$\|T(x)\| \leq k\|x\| \quad \forall x \in N$$

Suppose there is no such k to satisfy the above condition.

ie) For all $k \geq 0$ the inequality does not hold for some $x \in N$

For all +ve integer n, there exist $x_n \in N$ such that

$$\|T(x_n)\| \geq n\|x_n\|$$

$$\Rightarrow \frac{\|T(x_n)\|}{n\|x_n\|} > 1, \forall n$$

$$\Rightarrow \left\| \frac{1}{n\|x_n\|} T(x_n) \right\| > 1 \to (*), \forall n.$$

$$\Rightarrow \left\| T\left(\frac{x_n}{n\|x_n\|}\right) \right\| > 1 \to (1)$$

put $y_n = \dfrac{x_n}{n\|x_n\|}$

$$\|y_n\| = \frac{\|x_n\|}{n\|x_n\|} = \frac{1}{n}$$

$\therefore \|y_n\| \to 0 \quad n \to \infty$

$T(y_n) = T\left(\frac{x_n}{n\|x_n\|}\right)$

$$\|T(y_n)\| = \left\| T\left(\frac{x_n}{n\|x_n\|}\right)\right\| > 1$$

$\Rightarrow$ T is not continuous at the origin

$\Rightarrow$ This is contradiction

Hence, there must exist a $k > 0$

Such that $\|T(x)\| \leq k\|x\| \;\forall x \in N \rightarrow$ ② holds.

$\Rightarrow$ To prove, T is continuous at the origin

Let $\{x_n\}$ be a sequence in N.

Such that $x_n \rightarrow 0$

$\therefore \|x_n\| \rightarrow 0 \;\; \|0\| = 0$

$\Rightarrow \|x_n\| \rightarrow 0$

$\|T(x_n)\| < k\|x_n\| <$ by ②

$\therefore \|x_n\| \rightarrow 0 \Rightarrow \|T(x_n)\| \rightarrow 0$

$\Rightarrow T(x_n) \rightarrow 0$

T is cts at the origin

(iii) $\Rightarrow$ (iv)

Suppose $k \geq 0$ such that ② holds.

$S = \{x : \|x\| \leq 1\}$ is a closed unit sphere in N ② true for $x \in N'$

$\therefore \forall x \in S \; \|T(x)\| \leq k\|x\|$

$\therefore \|T(x)\| \leq k$ ($\because \|x\| \leq 1$ for $x \in S$).

$\therefore \{T(x) : x \in S\}$ is bounded above by k in N'

$\Rightarrow T(S)$ is a bounded set in N'

Conversely,

$T(S)$ be bounded set in N'

There exist $k \geq 0$ such that.

$\| T(x) \| < k \ \forall x \in S.$

If $x = 0$ then $\| x \| = 0$

$\| T(x) \| = \| T(0) \| = 0$

$\therefore \| T(x) \| \leq k \| x \| \rightarrow$ ③

If $x \neq 0$ there exists $y = \frac{x}{\|x\|}$

$\| y \| = 1 \Rightarrow y \in S.$

$\therefore \| T(y) \| < k.$

$\Rightarrow \left\| T\left(\frac{x}{\|x\|}\right) \right\| \leq k$

$\Rightarrow \left\| \frac{1}{\|x\|} T(x) \right\| < k$ $\quad (\because \| \alpha x \| = \alpha \| x \|)$

$\Rightarrow \frac{1}{\|x\|} \| T(x) \| \leq k$

$\Rightarrow \| T(x) \| \leq k \|x\| \rightarrow$ ④

From ③ and ④ we get.

$\| T(x) \| \leq k \| x \| \ \forall x \in N'$

$T(S)$ is bounded in N' iff $\| T(x) \| \leq k.$

Definition:

Let T be a linear transformation from a normed linear space N into a normed linear space N'

If $\| T(x) \| \leq k \| x \| \ \forall x \in N$ for some $k \geq 0$ then T is said to be bounded linear transformation.

Result:

A linear transformation T from normed linear space N into a normed linear space N' is continuous iff it is bounded.

Proof:

T is continuous

$\Leftrightarrow$ T is continuous at the origin

$\Leftrightarrow$ There exist $k \geq 0$ such that $\|T(x)\| \leq k\|x\|$

$\Leftrightarrow$ T is bounded

Defn:- Suppose, $T: N \to N'$ is continuous.

define its norm by $\|T\| = \sup\{\|T(x)\|; x \in N, \|x\| \leq 1\}$

Theorem:

Let N and N' be Normed linear space and T be a bounded linear transformation from N into N'.

Let $a = \sup\{\|T(x)\|; x \in N, \|x\| = 1\}$

$$b = \sup\left\{\frac{\|T(x)\|}{\|x\|}, x \in N, x \neq 0\right\}$$

$c = \inf\{k : k \geq 0\ \|T(x)\| \leq k\|x\|\ \forall x \in N\}$

Then, $\|T\| = a = b = c$

Proof:-

Let $S = \{\|T(x)\|; x \in N, \|x\| = 1\}$

$S_a = \{\|T(x)\|; x \in N, \|x\| = 1\}$

$$S_b = \left\{\frac{\|T(x)\|}{\|x\|}; x \in N, x \neq 0\right\}$$

$S_c = \{k, k \geq 0, \|T(x)\| \leq k\|x\|, \forall x \in N\}$

clearly, $S_a \subseteq S$

$\therefore \sup S_a \leq \sup S$

$a \leq \|T\| \to (1)$

$c = \inf\{k : k \geq 0, \|T(x)\| \leq k\|x\|, \forall x \in N\}$

$\therefore \|T(x)\| \le c \cdot \|x\| \;\forall\, x \in N.$

ie) $\|x\| \le 1$. then $\|T(x)\| \le c$.

$\therefore$ c is the upperbound of S.

$\therefore \|T\| \le c \rightarrow$ ②

For all $x \neq 0$. $\dfrac{\|T(x)\|}{\|x\|} \le b$

$\|T(x)\| \le b\|x\| \;\forall\; x \neq 0$ in N

Suppose, $x = 0 \Rightarrow T(x) = 0$

$\Rightarrow \|T(x)\| = 0$ and $b\|x\| = 0$

$\therefore$ If $x = 0$ then $\|T(x)\| \le b\|x\|$

$\therefore \|T(x)\| \le b\|x\| \;\forall\, x \in N$

$\Rightarrow b \in S_c \Rightarrow c \le b \rightarrow$ ③

If $x \neq 0$; $\dfrac{\|T(x)\|}{\|x\|} \in S_b$

$\Rightarrow \left\| T\left(\dfrac{x}{\|x\|}\right) \right\| \in S_b$

consider, $\dfrac{x}{\|x\|}$ then $\left\| \dfrac{x}{\|x\|} \right\| = 1$

$\Rightarrow \left\| T \cdot \dfrac{x}{\|x\|} \right\| \in S_a$

$\Rightarrow S_b \subseteq S_a$

$\Rightarrow b \le a \rightarrow$ ④ {$\therefore$ using ①, ②, ③ & ④}

$a \le \|T\| \le c \le b \le a$

$\Rightarrow a = b = c = \|T\|$

Theorem:

If N and N' are normed linear Space. Then the Space $\mathcal{B}(N,N')$ of all continuous transformations of N into N' is itself is normed linear space with respect to the pointwise linear operations and the norm define by.

$$\|T\| = \sup\{\|T(x)\| ; x \in N, \|x\| \leq 1\}$$

Further, N' is a banach space then $\mathcal{B}(N,N')$ is also a banach space.

Proof: If $T_1, T_2 \in \mathcal{B}(N,N')$

Then, $\alpha T_1 + \beta T_2$ is a linear transformation

$\Rightarrow T_1 + T_2$ are bounded

There exist $K_1 \geq 0$ and $K_2 \geq 0$ such that,

$$\left.\begin{array}{l} \|T_1(x)\| = K_1\|x\| \\ \|T_2(x)\| = K_2\|x\| \end{array}\right\} \forall x \in N$$

$$\|\alpha T_1 + \beta T_2\| = \|\alpha T_1(x) + \beta T_2(x)\|$$

$$\leq |\alpha| \, \|T_1(x)\| + |\beta| \, \|T_2(x)\|$$

$$\leq \alpha . K_1\|x\| + \beta . K_2\|x\|$$

$$\alpha T_1(x) + \beta T_2(x) \leq (\alpha K_1 + \beta K_2)\|x\| \geq 0$$

$$\alpha T_1 + \beta T_2 \in \mathcal{B}(N,N')$$

$\therefore \mathcal{B}(N,N')$ linear Space

$$\|T\| = \sup\{\|T(x)\| ; x \in N, \|x\| \leq 1\}$$

i) $\|T(x)\| \geq 0, \forall x \in N$

$$\|T(x)\| \geq 0$$

$$\|T\| \geq 0$$

ii) $\|x\| = 0 \Leftrightarrow x = 0$

$$\|T\| = 0 \Leftrightarrow \operatorname{Sup}\{\|T(x)\|, x \in N, \|x\| \leq 1\} = 0$$

$$\Leftrightarrow \operatorname{Sup}\left\{\frac{\|T(x)\|}{\|x\|}; \|x\| \leq 1\right\} = 0$$

$$\Leftrightarrow \frac{\|T(x)\|}{\|x\|} = 0 \ \forall x \neq 0$$

$$\Leftrightarrow \|T(x)\| = 0$$

$$\Leftrightarrow \|T(x)\| = 0 \ \forall x$$

$$\Leftrightarrow T = 0$$

iii) $\|\alpha T\| = \operatorname{Sup}\{\|T(x); x \in N, \|x\| \leq 1\}$

$$= \operatorname{Sup}\{\|\alpha T(x)\|; x \in N, \|x\| \leq 1\}$$

$$= \operatorname{Sup}\{|\alpha| \, \|T(x)\|, x \in N, \|x\| \leq 1\}$$

$$= |\alpha| \operatorname{Sup}\{\|T(x)\|, x \in N, \|x\| \leq 1\}$$

$$= |\alpha| \, \|T\|$$

iv) $\|T_1(x) + T_2(x)\| = \|T_1\| + \|T_2\|$

triangle inequality

$$\leq \operatorname{Sup}\{\|T_1(x)\| + \|T_2(x)\|, x \in N, \|x\| \leq 1\}$$

$$+ \operatorname{Sup}\{\|T_1(x)\| + \|T_2(x)\|, x \in N, \|x\| \leq 1\}$$

$$= \|T_1(x)\| + \|T_2(x)\|$$

$\therefore$ $\mathbb{B}(N, N')$ is a normed linear space

Let $\{T_n\}$ be a cauchy sequence in $\mathbb{B}(N, N')$

To prove: Banach space is complete every cauchy sequence is convergence.

$\mathbb{B}(N, N')$ is a banach space when N' is a banach space.

Let x be any vector in N.

Then $\{T_n(x)\}$ is a sequence in N'

Since, T_n is a cauchy sequence in $\mathbb{B}(N, N')$

There exist a real number no such that,

$\|T_n - T_m\| < \epsilon, \ \forall n, m \geq n_0$ (or)

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$\|T_n - T_m\| \to 0$ as $n, m \to \infty$

consider, $\|T_n(x) - T_m(x)\| = \|(T_n - T_m)(x)\|$

$\leq \|T_n - T_m\|$

$\|T_n(x) - T_m(x)\| \to 0$

$\therefore \{T_n(x)\}$ is a cauchy sequence in N'

$\therefore \{T_n(x)\}$ is a cauchy sequence in N'

$\lim_{n\to\infty} T_n(x) = T(x) \ \forall x \in N$

To prove :

T is linear

$T(\alpha x_1 + \beta x_2) = \lim_{n\to\infty} T_n(\alpha x_1 + \beta x_2)$

$= \lim T_n \cdot \alpha x_1 + \lim T_n \beta x_2$

$= \alpha \lim T_n(x_1) + \beta \lim T_n(x_2)$

$= \alpha T_1(x) + \beta T_2(x)$

$\Rightarrow$ T is linear

To prove:

T is bounded

Let us consider,

$\|T\| = \sup\{\|T(x)\|, x \in N, \|x\| \leq 1\}$

$\|T\| = \sup\{\lim T_n(x), x \in N, \|x\| \leq 1\}$

$\leq (\sup \|T_n\|) \|x\|$

$\|T\| \leq k\|x\|$ for some $k \geq 0$

$\Rightarrow$ T is bounded

$\Rightarrow$ T is continuous

To prove, $T_n \to T$

$\| T_n(x) - T(x) \| = \| T_n(x) - T_m(x) + T_m(x) - T(x) \|$

$\leq \| T_n(x) - T_m(x) \| + \| T_m(x) - T(x) \| \;\forall x \in N$

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$\leq \epsilon/2 + \epsilon/2$

$= \epsilon$

$\Rightarrow T_n \to T$

$\therefore \mathbb{B}(N, N')$ is a banach Space.

10m **Hahn Banach Theorem:**

Let M be a linear subspace of a normed linear space N and let f be a functional defined on M. Then f can be extended to a functional to defined on the whole space N. such that $\|f\| = \|f_0\|$

Proof:

Before proving this theorem we have To Prove the following lemma.

Lemma:

Let M be a linear subspace of a normed linear Space N and let f be a functional defined on M. If x_0 is a vector not in M and if $M_0 = M + [x_0]$ is the linear subspace spanned by M and x_0, then f can be extended to a functional f_0 on M_0 such that $\|f\| = \|f_0\|$

Proof: case (i):

By assuming N be a real linear space without loss of generality.

Assume that $\|f\| = 1$

Since, $x_0 \notin M_0$, each vector $y \in M_0$ is uniquely expressed as $y = x + \alpha x_0$ where $x \in M$

Define $f_0(x + \alpha x_0) = f_0(x) + f_0(\alpha x_0)$

Put $f_0 x_0 = r_0$

$$= f(x) + \alpha r_0$$

Let us show that the definition of extence f linearly to M_0 for every choice of the real numbers

$$r_0 = f_0(x_0)$$

Let $w_1, w_2 \in M_0$

Then, $w_1 = x + \alpha_1 x_0$, $w_2 = y + \alpha_2 x_0$ where $x, y \in M$

$$\beta w_1 + \gamma w_2 = \beta(x + \alpha_1 x_0) + \gamma(y + \alpha_2 x_0)$$

$$= \beta(x) + \alpha_1 \beta(x_0) + \gamma(y) + \alpha_2 \gamma(x_0)$$

$$f_0(\beta w_1 + \gamma w_2) = f_0[\beta(x) + \alpha_1 \beta(x_0) + \gamma(y) + \alpha_2 \gamma(x_0)]$$

$$= f_0[\beta x + \gamma y + (\alpha_1 \beta + \alpha_2 \gamma) x_0]$$

$$= f_0[\beta x + \gamma y] + f_0[\alpha_1 \beta + \gamma \alpha_2] x_0$$

$$f_0(\beta w_1 + \gamma w_2) = f[(\beta x + \gamma y) + (\alpha_1 \beta + \alpha_2 \gamma) f_0(x_0)]$$

$f[(\beta x + \gamma y) + \alpha_1 \beta f_0(x_0) + \alpha_2 \gamma f_0(x_0)$

$$= \beta f(x) + \gamma f(y) + \alpha_1 \beta \cancel{x_0}\, r_0 + \alpha_2 \gamma \cancel{x_0}\, r_0]$$

$$= \beta[f(x) + \alpha_1 \cancel{x_0}\, r_0] + \gamma[f(y) + \alpha_2 \cancel{x_0}\, r_0]$$

$= \beta[f_0(x) + \alpha_1 f_0(x_0)] + \gamma[f_0(y) + \alpha_2 f_0(x_0)]$

$$\therefore \gamma_0 = f_0(x_0) = \beta[f_0(x + \alpha_1 x_0) + \gamma(f_0(y + \alpha_2 x_0))]$$

$$f_0(\beta w_1 + \gamma w_2) = \beta f_0(w_1) + \gamma f_0(w_2)$$

$\therefore$ f_0 is linear on M_0 for $x \in M$

$x.0 = x.(0+0)$
$= x.0 + x.0$

$$x = x + 0.x_0$$

$$\Rightarrow f_0(x) = f_0(x + 0.x_0)$$

$$f_0(x) = f(x) + 0.x_0$$

$$f_0(x) = f(x)$$

(27)

Now to prove

$$|f_0(x+\alpha x_0)| \le \|x+\alpha x_0\| \quad \forall x \in M, \forall \alpha \neq 0 \rightarrow (1)$$

The above inequality can be written as,

$$-\|x+\alpha x_0\| \le f_0(x+\alpha x_0) \le \|x+\alpha x_0\|$$

$$\Rightarrow -\|x+\alpha x_0\| \le f(x) + \alpha \gamma_0 \le \|x+\alpha x_0\|$$

$$\Rightarrow -f(x) - \|x+\alpha x_0\| \le \alpha.\gamma_0 \le -f(x) + \|x+\alpha x_0\|$$

$$\Rightarrow -f(x) - \|\alpha(x/\alpha + x_0)\| \le \alpha.\gamma_0 \le -f(x) + \|\alpha(x/\alpha + x_0)\|$$

$$\Rightarrow -f(x) - |\alpha|\,\|x/\alpha + x_0\| \le \alpha.\gamma_0 \le -f(x) + |\alpha|\,\|x/\alpha + x_0\|$$

$$\Rightarrow -1/\alpha\, f(x) - \|x/\alpha + x_0\| \le \gamma_0 \le -1/\alpha\, f(x) + \|x/\alpha + x_0\|$$

$$\Rightarrow -f(x/\alpha) - \|x/\alpha + x_0\| \le \gamma_0 \le -f(x/\alpha) + \|x/\alpha + x_0\| \rightarrow (2)$$

To prove (1) it is enough to prove

(2) Now for any $x_1, x_2 \in M$.

$$\begin{aligned} f(x_2) - f(x_1) &\le |f(x_2 - x_1)| \\ &\le \|f(x_2 - x_1)\| \\ &\le \|f\|\,\|x_2 - x_1\| \\ &\le \|x_2 - x_1\| \\ &\le \|x_2 + x_0 - x_0 - x_1\| \end{aligned}$$

$$f(x_2) - f(x_1) \le \|x_2 + x_0\| + \|x_0 + x_1\| \rightarrow (3)$$

Define, two real numbers a and b such that

$$a = \sup\{-f(x) - \|x + x_0\| ; x \in M\}$$

$$b = \inf\{-f(x) + \|x + x_0\| ; x \in M\}$$

Since, x_1 is arbitary.

eqn. (3) becomes.

$$-f(x_1) - \|x_1 + x_0\| \leq -f(x_2) + \|x_2 + x_0\|$$ (28)

$$\Rightarrow -f(x) - \|x - x_0\| \leq -f(x_2) + \|x_2 + x_0\|$$

$$\text{Sup}\{-f(x) - \|x + x_0\| ; x \in M\} \leq -f(x_2) + \|x_2 + x_0\|$$

$$\Rightarrow a \leq -f(x_2) + \|x_2 + x_0\|$$

Also, x_2 is arbitary

$$\therefore a \leq -f(x) + \|x + x_0\|$$

$$\leq \inf\{-f(x) + \|x + x_0\|\}$$

$$a \leq b$$

$\Rightarrow$ choose a real number r_0 such that $a \leq r_0 \leq b$

$$\text{Sup}\{-f(x)\} - \|x + x_0\|, x \in M\} \leq r_0 \leq \inf \{-f(x) + \|x + x_0\| ; x \in M\}$$

$$\Rightarrow -f(x) - \|x + x_0\| \leq r_0 \leq -f(x) + \|x + x_0\| \ \forall x \in M$$

$\Rightarrow$ (2)

$\Rightarrow$ (1)

ie) $|f_0(x + \alpha x_0)| \leq \|x + \alpha x_0\| \ \forall x \in M, \alpha \neq 0$

Now, $\|f_0\| = \text{Sup}\{|f_0(y)| ; y \in M_0, \|y\| \leq 1\}$

$$= \text{Max}\{\text{Sup}\{|f_0(y)|, y \in M, \|y\| \leq 1\}$$

$$\{\text{Sup}\, |f_0(y)| ; y \in M_0\ y \notin M, \|y\| \neq 1\}$$

$$= \text{Max}\{\|f\|, \text{Sup}\{|f_0(x + \alpha x_0) ; x \in M, \alpha \neq 0\ \|x + \alpha x_0\| < 1\}$$

$$= \text{Max}\{1, 1\}$$

$$\|f_0\| = 1 = \|f\|$$

$$\|f_0\| = \|f\|$$

Hence, the real space.

Case (ii) :

Let N be a complex normal linear space, f is a complex valued functional defined on M ∋ $\|f\| = 1$

$$f(x) = g(x) + i h(x)$$

where g and h are real and imaginary parts

Here 'g' and 'h' are real valued funs on M.

Since, $\|f\| = 1$ we have $\|g\| \leq 1$

[for $\|g\| = \sup\{|g(x)|;\ x \in M\ \|x\| \leq 1\}$

$\leq \sup\{|f(x)|;\ x \in M,\ \|x\| \leq 1\}$ $\quad \therefore |g| \leq \|f\|$

$= \|f\|$

$\therefore \|g\| \leq 1$].

Now $f(ix) = i f(x)$ $\quad i^2 = -1$

$\Rightarrow g(ix) + i h(ix) = i[g(x) + i h(x)]$

$\Rightarrow g(ix) + i h(ix) = i g(x) - h(x)$

$\Rightarrow g(ix) = -h(x) + h(ix) = g(x)$

$\therefore f(x) = g(x) + i h(x)$

$= g(x) + i[-g(ix)]$

$f(x) = [g(x) - i g(ix)]$

Here, g is a real valued function on M.

By case (i) it can be extended to a real valued functional g_0 on M_0 ∋ $\|g_0\| = \|g\|$

$f_0(x) = g_0(x) - i g_0(ix)$ $\forall x \in M_0$

Clearly, we have $f_0(x+y) = f_0(x) + f_0(y)$

$$f_0(\alpha x) = \alpha f_0(x) \quad \forall \text{ real } \alpha\text{'s}$$

Suppose α is complex, $\alpha = 0+i = i$ (30)

$$f_0(\alpha x) = f_0(ix)$$

$$f_0(\alpha x) = g_0(ix) - i\, g_0[i(ix)]$$

$$= g_0(ix) + i\, g_0(x)$$

$$= i[g_0(x) - i\, g_0(ix)]$$

$$= i\, f_0(x)$$

$$f_0(\alpha x) = \alpha . f_0(x)$$

where α is complex.

$\therefore$ f_0 is a complex valued functional in the complex space M_0.

Let $x \in M_0$ be any vector for which $\|x\| = 1$

Then to prove $|f_0(x)| \leq 1$.

$$\|g_0\| = \|g\| = \sup\{|g(x)| ; x \in M, \|x\| \leq 1\}$$

$$\leq \sup\{|f(x)| ; x \in M, \|x\| \leq 1\}$$

$$= \|f\|$$

$$= 1$$

$$\|g_0\| \leq 1$$

[If f_0 is real than $\rightarrow$ (*)

$$f_0(x) = g_0(x)$$

$$\Rightarrow |f_0(x)| = |g_0(x)|$$

$$\Rightarrow |f_0(x)| \leq 1]$$

If f_0 is complex, then let $f_0(x) = re^{i\theta}$ with $r>$

$|f_0(x)| = |re^{i\theta}| = re^{i\theta}e^{-i\theta}$ $\quad re^{i\theta}$

$= e^{-i\theta}\cdot f_0(x)$ $\quad = |r||e^{i\theta}|$

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$= r|(\cos\theta + i\sin\theta)|$

Now,

$= r\sqrt{\cos^2\theta + \sin^2\theta}$

$\|e^{-i\theta}(x)\| = |e^{-i\theta}|\,\|x\|$ $\quad |z| = x^2+y^2$

$= \|x\| = 1$ $\quad = re^{i\theta}e^{-i\theta}$

$\therefore e^{-i\theta}x$ is a vector in M such that $\|e^{-i\theta}x\| = 1$

$\therefore |f_0(e^{-i\theta}x)| \le 1 \Rightarrow |f_0(x)| \le 1$

$\{[f_0(e^{-i\theta}x) = e^{-i\theta}f_0(x) = e^{-i\theta}re^{i\theta} = r = \text{real}]$

$\therefore$ by $(*)$, $|f_0(e^{-i\theta}x)| \le 1\}$

$\therefore |f_0(x)| \le 1$ whenever $\|x\| = 1$

Now, $\|f_0\| = \sup\{|f_0(x)| ; \|x\| \le 1\}$

$= \sup\{f_0(x) ; \|x\| \le 1\}$

$\|f_0\| = 1$ Also, $\|f\| = 1$

$\|f_0\| = \|f\|$

Hence proved the lemma.

Proof of the theorem:

let $x_0 \in M$ and $M_0 - M + [x_0]$ Then by the lemma f can be extended to a functional of f_0 on M_0 Such that $\|f\| = \|f_0\|$

we can defined the set S, such that $S = \{g$ | g is a function on $M' \supset M$ and $\|g\| = \|f\|$ where f is extended to g on $M'\}$

To prove:

S is a partially ordered set w.r.to the relation $\leq$ which is defined below

$g_1 \leq g_2 \Leftrightarrow$ domain of $g_1 \subseteq$ domain of g_2

i) $\forall\ g \in S$ domain of $g \subseteq$ domain of g 32

$g \leq g\ \forall\ g \in S$

$\leq$ is reflexive

ii) Let $g_1 \leq g_2$ and $g_2 \leq g_1$

$\Rightarrow$ domain of $g_1 \subseteq$ domain of g_2 and

domain of $g_2 \subseteq$ domain of g_1

$\Rightarrow$ domain of $g_1 \subseteq$ domain of g_2

$g_1 = g_2$.

$\Rightarrow$ $\leq$ is ~~anti~~symmetric

iii) Let, $g_1 \leq g_2$ and $g_2 \leq g_1$

$\Rightarrow$ domain of $g_1 \subseteq$ domain of g_2 and

domain of $g_2 \subseteq$ domain of g_2

$\Rightarrow$ domain of $g_1 \subseteq$ domain of g_2

$\Rightarrow g_1 \leq g_2$

'$\leq$' is transitive

$\therefore S$ is partially ordered set.

Let $\mathcal{Q}$ be any chain in S

Let $F = \bigcup_{g_i \in \mathcal{Q}} g_i$ defined on $\cup M_i$

where, M_i domain of g_i and g_i's are linear

clearly $\cup M_i \supset M$

clearly, f is a linear functional on $\cup M_i$ for

all $g_i \in \mathcal{Q}$, $g \leq f$

$\Rightarrow$ $\mathcal{Q}$ has an upper bound.

$\Rightarrow$ Since, $\mathcal{Q}$ is an arbitrary.

$\Rightarrow$ Every chain of $\mathcal{Q}$ is bounded above

$\Rightarrow$ S has a maximal element say to < By zero lemma >

domain of $f_0 = N$

If not (i.e.) domain of $f_0 = M_0$ such that, $M_0 \neq N$.

$\exists$ M_{00} such that $M_0 \supset M_0$ and f_0 extended f_{00} such that

$\|f_0\| = \|f_{00}\|$

$\Rightarrow f_0 \leq f_{00}$

$\Rightarrow \Leftarrow$

$\therefore f_0$ is maximal

$\therefore M_0 = N$

$\therefore f$ extended to f_0 on N such that, $\|f\| = \|f_0\|$

Theorem :-

Let N be a normed linear space and x_0 a non zero vector in N. Then, there exist a function f_0 in N^* such that $f_0(x_0) = \|x_0\|$ and $\|f_0\| = 1$

Proof :-

Let $M = \{\alpha x_0\}$ be the space span by x_0

Suppose f is a functional on N Then, define

$f(\alpha x_0) = \alpha \|x_0\|$

Let, $w_1, w_2 \in M$ then $w_1 = \alpha_1 . x_0$, $w_2 = \alpha_2 x_0$

$\therefore f[(\alpha w_1) + \beta w_2] = f[\alpha(\alpha_1 x_0) + \beta(\alpha_2 x_0)]$

$= f[(\alpha\alpha_1)x_0 + \beta(\alpha_2)(x_0)]$

$= \alpha\alpha_1\|x_0\| + \beta\alpha_2\|x_0\|$

$= \alpha f(\alpha_1 x_0) + \beta f(\alpha_2 x_0)$

$= \alpha f(w_1) + \beta f(w_2)$

$\therefore f$ is linear

$|f(\alpha x_0)| = |\alpha \| x_0\||$

$\leq |\alpha| \|x_0\|$

f is bounded

$\therefore$ f is functional on M

By Hahn Banach's theorem,

f can be extend to fo on N

Such that $\|f\| = \|f_0\|$

ie) $f(x_0) = f_0(x_0)$

$f_0(x_0) = f(1.x_0) = 1.\|x_0\|$

$\|f\| = \sup\{|f(x)|; x \in M, \|x\| \leq 1\}$

$= \sup\{\|x\|; x \in M, \|x\| \leq 1\}$

$= \sup\{\|x\|; x \in M; \|x\| = 1\}$

$\|f\| = 1$

Hence, proved

Result :

N^* Seperates the vector N, if $x, y \in N$ and $x \neq y$. Then there exist a functional f in N such that $f_0(x) \neq f_0(y)$

Given, $x \neq y$

$\Rightarrow x - y \neq 0 \in N$

$\Rightarrow \|x-y\| \neq 0$ (35)

f_0 in N^* such that,

~~$f_0(x-y) = \|x-y\| \|f_0\|$~~

$f_0(x) - f_0(y) = f_0(x-y)$

$= \|x-y\|$

$f_0(x) - f_0(y) \neq 0$

$\therefore f_0(x) \neq f_0(y)$

$\therefore$ Hence, proved.

Theorem :-

Let M is a closed linear subspace of a normed linear space, N and x_0 is a vector not in M. Then there exist a functional f_0 in N^* such that f_0 of M is equal to zero

ie) $f_0(M) = 0$ and $f_0(x_0) \neq 0$

Proof :-

$T(M) = 0$

$x_0 \notin M$

$\Rightarrow x_0 + M = 0$ in N/M

By theorem, there exist a functional f_0 in $(N/M)^*$

$f_0(x_0 + M) = \|x_0 + M\| \quad \|f_0\| = 1$

$f_0(x) = f\,T(x)$

$f_0(\alpha x + \beta y) = f[\alpha(d_1 x) + \beta(d_2 y)]$

$= f[(\alpha T)x + (\beta T)y]$

$$= \alpha f[T(x)] + \beta f[T(y)]$$

$$fo(\alpha x + \beta y) = \alpha\, fo(x) + \beta\, fo(y)$$

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$\therefore fo$ is linear

Now,

$$|fo(x)| = |f(T(x))|$$
$$\leq \|f\|\,\|T(x)\|$$
$$\leq \|f\|\,\|T\|\,\|x\|$$
$$\|fo(x)\| \leq \|f\|\,\|x\| \qquad \therefore \|T\| \leq 1$$

Since, f is bounded and fo is bounded

$\therefore fo$ is functional in N^*

$$fo(M) = f(T(M))$$
$$= f(0)$$
$$fo(M) = 0$$
$$fo(x_0) = f(T(x_0))$$
$$= f(x_0 + M)$$
$$= \|x_0 + M\|$$
$$\therefore fo(x_0) \neq 0$$

Definition :- Second conjugate Space :

Let N be a normed linear space Then it's conjugate space N^* is also a normed linear space Hence, the conjugate Space of N^* can be found It is denoted by, N^{**} and called the Second conjugate Space of N.

ie) $f \in N^*$; $f : N \to R$ or C.

$F \in N^{*}$; $f : N^{*} \to R$ or C.

Theorem:

Each vector x in N induces the functional F_x in N^{**} defined by $F_x(f) = f(x)$ $\forall f \in N^{*}$ and the mapping $x \to F_x$ is an isometric isomorphism of N into N^{**}. F_x is called the induced functional on N^{**}

Proof:

i) $F_x(\alpha f + \beta g) = (\alpha f + \beta g)(x)$

$$= \alpha f(x) + \beta g(x)$$

$$= \alpha F_x(f) + \beta F_x(g)$$

F_x is linear

ii) $|F_x(f)| = |f(x)| \leq \|f\| \, \|x\|$

$\because$ f is bounded

$\Rightarrow$ F_x is bounded

F_x is a functional on N^{**}

$J : N \to N^{**}$ defined by $J(x) = F_x$

$$J(x+y) = f(x+y)$$

$$= f(x) + f(y)$$

$$F_{x+y}(f) = F_x(f) + F_y(f) \quad \forall f$$

$$F_{x+y} = F_x + F_y$$

$$J(x+y) = F_{x+y}$$

$$= F_x + F_y$$

$$J(x+y) = J(x) + J(y)$$

$$J \alpha x = F_{\alpha x}$$

$$F_{\alpha x}(f) = f(\alpha x)$$

$= \alpha f(x)$

$= \alpha \cdot F_x(f) \ \forall f$

$= \alpha \cdot F_\alpha$

(38)

$J(\alpha x) = \alpha \cdot F_x$

$= \alpha \cdot J(x)$

J is linear

$J(x) = J(y)$

$J(x) - J(y) = 0$

$J(x-y) = 0$

$x - y = 0$

$x = y$

$\therefore J$ is one to one

$F_x(f) = F(x) \ \forall \ f \in N^*$

$\| F_x \| = \text{Sup} \{ | F_x(f) | ; f \in N^* \text{ and } f \leq 1 \}$

$= \text{Sup} \{ | f(x) | ; f \in N^* \text{ and } \| f \| \leq 1 \}$

$\leq \text{Sup} \{ \| f \| \| x \| , f \in N^*, \| f \| \leq 1 \}$

$\leq \text{Sup} \{ \| x \| \}$

$= \| x \|$

$\| F_x \| \leq \| x \| \rightarrow (1)$

Suppose, $x = 0 \Rightarrow \| x \| = 0$

$F_x f = F_0(f) = f(0) = 0 \quad \forall x \in N^*$

$F_x = 0 \Rightarrow \| F_x \| = 0$

$\| x \| = \| F_x \| \quad \therefore$ if $x = 0$

$x \neq 0 \ \therefore x$ is a non-zero vector in N,

lies $\mathcal{F}_1$, $f_0 \in N^*$

Such that $f_0(x) = \| x \| \quad \| f_0 \| = 1$

$\|x\| = f_0(x)$

$= |f_0(x)| \quad \{ \|x\| > 0 \}$

$\leq \sup \{ |F(x)| ; x \in N, \|x\| \leq 1 \}$

$= \sup \{ |F_x(f)| ; f \in N^*, \|f\| \leq 1 \}$

$\|x\| \leq \|F_x\| \rightarrow (2)$

From (1) and (2), we have

$\|F_x\| = \|x\|$.

$J : N \rightarrow N^*$ is an isometric isomorphism.

Definition:

i) If 'A' is a subset of a metric space x and if $A = x$ then, A is said to be tends to X

ii) A is said to be no where iff $(\bar{A})^0 = \phi$, where A^0 is the inferior point of A

$\Rightarrow A^0 - \{$ interior point of A $\}$

$x \in A^0 \Rightarrow x$ is an interior point of A.

Definition:

Let x and y be two metric space and $f : X \rightarrow Y$ be continuous if $f^{-1}(G)$ is open for every open set G in Y.

f is called an open mapping $f(G)$ is open in y whenever G is open in x The homogeneous one to one continuous mapping of one topological space onto another is also an open mapping.

Note:

Let B and B' are Banach space. Then $S(x,r)$ & $S'(x,r)$ denote the open sphere centred on x with radius 'r' in B and B' respectively (40)

For $x \in B$; $S(x,r) = \{y \mid y \in B, \|y-x\| < r\}$

$x \in A$; $S'(x,r) = \{y \mid y \in B', \|y-x\| < r\}$

S_r, S_r' denote the open spheres centred at the origin with radius 'r' in B and B' respectively.

Result:

$S(x,r) = x + S_r$ and $S_r = r S_1$

$y \in S(x,r) \Leftrightarrow \|y-x\| < r$

$\Leftrightarrow \|z\| < r$, $z = y - x$

$\Leftrightarrow \|z\| < r$, $y = z + x$

$\Leftrightarrow z \in (0,r)$, $y = z + x$

$\Leftrightarrow z \in S_r$, $y = z + x$

$\Leftrightarrow y \in S_r + x$

$S(x,r) = x + S_r$

$S_r = \{x, \|x\| < r\}$

$= \{x, \frac{\|x\|}{r} < 1\}$

Put $x/r = y \Rightarrow x = ry$

$\|y\| = \|x/r\|$

$= \frac{1}{r}\|x\|$

$= \{x, \frac{\|x\|}{r} < 1\}$

$S_r = \{ry, \|y\| < 1\} \Rightarrow r\{y, \|y\| < 1\}$

$= r S_1$

Barie category theorem:

Every complete metric space is of second category as a subset to itself (41)

Definition:

A subset of a metric space is said to be is of it can be written as the union of countable family of no where dense set otherwise, it is called a set of second category

1 st category $A = \bigcup_{1}^{n} \{\text{nowhere dense set}\}$

Lemma:

If B and B' are Banach space and if r is a continuous linear transformation of B onto B' then the image of each open sphere centred on the origin in B contains on open sphere centred on the origin in B'

Proof:

Let S_r and S_r' denote the open sphere centred on the origin in B and B' respectively

$T(S_r) = T(r S_1) = r T(S_1) \rightarrow (1)$

For every positive integer n,

Let us define $B' = \cup T(S_n)$

Since B' is complete.

$\therefore$ By Barie category theorem B' is of 2nd category

$\therefore T(S_n)$ is not nowhere fon for some.

i.e. $T(\overline{S_{n_0}})^{\circ} \neq \phi$

Scanned with CamScanner

Claim: (i)

There exist $y_0 \in T(\overline{S_{n_0}})$ such that $y_0 \in T(\overline{S_{n_0}})$ and y_0 is an interior point of $T(S_{n_0})$. (42)

$T(\overline{S_{n_0}})^{\circ} \neq \phi \Rightarrow [T(\overline{S_{n_0}})$ has an interior point]

$\Rightarrow$ There exist an open set G such that,

$$y \in G \subset T(\overline{S_{n_0}})$$

$\Rightarrow y \in T(\overline{S_{n_0}})$, (If $P \subset M$ and N, M^* intersects N where $M \cap N$ neighbourhood)

$\Rightarrow$ Every neighbourhood y intersects $T(S_{n_0})$

G is a neighbourhood of $y \Rightarrow$ (G intersects $T(\overline{S_{n_0}})$

Let, $y_0 \in G \cap T(\overline{S_{n_0}}) \rightarrow$ (2)

$y_0 \in G \Rightarrow$ and $y_0 \in T(\overline{S_{n_0}})$

$y_0 \in G \subset T(\overline{S_{n_0}})$

$\Rightarrow y$ is an interior point of $T(\overline{S_{n_0}})$ now from (3)

y_0 is an interior point of $T(\overline{S_{n_0}}) \rightarrow$ (3)

Such that,

Let $f : B' \rightarrow B'$ by $f(y) = y - y_0 \rightarrow$ (4)

Claim: (ii)

f is homomorphism

$$f(y_1) = f(y_2)$$

$$y_1 - y_0 = y_2 - y_0$$

$$y_1 = y_2$$

$\therefore f$ is one to one

Let $x \in B'$, $x + y_0 \in B'$

$f(x + y_0) = x + y_0 \ldots$

$= x$

f is onto.

Suppose, $\{y_n\}$ be a sequence in $B' \to y \in B'$

Then, $F(y_n) = y_n - y_0 \to y - y_0 = F(y)$

$\Rightarrow F(y_n) = F(y)$

f is continuous

If $g : B' \to B'$ such that, $g(y) = y + y_0$. Then $g = f^{-1}$

$$f(g(y)) = F(g(y))$$
$$= F(y + y_0)$$
$$= y + y_0 - y_0$$
$$= y$$
$$= g(f(y))$$
$$= g(y - y_0)$$
$$= y - y_0 + y_0$$
$$= y$$

$f^{-1} = g$

Now, $y_n \to y$ and $g(g_n) = y_n + y_0 \to y + y_0 = g(y)$

$\therefore$ g is continuous

$\Rightarrow$ f is a homomorphism.

ie) The mapping $y \to y - y_0$ is a homomorphism.

case-(iii)

'0' is an interior point of $T(\overline{S_{n_0}}) - y_0$

y_0 interior point of $T(\overline{S_{n_0}})$ ($\because$ from (3))

$\therefore$ There exist an open set G_0 in B' such that

$\Rightarrow F(y_0) \in F(G_0) \subset F(\overline{T(S_{n_0})})$

$\Rightarrow y - y_0 \in F(G_0) \subset \overline{T(S_{n_0})} - y_0$

$\Rightarrow 0 \in F(G_0) \subset \overline{T(S_{n_0})} - y_0$

f is homomorphism

$\Rightarrow f$ is an open mapping from $B' \to B'$

Since, G_0 is open in B and (G_0) is an open set in B' '0' is an interior point of $\overline{T(S_{n_0})} - y_0$

claim : (iv)

$T(S_{n_0}) - y_0 \subset T(S_2 n_0)$.

$\Rightarrow$ There exist $x \in S_{n_0}$ such that, $y = T(x) = y_0$

$y_0 \in T(S_{n_0})$ (by claim)

$y_0 \in T(x_0)$ for some $x_0 \in S_{n_0}$

$y = T(x) - T(x_0) = T(x - x_0)$

$x_1, x_0 \in S_{n_0} \Rightarrow \|x\| < n_0$ and $\|x_0\| < n_0$

$\Rightarrow \|x - x_0\| \leq \|x\| + \|x_0\| < 2n_0$

$\Rightarrow (x - x_0) \in (S_2 n_0)$

$\Rightarrow T(x - x_0) \in T(S_2 n_0)$

$\Rightarrow y_0 \; T(S_2 n_0)$

$\Rightarrow T(S_{n_0}) - y_0 \subset T(S_2 n_0)$

$\Rightarrow T(S_{n_0}) - y_0 \subset 2n_0 \cdot T(S_1)$

$\Rightarrow T(\overline{S_{n_0} - y_0}) \subset \overline{2n_0 \cdot T(S_1)} \rightarrow (5)$

Since, F is a homomorphism.

$f(\overline{T(S_{n_0})}) = T\overline{(S_{n_0})} - y_0 \subset \overline{(2n_0 T(S_1))}$

$\Rightarrow T\overline{(S_{n_0})} - y_0 \subset 2n_0 \, T\overline{(S_1)} \rightarrow (6)$

{by (4) and (5)}

By claim S_1, '0' is an interior point of

$\therefore$ 0 is an interior point of $T(\overline{S_1})$

$\therefore$ There exist $\epsilon > 0$ such that $S'_\epsilon \subset T(\overline{S_1}) \to$ 7

Claim : (v)

$S_\epsilon' \subset T(S_1)$

Let y be an arbitrary point of S_ϵ'

$\therefore \|y\| < \epsilon$

$\Rightarrow$ Every neighbourhood of y intersects $T(S_1)$

Let $N_{\epsilon/2}(y)$ be a neighbourhood of y in B'.

$N_{\epsilon/2}(y) \cap T(S_1) \neq \phi$

$y_1 \in N_{\epsilon/2}(y) \Rightarrow \|y - y_1\| < \epsilon/2$

$y_1 \in T(S_1) \Rightarrow y_1 = T(x_1)$ for some $x_1 \in S_1$

$\|x_1\| < 1$

From 7,

$S'_{\epsilon/2} \in T(\overline{S_{1/2}})$

Now $\|y - y_1\| < \epsilon/2$

$\Rightarrow y - y_1 \in S'_{\epsilon/2} \subset T(\overline{S_{1/2}})$

$\therefore y - y_1$ is a point of $T(S_{1/2})$

$\therefore$ Every nbhd of $y - y_1$ intersects $T(S_{1/2})$.

$\|y - y_1 - y_2\| < \epsilon/2^2$ and $y_2 \in T(S_{1/2})$ for some $y_2 \in B'$

$\Rightarrow y_2 = T(x_2)$ for some $x_2 \in S_{1/2}$

$\therefore \|x_2\| < 1/2$

Continuing this we get a sequence $\{x_n\}$ in B.

$\Rightarrow \|x_n\| < \frac{1}{2^{n-1}}$ and $\|y - (y_1 + y_2 + \cdots + y_n)\| < \epsilon/2^n$

$\to$ 8

where, $y_n = T(x_n)$

Define, $S_n = x_1 + x_2 + \cdots + x_n$

$$\|S_n\| = \|x_1 + x_2 + \cdots + x_n\|$$
$$\leq \|x_1\| + \|x_2\| + \cdots + \|x_n\|$$
$$< \{1 + 1/2 + \cdots + 1/2^{n-1}$$
$$= \frac{1-(1/2)^n}{1-1/2}$$

$$\|S_n\| \leq [2(1-1/2)^n]' < 2$$

$\|S_n\| \leq 2 \rightarrow (2)$ fn $n > m$

$$\|S_n - S_m\| = \|x_1 + x_2 + \cdots + x_n - x_1 + x_2 + \cdots + x_m\|$$
$$= \|x_{m+1} + x_{m+2} + \cdots + x_n\|$$
$$\leq \|x_{m+1}\| + \|x_{m+2}\| + \cdots + \|x_n\|$$
$$< \frac{1}{2}m + \frac{1}{2}m+1 + \cdots + \frac{1}{2n-1}$$
$$= \frac{1}{2}m\left[1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2n-m+1}\right]$$
$$= \frac{1}{2^m}\left[\frac{1 - 1/2^{n-m}}{1-1/2}\right]$$
$$= \frac{1}{2^{m-1}}\left(1 - 1/2^{n-m}\right)$$
$$= \frac{1}{2^{m-1}} - \frac{1}{2^{n-1}}$$

$\|S_n - S_m\| \to 0$ as $n \to \infty$

$\therefore \{S_n\}$ is a cauchy sequence in B and hence, converges $(S_n \to x)$

Let $\lim S_n = x$, $\|x\| = \|\ell + S_n\|$

$$= \lim \|S_n\|$$

≤ 2

< 3

$\|x\| < 3$

$\Rightarrow x \in S_3 = 3 - S_1$

$\Rightarrow x \in S_1$

$y_1 + y_2 + \cdots + y_n = T(x_1) + T(x_2) + \cdots + T(x_n)$

$= T(x_1 + x_2 + \cdots + x_n)$

T is continuous and $s_n \to x$

$\Rightarrow T(s_n) \to T(x)$

$T(x) = \lim T(s_n)$

$= \lim T(x_1 + x_2 + \cdots + x_n)$

$= \lim (y_1 + y_2 + \cdots + y_n)$

$= y$ (from (S1))

$y = T(x)$ for some $x \in S_3$

$\therefore y \in T(S_3)$

Thus, $y \in S_\epsilon' \Rightarrow y \in T(S_3)$

$S_\epsilon' \subset T(S_3)$

$\Rightarrow S_\epsilon' \subset 3T(S_1)$

$\Rightarrow S'_\epsilon \subset T(S_1)$ ($\because$ hence claim)

From (1)

$S_\epsilon \subset T(S_1) = T(S_r)$.

$\Rightarrow S_\epsilon \subset T(S_r)$

Hence, the image of every open sphere centered on the origin in B under T contains an open sphere centered on the origin in S.

Theorem: [open mapping theorem] (X) 10M

If B and B' are Banach spaces and if T is continuous linear transformation B onto B' Then T is open mapping. (48)

Proof:

To prove:

$T(G)$ is open in B'

For every open set G in B

Let $y \in T(G)$

There exist $x \in G$ such that $y = T(x)$

$x \in G$, G is open in B.

There exist an open sphere on x say $S(x,r)$ Such that $S(x,r) \subset G$

$\Rightarrow x + S_r \subset G$

$\Rightarrow T(x + S_r) \subset T(G)$

$\Rightarrow T(x) + T(S_r) \subset T(G)$

$\Rightarrow y + T(S_r) \subset T(G)$

$\Rightarrow S_r$ is an open sphere (on) the origin in B

By Lemma,

$T(S_r)$ contains an open sphere S'_ϵ, such that

$S'_\epsilon \subset T(S_r)$

$y + S'_\epsilon (y + T(S_r) \subset T(G))$

$\Rightarrow S'(y, \epsilon) \subset T(G)$.

$\Rightarrow T(G)$ is an open set in B'

(T is an open mapping)

Theorem :-

A H continuous linear transform of are banach Space onto another banach space is a homomorphism In particular of a linear transform (T) of a banach space onto itself in continuous then the inverse (T^{-1}) is automatically continuous.

(49)

Proof:

By open mapping theorem.

" T is an open mapping".

$\Rightarrow$ T is homomorphism.

If $B = B'$ then T^{1} is continuous.

Projection :-

A vector space V be the direct some of the two surface ie) $M \cap N = 0$

Then, each elements $z \in V$ can be uniquely expressed as $z = x + y$, where $x \in M$ and $y \in M$

The projection of M along N is a linear operator "E" define by $E(z) = x$

similarly, the profection on N along M is a linear operator 'E' define by $E'(z) = y$

A linear transformation E on V is,

ie) E is projection, iff $E^2 = E$

Range and Null Space of E :-

Let us define two subspaces M and N. Such that,

$$M = \{ z \in V ; E(z) = z \}$$

$$N = \{ z \in V : E(z) = 0 \}$$

Sphere :

M is the range of E and N is the Null space

Projection on a Banach Space :

A projection p on a Banach space B is an independent linear operator on B is also continuous

Theorem : [closed Graph Theorem] 5m

If B and B' are Banach space and iff T is a linear transformation of B into B'. Then, T is continuous iff graph is closed.

Proof :

Let us show that, $B \times B'$ is a Banach space under the norm define by,

$$\|(x,y)\|_1 = \|x\| + \|y\|$$

To prove,

$\|(x,y)\|_1$ is a Banach space

i) $\|(x,y)\|_1 \geq 0$ $\|x\| \geq 0$ and $\|y\| \geq 0$

$$\|x\| + \|y\| \geq 0$$

ii) $\|(x,y)\|_1 = 0 \Leftrightarrow \|x\| + \|y\| = 0$

$$\Leftrightarrow \|x\| = 0 \text{ and } \|y\| = 0$$

iii) $\|\alpha(x,y)\|_1 = \|\alpha x, \alpha y\|$

$$= \|\alpha x\| + \|\alpha y\|$$

$$= |\alpha|\|x\| + |\alpha|\|y\|$$

$$= |\alpha| \|(x,y)\|$$

iv) $\|(x,y)\| = \|x_1 + x_2, y_1 + y_2\|$

$$\leq \|x_1 + x_2\| + \|y_1 + y_2\|$$

$$\leq \|x_1\| + \|x_2\| + \|y_1\| + \|y_2\|$$

$= [\|x_1\| + \|y_1\|] + [\|x_2\| + \|y_2\|]$

$\|(x_1, y_1) + (x_2, y_2)\| \leq \|(x_1, y_1)\|_1 + \|(x_2, y_2)\|_1$

B and B' is a normed linear space Let, TG is continuous

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To prove:

TG is closed

Let (x, y) be any limit point of TG.

There exist a sequence $(x_n, T(x_n))$ such that

$\{x_n, T(x_n)\} \to (x, y)$

ie) limit $\{x_n, T(x_n)\} = (x, y)$

$\therefore \|x_n, T(x_n) - (x, y)\| \to 0$ as $n \to \infty$

$\|(x_n - x), T(x_n - y)\| \to 0$

$\Rightarrow \|x_n - x\| + \|T(x_n) - y\| \to 0$

$\|x_n - x\| \to 0$ and $\|T(x_n) - y\| \to 0$

$x_n \to x$, $T(x_n) \to y$

But T is continuous, and $x_n \to x$

$\therefore T(x_n) \to T(x)$

we have $y = T(x)$

$\therefore (x, y) = (x, T(x)) \in TG$

$\therefore (x, y) \in TG$

$\therefore$ TG is closed

Conversely,

Assume that, TG is closed

Let B_0 be the space B removed by,

$\|x\|_0 = \|x\| + \|T(x)\|$

Clearly, B_0 is a normed linear space Let us show that B_0 is a Banach space

Let $\{x_n\}$ be a cauchy sequence in B_0

$\|x_n - x_m\|_0 \to 0$ as $n, m \to \infty$ (52)

$\Rightarrow \|x_n - x_m\| + \|T(x_n - x_m)\| \to 0$

$\Rightarrow \|x_n - x_m\| \to 0$ and $\|T(x_n - x_m)\| \to 0$

$\Rightarrow$ $\{x_n\}$ be a cauchy sequence in B and $\{T(x_n)\}$ is a cauchy sequence in B'

$\therefore$ B is complete

$\Rightarrow$ $\{T(x_n)\}$ is converges in B'

Let $\{T(x_n)\} \to y \in B'$

Now, $x_n \to x$, $T(x_n) \to y$

$\therefore$ $\{x_n + T(x_n)\}$ is a sequence in T_G converges to (x, y)

$T_G = \{x, T(x) \mid x \in B\}$

Given,

T_G is closed

$\Rightarrow (x, y) \in T_G$

$y = T(x)$

$\|T(x)\| \leq \|x\| + \|T(x)\| = \|x\|_0$

T is a mapping of B_0 into B'

Now, $y = T(x)$

$\Rightarrow T(x_n) \to T(x)$

$\therefore x_n \to x$

$\Rightarrow T(x_n) \to T(x)$

$\therefore$ T is continuous mapping of B into B'

T is cts $B \to B'$

Let T be the identify map

$I : B_0 \to n$

$I(x) = x$

$\therefore$ I is one to one - onto

$\|I(x)\| \leq \|x\| + \|T(x)\| = \|x_0\|$

I is continuous as a mapping of B_0 onto B since B and B_0 are Banach space

I is homomorphism

The uniform Boundedness theorem:

Let B be a Banach space and N be a normed linear space. If $\{T_i\}$ is a non-empty set of cts linear transformation of B into N with the property that $\{T_i(x)\}$ is a bounded subset of N for each vector x in B. Then $\{\|T_i\|\}$ is a bounded set of numbers, that is $\{T_i\}$ is bounded as a subset of $\mathcal{B}(B,N)$

Proof:

For each +ve integer 'n'

Define $F_n - S_n;\ x \in B$ and $\|T_i(x)\| \leq n\ \forall i$

claim (i):

F_n is a closed subset of B $\forall n$

Fix, n

$x \in F_n \Leftrightarrow \|T_i(x)\| \leq n\ \forall i$

$\Leftrightarrow T_i(x) \in S_n^c$, where S_n^c

denotes the closed sphere centred on the origin with reached n in, N, iff

$x \in F_n \Leftrightarrow T_i^{-1}(S_n^c)\ \forall i$

$\Leftrightarrow x \in \cap T_i^{-1} S_n^c$

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$$F_n = \cap T_i^{-1}(S_n^c)\ \forall i$$

T_i is cts and S_n^c closed

$\therefore T_i^{-1}(S_n^c)$ is a closed set

But, arbitrary intersects of closed set is closed

$\therefore F_n$ is closed

claim (ii):

$$B = \bigcup_{1}^{\infty} F_n$$

Suppose, $B \neq \cup F_n$

Then there exist $x \in B$ such that, $x \notin \cup F_n \Rightarrow$

$x \notin F_n\ \forall n$.

For each n, there is one to one in such that,

$\| T_{in}(x) \| > n$

Given $\{T_i(x)\}$ is bounded

$\therefore$ There exist k such that $\| T_i(x) \| \leq k\ \forall i$

Let n_0 be the +ve integer such that $k \leq n_0$, then

$\| T_i(x) \| \leq k \leq n_0\ \forall i$

$\Rightarrow \| T_i(x) \| \leq n_0\ \forall i$

which is $\Rightarrow \Leftarrow$ to the fact that for n_0

there is sum "i" such that $\| T_i(x) \| > n_0$

$\Rightarrow B = \cup F_n$.

B is expressed as a countable union of

Sets.

B is complete hence, by ~~lemma~~ baire category theorem B is of 2nd category

Suppose that, Fn for every n Fn is nowhere dense

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Then, B is of 1^{st} category

which is $\Rightarrow\Leftarrow$

There exist some n_0 such that F_{n_0} is not nowhere dense $\overline{F_{n_0}} \neq \phi$

Let $x_0 \in \overline{F_{n_0}}$

x is an interior point of $\overline{F_{n_0}}$

There exist an open sphere S centred on x_0 and $S \subset \overline{F_{n_0}}$ from

claim (i)

F_n is closed $\forall n$

$\therefore F_n = \overline{F_n} \; \forall n$

$\Rightarrow \overline{F_{n_0}} = F_{n_0}$

$\Rightarrow x_0 \in S \subset F_{n_0} \rightarrow (1)$

Let 'S' be a radius r_0

Then, $S = \{x : \|x - x_0\| < r_0\} \subset F_{n_0}$

$= \{x : \|x - x_0\| < r_0\} \subset \overline{F_{n_0}} = F_{n_0} \rightarrow (2)$

$\Rightarrow \bar{S} = \{x : \|x - x_0\| \leq r_0\}$

If $\|y\| \leq 1$, then for any arbitrary

But for fixed i

$\|T_i(y)\| = \|T_i(z/r_0)\| \quad z = r_0 y$

$= \|\frac{1}{r_0} T_i(z)\|$

$= \frac{1}{r_0}\|T_i(z + x_0 - x_0)\|$

$$= \frac{1}{r_0} \| T_i(z+x_0) - T_i(x_0) \|$$

$$\leq \frac{1}{r_0} \left[\| T_i(z+x_0) \| + \| T_i(x_0) \| \right] \rightarrow (3)$$

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$x_0 \in F_{n_0}$ from (1)

$$\Rightarrow \| T_i(x_0) \| \leq n_0$$

$$\| z + x_0 - x_0 \| = \|z\| = \| r_0 y \| = r_0 \|y\| \leq r_0$$

$$\Rightarrow \| z + x_0 - x_0 \| \leq r_0$$

$$\therefore z + x_0 \in \bar{S} \subset F_{n_0} \quad \{ \therefore \text{From } (3) \}$$

$$(3) \Rightarrow \| T_i(y) \| \leq \frac{1}{r_0} \left[\| T_i(z+x_0) \| + \| T_i(x_0) \| \right]$$

$$\leq \frac{1}{r_0} (n_0 + n_0)$$

$$\| T_i(y) \| \leq \frac{2n_0}{r_0} \text{ if } \|y\| \leq 1$$

$$\| T_i \| = \sup \{ \| T_i(y) \|, \|y\| \leq 1 \} \leq \frac{2n_0}{r_0}$$

$\{ \| T_i \| \}$ is a bounded set of numbers.

Continuous Linear transformation:

Let N and N' be two normed linear space with the same scalar and let T be the linear transformation of N into N'. Then, T is said to be continuous if it is cts as a mapping of the metric space N into the metric space N'

Note:

i) we denote the set of all continuous (or) bounded linear transformation of N into N' by $\mathcal{B}(N, N')$

$\|T\| = \text{Inf} \{ k ; k \geq 0, \|T(x)\| \leq k\|x\| \ \forall x \}$

ie) $\|T(x)\| \leq \|T\| \, \|x\|$

ii) we can consider the set of all linear transformation from N into N' is denoted by $\mathcal{B}(N,N')$ in short. $\mathcal{B}(N)$ and every such transformation is called an operator N.

Definition:

The set of all continuous linear transformation from N into R (or) C according N is real (or) complex and is denoted by N^*

The Set N^* is called the conjugate space of N and the element of the N^* are called linear functional (or) simply functional.

Note:

i) N^* is always a banach space

ii) Norm of a functional f in N is defined by $\|f\| = \text{Sup} \{ \|f(x)\| ; x \in N \ \|x\| \leq 1 \}$

Definition:

Let N and N' be normed linear space. An isometric and isomorphism and N into N' is a onto one linear transformation N into N' such that

$$\|T(x)\| = \|x\| \quad \forall x \in N$$

and N is said to be isomatrically isomorphism to N'

If there exist an isometric isomorphism from N into N'

Theorem :

A non-empty subset X of a normed linear Space N is bounded iff $F(X)$ is a bounded set of numbers for each f in N^*

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Proof :

Let X be a bounded subset of N. There exist a real number k, such that,

$$\|x\| \leq k, \ \forall x \in X$$

$f \in N^*$, F is a continuous linear transformation

$\Rightarrow$ f is bounded

$k_2 > 0$ such that $\| F(x) \| \leq k_2 \|x\|$

$\Rightarrow$ $F(X)$ is bounded

conversely,

Suppose $F(X)$ is a bounded set of numbers for each $F \in N^*$

let, $X = \{x_i\}$

N is a normed linear space.

Each x_i induces a functional (F_{x_i} on N^*) in N^{**} such that

$$F_{x_i}(F) = f(x_i) \ \forall f \in N^*$$

$$\| F_{x_i} \| = \|x_i\|$$

$\Rightarrow$ $F(x)$ is bounded Set

$\Rightarrow$ $\{ F(x_i) \}$ is a bounded Set

$\Rightarrow$ $\{ F_x(f) \}$ is a bounded Set

we have N^* is a banach space and C is a normed linear space.

$F_{x_i} : N^* \to C$

and each F_{x_i} is continuous

$\{F_{x_i}\}$ is a cts $Fn : N^* \to C \ \forall \ F \in N^*$

$\{F_{x_i}(f)\}$ is a bounded set

All the conditions of uniform bounded theorem are satisfied

$\Rightarrow \{\|F_{x_i}\|\}$ is a bounded set

$\Rightarrow \{\|x_i\|\}$ is a bounded set

$\therefore X$ is a bounded set

UNIT –II

UNIT - II

(1)

Hilbert Space :- A-19 8m

A Hilbert space is a complex banach space whose norm ~~atex~~ araises from an inner product inwhich there is define a complex function (x, y) of vector x and y with the following properties

i) $(\alpha x+\beta y, z) = \alpha(x, z)+\beta(y, z)$

ii) $\overline{(x, y)} = (y, x)$

iii) $(x, x) = \|x\|^2$

Result :-

i) $(\alpha x-\beta y, z) = \alpha(x, z)-\beta(y, z)$

ii) $(x, \alpha y+\beta z) = \bar{\alpha}(x, y)+\bar{\beta}(x, z)$

iii) $(x, \alpha y-\beta z) = \bar{\alpha}(x, y)-\bar{\beta}(x, z)$

iv) $(x, 0) = (0, x) = 0 \quad \forall x \in H$

Proof:

(ii) $(x, \alpha y+\beta z) = \overline{(\alpha y+\beta z, x)}$

$= \overline{\alpha(y,x)+\beta(z,x)}$

$= \bar{\alpha}\,\overline{(y,x)} + \bar{\beta}\,\overline{(z,x)}$ (2)

$= \bar{\alpha}(x,y) + \bar{\beta}(x,z)$

iv) $(0,x) = (0\cdot\bar{0}, x)$

$= 0(\bar{0}, x)$

$= 0$

Schwartz's inequality: (X) 5m

(X) 10M If x and y any two vectors in a hilbert space H then $|(x,y)| \le \|x\|\,\|y\|$

Proof:

$y=0 \Rightarrow \|y\|=0$

$\Rightarrow \|x\|\,\|y\|=0$

$(x, y) = (x, 0) = 0 \quad \forall x \in H$

claim: $|(x,y)| \le \|x\|\,\|y\|, \; y \neq 0$

If $\|y\|=1$ then $|(x,y)| \le \|x\|$

$\|x-(x,y)\|^2 = (x-(x,y)y, x-(x,y)y)$

$= (x,x) - (x,y)(y,x) - \overline{(x,y)}(x,y) + (x,y)\overline{(x,y)}(y,$

$= \|x\|^2 - (x,y)\overline{(x,y)} - \overline{(x,y)}(x,y) + (x,y)\overline{(x,y)}$

$= \|x\|^2 - |(x,y)|^2$

≥ 0

$\|x\|^2 \geq |(x,y)|^2$.

$\Rightarrow |(x,y)| \le \|x\|$

Now, $\left|\frac{y}{\|y\|}\right| = \frac{\|y\|}{\|y\|} \Rightarrow 1$

$$\Rightarrow \frac{1}{\|y\|} |(x,y)| \leq \|x\|.$$

$$\Rightarrow |(x,y)| \leq \|x\| \|y\|$$

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Result:

Inner product is jointly continuous in a hilbert space. If $x_n \to x$ and $y_n \to y$

Then,

$$\|(x_n, y_n) - (x,y)\| = |(x_n, y_n) - (x_n, y) + (x_n, y) - (x,y)|$$

$$= |x_n, (y_n - y) + (x_n - x) y|$$

$$\leq |x_n, y_n - y| + |x_n - x, y|$$

Parallelogram law in a Hilbert space:

If x and y are any two Hilbert space then

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

Proof:

$$\|x+y\|^2 + \|x-y\|^2 = (x+y, x+y) + (x-y, x-y)$$

$$= (x,x) + (y,x) + (x,y) + (y,y) + [(x,x) - (x,y) - (y,x) + (y,y)]$$

$$= 2\|x\|^2 + 2\|y\|^2$$

Theorem:

A closed convex subset C of a H.S. contain a unique vector of the smaller norm.

Proof:

Let $d = \inf \{\|x\|, x \in C\}$

Then there exist $\{x_n\}$ in C such that $\|x\| \to d$

$x_n, x_m \in C \Rightarrow \|x\| \geq d$

where C is convex

$$\frac{x_n + x_m}{2} \in C$$

$$\left\| \frac{x_n + x_m}{2} \right\| \geq d$$

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$$\Rightarrow \|x_n + x_m\| \geq 2d$$

$$\|x_n + x_m\|^2 + \|x_n - x_m\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2$$

$$\|x_n - x_m\|^2 = 2\|x_n\|^2 + 2\|x_m\|^2 - \|x_n + x_m\|^2$$

$$= 2d^2 + 2d^2 - 4d^2$$

$\to 0$ as $n, m \to \infty$

$\Rightarrow \{x_n\}$ cauchy sequence in c

Here, c is a closed subset of H. But H is a Hilbert space

$\therefore H$ is complete

Since every closed subset of a complete space is complete

$\therefore c$ is complete

Every cauchy sequence in c converges

Let $\{x_n\} \to x \in c$

Now, $\|x\| = \|\lim x_n\|$

$= \lim \|x_n\| = d$

There exist $x \in c$ such that $\|x\| = d$

$\therefore c$ contains the vector of the smallest number

To prove, x is unique

There exist $x \in c$ such that $\|x\| = d$ Suppose there exist $x' \in c$ such that $\|x'\| = d$

By parallelogram law,

$$\|x + x'\|^2 + \|x - x'\|^2 = 2\|x\|^2 + 2\|x'\|^2$$

$$\|x - x'\|^2 = 2\|x\|^2 + 2\|x'\|^2 - \|x + x'\|^2$$

$= 2d^2 + 2d^2 - \|x + x'\|^2$

$\leq 4d^2 - 4d^2$

$\|x - x'\|^2 \leq 0$ (5)

$\|x - x'\| \leq 0$

But, $\|x - x'\| \geq 0$

$\|x - x'\| = 0$

$\Rightarrow x = x'$

$\therefore x$ is unique

Theorem:

If B is a Banach space whose norm obey the parallelogram law and if and inner product defined by

$4(x, y) = \|x+y\|^2 - \|x-y\|^2 + \|x+iy\|^2 - \|x-iy\|^2 \rightarrow (1)$

then B is a Hilbert space

Proof:

B obey parallelogram law.

$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \rightarrow (2)$

$4(x, x) = \|x+x\|^2 - \|x-x\|^2 + i\,\|x+ix\|^2 - i\,\|x-ix\|^2$

$= \|2x\|^2 - 0 + i\,\|(1+i)x\|^2 - i\|(1-i)x\|$

$= 4\|x\|^2 + i\,\|1+i\|^2\,\|x\|^2 - i\,\|(1-i)\|^2\,\|x\|^2$

$= 4\,\|x\|^2 + i2\,\|x\|^2 - i2\|x\|^2$

$4(x, x) = 4\|x\|^2$

$\therefore (x, x) = \|x\|^2$

$4\overline{(x, y)} = \|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2$

$= \|y+x\|^2 - \|-(y-x)\|^2 - i\|(y-ix)\|^2 + \|-i(y+ix)\|^2$

$$= \|y+x\|^2 - \|y-x\|^2 - i\|y - i(x)\|^2 + i\|y + ix\|^2$$

$$4(\overline{x,y}) = 4(y,x) \Rightarrow (\overline{x,y}) = (y,x)$$

$$4(x+y,z) = \|x+y+z\|^2 - \|x+y-z\|^2 + i\|x+y+iz\|^2 - i\|x+y-iz\|^2$$

$$\|x+y+z\|^2 = \|x+y+z\|^2 \rightarrow (3) \quad (6)$$

In (2) replace x by $x+z$

$$\|x+z+y\|^2 = 2\|x+z\|^2 + 2\|y\|^2 - \|x+z-y\|^2 \rightarrow (4)$$

Again

$$\|x+z-y\|^2 = \|(z-y)+x\|^2$$

$$= 2\|z-y\|^2 + 2\|x\|^2 - \|(z-y)-x\|^2 \rightarrow (5)$$

Sub. (5) in (4)

$$\|x+y+z\|^2 = 2\|x+z\|^2 + 2\|y\|^2 - 2\|z-y\|^2 - 2\|x\|^2 + \|z-y-x\|^2$$

$$= 2\|x+z\|^2 + 2\|y\|^2 - 2\|(y-z)\|^2 - 2\|x\|^2 + \|x+y-z\|^2$$

$$\|x+y+z\|^2 - \|x+y-z\|^2 = 2\|x+z\|^2 + 2\|y\|^2 - 2\|y-z\|^2 - 2\|x\|^2 \rightarrow (6)$$

Interchange x and y in (6)

$$\|x+y+z\|^2 - \|x+y-z\|^2 = 2\|y+z\|^2 + 2\|x\|^2 - 2\|x-z\|^2 - 2\|y\|^2 \rightarrow (7)$$

add. (6) and (7)

$$2\|x+y+z\|^2 - 2\|x+y-z\|^2 = 2\|x+z\|^2 - 2\|x-z\|^2 + 2\|y+z\|^2 - 2\|y-z\|^2$$

$$\therefore \|x+y+z\|^2 - \|x+y-z\|^2 = \|x+z\|^2 - \|x-z\|^2 + \|y+z\|^2 - \|y-z\|^2 \rightarrow (8)$$

Replace z by iz

$$\|x+y+iz\|^2 - \|x+y-iz\|^2 = \|x+iz\|^2 - \|x-iz\|^2 + \|y+iz\|^2 - \|y-iz\|^2$$

$$i\|x+y+iz\|^2 - i\|x+y+iz\|^2 = i\|x+iz\|^2 - i\|x-iz\|^2 + i\|y+iz\|^2 - i\|y-iz\|^2 \rightarrow (9)$$

Adding ⑧ and ⑨

$$\|x+y+z\|^2-\|x+y-z\|^2+i\|x+y+iz\|^2-i\|x+y-iz\|^2$$

$$= (\|x+z\|^2+\|x-z\|^2+\|x+iz\|^2-i\|x-iz\|^2)+$$

$$(\|y+z\|^2-\|y-z\|^2+i\|y+iz\|^2-i\|y-iz\|^2) \quad ⑦$$

From ③,

$$4(x+y,z)=4(x,z)+4(y,z)$$

$$(x+y,z) = (x,z)+(y,z)$$

To prove $(\alpha x, y) = \alpha(x,y)$ where α- scalar

case (i) :

α is a +ve integer

Let $\alpha = 2$

$$(2x,y) = (x+x,y) = (x,y)+(x,y)$$

$$= 2(x,y)$$

It is true for $\alpha = 2$

Assume that it is true for $\alpha = n$

$$(nx,y) = n(x,y)$$

$$((n+1)x,y) = (nx+x,y)$$

$$= (nx,y)+(x,y)$$

$$= n(x,y)+(x,y)$$

$$= (n+1)(x,y)$$

$$(\alpha x,y) = \alpha(x,y)$$

If α is a +ve integer.

case (ii) :

α is a -ve integer

Replace x by $-x$ in ①

$$4(-x,y) = \|-x+y\|^2-\|x-y\|^2+i\|-x+iy\|^2-i\|-x-iy\|^2$$

$= \|x-y\|^2 - \|x+y\|^2 + i\,\|-(x-iy)\|^2 - i\,\|(x+iy)\|^2$

$= -\left[-\|x-y\|^2 + \|x+y\|^2 - i\,\|x-iy\|^2 + i\,\|x+iy\|^2\right]$

$4(-x,y) = -\left[4(x,y)\right]$ 8

$(-x,y) = -(x,y)$

Since x is -ve integer

Let, $\alpha = -m$, where m is +ve integer

$(\alpha x, y) = (-mx, y)$

$= -(mx, y)$

$= -m(x,y)$

$\therefore (\alpha x, y) = \alpha(x,y)$.

Case iii)

α is a rational number

$\alpha = p/q$; $q \neq 0$ p and q are integer

$(\alpha x, y) = (p/q\, x, y)$

$= (pz, y)$

$(\alpha x, y) = p(z,y)$

$(qz, y) = q(z,y)$

$(z,y) = \frac{1}{q}(qz, y)$

$(\alpha x, y) = \frac{p}{q}(qz, y)$

$(\alpha x, y) = \alpha(x,y)$

similarly, α is a real number

$(\alpha x, y) = \alpha(x,y)$ holds

case (iv) :

α is a complex number

Let $\alpha = \alpha_1 + i\alpha_2$

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$$(\alpha x, y) = ((\alpha_1 + i\alpha_2) x, y)$$
$$= (\alpha_1 x + i\alpha_2 x, y)$$
$$= (\alpha_1 x, y) + i(\alpha_2 x, y)$$
$$= \alpha_1 (x, y) + i\alpha_2 (x, y)$$
$$= (\alpha + i\alpha_2)(x, y)$$

$\therefore (\alpha x, y) = \alpha (x, y) \quad \forall \alpha$

Orthogonal vectors :

Two vectors x and y in a H.S "H" are said to be orthogonal if inner product x and y and written as $x \perp y$ ie) $x \perp y \Rightarrow (x, y) = 0$

A vector is said to be orthogonal to a non-empty set S if, $x \perp y \ \forall \ y \in S$

ie) $(x, y) = 0 \ \forall \ y \in S$

Orthogonal complement :

orthogonal complement of a set 'S' is denoted by $S^\perp$ is a set of all vectors orthogonal to S

$$S^\perp = \{x : x \perp S\}$$
$$= \{x, x \perp y \ \forall \ y \in S\}.$$

Property :

If S is a non-empty, subset of H Then $S^\perp$ is a closed linear subspace of H

Proof :

$$S^\perp = \{x, x \perp S\}$$

$= \{x ; x \perp y \ \forall \ y \in S\}$

To prove : $S^{\perp}$ is non-empty.

Let $y \in S$

(10)

$(0, y) = 0$

$\Rightarrow 0 \perp y \quad \forall \ y \in S$

$\Rightarrow 0 \in S^{\perp}$

$\therefore S^{\perp}$ is non-empty.

Then $x_1 \perp S$ and $x_2 \perp S$

Let $x_1, x_2 \in S^{\perp}$

$\Rightarrow (x_1, y) = 0$ and $(x_2, y) = 0 \quad \forall \ y \in S$

$$(\alpha x_1 + \beta x_2, y) = (\alpha x_1, y) + (\beta x_2, y)$$

$$= \alpha (x_1, y) + \beta (x_2, y)$$

$$= 0 + 0$$

$(\alpha x_1 + \beta x_2, y) = 0 \quad \forall \ y \in S$

$\Rightarrow (\alpha x_1 + \beta x_2, y) \in S^{\perp}$

$S^{\perp}$ is linear

Let $y \in S$. and Let x be the limit point of $\{x_n\}$

$x_n \to x$ $\quad x_n \in S^{\perp}$

$\Rightarrow x_n \perp S$

$\Rightarrow x_n \perp y \ \forall \ y \in S$

$$(x, y) = (\lim x_n, y)$$

$$= \lim (x_n, y)$$

$$= 0 \quad \forall \ y \in S$$

$x \perp y, \ \forall \ y \in S,$

$x \perp S$

$\Rightarrow x \in S^{\perp}$

5m

(8) Theorem :

Let M be a closed linear subspace of H and let x be a vector not in M and let d be the distance from x to M. Then there exist a unit vectors y_0 in M such that $\|x_0 - y_0\| = d$ (11)

Proof :

M- closed linear $\subseteq$ H - HS

$x \notin M$, $x \in H$

$x \in H - M$

$d = \{\|x - m\| ; m \in M\}$

There exist y_0 in M such that $\|x - y_0\| = d$

Let, $d = \{\|x - m\|, m \in M\}$

$c = x + M$

$x_1, x_2 \in c$

$x_1 = x + m_1$; $x_2 = x + m_2$; $m_1, m_2 \in M$

$\alpha x_1 + (1-\alpha) x_2 = \alpha(x + m_1) + (1-\alpha)(x + m_2)$

$= \alpha x + \alpha m_1 + x + m_2 - \alpha x_2 - \alpha m_2$

$= x + m_2 + \alpha(m_1 - m_2)$

$\in x + M$ $\subseteq$ (M is linear)

c is convex.

Let y_n be the limit point of $\{y_n\}$

Let $y_n = y$ there exist a sequence $\{y_n\}$ in C

Such that $y_n \to y$

$y_n \in C \Rightarrow y_n = x + M_n$

$= \lim (x + m_n)$

$= x + \lim m_n$

$y - x = \lim m_n$

$\Rightarrow y - x \in M$ (M is closed)

$\Rightarrow x+y-x \in x+M$

$y \in x+M = C$

Let, $z_0 \in C$ 12

$z_0 = x+m, m \in H$

$\Rightarrow -m = x - z_0$

put, $y_0 = x - z_0$

$z = x - y_0$

put $\|z_0\| = \|x - y_0\|$

$\|z_0\| = d$

$\Rightarrow \|x - y_0\| = d$

There exist y_0 in M such that $\|x-y_0\| = d$.

y_0 is unique

There exist y_1 such that,

$\|x - y_1\| = d$ and $y_0 \neq y_1$

Suppose there exist y_1 such that $\|x - y_1\| = d$.

$y_0 \neq y_1$

$\Rightarrow x - y_0 \neq x - y_1$

$z_0 \neq z_1 \quad (z_1 = x - y_1)$

Put $\|z_0\| = \|x - y_0\| = d$ and

$\|z_1\| = \|x - y_1\| = d$

$\|z_0\| = \|z\| = d$ and $z = z_0$

$\Rightarrow\Leftarrow$ to the uniqueness of z_0

$\therefore y_0 \neq y_1$ is wrong

$\therefore y_0 = y_1$

$\therefore y_0$ is unique

Theorem :

If M is a proper closed linear subspace of a Hilbert H. Then there exist a non-zero vector z_0 in H such that z_0 in orthogonal to M ($z_0 \perp M$)

Proof :

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Since, M is proper

we have, $M \neq H$

There exist $x \in H$ such that $x \notin M$

Since M is linear

$0 \in M$

$\therefore x \neq 0$

ie) x is the non-zero vector not in H

Let d be the distance for x to M

$\therefore$ The above theorem there exist a unit vector y_0 in M. Such that $\|x - y_0\| = d$ define z_0 by $z_0 = x - y_0$

$y_0 \in M$ and $x \notin M$

$x - y_0 \neq 0$

$\therefore x - y_0$ is a non-zero vector in H.

claim :

If $y \in M$ then $z_0 \perp y$ for any scalar α $\|z_0 - \alpha y\|$

$= \|x - y_0 - \alpha y\|$

$= \|x - (y_0 + \alpha y)\|$

Since, $y_0, y \in M$

$\Rightarrow y_0 + \alpha y \in M$ ($\because M$ is closed)

Also, $d = \inf \{\|x - m\| ; m \in M\}$

$\therefore \|x - (y_0 + \alpha y)\| \geq d = \|z_0\|$

$\|x - (y_0 + \alpha y\|^2 \geq \|z_0\|^2$

$$(z_0 - \alpha y,\ z_0 - \alpha y) \geq (z_0, z_0)$$

$$(z_0, z_0) - \bar{\alpha}(z_0, y) - \alpha(y, z_0) + \alpha\bar{\alpha}(y, y) \geq (z_0, z_0)$$

$$\Rightarrow -(z_0, y) - \alpha(y, z_0) + |\alpha|^2 (y, y) \geq 0 \quad (14)$$

Put $\alpha = \beta(z_0, y)$ for any real number β

$$-\beta(\overline{z_0, y})(z_0 + y) - \beta(x_0, y)(y, z_0) + \beta^2 (z_0, y)^2 \|y\|^2$$

$$\Rightarrow -\beta|(z_0, y)|^2 - \beta|(z_0, y)|^2 + \beta^2 |(z_0, y)|^2 \|y\|^2 \geq 0$$

Put $a = |(z_0, y)|^2$ and $b = \|y\|^2$

$$-2\beta a + \beta^2 ab \geq 0$$

$$a\beta + (b\beta - 2) \geq 0$$

This is true for any real β

Let, $a > 0$. $b \geq 0$ as $\|y\|^2 \geq 0$

$a > 0$ and $\beta > 0$

$a\beta > 0$

$\beta b - 2 \geq 0$

β is the real number such that $0 < \beta < 2/b$

$$\beta b < b \times 2/b = 2$$

$$\Rightarrow \Leftarrow$$

$$\therefore a = 0$$

ie) $|(z_0, y)|^2 = 0$

$$\Rightarrow (z_0, y) = 0$$

$$\Rightarrow z_0 \perp y \ \forall\ y \in M$$

$$\therefore z_0 \perp M$$

Hence, the proof.

5m

(x) Theorem :

If M and N are closed linear sub spaces of a Hilbert space H such that $M \perp N$ then $M+N$ is also closed set (linear subspace) (15)

Proof :

Let z be a limit point of $M+N$

$\therefore$ There exist a sequence $\{z_n\}$ in $M+N$

such $z_n \to z$.

Since M and N are subspace of H

$0 \in M \cap N$

$M \cap N \supseteq \{0\}$

But $M \perp N$

$M \cap N = \{0\}$

$\therefore$ Any element of $M+N$ can be uniquely expressed by $m+n$ where $m \in M$ and $n \in N$

Let $z_n = x_n + y_n$ where $x_n \in M$ and $y_n \in N$

Consider,

$z_m - z_n = (x_m + y_m) - (x_n + y_n)$

$= (x_m - x_n) + (y_m - y_n)$

$\|z_m - z_n\| = \|(x_m - x_n) + (y_m - y_n)\|$

$\leq \|x_m - x_n\| + \|y_m - y_n\|$

Given $z_n \to z$

For $\epsilon > 0$ there is a positive integer $n_0 \ni$

$\|z_n - z\| < \epsilon/2 \ \forall n \geq n_0$

$\|z_m - z\| < \epsilon/2 \ \forall m \geq n_0$

Consider $\|z_m - z_n\| = \|z_m - z_n + z - z\|$

$$\leq \|z_m - z\| + \|z_n - z\|$$

$$< \epsilon/2 + \epsilon/2$$

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$$\|z_m - z_n\| < \epsilon$$

$\therefore z_m$ is a cauchy sequence

By pythegorcan theorem

$$x \perp y \Rightarrow \|x+y\|^2 = \|x-y\|^2 = \|x\|^2 + \|y\|^2$$

$$\|x_m - x_n + y_m - y_n\|^2 = \|x_m - x_n\|^2 + \|y_m - y_n\|^2$$

$$\|z_m - z_n\|^2 = \|x_m - x_n\|^2 + \|y_m - y_n\|^2$$

$\|x_m - x_n\|^2 \leq \|z_m - z_n\|^2$ and

$$\|y_m - y_n\|^2 \leq \|z_m - z_n\|^2$$

Since, $\{x_n\}$ is a cauchy sequence

$\|z_m - z_n\| \to 0$ as $m, n \to \infty$

$\|x_m - x_n\| \to 0$ as $m, n \to \infty$ and

$\|y_m - y_n\| \to 0$ as $m, n \to \infty$

Theorem :-

If M is a closed linear subspace of H S H, then

$H = M \oplus M^\perp$

Proof :-

By above theorem ($z_0 \perp H$).

$M + M^\perp$ is closed linear subspace of H

If, $H = M + M^\perp$ then the theorem is proved

So let $H \neq M + M^\perp$

$\Rightarrow M + M^\perp$ is a proper closed linear subspace of

Then by theorem ($z_0 \perp H$) there exist a non-zero

vector z_0 in H

z_0 such that, $z_0 \perp M+M^\perp$

Now, $z_0 \perp M+x, \forall x \in M^\perp$

$\Rightarrow z_0 + M+0$ $(0 \in M^\perp)$ $(\because 0 \in M$

$\Rightarrow z_0 \perp M^\perp \rightarrow$ ①

(17)

$\Rightarrow z_0 \in (M^\perp)^\perp \rightarrow$ ②

From ① and ② we have

$z_0 \in M^\perp \cap M^{\perp\perp} \subset \{0\}$

$\Rightarrow z_0 \in \{0\}$

$\Rightarrow z_0 = 0$

$\Rightarrow \Leftarrow$ to that z_0 in a non-zero vector

$H = M+M^\perp$

Orthogonal set:

Definition:

In a Helbert space a set $\{e_i\}$ of vector said to be mutually orthogonal if $e_i \perp e_j \ \forall i \neq j$

ii) Definition:

An orthonormal set in a Hilbert space H is a non-empty subset of H which consists of mutually orthogonal unit vectors.

ie) It is a non-empty subset, $\{e_i\}$ of H with the following properties.

i) $e_i \neq 0 \ \forall i$

ii) $e_i \perp e_j \ \forall i \neq j$

iii) $\|e_i\| = 1 \ \forall i$

Example:

Suppose $\{x_i\}$ is an ~~orthogonal~~ Orthonormal set of non-zero vector then $\{x_i / \|x_i\|\}$ is an orthonormal set of vectors.

Example :-

The subset $\{e_1, e_2, \dots e_n\}$ of l_2^n where e_i is n tuple with one in the i^{th} place and zero's where is an orthonormal set in the space

Proof :

$$l_2^n = \{(x_1, x_2 \dots x_n) / x_1, x_2 \dots x_n \in C\}$$

$$\text{Given } e_i = \begin{cases} 1 & ; i^{th} \text{ place} \\ 0 & ; \text{ elsewhere} \end{cases}$$

$\therefore$ ie) $e_1 = 1, 0, 0 \dots 0$

$e_2 = 0, 1, 0 \dots 0$

$\vdots$

$e_n = 0, 0, 0, \dots 1$

$(e_1, e_2) = (1, 0, 0 \dots 0)\ (0, 1, 0, \dots 0)$

$= 0 + 0 + \dots + 0$

$(e_1, e_2) = 0$

$\therefore 1 \neq 2$

In general

$(e_i, e_j) = 0$ if $i \neq j$

$e_i \perp e_j$

and $(e_1, e_1) = (1, 0, 0 \dots 0)\ (1, 0, 0 \dots 0)$

$= (1 + 0 + \dots + 0)$

$(e_1, e_1) = 1$

similarly,

$(e_2, e_2) = 1$, $(e_3, e_3) = 1$

$(e_i, e_i) = 1 \ \forall i$

$\|e_i\|^2 = 1$

$\|e_i\| = 1 \ \forall i$

Theorem :

Let $\{e_1, e_2 \dots e_n\}$ be finite orthonormal set in a Hilbert space H if x is any vector in H, then $\sum_{i=1}^{n} |(x, e_i)|^2 \le \|x\|^2$ Further, $x - \sum_{i=1}^{n} (x, e_i) e_i \perp e_j \ \forall j$

Proof :

Given, $\sum_{i=1}^{n} |(x, e_i)|^2 \le \|x\|^2$

Let $y = x - \Sigma (x, e_i) e_i$

$\|y\|^2 = \|x - \Sigma (x, e_i) e_i\|^2 \ge 0$

$\Rightarrow (x - \Sigma (x, e_i) e_i, x - \Sigma (x, e_i) e_i) \ge 0$

$\Rightarrow (x, x) - \Sigma (x, e_i)(e_i, x) - \Sigma \overline{(x, e_i)} (x, e_i)$

$+ \sum_i \sum_i (x, e_i) \overline{(x, e_i)} (e_i, e_i) \ge 0$

$\Rightarrow \|x\|^2 - \Sigma |(x, e_i)|^2 - \Sigma |(x, e_i)|^2 + \Sigma |(x, e_i)^2| \ge 0$

$\Sigma |(x, e_i)|^2 < \|x\|^2$

$(x - \Sigma (x, e_i) e_i, e_j) = (x, e_j) - \sum_{i=1}^{n} (x, e_i)(e_i, e_j)$

$= (x, e_j) - (x, e_j)$

$(x - \Sigma (x, e_i) e_i, e_j) = 0$

Theorem :

If $\{e_i\}$ is an orthonormal set in a Hilbert Space H and if x is any vector of H, Then the set, $S = \{e_i, (x, e_i) \neq 0\}$ either empty, (or) countable.

Proof :

If, $S = \phi$ Then there is nothing to prove

Suppose $\epsilon \neq \phi \ \forall n$ define

$$S_n = \{ e_i \mid |(x, e_i)|^2 > \frac{\|x\|^2}{n} \}$$

Suppose S_n contains n (or) more elements

Let. $|S_n| = k$, $k \geq M$

(20)

$$\sum_{e_i \in S_n} |(x_i, e_i)|^2 = \sum \frac{\|x\|^n}{n}$$

$$= \sum \frac{k\|x\|^2}{n}$$

$$\geq n_0 \frac{\|x\|^2}{n}$$

$$\geq \|x\|^2$$

Now, S_n is a set of k orthogonal normal vectors

S_n contains finite orthonormal set of vectors

$\therefore \sum |(x, e_i)|^2 \leq \|x\|^2$ (by previous theorem)

our assumption is wrong

S_n contains at most is countable $\forall n$

Since, $S \neq 0$ then $e_i \in S$ for some i

$(x, e_i) \neq 0$

$|(x, e_i)|^2 > 0$

we can find a +ve integer n_0 such that

$$|(x, e_i)|^2 > \frac{\|x\|^2}{n_0}$$

$e_i \in S_{n_0}$

$$S = \bigcup_{n=1}^{\infty} S_n$$

$\therefore S$ is countable.

priya

Theorem : (Bessel's Inequality) A-19 5M

If $\{e_i\}$ is a orthogonal set in a Hilbert space then, $\sum |(x, e_i)|^2 \leq \|x\|^2$ for every vector x in H

Proof :-

Let, $x \in H$

Let $S = \{ e_i, (x, e_i) \neq 0 \}$

By previous theorem, (21)

S is either empty (or) countable

Case (i) :-

Suppose $S = \phi$ such that

$$(x, e_i) = 0$$

$$\Rightarrow |(x, e_i)|^2 = 0$$

$$\|x\|^2 \geq 0 = |(x, e_i)|^2$$

$$\Sigma |(x, e_i)|^2 \leq \|x\|^2$$

case (ii) :-

$S \neq \phi$ and finite

Let $S = \{ e_1, e_2 \dots e_n \}$

$$\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2$$

case (iii) :-

$S \neq \phi$ and countably infinite

Then $S = \{ e_1, e_2 \dots e_n \}$

Let $S_n = \sum_{i=1}^{n} |(x, e_i)|^2$

$\{ S_n \}$ is the sequence of partial sum of the series. $\sum_{i=1}^{n} |(x, e_i)|^2$

The increasing sequence S_n is bounded above and hence, it is convergent

By case (ii) $S_n \leq \|x\|^2$

$$\lim_{n \to \infty} S_n \leq \lim \|x\|^2$$

$\lim_{n \to \infty} \sum^{n} |(x, e_i)|^2 \leq \|x\|^2$ i.e. $\sum_{i=1}^{n} |(x, e_i)|^2 \leq \|x\|^2$

Hence proved

Orthonormal Set :

An orthonormal set $\{e_i\}$ in a S Hilbert space H is said to be complete if any vector e in H such $e \perp \{e_i\}$, $\|e\| = 1$ cannot be adjoint sequence $\{e_i\}$ gives an orthonormal set $\{e, e_i\}$) priya

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Theorem :

uma (Every non-zero Hilbert Space contains a complete (or) orthonormal set.

Proof :

$$H \neq \{0\}$$

Then it contains in orthonormal set consider, the classes of all orthonormal sets with non-empty

This class is partially order set with respect to set inclusion.

Every chain in the partially order set has an upper bound.

By zorn's lemma, it has an maximal element set A.

$\therefore$ Number of elements of H can be adjoint with H to find an orthonormal set

$\therefore$ A is complete.

By zorn's lemma,

Every non-empty partially order set in which each chain has an upper bound has a maximal element.

Theorem :

Let H be H.S and let $\{e_i\}$ be an orthonormal in H Then the following condition are equalent to one another

i) $\{e_i\}$ is complete

ii) $x \perp \{e_i\} \Rightarrow x = 0$

iv) If x is any arbitrary vectors in H

then $\|x\|^2 = \Sigma |(x, e_i)|^2$

iii) If x is any arbitrary vectors in H Then

$x = \Sigma (x, e_i) e_i$

Proof :

(i) → (ii)

Assume that $\{e_i\}$ is complete

Let $x \perp \{e_i\}$

To prove $x = 0$

An the contrary at $x \neq 0$

then $\left\{\frac{x}{\|x\|}, e_i\right\}$ is an orthonormal, set which is

contradiction to the fact that $\{e_i\}$ is complete

$\therefore x = 0$

(ii) → (iii)

Assume that, $x \perp \{e_i\} \Rightarrow x = 0$

To prove, $x \in H$, then

$x - \Sigma (x, e_i) e_i \perp e_j \ \forall j$

$x - \Sigma (x, e_i) e_i \perp \{e_i\}$

By (ii) $x - \Sigma (x, e_i) e_i = 0$

$\Rightarrow x = \Sigma (x, e_i) e_i$

(iii) → (iv)

Assume that, $x = \Sigma (x, e_i) e_i$ where $x \in H$

$\|x\|^2 = (x, x)$

$$= \left[\Sigma (x, e_i) e_i , \ \Sigma (x, e_i) e_j \right]$$

$$= \Sigma (x, e_i) \left(e_i \ \Sigma (x, e_i) e_j \right) -$$

$$\|x\|^2 = \sum_i \sum_j (x, e_i) \overline{(x, e_i)} (e_i, e_j) \quad (24)$$

$$= \Sigma |(x, e_i)|^2$$

(iv) $\rightarrow$ (i)

Assume that $\|x\|^2 = \Sigma |(x, e_i)|^2$ where $x \in H$

Suppose $\{e_i\}$ not complete

There exist $e \in H$ such that $e \perp \{e_i\}$ and $\|e\| =$

$\Rightarrow (e, e_i) = 0 \ \forall i$

$\|e\| = 1$

$y \in H$ we define f_y on H such that there is a natural correspondence between H and H^*

$$f_y(x) = (x, y) \quad \forall x \in H$$

$$f_y(x_1 + x_2) = (x_1 + x_2, y)$$

$$= (x_1, y) + (x_2, y_2)$$

$$= f_y(x_1) + f_y(x_2)$$

$$f_y(\alpha x) = (\alpha x, y)$$

$$= \alpha (x, y)$$

$$= \alpha f_y(x)$$

$\therefore f_y$ is linear

$$|f_y(x)| = |(x, y)| \leq \|x\| \|y\|$$

$$\therefore |f_y(x)| \leq \|x\| \|y\|$$

$\Rightarrow f_y$ is bounded

f_y is continuous

Now, $\|f_y\| = \sup\{f_y(x) ; \|x\| \le 1\}$

$\|f_y\| \le \sup\{\|x\| \|y\|, \|x\| \le 1\}$

$\le \sup\{\|y\|\}$

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$= \|y\|$

$\|f_y\| \le \|y\| \rightarrow (1)$

If $y = 0$ then $\|y\| = 0$

Also, $\|f_y\| = \|(x, y)\|$

$= \|(x, 0)\|$

$= 0$

$\therefore \|f_y\| = \|(x, y)\|$

Suppose, $y \neq 0$

$\|f_y\| = \sup\{|f_y(x)| ; \|x\| = 1\}$

$\ge |f_y(x)|$

$y \neq 0 \Rightarrow \left\| \frac{y}{\|y\|} \right\| = 1$

$\|f_y\| \ge |f_y(x)|$

$= \left| f_y\left(\frac{y}{\|y\|}\right) \right|$

$= \frac{1}{\|y\|} |f_y(y)|$

$= \frac{1}{\|y\|} |(y, y)|$

$= \frac{1}{\|y\|} \|y\|^2$

$\|f_y\| \ge \|y\| \rightarrow (2)$

From (1) and (2) we have

$\|f_y\| = \|y\|$

Theorem :-

Engg Let H be a Hilbert space and let f be a functional in H^*. Then there exist a unique vector y in H. Such that $f(x,y) = (x,y)$ $\forall\ x \in H$

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Proof :

To prove $f(x) = (x,y)$ $\forall x \in H \rightarrow$ (1)

1^{st} let us show that if such la exist (satisfyingin) then it must be unique

Let $y_1, y_2 \in H$ such that

$f(x) = (x, y_1)$ and

$f(x) = (x, y_2)$ $\forall\ x \in H$

$\therefore (x, y_1) = (x, y_2) = 0$

$\Rightarrow (x, y_1) - (x, y_2) = 0$

$(x, y_1 - y_2) = 0$

$y_1 - y_2 \perp x$; $\forall x \in H$

$y_1 - y_2 \perp 0$; $0 \in H$

(Since 0 is the only vector orthogonal is itself)

$y_1 - y_2 = 0$

$\Rightarrow y_1 = y_2$

$\therefore y_1$ is unique

i) Suppose $f = 0$ then $f(x) = 0$ $\forall x$

Take $y = 0 \Rightarrow (x, y) = 0$

$f(x) = (x, y)$ $\forall x$

ii) Let $f \neq 0$ then there exist $x \in H$

Such that $f(x) \neq 0$

Let M be the null space of f

i.e. $M = \{y, f(y) = 0\}$ (27)

$\therefore$ M is a closed linear subspace of H

since, $f(x) \neq 0$ $x \notin M$

$\Rightarrow$ M is a proper closed linear subspace of H

Also $x \notin M$

Then there exist a non-zero vector y_0 (say)

in H such that

$y_0 \perp y \Rightarrow y_0 \perp y \quad \forall y \in M \rightarrow (*)$

$\therefore y_0 \in M^\perp$

Let $y = \alpha y_0$ and $\alpha \neq 0$

consider $(x, \alpha y_0) = \bar{\alpha}(x, y_0)$

$= \bar{\alpha}(0) \quad \forall x \in M$ using α

$(x, \alpha y_0) = 0$

Also, $f(x) = 0 \quad \forall x \in M$

$\therefore f(x) = (x, \alpha y_0) \quad \forall x \in M, \alpha \neq 0 \rightarrow ②$

In these any α, such that ① is then for

$x = y_0, \quad y = \alpha y_0$

i.e. $f(y_0) = (y_0, \alpha y_0)$

$= \bar{\alpha}(y_0, y_0)$

$= \bar{\alpha}\|y_0\|^2$

$y_0 \neq 0 \Rightarrow \bar{\alpha} = \dfrac{f(y_0)}{\|y_0\|^2}$

$\Rightarrow \alpha = \dfrac{\overline{f(y_0)}}{\|y_0\|^2}$

hence, when $\alpha = \dfrac{\overline{f(y_0)}}{\|y_0\|^2}$

then $f(x) = (x, y)$ if $x = y_0$ and $y = \alpha y_0$ from

we have.

$$f(x) = (x, y) \ \forall\ x \in M \text{ and } \alpha = \frac{\overline{f(y_0)}}{\|y_0\|^2}$$

Also, $f(x) = (x, y)$, $x = y_0$ and $\alpha = \dfrac{\overline{f(y_0)}}{\|y_0\|^2}$

combaining these two, we have

(28)

$$f(x) = (x, y) \ \forall\ x \in M \cup \{y_0\}$$

$$y = \alpha y_0, \quad \alpha = \frac{\overline{f(y_0)}}{\|y_0\|^2} \rightarrow (3)$$

$$M \cap M^{\perp} = \{0\}$$

Since, $y_0 \in M^{\perp}$

$$\Rightarrow y_0 \notin M$$

$$\Rightarrow f(y_0) \neq 0$$

consider any $x \in H$

$$f(x) = \frac{f(x)}{f(y_0)} \cdot f(y_0)$$

$$= \beta f(y_0) \text{ where } \beta = \frac{f(x)}{f(y_0)}$$

$$f(x) - \beta f(y_0) = 0$$

$$f(x) - f(\beta y_0) = 0$$

$$f(x - \beta y_0) = 0$$

$$\Rightarrow x - \beta y_0 \in M$$

Let, $x - \beta y_0 = m$ for some $m \in M$

$$\Rightarrow x = m + \beta y_0$$

$$f(x) = f(m + \beta y_0) = f(m) + \beta f(y_0)$$

$f(x) = (x, y) \ \forall\ f \in H^*$ & where, $y = \alpha y_0$, $\alpha = \dfrac{\overline{f(x)}}{\|y_0\|^2}$

Theorem:

The adjoint operation $T \to T^*$ on $B(H)$ has the following properties

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i) $(T_1 + T_2)^* = T_1^* + T_2^*$

ii) $(\alpha T)^* = \bar{\alpha} T^*$

iii) $(T_1, T_2)^* = T_2^*, T_1^*$

iv) $(T^{**}) = T$

5m v) $\| T^* \| = \| T \|$

vi) $\| T^* T \| = \| T \|^2$

Proof:

Let, $x, y \in H$

i) $(x, (T_1 + T_2)^*, y) = ((T_1 + T_2) x, y)$

$= (T_1(x), y) + (T_2(x), y)$

$= (x, T_1^* y) + (x, T_2^* y)$

$= (x, (T_1^* + T_2^*), y) \ \forall x, y \in H$

$(T_1 + T_2)^* = T_1^* + T_2^*$

ii) $(x, (\alpha T)^* y) = ((\alpha T) x, y)$

$= \alpha (Tx, y)$

$= \alpha (x, T^* y)$

$= (x, \bar{\alpha} T^* y) \ \forall x, y \in H$

$(\alpha T)^* = \overline{\alpha} T^*$

iii) $(x, (T_1, T_2)^* y) = (T_1, T_2(x), y)$

$= (T_1, (T_2(x)), y)$

$= (T_2(x), T_1^* y)$

$= (x, T_2^* T_1^* y) \ \forall x, y \in H$

$$(T_1 T_2)^* = T_2^* T_1^*$$

iv) $(x, T^{**}y) = (T^* x, y)$

$$= \overline{(x, T^* y)}$$

$$= \overline{(Ty, x)}$$

$$= (x, Ty) \quad \forall\, x, y \in H$$

$$T^{**} = T$$

iv) For every $y \in H$

$$\|T^* y\|^2 = (T^* y, T^* y)$$

$$= (TT^* y, y)$$

$$= |(TT^* y, y)|$$

$$= \|T\| \, \|T^* y\| \, \|y\|$$

$$\|T^* y\|^2 \leq \|T\| \, \|y\|$$

if $\|y\| < 1 \Rightarrow \|T^* y\| \leq \|T\|$

$$\text{Sup}\, \{ \|T^* y\| \, ; \|y\| \leq 1 \} \leq \|T\|$$

$$\|T^*\| \leq \|T\| \rightarrow (1)$$

$$\|(T^*)^*\| \leq \|T^*\|$$

$$\Rightarrow \|T\| \leq \|T^*\| \rightarrow (2)$$

From (1) and (2)

$$\|T\| = \|T^*\|$$

vi) Let us consider

$$\|T^* T\| \leq \|T^*\| \, \|T\|$$

$$= \|T\| \, \|T\|$$

$$\leq \|T\|^2 \rightarrow (1) \quad \forall\, x \in H$$

$$\|Tx\|^2 = (Tx, Tx)$$
$$= (T^*Tx, x)$$
$$\leq \|T^*T\| \, \|x\| \, \|x\|$$ 31
$$\leq \|T^*T\| \, \|x\|^2$$

$\forall \|x\| \leq 1, \; \|Tx\|^2 \leq \|T^*T\|$

$\sup\{\|Tx\|^2 ; \|x\| \leq 1\} \leq \|T^*T\|$

$$\|T\|^2 \leq \|T^*T\| \rightarrow (2)$$

From (1) and (2) $\|T^*T\| = \|T\|^2$) Sangee

Imp Self adjoint operators :-

An operation A is said to be ~~self~~ self adjoint if $A^* = A$.

Theorem :-

If A_1 and A_2 are self adjoint operators on H ~~The~~ their product $A_1 A_2$ is self adjoint if $A_1A_2 = A_2A_1$

Proof :-

Given that, A_1 and A_2 are self disjoint

$\therefore A_1 = A_1^*$ and $A_2 = A_2^*$

Assume that $A_1 A_2$ is self adjoint

$$(A_1 A_2)^* = A_1 A_2$$

To prove. $A_2 A_1 = (A_1 A_2)^*$

$$= A_2^* A_1^*$$
$$= A_2 A_1$$

conversely, Assume that

$$A_1 A_2 = A_2 A_1$$
$$(A_1 A_2)^* = A_2^* A_1^*$$
$$= A_2 A_1$$

(8)
5m Theorem:

If T is an operator on H then $(T\alpha, x) = 0$ iff $T = 0 \ \forall x$.

Proof:

(32)

T is an operator on H

Assume that $T = 0$

$(Tx, x) = (0x, x) = (0, x)$

$(Tx, x) = 0 \ \forall x$

Assume that $(Tx, x) = 0 \ \forall x \in H$

Clearly, $\alpha x + \beta y \in H$

$(T(\alpha x + \beta y), \alpha x + \beta y) = 0$

$\Rightarrow (\alpha(T(x) + \beta T(y)), \alpha x + \beta y) = 0$

$(\alpha T(x), \alpha x + \beta y) + (\beta T(y), \alpha x + \beta y) = 0$

$(\alpha T(x), \alpha x) + (\alpha T(x), \beta y) + (\beta T(y), \alpha x) + (\beta T(y), \beta y)$

$\Rightarrow \alpha\bar{\alpha}(Tx, x) + \alpha\bar{\beta}(Tx, y) + \beta\bar{\alpha}(Ty, x) + \beta\bar{\beta}(Ty, y) = 0$

$\alpha = 1, \ \beta = 1$

$\therefore (Tx, x) = 0$
$(Ty, y) = 0$
$\forall x, y \in H.$

$(Tx, y) + (Ty, x) = 0$ and

$(Tx, y) + i(Ty, x) = 0 \rightarrow (2)$

Put $\alpha = i, \ \beta = 1$ in (1)

$i(Tx, y) - i(Ty, x) = 0 \rightarrow (3)$

(2) + (3) $\Rightarrow 2i(Tx, y) = 0$

$(Tx, y) = 0 \ \forall y \in H$

Put $y = Tx$

$(Tx, Tx) = 0$

$\Rightarrow \|Tx\|^2 = 0 \ \forall x \in H$

$$\|Tx\| = 0$$
$$Tx = 0 \quad \forall x$$
$$T = 0$$

Theorem:

An operator T on H self adjoint iff (Tx, x) is real for α all x.

Proof:

Given an operator T on H is a self adjoint

To prove (Tx, x) is real for all x

$$T = T^* \Leftrightarrow \overline{(Tx, x)} = (Tx, x) \quad \forall x$$

Assume that, T is self adjoint

$$\therefore T = T^*$$

$$\overline{(Tx, x)} = (x, T^*x) \quad \forall x$$

$$\overline{(Tx, x)} = \overline{(x, Tx)}$$
$$= (Tx, x) \quad \forall x$$

conversely,

Assume that $\overline{(Tx, x)} = (Tx, x)$

To prove, T is self adjoint

$$(T^*x, x) = (\bar{x}, T^*(x))$$
$$= \overline{(Tx, x)}$$

$$(T^*x, x) = (x, Tx)$$
$$\Rightarrow T^* = T \quad (T^* \text{ is unique})$$

T is self adjoint.

Suppose

Normal and unitary operators:

Definition:

An operator N on H said to be normal if it computs with its adjoint

ie) $NN^* = N^*N$

Theorem :

If N_1 and N_2 are normal operators on H with the property that either computs the self adjoint of the other then $N_1 + N_2$ and $N_1 \times N_2$ (or) $N_1 N_2$ are no

Proof :

Given N_1 and N_2 are normal operators (34)

we have $N_1 N_1^* = N_1^* N_1$

$$N_2 N_2^* = N_2^* N_2$$

Also given that the operators compute with the self adjoint of the other

$$N_1 N_2^* = N_2^* N_1$$

$$N_2 N_1^* = N_1^* N_2$$

$$(N_1 + N_2)(N_1 + N_2)^* = (N_1^* + N_2^*)(N_1 + N_2)$$

$$(N_1 + N_2)(N_1 + N_2)^* = N_1 N_1^* + N_1 N_2^*$$

$$(N_1 + N_2)^*(N_1 + N_2) = (N_1^* + N_2^*)(N_1 + N_2)$$

$$= N_1^* N_1 + N_2^* N_2 + N_1^* N_2 + N_2^* N_1$$

$$= N_1 N_1^* + N_2 N_2^* + N_2 N_1^* + N_1 N_2^*$$

$$= (N_1 + N_2)(N_1 + N_2)^*$$

$\therefore$ $(N_1 + N_2)$ is normal

$$(N_1 \cdot N_2)(N_1 \cdot N_2) = (N_1 \cdot N_2)(N_2^* N_1^*)$$

$$= (N_1 N_2^* \; N_2 N_1^*)$$

$$= (N_2^* N_1 , N_2^* N_2)$$

$$= N_2^* N_1^* \cdot N_1 N_2$$

$$= (N_1 , N_2)^* (N_1 \cdot N_2)$$

$\therefore$ $N_1 N_2$ is normal

Theorem:

An operator T on H is normal iff $\|T^*(x)\| = \|T(x)\| \; \forall x$

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Proof:

For all x

$$\|T^*(x)\| = \|T(x)\| \Leftrightarrow \|T^*(x)\|^2 = \|T(x)\|^2$$

$$\Leftrightarrow (T^*x, T^*x) = (Tx, Tx)$$

$$\Leftrightarrow \cancel{(TT^*x, x) = (Tx, (T^*)^*, x)}$$

$$\Leftrightarrow (TT^*x, x) = (T^*Tx, x)$$

$$\Leftrightarrow ((TT^* - T^*T)x, x) = 0$$

$$\Leftrightarrow TT^* - T^*T = 0$$

$$\Leftrightarrow TT^* = T^*T$$

$$\Leftrightarrow T \text{ is normal}$$

Since the operator T is unique

Projection:

P is a projection iff $P^* = P$ and $P^2 = P$

$M = \{px \,/\, px = x\}$, $N = \{x \,|\, px^*\}$

Theorem:

If p is a projection on a Hilbert space H with range M and null space N. Then $M \perp N$ iff P is self adjoint and in this case $N = M^\perp$

Proof:

case i):

Let p be a projection on H with range M and null space N

Then $M = \{px \,/\, px = x\}$, $N = \{x \,|\, px = 0\}$

Then $M \oplus N = H$

ie) $H = M + N$ and $M \cap N = \{0\}$ and $x \perp y$

$\Rightarrow (x, y) = 0$

$$\left.\begin{aligned} x \in M &\Leftrightarrow px = x \\ y \in N &\Leftrightarrow py = 0 \end{aligned}\right\} \rightarrow (*)$$

Now let us suppose that $M \perp N$ 36

Then to prove p is self adjoint

For let z be any vector in H

Then $z = x+y$ where $x \in M$, $y \in N$

$$\therefore pz = p(x+y) = px + py$$

$$= x + 0$$

$$pz = 0$$

Now consider,

$$(pz, z) = (x, x+y)$$

$$= (x, x) + (x, y)$$

$$= (x, x) + 0$$

$$(pz, z) = (x, x) \rightarrow (1)$$

$\because M \perp N$ then $(x, y) = 0$ $\forall x \in N, y \in M$

and,

$$(p^* z, z) = (z\ (p^*)^* z)$$

$$= (z, pz)$$

$$(p^* z, z) = (x+y, x)$$

$$= (x, x) + (y, x)$$

$$= (x, x) + 0$$

$$(p^* z, z) = (x, x) \rightarrow (2)$$

From (1) and (2)

$$(pz, z) = (p^* z, z)$$

$$\Rightarrow ((p, p^*) z, z) = 0 \quad \forall x \in H$$

$$\Rightarrow p - p^* = 0$$

$$p^* = p$$

P is self adjoint.

case ii) :

Let P be any self adjoint operator

To prove, $M \perp N$

Since, P is self adjoint $P^* = P$ (37)

Let $x \in M$ and $y \in N$ then $Px = x$ and $Py = 0$

Now consider,

$(x, y) = (Px, y) \ \forall \ x, y \in H$

$= (x, P^* y) \ \forall \ x, y \in H$

$= (x, Py) \ \forall \ x, y \in H$

$= (x, 0)$

$\therefore (x, y) = 0 \ \forall \ x \in M$ and $y \in N$

$x \perp y \Rightarrow M \perp N$

From ① and ②

$M \perp N$ iff P is self adjoint

Next to prove $N = M^\perp$

$M \perp N = x \perp M \ \forall \ x \in N$

ie) $x \in N \Rightarrow x \in M^\perp$

$\therefore N \subseteq M^\perp$

Suppose, $N \neq M^\perp \rightarrow (**)$

Then we having $N \subset M^\perp$ which means N is a closed subspace proper subspace of $M^\perp$ and it is a closed subspace of $M^\perp$ where $M^\perp$ is a

Hilbert space

Hence, there exist a non-zero vector z_0

Such that

$z_0 \in M^\perp$ and $z_0 \perp z$

$z_0 \in M^{\perp}$ and $z_0 \perp M$

Also, $z_0 \perp N$ (38)

$z_0 \perp M+N = H$ $\therefore$ ($z_0 + M \cap N = \{0\}$

$z_0 \perp H$ $z_0 \perp 0$

$\Rightarrow z_0 \in H^{\perp} = \{0\}$ $\Rightarrow (z_0, 0) = 0$

$z_0 = 0$ $\therefore \Rightarrow z_0 = 0$

$\Rightarrow \Leftarrow$ to $z_0 \neq 0$ $\Rightarrow \Leftarrow$

$N = M^{\perp}$ $N = M^{\perp}$)

Theorem:-

Sujitra (If A is a +ve operator on H. Then $I+A$ is non-singular. In operator $I + T^* T$ and $I + TT^*$ are non-singular for any arbitrary operators in H

Proof:

To prove $(I+A) \cdot x = 0 \Rightarrow x = 0$

Let $(I+A)x = 0$

$\Rightarrow (Ix + Ax) = 0$

$\Rightarrow x + Ax = 0$

$\Rightarrow Ax = -x$

consider,

$(Ax, x) = (-x, x)$

$= -(x, x)$

$= -\|x\|^2$

Given A is +ve $\Rightarrow (Ax, x) = 0$

$\therefore -\|x\|^2 \overset{0}{\geq} 0$

But $\|x\|^2 \geq 0$

$\|x\|^2 = 0$

$x = 0$

Let, $(I+A)x = (I+A)y$

$(I+A)x - (I+A)y = 0$

$(I+A)(x-y) = 0$

$\Rightarrow x = y$

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$(I+A)$ is one to one

Let M be the ~~image~~ range of $I+A$ for any $x \in H$

$\|(I+A)x\|^2 = \|x + Ax\|^2$

$= (x+Ax, x+Ax)$

$= (x,x) + (Ax,x) + (x,Ax) + (Ax, Ax)$

$= \|x\|^2 + (Ax,x) + (Ax,x) + \|Ax\|^2$

$= \|Ax\|^2 + \|x\|^2 + 2(Ax,x)$

we know that

$\|Ax\|^2 \geq 0$

$\therefore A$ is positive $\Rightarrow (Ax,x) \geq 0$

$\|(I+A)x\|^2 \geq \|x\|^2$

Let $\{(I+A)x_n\}$ be a cauchy sequence in M

for $m,n \in Z^+$

$\|x_n - x_m\| \leq \|(I+A)(x_n - x_m)\|$

$= \|(I+A)x_n - (I+A)x_m\|$

$\therefore \|x_n - x_m\| \to 0$

$\{x_n\}$ is a cauchy sequence in H

H is a complete

$\Rightarrow \{x_n\}$ converges

Let, $x_n \to x \in H$

$(I+A)x_n = (I+A) \lim x_n$

$= (I+A)x$

$x \in H \Rightarrow (I+A)x \in M$

$\{(I+A)x\}$ converges to $(I+A)x$ in M.

M is a closed subspace of H

If $M=H$ then $I+A$ is onto, then there is no[t]

to prove (40)

So let $M \neq H$

$\therefore$ M is a proper closed linear subspace of H, There exist a non-zero vector x_0 in H

Such that $x_0 \perp M$

$x_0 \in M \Rightarrow (I+A)x_0 \in M$

$x_0 \in M \Rightarrow$ ie, $(x_0 + Ax_0) \in M$

$\Rightarrow (x_0, x_0 + Ax_0) = 0$

$(x_0, x_0) + (x_0, Ax_0) = 0$

$\Rightarrow \|x_0\|^2 + \overline{(Ax_0, x_0)} = 0$

$\Rightarrow \|x_0\|^2 + (Ax_0, x_0) = 0$

$\Rightarrow (Ax_0, x_0) = -\|x_0\|^2$

Since, $(Ax_0, x_0) \geq 0$

$-\|x_0\|^2 \geq 0$

$\Rightarrow \|x_0\|^2 = 0$

$\therefore x_0 = 0$

which is a contradiction

$M = H$

$\Rightarrow (I+A)$ is onto.

$(I+A)^{-1}$ exist

$\Rightarrow (I+A)$ is non-singular

Let T be any operator on H, then TT^* and T^*T are +ve operator.

$\therefore$ If TT^* and T^*T are non-singular

Theorem :-

An operator T on H is unitary iff it is an isometric isomorphism of H onto itself.

Proof :-

Suppose T is unitary 41

$$TT^* = T^*T = I$$

$$T^{-1} = T^*$$

Hence, T^{-1} exists

$\therefore$ T is one to one and onto

$$T^*T = I$$

$\Rightarrow \|Tx\| = \|x\|$ < by theorem

$\therefore$ Norm is preserved

T is isometric isomorphism

conversely,

Suppose T is an isometric isomorphism

$\therefore \|Tx\| = \|x\|$ and T^{-1} exist

By theorem

$$T^*T = I$$

$$\Rightarrow (T^*T)T^{-1} = IT^{-1}$$

$$T^*I = T^{-1}$$

$$\Rightarrow T^* = T^{-1}$$

$$\Rightarrow TT^{-1} = T^{-1}T = I$$

Defintion :-

Let T be a operator on H, then a closed subspace M of H is inverient under T if $T(M) \subseteq M$

Definition :-

A closed linear subspace M of H reduced an operator T if both M and $M^\perp$ are invarient under T

Theorem:-

If P and Q are the projections on closed linear subspace N and M of H then $M \perp N$ iff P iff $QP = 0$

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Proof:-

P and Q are projections on M and N respectively

$\therefore P^2 = P = P^*$

$Q^2 = Q = Q^*$

$PQ = 0 \Leftrightarrow (PQ)^* = 0$

$\Leftrightarrow Q^* P^* = 0$

$QP = 0 \Leftrightarrow PQ = 0$

Let, $M \perp N$

Let $y \in N$, $y \perp M$

$y \perp M^\perp$

$N \subseteq M^\perp$

$z \in H$ $(\because PQ = N)$

$Qz \in H$

$Qz \in N \subseteq M^\perp$

$Qz \in M^\perp$ $(\because M^\perp$ is a null space of P

$P(Qz) = 0$

$PQ(z) = 0 \ \forall z \in H$

$PQ = 0$

Conversely,

Let, $PQ = 0$

$x \in M$ and $y \in N$

$\Rightarrow Px \in M$ / $Qy \in N$

$Px = x$ $Qy = y$

$$(x, y) = (Px, Qy)$$
$$= (x, P^* Qy)$$
$$= (x, PQy)$$
$$= (x, Qy)$$

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$$(x, y) = 0$$

$$x \perp y \quad \forall\, x \in M,\ y \in N$$

$$\therefore M \perp N$$

Theorem :-

If T is an operator on H then the following condition are equivalent to one another

i) $T^*T = T$

ii) $(Tx, Ty) = (x, y)$ for every $x, y \in H$

iii) $\|Tx\| = \|x\|$ for every $x \in H$

Proof :

(i) $\rightarrow$ (ii)

Assume that $T^* = T$

consider $(Tx, Ty) = (T^* Tx, y) \quad \forall\, x, y$ in H

$$= (Ix, y)$$

$$(Tx, Ty) = (x, y)$$

(ii) $\Rightarrow$ (iii)

Assume that

$$(Tx, Ty) = (x, y) \quad \forall\, x, y \in H$$

put $y = x$

$$(Tx, Tx) = (x, x) \quad \forall\, x \in H$$

By the definition

$$\|Tx\|^2 = \|x\|^2$$

$$\|Tx\| = \|x\|$$

(iii) $\Rightarrow$ (i)

Assume that

$$\|Tx\| = \|x\| \quad \forall x \in H$$

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$$\|Tx\|^2 = \|x\|^2$$

$$(Tx, Tx) = (x, x)$$

$$\Rightarrow (T^* Tx, x) = (Ix, x)$$

$$\Rightarrow (T^* Tx, x) - (Ix, x) = 0$$

$$(T^* T - I)x = 0 \quad \forall x \in H$$

$$\therefore (T^* T - I) = 0$$

$$\Rightarrow T^* T = I.$$

UNIT –III

13/2/2020

UNIT - III

(1)

Finite Dimensional spectral Theory :-

Definition:-

A non-zero vector x such that $T(x) = \lambda x$ is true for some λ. λ is called an eigen values of T. where T is an operator on H.

Definition :-

Let T be an operator on a Hillbert Space H, A scalar λ such that $Tx = \lambda x$ is true for some non-zero vector x is called an eigen vector of T.

Definition :-

Let M be a non-zero closed linear Subspace of H and it has the set of all vectors which satisfy the equation $(T-\lambda I)x = 0$. T is an operator on H. and λ is a scalar then M is said to be eigen space of an operator T on H

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· Spectral Theory :

If T satisfy the following three conditions then T has a spectral resolution

i) T has eigen values and there are finitely many of them say $\lambda_1, \lambda_2 \dots \lambda_m$ which are distinct with corresponding eigen spaces $M_1, M_2 \dots M_m$

ii) M_i's are pairwise orthogonal

iii) M_i's span H.

Every vector x in H can be expressed uniquely in the form,

$$x = x_1 + x_2 + \dots + x_m$$

Now, $Tx = T(x_1 + x_2 + \dots + x_m)$

$$= Tx_1 + Tx_2 + \dots + Tx_m$$

$$= \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_m x_m \rightarrow (1)$$

Now, if P_i's are the projections on M_i then P_i's are pairwise orthogonal

Since, $P_i(x) = x_i$

$$Ix = x = P_1 x + P_2 x + \dots + P_m x$$

$$Ix = (P_1 + P_2 + \dots + P_m)x$$

$$I = (P_1 + P_2 + \dots + P_m) = \Sigma P_i \rightarrow (2)$$

(1) $\Rightarrow Tx = \lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_m x_m$

$$= \lambda_1 P_1 x + \lambda_2 P_2 x + \dots + \lambda_m P_m x$$

$$Tx = (\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m) x$$

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m$$

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Example :-

consider T on l_2 defined by $T = \{x_1, x_2 \ldots\} = \{0, x_1, x_2 \ldots\}$. If it is finite dimensional then every operator has an eigen values.

Solution:

It is enough to prove that, T is normal

$$T = \lambda_1 P_1 + \cdots + \lambda_m P_m$$

$$T^* = (\lambda_1 P_1 + \cdots + \lambda_m P_m)^* \; ; \; P_1^* = P_1$$

$$= \bar{\lambda}_1 P_1 + \bar{\lambda}_2 P_2 + \cdots + \bar{\lambda}_m P_m$$

Now

$$TT^* = (\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m)(\bar{\lambda}_1 P_1 + \cdots + \bar{\lambda}_m P_m)$$

$$= \lambda_1 \bar{\lambda}_1 P_1^2 + \lambda_2 \bar{\lambda}_2 P_2^2 + \cdots + \lambda_m \bar{\lambda}_m P^2 m$$

$\langle \therefore P_i P_j = 0, \; i \neq j$

$$TT^* = |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \cdots + |\lambda_m|^2 P_m = 0 \rightarrow (1)$$

similarly we get

$$T^* T = |\lambda_1|^2 P_1 + |\lambda_2|^2 P_2 + \cdots + |\lambda_m|^2 P_m \rightarrow (2)$$

From (1) and (2) we get

$$T^* T = TT^* \rightarrow (3)$$

T is normal

If every normal operator on H satisfy the condition (1), (2) and (3)

$\therefore$ It has a specturral resolution here we will see it is finite dimensional with dimension $n > 0$.

Matrices :-

Lemma :-

(4)

Let $B = \{e_1, e_2 \dots, e_n\}$ be an ordered basis for a finite dimensional Hilbert space H. If $\{f_1, f_2 \dots f_n\}$ are any set of n-vectors in H. Then there exist a unique operator T on H. There exist $T(e_i) = f_i$, $i = 1, 2 \dots$

Proof :-

Existen of T for any x in H we have

$(x = \alpha_1 e_1 + \alpha_2 e_2 + \dots + \alpha_n e_n) \rightarrow (1)$

where $\alpha_1, \alpha_2, \dots, \alpha_n$ are unique scalar with respect to B knowing.

$\alpha_1, \alpha_2, \dots, \alpha_n$ associated with define T as

$Tx = \alpha_1 f_1 + \alpha_2 f_2 + \dots + \alpha_n f_n \rightarrow (2)$

$\therefore T(x)$ is well defined element of H and it associated with $x \in H$ as unique element $T(x) \in H$

So that T is mapping element

$T(x)' \in H$

So that T is mapping of H on to H.

If we take $e_i \in H$ it has an representation with respect to B. As,

$e_i = 0e_1 + 0e_2 + \dots + 0e_{i-1} + e_i + 0e_{i+1} + \dots + 0e_n$

Hence, using the defn. of T we get

$T(e_i) = f_i \quad \forall i = 1, 2 \dots$

i) To prove : T is linear

Let $x, y \in H$ and α, β be scalars with respect to the basis B we have

$$x = \alpha_1 e_1 + \dots + \alpha_n e_n$$
$$y = \beta_1 e_1 + \beta_2 e_2 + \dots + \beta_n e_n \quad (5')$$

Now,

$$\alpha x + \beta y = \alpha(\alpha_1 e_1 + \dots + \alpha_n e_n) + \beta(\beta_1 e_1 + \dots + \beta_n e_n)$$
$$= \alpha\alpha_1 e_1 + \dots + \alpha\alpha_n e_n + \beta\beta_1 e_1 + \dots + \beta\beta_n e_n$$
$$= (\alpha\alpha_1 + \beta\beta_1) e_1 + \dots + (\alpha\alpha_n + \beta\beta_n) e_n$$
$$T(\alpha x + \beta y) = (\alpha\alpha_1 + \beta\beta_1) f_1 + \dots + (\alpha\alpha_n + \beta\beta_n) f_n$$

∠ by (2)

$$= (\alpha\alpha_1 f_1 + \beta\beta_1 f_1) + \dots + (\alpha\alpha_n f_n + \beta\beta_n f_n)$$
$$= (\alpha\alpha_1 f_1 + \dots + \alpha\alpha_n f_n) + (\beta\beta_1 f_1 + \dots + \beta\beta_n f_n)$$
$$= \alpha(\alpha_1 f_1 + \dots + \alpha_n f_n) + \beta(\beta_1 f_1 + \dots + \beta_n f_n)$$
$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$

$\Rightarrow$ T is linear

ii) To prove : T is unique

Suppose that, T is not unique

Let, T_1 be an operator on H.

Such that $T_1(e_i) = f_i$, $i = 1, 2 \dots$

If $x = \alpha_1 e_1 + \dots + \alpha_n e_n \in H$. Then, we have

$$T_1(x) = T_1(\alpha_1 e_1 + \dots + \alpha_n e_n)$$
$$= \alpha_1 T_1(e_1) + \dots + \alpha_n T_1(e_n)$$
$$= \alpha_1 f_1 + \dots + \alpha_n f_n = T(x)$$

$\therefore T_1(x) = T(x)$

$\Rightarrow$ T is unique

Thus, T is completely determined by $T(e_i)$.

where e_i ∈

on H. Such

$T_1(e_i) =$

Thus, T_1

Thus the

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Theorem

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Then T is n

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Proof :

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T

Since

(1) ⇒

⇒

⇒

where $e_i \in B$. Thus if T_1, T_2 are two operator on H. Such that

$T_1(e_i) = T_2(e_i) \ \forall \ e_i \in B$ in H (6)

Thus, $T_1(x) = T_2(x) \ \forall \ x \in H$

Thus the two operators on H are equal of any agree on basic for H.

Theorem : KOW A-19 5M

Let, B be a basis for H and T is an operator. where matrix relative to B is $[\alpha_{ij}]$ Then T is non-singular iff $[\alpha_{ij}]$ is an non-singular and in this case.

$$[\alpha_{ij}]^{-1} = [T^{-1}]$$

Proof :

T is non-singular iff there exist an operator, such that

$$T T^{-1} = T^{-1} T = I \rightarrow (1)$$

Since, $T \ni [T]$ is one to one

$$(1) \Rightarrow [T][T^{-1}] = [T^{-1}][T] = [I] = [\delta_{ij}]$$

$$\Rightarrow [\alpha_{ij}][T^{-1}] = [T^{-1}][\alpha_{ij}] = [I] = [\delta_{ij}]$$

$$\Rightarrow [T^{-1}][\alpha_{ij}] = [I]$$

$$[T^{-1}] = \frac{[I]}{[\alpha_{ij}]}$$

$$= [I][\alpha_{ij}]^{-1}$$

$$[T^{-1}] = [\alpha_{ij}]^{-1}$$

Definition:-

Let A and B be square matrices of order n over the field of complex numbers then B is said to be similar to A if there exist $n \times n$ non-singular matrix C over the field of complex numbers such that, $B = C^{-1}AC$.

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Definition:

Let A and B be operators on a Hilbert Space H then B is said to be similar to A if there exist a non-singular operator C on H such that, $B = C^{-1}AC$

Theorem:-

Two matrices in An are similar $\Leftrightarrow$ They are matrices of a single operator on H relative to [possibly] different spaces.

Proof:-

1st, we shall establish that if T is an operator on n-dimensional space N and if B and B' are two ordered basic for H then the matrix of T relative to B'

Let $B = \{e_1, e_2 \dots e_n\}$ and $B' = \{e_1', e_2' \dots e_n'\}$

Let $[T]_B = [\alpha_{ij}]$ and $[T]_{B'} = [\beta_{ij}]$ $\forall$ $i, j = 1, 2 \dots n$

Then we have,

$$T(e_j) = \sum_{i=1}^{n} \alpha_{ij} e_i \qquad j = 1, 2 \dots n$$

$$T(e_j) = \sum_{i=1}^{n} \beta_{ij} e_i, \qquad j = 1, 2 \dots n$$

Let S be the operator on H defined by,

$S(e_j) = e_j'$ for $j = 1, 2 \dots n \rightarrow (1)$

Since S maps a basis B onto a basis B'

it is non-singular

Let, $[\gamma_{ij}]$ be the matrix of S relative to B.

Then $[\gamma_{ij}]$ is non-singular (by them. (8)

Since, $[S]_B = [\gamma_{ij}]$

$$\therefore S(e_j) = \sum_{i=1}^{n} \gamma_{ij} e_i \quad j = 1, 2 \dots n \rightarrow (2)$$

From (1) and (2)

$$T(e_j') = T(S(e_j)) = T\left(\sum_{k=1}^{n} \gamma_{kj} e_k\right)$$

$$= \sum_{k=1}^{n} \gamma_{kj} T(e_k) \quad (\because T \text{ is linear}$$

$$= \sum_{k=1}^{n} \gamma_{kj} \sum_{k=1}^{n} \alpha_{ik} e_i$$

$$T(e_j') = \sum_{i=1}^{n} \left[\left(\sum_{k=1}^{n} \alpha_{ik} \gamma_{kj} \right) e_i \right] \rightarrow (3)$$

Now,

$$T(e_j') = \sum_{k=1}^{n} \beta_{kj} e_k'$$

$$= \sum_{k=1}^{n} \beta_{kj} S(e_k) \quad \angle \text{ by } (1)$$

$$= \sum_{k=1}^{n} \beta_{kj} \left(\sum_{i=1}^{n} \gamma_{ik} e_i \right) \quad \angle \text{ by } (2)$$

$$= \sum_{i=1}^{n} \left(\sum_{k=1}^{n} (\gamma_{ik} \beta_{kj}) e_i \rightarrow (4)$$

From (3) and (4) we have

$$\sum_{i=1}^{n} \left(\sum_{k=1}^{n} \alpha_{ik} \gamma_{kj} \right) e_i = \sum_{i=1}^{n} \left(\sum_{k=1}^{n} \gamma_{ik} \beta_{kj} \right) e_i$$

Since, e_i is set of linearly independent vector, we get

$$[\alpha_{ij}][\gamma_{ij}] = [\gamma_{ij}][\beta_{ij}] \quad (9)$$

Pre multiply, by $(\gamma_{ij})^{-1}$ on both sides we have

$$[\gamma_{ij}]^{-1}[\alpha_{ij}][\gamma_{ij}] = [\beta_{ij}] \quad (\because \gamma_{ij} \text{ is non singular})$$

$\Rightarrow$ $[\alpha_{ij}]$ and $[\beta_{ij}]$ are singular matrices

consequences $[T]_B$ is similar to $[T]^{-1}_B$

conversely

Suppose that, $[\alpha_{ij}]$ and $[\beta_{ij}]$ are two $n \times n$ similar matrices

Then, there exist a non singular matrix $[\gamma_{ij}]$

Such that $[\gamma_{ij}]^{-1}[\alpha_{ij}][\gamma_{ij}] = [\beta_{ij}] \rightarrow (5)$

Let $B = \{e_1, e_2 \dots e_n\}$ be any ordered basis for 4.

Let T.S be the operator on H with

$$[T]_B = [\alpha_{ij}] \rightarrow (6)$$

$$[S]_B = [\gamma_{ij}] \rightarrow (7)$$

$\Rightarrow$ S is non-singular ⟨from hypothesis⟩

Let $B' = \{e_1', e_2', \dots e_n'\}$

Since S is a non-singular operator it maps a basis onto a basis on H

So that B' is a basis for H from hypothesis

$$[T]_{B'} = [\gamma_{ij}]^{-1}[T]_B[\gamma_{ij}]$$

$$= [\gamma_{ij}]^{-1}[\alpha_{ij}][\gamma_{ij}] \quad \langle \text{by } (6)$$

$= [\beta_{ij}]$ by (5)

Thus $[\alpha_{ij}]$ and $[\beta_{ij}]$ are matrices of T with respect to the basis B and B' respectively.

Determinates and the spectrum of an operator:

Let $[\alpha_{ij}]$ be an $n \times n$ matrix the determinant of this matrix is denoted by let $[\alpha_{ij}]$ and is a scalar valued function of matrices having the (10) properties.

i) let $([\alpha_{ij}]) = 1$

ii) let $([\alpha_{ij}][\beta_{ij}])$ = let $([\alpha_{ij}])$ let $([\beta_{ij}])$

iii) let $([\alpha_{ij}]) \neq 0 \Leftrightarrow [\alpha_{ij}]$ is an singular

iv) let $([\alpha_{ij}]) - \lambda([\delta_{ij}])$ is a polynomial with complex co-efficients of degree 'n' in the variable

The determinant of a matrix is usually written out with vertical base as follows

$$\text{let}([\alpha_{ij}]) = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2n} \\ \vdots & & & \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn} \end{bmatrix}$$

Determinant of the operator T :-

consider an operator T. If B and B' are basis for H. Then the matrix $([\alpha_{ij}][\beta_{ij}])$ of T relative to B and B' may be entirely different but we have the same determinant.

Theorem :-

If T is an arbitrary on H. Then the eigen values of T constitude a non-empty finite subset of the complex plane, further more the number of points in this set does not exceeds the dimension n of the space H.

(11)

Proof :- Lemma :-

A operator T on a finite dimensional Hilbert Space H is singular. iff such there exist a non-zero vector x in H such that $Tx = 0$

Proof of the lemma :-

Suppose that there exist $x \neq 0$ in H.

Such that, $Tx = 0$

we can write $Tx = 0$ as $Tx = T \cdot 0$

Since $x \neq 0$ and these two distinct elements have the same image

Therefore, the mapping T is not one to one

Hence T does not closed

T is singular

conversely,

Suppose that, T is singular

Suppose there exist no non zero vector x

Such that $Tx = 0 \Rightarrow x = 0$

Then T is one to one

Since Hilbert space H is finite dimensional and T is one to one also one.

$\Rightarrow$ T is non-singular

$\Rightarrow \Leftarrow$

There exist $x \neq 0$ such that $Tx = 0$

Hence, proved

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Proof of the main Theorem:

Let T be an operator on a finite dimensional Hilbert space H of dimension n.

$\lambda \in \sigma(T)$ if there exist $x \neq 0$

$$(T - \lambda I)x = 0$$

By the lemma.

$$(T - \lambda I)x = 0 \Leftrightarrow (T - \lambda I) \text{ is singular}$$

we know that

$(T - \lambda I)$ is singular $\Leftrightarrow$ det $(T - \lambda I) = 0$

Thus, $\lambda \in \sigma(T) \Leftrightarrow \lambda$ satisfy the equation

$$= \det\{[T]_B - \lambda [I]_B\}$$

$$= \det\{[T]_B - \lambda [\delta_{ij}]_B\} \quad (\because I = \delta_{ij}$$

But $\det(T - \lambda I) = 0$

$$\det\{[T]_B - \lambda[\delta_{ij}]\} = 0 \rightarrow (1)$$

If $[T]_B = [\alpha_{ij}]$ is the matrix of T

Then (1) gives

$$\begin{vmatrix} \alpha_{11} - \lambda & \alpha_{12} & \dots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22} - \lambda & \dots & \alpha_{2n} \\ \vdots & & & \\ \alpha_{n1} & \alpha_{n2} & \dots & \alpha_{nn} - \lambda \end{vmatrix} = 0 \rightarrow (2)$$

The expansion of the determinant of (2) gives a polynomial equation of degree n in λ with complete co-efficient in the variable.

By the fundamental theorem of algebra (3) must have exactly n roots in the field of complex number.

(13)

Hence, every operators T on H has an eigen value so that T has n eigen values and the number of distinct eigen value of T is $\leq n$.

Hence, the number of element of $\sigma(T)$ is less than or equal

Hence, proved.

Theorem :-

Let T be an operator on H. Then prove that, i) T is singular $\Leftrightarrow$ $0 \in \sigma(T)$

ii) If T is non-singular, then $\lambda \in \sigma(T)$ $\Leftrightarrow \lambda^{-1} \in \sigma(T^{-1})$

iii) If A is non-singular then

$$\sigma(ATA^{-1}) = \sigma(T)$$

iv) If $\lambda \in \sigma(T)$ and if p is a polynomial then $p(\lambda) \in \sigma[p(T)]$

Proof :-

i) $\Rightarrow$ T is singular $\Leftrightarrow$ There exist $x \neq 0$ Such that, $Tx = 0$ (by above lemma)

ie) $Tx = 0.x$

Hence, T is singular $\Leftrightarrow$ 0 is the eigen value of T

ie) $0 \in \sigma(T)$

T is singular $\Leftrightarrow$ $0 \in \sigma(T)$.

ii) $\Rightarrow$ T to be a non-singular and $\lambda \in \sigma(T)$

$\lambda \neq 0$ and T^{-1} exists

[" Since λ is an eigen value of T. There exist a non zero vector $x \in H$. Such that $Tx = \lambda x$ (by (iii)(2))

pre multiply by T^{-1} on both sides, we get

$$T^{-1}Tx = T^{-1}\lambda x$$

$$x = T^{-1}(\lambda x)$$

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$\{ \exists\, x \neq 0 \ni Tx = \lambda x$

similarly,

$\lambda^{-1}x = T^{-1}(x)$

$\Rightarrow \lambda^{-1}(x) = T^{-1}(x)$ for $x \neq 0$

$\Rightarrow \lambda^{-1} \in \sigma(T^{-1})$

conversely,

If λ^{-1} is an eigen value of T^{-1} Then, $(\lambda^{-1})^{-1} = \lambda$ is an eigen value of $(T^{-1})^{-1} = T$

Hence $\lambda \in \sigma(T)$

iii) $\Rightarrow$ Let $S = ATA^{-1}$

$$S - \lambda I = ATA^{-1} - \lambda I$$

$$= ATA^{-1} - (A)(\lambda I)A^{-1}$$

$$= A(T - \lambda I)A^{-1}$$

let $(S - \lambda I) =$ let $A(T - \lambda I)A^{-1} = 0$ let $(T - \lambda I)$

$\therefore$ let $(S - AI) \Leftrightarrow$ let $(T - \lambda I) = 0$

$\Rightarrow S$ and I have the same eigen values

$\therefore \sigma(ATA^{-1}) = \sigma(T)$

$\Rightarrow$ Let $\lambda \in \sigma(T)$

$\Rightarrow A$ is an eigen value of T

$\Rightarrow$ There exist a non-zero vector x.

Such that $Tx = \lambda x$

$\therefore \sigma(Tx) = T(\lambda x)$

$T(Tx) = \lambda T(x)$

$= \lambda \lambda x$ (15)

$T^2 x = \lambda^2 x$

If λ is an eigen value of T

Then λ^2 is an eigen value of T^2

Similarly, If λ is an eigen value of T

Then, λ^n is an eigen value of T^n for any positive integer n.

Let $p(T) = a_0 + a_1 + \dots + a_m T^m$

where, $a_0, a_1, a_2 \dots a_m$ are scalars

$[p(T)]_x = (a_0 + a_1 T + a_2 T^2 + \dots + a_m T^m) x$

$= (a_0 x + a_1 T x + \dots + a_m T^m x)$

$= (a_0 \lambda x + a_1 \lambda x + \dots + a_m \lambda m x)$

$= (a_0 + a_1 \lambda + \dots + a_m \lambda m) x = [p(\lambda) x]$

$\therefore p(\lambda) = a_0 + a_1 \lambda + \dots + a_m \lambda^m$ is an eigen value of $p(T)$

Thus if $\lambda \in \sigma(T)$ then $p(\lambda) \in \sigma[p(T)]$

The Spectral Theorem: (A)

If T is normal then x is an eigen vector of T with eigen value λ iff x is an eigen vector of T^* with eigen value $\bar{\lambda}$

Proof:

Given that T is normal operator on H

Then $T - \lambda z$ is also normal where λ is any scalar

$(T - \lambda z)^* = T^* - \bar{\lambda} z = T^* - \bar{\lambda} z$ $\quad \angle \therefore z^* = z$

Since, $T-\lambda I$ is normal

we have

$$\|(T-\lambda I)\alpha\| = \|(T-\lambda I)^*\alpha\|$$ 16

(By theorem T is normal)

ie) $\|T\alpha\| = \|T^*(\alpha)\| \quad \forall \alpha \in H$

$$= \|T\alpha - \bar{\lambda} I\alpha\|$$

$$\Rightarrow T\alpha - \lambda\alpha = 0 \Leftrightarrow T^*\alpha = \bar{\lambda}\alpha$$

$$\Rightarrow T\alpha = \lambda\alpha \Leftrightarrow T^*\alpha = \bar{\lambda}\alpha$$

ie) α is an eigen vector of T with eigen value of λ iff α is an eigen vector at T^* with value $\bar{\lambda}$.

Theorem : (6)

If T is normal then the eigen spaces M_i's are pairwise orthogonal.

Proof :

Let M_i, M_j be eigen space of a normal operator T on H corresponding to the distinct eigen values λ_i and λ_j then to prove $M_i \perp M_j$ if $i \neq j$ for let x_i be any vector in M_i and let x_j be any vector in M_j

we have $T x_i = \lambda_i x_i \rightarrow (1)$

$$T x_j = \lambda_j x_j \rightarrow (2)$$

Given that T is normal

Then by theorem

$$T\alpha = \lambda\alpha \Leftrightarrow T^*\alpha = \bar{\lambda}\alpha$$

$$T x_j = \lambda_j x_j \Leftrightarrow T^* x_j = \bar{\lambda}_j \bar{x}_j \rightarrow (3)$$

$$\lambda_i (x_i, x_j) = (\lambda_i x_i, x_j)$$
$$= (T x_i, x_j)$$
$$= (x_i, T^* x_j)$$

$$= (x_i, \bar{\lambda}_j x_j)$$
$$= \bar{\bar{\lambda}}_j (x_i, x_j)$$
$$= \lambda_j (x_i, x_j)$$

$$\lambda_i (x_i, x_j) = \lambda_j (x_i, x_j)$$

$$\Rightarrow (\lambda_i, \lambda_j)(x_i, x_j) = 0$$

But $\lambda_i - \lambda_j \neq 0$ because $i \neq j$

we have $\lambda_i \neq \lambda_j$

$\lambda_i - \lambda_j \neq 0$

Hence $(x_i, x_j) = 0$

$\Rightarrow x_i \perp x_j \;\forall\; x_i \in M_j$ and $x_j \in M_i$

$\Rightarrow M_i \perp M_j \;\forall\; i \neq j$

$\therefore M_i$'s are piecewise orthogonal

Theorem : (c)

If T is normal then M_i reduces T

Proof :

Let M_i be an eigen space of T corresponding to the eigen value λ_i

Now in order to prove that,

M_i reduces T it is enough to prove that,

M_i is invariant under both T and T^*

Since, M_i is an space of T and let x to any

... where M_i is a linear subspace of H

and $\alpha \in M_i$ for the scalar λ_i

$\lambda_i \in M_i$

(18)

ie) $T\alpha \in M_i \rightarrow (*)$

which means that M_i is invariant under T.

similarly, M_i is invariant under T^* also Because T is normal

$T\alpha = \lambda_i \alpha$ iff $T^*\alpha = \bar{\lambda}\alpha$ (previous theorem)

and hence, $\alpha \in M$ for any scalar $\bar{\lambda}_i$

we have, $\bar{\lambda_i}\alpha$ also belongs to M_i

$T^*\alpha \in M_i$

ie) $\alpha \in M_i \Rightarrow T^*\alpha = M_i$ means $T^*(M_i) \subseteq M_i$

(or) in otherwords

M_i is invariant under $T^* \rightarrow (**)$

From $(*)$ and $(**)$

M_i is invariant under T and T^* hence M_i reduces T

whenever T is normal

Theorem : (D) A-19 5M

If T is normal then M_i's span H

Proof:

Let M_i's are eigen space of T

Since T is normal

M_i's are pairwise orthogonal

ie) M_i's orthogonal to M_j $\forall i \neq j$

ie) $M_i \perp M_j$ $\forall i \neq j$

Let $M = M_1 + M_2 + \cdots + M_m$

Then M is a closed linear subspace of $^{*}H$

and it's associated projection is ie $\begin{pmatrix} P^* = P \\ P^2 = P \end{pmatrix}$

$$P = P_1 + P_2 + \cdots + P_m$$

Since, T is normal on H 19

M_i reduces T

Also, P_i's are projection on M_i

M_i reduces $T = P_i T = T P_i \ \forall i$

$$Tp = T(P_1 + P_2 + \cdots + P_m)$$

$$= TP_1 + TP_2 + \cdots + TP_m$$

$$= P_1 T + P_2 T + \cdots + P_m T$$

$$= (P_1 + P_2 + \cdots + P_m) T$$

$TP = PT$ and p is projection on H

$\Rightarrow M$ reduces T

$\Rightarrow M^\perp$ is invariant under T

Let U be the restriction of T to $M^\perp$ Then U is an operator on finite dimensional H S $M^\perp$

$\therefore \forall x$ in $M^\perp$ $Ux = Tx$

If x is an eigen vector for U corresponding to the eigen value then, $x \in M^\perp$ and $Ux = \lambda x$

$Tx = \lambda x$ and so it is also an eigen vector for T

Since, all the eigen vectors of T are in M

and $M \cap M^\perp = 0$

So T has no eigen vector in $M^\perp$

So U is an operator on a finite dimensional

Hilbert space $M^\perp$ and U has no eigen vector in M so no eigen value

we must have $M^\perp = \{0\}$ (20)

{∴ because if $M^\perp \neq 0$ then every operator on a non-zero finite H.S must have an eigen value}

∴ M's span H

Problem: 1

If P and Q are the projection on closed linear subspace M and N of H prove that $Q-P$ is a projections $\Leftrightarrow P \leq Q$. In this case show that $Q-P$ is projection on $N \cap M^\perp$

Solution:

If P and Q are the projection on closed linear subspaces M and N of H Then $P^* = P$, $P^2 = P$

$Q^* = Q$, $Q^2 = Q$

Suppose $Q-P$ is projection

$(Q-P)^2 = (Q-P)$ and

$(Q-P)^* = (Q-P)$

$(Q-P) = (Q-P)^2 \geq 0$

$Q \geq P \Rightarrow P \leq Q$

Conversely, Assume that $P \leq Q$

To prove $Q-P$ is projection

$(Q-P)^2 = (Q-P)(Q-P) = Q^2 - 2QP + P^2$

$= Q^2 + P^2 - 2P$

$= Q + P - 2P$

$= Q - P$

$(Q-P)^* = Q^* - P^* = Q - P$

$(Q-P)$ is a projection $\Leftrightarrow P \leq Q$

Now let us prove that

$Q-P$ is a projection on $N \cap M^\perp$ (21)

i.e. $R.(Q-P) = N \cap M^\perp$

Let $x \in N \cap M^\perp$

$x \in N$ and $x \in M^\perp$

$x \in N \Rightarrow Qx = x$ $\quad$ $\angle \therefore R(Q) = N$

$\angle \therefore R(P) = M^\perp$

$x \in M^\perp \Rightarrow Px = 0$

$(Q-P)x = Qx - Px$

$= x - 0$

$(Q-P)x = x$

$x \in R(Q-P)$

$x \in N \cap M^\perp \Rightarrow x \in R(Q-P)$

$\Rightarrow N \cap M^\perp \subseteq R(Q-P) \rightarrow$ (1)

$\Rightarrow (Q-P)x = x$

$Qx - Px = x$ $\quad$ $\angle \therefore Q^2 = R$

$Q(Qx - Px) = Qx$ $\quad$ $QP = P$

$Q^2x - QPx = Qx$

$Qx - Px = Qx$

$\Rightarrow Px = 0 \Rightarrow x \in$ Null space of P

i.e. $x \in M^\perp \rightarrow$ (1)

$(Q-P)x = x$

$Qx - Px = x$

$Qx - 0 = x$

$$Qx = x$$

$$x \in N \rightarrow \text{(ii)}$$

From (i) and (ii) we have

$$x \in N \cap M^{\perp}$$ (22)

$$x \in R(Q-P) \Rightarrow x \in N \cap M^{\perp}$$

$$\Rightarrow R(Q-P) \subseteq N \cap M^{\perp} \rightarrow ②$$

From ① and ②

$$R(Q-P) = N \cap M^{\perp}$$

$(Q-P)$ is projection on $N \cap M^{\perp}$

Definition :-

Let T be an normal operator on H and $\lambda_1, \lambda_2 \dots \lambda_m$ be distinct eigen value of T and $M_1, M_2 \dots M_m$ be their corresponding eigen space then T has an expression of the form $T = \lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_m P_m \rightarrow ①$ where $P_1, P_2 \dots P_m$ are projections on $M_1, M_2 \dots M_m$. The expression ① is called "Spectral resolution of T"

Theorem :-

Spectral resolution of an operator is unique

Proof:

$$T^2 = \left(\sum_{i=1}^{m} \lambda_i P_i\right)\left(\sum_{j=1}^{m} \lambda_j P_j\right)$$

$$= \left(\sum_{i=1}^{m} \lambda_i^2 P_i^2 + \sum_{i,j=1}^{m} \lambda_i \lambda_j P_i P_j\right)$$

$(\because P_i P_j = 0, \ i \neq j)$ $\qquad (\because P_i^2 = P_i$

$$T^2 = \sum_{i=1}^{m} \lambda_i^2 P_i$$

In general

$$T^n = \sum_{i=1}^{m} \lambda_i^n p_i \rightarrow ②$$

we have, ㉓

$$T^n = I = p_1 + p_2 + \dots + p_n$$

$$I = [\lambda_1]^n p_1 + [\lambda_2]^0 p_2 + \dots + [\lambda_n]^0 p_n$$

eqn. ② is true when $n=0$

consider $p(x)$ be a polynomial with complex co-efficien

in the complex variable z

Let $p(z) = a_0 + a_1 z + a_2 z^2 + \dots + a_r z^r \rightarrow ③$

$$p(\lambda_i) = a_0 + a_1 \lambda_i + a_2 \lambda_i^2 + \dots + a_r \lambda_i^r \rightarrow ④$$

$$p(T) = a_0 T^0 + a_1 T + a_2 T^2 + \dots + a_r T^r$$

$$= a_0 \sum_{i=1}^{m} p_i + a_1 \sum_{i=1}^{m} \lambda_i p_i + \dots + a_r \sum_{i=1}^{m} (\lambda_i)^r p_m$$

$$= (a_0 + a_1 \lambda + a_2 \lambda^2 + \dots + a_r \lambda^r) p_1 +$$

$$(a_0 + a_1 \lambda_2 + \dots + a_r \lambda^r) p_1 + \dots +$$

$$(a_0 + a_1 \lambda_m + a_2 \lambda_m^2 + \dots + a_2 \lambda^r_m) p_m$$

$$p(T) = p(\lambda_1) p_1 + p(\lambda_2) p_2 + \dots + p(\lambda_m) p_m$$

$$p(T) = \sum_{i=1}^{m} p(\lambda_i) p_i \rightarrow ⑤$$

Let p_i be a polynomial such that

$$p_j \lambda_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

In ⑤ replacing p by p_i we have

$$p_i(T) = \sum_{i=1}^{m} p_j(\lambda_i) p_j \qquad \angle \because p_j(\lambda_j) = 0,\ i \neq j$$

$$= p_j(\lambda_j) p_j$$

$P_i(T) = P_j \rightarrow (6)$

(24)

$\therefore P_j(T)$ is a projection and a polynomial in T

Let $$P_i(x) = \frac{(x-\lambda_1)(x-\lambda_2)\ldots(x-\lambda_{j-1})(x-\lambda_{j+1})(x-\lambda_m)}{(\lambda_j-\lambda_1)(\lambda_j-\lambda_2)\ldots(\lambda_j-\lambda_{j-1})(\lambda_j-\lambda_{j+1})(\lambda_j-\lambda_m)}$$

$$= \frac{(\lambda_i-\lambda_1)(\lambda_i-\lambda_2)\ldots(\lambda_i-\lambda_{j-1})(\lambda_i-\lambda_{j+1})(\lambda_i-\lambda_m)}{(\lambda_j-\lambda_1)(\lambda_j-\lambda_2)\ldots(\lambda_j-\lambda_{j-1})(\lambda_j-\lambda_{j+1})(\lambda_j-\lambda_m)}$$

$= 1$

$\therefore$ Such a polynomial exists, suppose the spectral resolution is not unique

Let $T = \alpha_1 Q_1 + \alpha_2 Q_2 + \ldots + \alpha_k Q_k \rightarrow (7)$

be a spectral resolution of T where $\alpha_1, \alpha_2 \ldots \alpha_k$ are distinct complex numbers $Q_1, Q_2, \ldots Q_k$ are pairwise orthogonal projection.

$I = Q_1 + Q_2 + \ldots + Q_k \rightarrow (8)$

For each i $Q_i \neq 0$

$R(Q_i) = \{0\}$ for a fixed i

$\therefore$ There exists $x \in R(Q_i)$

$\Rightarrow Q_i x = x$

and if $j \neq i$ Then $Q_i x = 0$

$Tx = (\alpha_1 Q_1 + \alpha_2 Q_2 + \ldots + \alpha_k Q_k) x$

$= \alpha_1 Q_1 x + \alpha_2 Q_2 x + \ldots + \alpha_k Q_k x$

$= \sum_{i=1}^{k} \alpha_i Q_i x \rightarrow (9)$ $\quad \langle \therefore Q_i x = x$

$Tx = \alpha_i x$

∴ α_i is an eigen value of T and x is corresponding eigen vector.

If λ is an eigen value of T. Then $Tx = \lambda_i x$ for some $x \neq 0$

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$$Tx = \lambda_i x = \lambda_i I x$$
$$= \lambda_i (Q_1 + Q_2 + \dots Q_k) x$$
$$= \sum_{i=1}^{k} \lambda_i Q_i x$$

From (9), $Tx = \sum_{i=1}^{k} \alpha_i Q_i x$

$$\Rightarrow \sum_{i=1}^{k} \lambda_i Q_i x = \sum_{i=1}^{k} Q_i \alpha_i x$$

$$\Rightarrow \sum_{i=1}^{k} (\lambda_i - \alpha_i) Q_i x = 0$$

Q_i's are pairwise orthogonal

∴ $Q_i x$'s are pairwise orthogonal

$Q_i x \perp Q_j x \quad i \neq j$

The non-zero $Q_i x$ are linearly independent

∴ Atleast one $x \neq 0$ ($Q_i x = 0$)

$\{Q_i x\}$ is linearly independent ⟨∵ $Q_i x = x$, x is non-zero ∴ $Q_i x \neq 0$

$(\lambda_i - \alpha_i) = 0 \quad (Q_i x \neq 0)$

$$\Rightarrow \lambda_i = \alpha_i$$

∴ Any eigen value of $T_i = \alpha_i$ for some i

∴ $(\lambda_1, \lambda_2 \dots \lambda_m) = \{\alpha_1, \alpha_2 \dots \alpha_k\}$

From (6)

$$P_j(T) = Q_j$$
$$P_j(T) = Q . P_j$$
$$\Rightarrow P_j = Q_j$$

The Spectral resolution is unique

The Spectral theorem:- A-19

Let T be an arbitrary operator on H. Let $\lambda_1, \lambda_2 \ldots \lambda_m$ be the distinct eigen value of T and $M_1, M_2 \ldots M_m$ be their corresponding eigen spaces. Let $P_1, P_2 \ldots P_m$ be the projections on these eigen spaces. Then the following statement are equivalent. 26

i) M_i's are pairwise orthogonal and span H

ii) The P_i's are pairwise orthogonal

$$I = \sum_{i=1}^{n} P_i \quad \text{and} \quad T = \sum_{i=1}^{m} \lambda_i P_i$$

iii) T is normal

Proof: (i) $\Rightarrow$ (ii)

Let M_i's be pairwise orthogonal and span H

$\therefore M_i \perp M_j$ for $i \neq j$

Let $x \in H$ each $x \in H$ can be uniquely expressed as

$x = x_1 + x_2 + \ldots + x_m, \ x_i \in M_i$

we know that, If P and Q are projections on closed linear subspaces M and N then $M \perp N \Leftrightarrow PQ = 0$

Here P_i's are projections on M_i and P_j's are projections on M_j

$\therefore x_1 \in M$

$R(P_1) = M_1$

$P_1 x_1 = x_1$

$\therefore M_i \perp M_j$ for $i \neq j$

$\Rightarrow P_i P_j = 0$ for $i \neq j$

$\Rightarrow P_i$'s are pairwise orthogonal

Now,

$P_i x = P_i(x_1 + x_2 + \cdots + x_m)$

$= P_i x_1 + P_i x_2 + \cdots + P_i x_m$

$= P_i x_1 + P_i x_2 + \cdots + P_i x_i + \cdots + P_i x_m$

$= P_i P_1 x_1 + P_i P_2 x_2 + \cdots + P_i P_i x_i + \cdots + P_i P_m x_m$

$= 0 + 0 + \cdots + P_i^2 x + \cdots + 0$ 27

$P_i x = P_i x_i$ $\quad$ $\langle P_i x_i = x_i, x \in M$

$P_i x = x_i \ \forall i \rightarrow (1)$ $\quad$ $P_i x_i = 0, i \neq j$

$\forall x \in H, \ Ix = x$

$= x_1 + x_2 + \cdots + x_m$

$= P_1 x + P_2 x + \cdots + P_m x$ $\quad$ $\langle$ by (1)$\rangle$

$Ix = (P_1 + P_2 + \cdots + P_m)x$

$I = P_1 + P_2 + \cdots + P_m$

$I = \sum_{i=1}^{m} P_i \rightarrow (2)$ $\quad$ $\langle \therefore$ characteristic eqn. $\lambda x = Tx$

$Tx = T(x_1 + x_2 + \cdots + x_m)$ $\quad$ $\langle \therefore Tx_i = \lambda_i x_i \ \forall i$

$= Tx_1 + Tx_2 + \cdots + Tx_m$

$= \lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_m x_m$

$= \lambda_1 P_1 x + \lambda_2 P_2 x + \cdots + \lambda_m P_m x$

$Tx = (\lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m)x$ $\quad$ $\langle$ by (1)$\rangle$

$\underset{\sim}{T} = \sum_{i=1}^{m} \lambda_i P_i \rightarrow (3)$

(ii) $\Rightarrow$ (iii)

Assume that (2) and (3) are hold

$T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m$

$T^* = \bar{\lambda}_1 P_1 + \bar{\lambda}_2 P_2 + \dots + \bar{\lambda}_m P_m$

$TT^* = (\lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_m P_m)$
$(\bar{\lambda}_1 P_1 + \bar{\lambda}_2 P_2 + \dots + \bar{\lambda}_m P_m)$

$= \sum_{i=1}^{m} \bar{\lambda}_i P_i^2 + \sum_{i,j=1}^{m} \lambda_i \bar{\lambda}_j P_i P_j$ $\langle \because P_i P_j = 0$ for $i \neq j$:

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$\Rightarrow TT^* = \sum_{i=1}^{m} |\lambda_i|^2 P_i$

similarly, we can prove

$T^*T = \sum_{i=1}^{m} (\lambda_i)^2 P_i$

$\therefore TT^* = T^*T$

$\Rightarrow T$ is normal

(iii) $\Rightarrow$ (i)

Suppose T is normal

Then x is an eigen vector of T with λ. $\Leftrightarrow$ x is an eigen vector of T^* with $\bar{\lambda}$. $\langle$Theorem A $\to$ Theorem B$\rangle$

To prove M_i's are pairwise orthogonal write the proof of the theorem B of (62)

To prove: M_i's span H

write the proof of theorem D of (62)

Hence, proved.

UNIT –IV

UNIT - IV

(1)

Banach Algebra :-

Definition :-

A banach algebra is a complete banach Space which is also an algebra with identity and in which the multiplication is related to the norm with following conditions.

i) $\|xy\| \leq \|x\| . \|y\|$

ii) $\|1\| = 1$

Properties :-

The multiplication is jointly continuous its on any Banach space.

Proof:

If $x_n \to x$, $y_n \to y$ as $n \to \infty$

Then it is to be proved that $x_n y_n \to xy$ as $n \to \infty$

consider,

$$\|x_n y_n - xy\| = \|x_n y_n + x_n y - x_n y - xy\|$$

$$= \| x_n(y_n - y) + y(x_n - x)\|$$

$$\leq \|x_n(y_n - y)\| + \|(y(x_n - x))\|$$

$$\leq \|x_n\| \, \|y_n - y\| + \|y\| \, \|x_n - x\|$$

$x_n \to x$ as $n \to \infty$

Given $\epsilon > 0$ There exist $n > 0$ (on t)

Such that $\|x_n - x\| < \frac{\epsilon}{2\|y\|}$ $\forall \, n \geq n_1$ $\to$ (2)

$y_n \to y$ as $n \to \infty$

for $\epsilon > 0$ there exist $n_2 > 0$ such that

$\| y_n - y\| < \frac{\epsilon}{2K}$ $\to$ (3)

where $\{x_n\}$ bounded, $\|x_n\| \leq k$, $\forall\, n \geq n_0$.

Then (2) and (3) are true for every $n \geq n_0$.

$$\|x_n y_n - xy\| = \|x_n y_n - x_n y + x_n y - xy\| \quad (2)$$

$$< k \cdot \frac{\epsilon}{2k} + \|y\| \cdot \frac{\epsilon}{2\|y\|}$$

$$< \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \forall\, n \geq n_0$$

$$< \epsilon$$

Given $\epsilon > 0$ there exist an continuous $n_0 > 0$

Such that $\|x_n y_n - xy\| < \epsilon$, $\forall\, n \geq n_0$

$\Rightarrow x_n y_n \to xy$, as $n \to \infty$

This is the multiplication is jointly continuous is any banach algebra.

Banach sub algebra :

A Banach subalgebra of a banach algebra 'A' is a closed subalgebra of that contain the identity one.

Remark :

The banach algebra of A are precisely with a the same identity to the same algebraic operator the same identity and the same norm.

Example :

a) one of the most important Banach algebra in the set $y(x)$ of all bounded continuous complex function defined on a topological space X. The case in which X is a compact H.S will have particular can be identical with the

[illegible] For $g(x)$ can be identical with the

Simplest of all Banach algebra the algebra of complex numbers.

(3)

b) consider the closed unit disc $D = \{ \frac{z}{\|z\| \le 1} \}$ in the complex plane the subset of $C(D)$ which consist of all funs analytic in the interior expressed on the four way.

If two funs f and g in $L_1(G)$ are given that this which is defined by $f+g$ are called their convolution is that function whose value of g_k is

$$(f * g)(g_k) = \sum_{g_i g_j = g_k} f(g_i)\, g(g_j)$$

we note that if each element of G identified with the funl. whose value is one at that element and zero elsewhere then G because a subset of $L_1(G)$

Further multiplication on G agree with convolution in $L_1(G)$ and the element of $L_1(G)$ which corresponds its the identity in G is an identity element of G has norm and that the basis norm inequality for a Banach algebra is satisfied

$$\|f * g\| = \sum_{k=1}^{m} |(f * g)\, g_k|$$

$$= \sum_{k=1}^{n} \left| \sum_{j=1}^{n} f(g_k g_j^{-1})\, g(g_j) \right|$$

$$= \sum_{k=1}^{n} \sum_{j=1}^{n} |f(g_k g_j^{-1})| \cdot |g(g_j)|$$

$$= \sum_{j=1}^{n} |g(g_j)| \cdot \sum_{k=1}^{n} |f(g_k \cdot g_j^{-1})| \qquad (4)$$

$$= \sum_{j=1}^{n} |g(g_i)| \cdot \|f\| \Rightarrow \|f\| \sum_{j=1}^{n} |g(g_i)|$$

$$\|f * g\| = \|f\| \, \|g\|$$

Group of integer. Its group algebra $L(G)$ is the set of all complex function f defined on G for which $\sum_{n=\infty}^{\infty} |f(n)|$ converges.

The linear operation are defined pointwise the norm by, $\|f\| = \sum_{n=\infty}^{\infty} |f(n)|$ and

$$(f * g)\, n = \sum_{m=-\infty}^{\infty} f(n-m)\, g(m)$$ just as in (G)

G is contained in $L_1(G)$ in a natured way of non discrete topological group like the Goal line must Clearly. The based on an adequate theory of integration.

It should also have available theory of banach algebra. In which no identity is assumed to be present these ideas analysis theory however outside the slope of this work.

The banach algebra described above are many and diverse and these and yet other which we have not maintained out attention. In the following chaplers centred on $y(x)$

commutative C^* algebras but the general theory we develop is equally applicable to all

It is worthy to notice that an arbitrary banach algebra. A can be regarded as a banach Subalgebra of B(A). In a sense. (5)

Examples:

z(a) its banach subalgebra include all banach algebra to see this we recall from problem that $a \to M_\alpha$ where $M_\alpha(x) - \alpha x$ is an isomorphism of A into B(A). It is easy to see that M is the identity operator on A so all that remains is to observe that, $\|\alpha\| = \|M_\alpha\| \ \forall \alpha$.

Proof:

$$\|M_\alpha(x)\| = \|\alpha x\| \leq \|\alpha\| \, \|x\|$$

Show that, $\|M_\alpha\| = \|\alpha\|$ of D is obviously Subalgebra which contain the identity.

A simple application of Morera's theorem from complex analysis theorem from shows that it is closed and is since banach subalgebra of y(D).

This banach algebra as called the disc algebra. It has a number of interesting properties which are of course ~~in~~ intimately related to the special character of its function.

a) If B is an non trivial complex banach space Then the set B(B) of all operator on B is a Banach algebra

we assume that B is non-trivial an order to generate that the identity operator is an identity in the algebraic sense. (6)

b) If we consider a non-trivial H.S 'H'. then B(A) is banach space. This is special case and B(B) and it is important to observe that additional structure is present here, namely the adjoint operator $T \to T^*$

c) A sub algebra of B(H) is said to be self adjoint if it contains the adjoint of each of its operators Banach subalgebra of B(H) is which race self-adjoint are called c^*, algebra we shall return to the subject of commutative c^* algebra we shall return to the subject of commutative c^* algebra in chapter (4).

Topology generated by all function of the from $T \to (Tx, y)$

(e) It is the weekest topology w.r.to which all these functions are continuous.

It is easy to see from the in equaility

$$|(Tx, y) - (T_0x, y)| \leq \| T - T_0 \| . \|x\| \|y\|$$

That this topology is weeker than the usual norm topology so that its closed sets are also closed in the usual sense.

A C^*- algebra with the farther property of being closed in the weak operator topology is called W^*- algebra. (7)

Algebra's of this kind are also called "sings of an operator or non-newmann algebra.

They are among the most interesting of all Banach algebra But this theory is quite beyond the slope of this Book.

prove that $L_1(G)$ is a BA

If $G = \{g_1, g_2 \ldots g_n\}$ is a finite group then its group $L_1(G)$ is the set of all complex fun's defined on G.

Addition and scalar multiplication are defined Pointwise and the norm by.

$$\|F\| = \sum_{i=1}^{n} |f(g_i)| \rightarrow (1)$$

In order to see that under lies definition(1) multiplication it is convenient to regard a typical element of $L_1(G)$ as a formal sum $\sum_{i=1}^{n} \alpha_i g_i$.
where α_i is the value of 'f' at g_i with this interpedation multiplication in $L_1(G)$ as follows.

$$\left(\sum_{i=1}^{n} \alpha_i g_i\right)\left(\sum_{i=1}^{n} \beta_i g_i\right) = \sum_{k=1}^{n} \gamma_k g_k, \text{ where}$$

$$\gamma_k = \sum \alpha_i \beta_i \rightarrow (2)$$

The meaning of the sum is $g_i g_j = g_k$

(2) that the symmetric is to be extended over

all subscripts i and j (8)

$$g_i g_j = g_k$$

In effect, ∴ we formally, multiply get the sums on the left of ① and we then together call the resulting terms which contains the same element of a with these Ideas our 1st point of view to which that element of $L_1(G)$ are funsl. and we see that our defn. can be and the fact

that, $\|\alpha\| \leq \|M\alpha\|$

follows that fom,

$$\|M\alpha\| > \sup\{\|M\alpha(x)\| / \|x\| < 1\}$$

$$> \|M\alpha(1)\| = \|\alpha\|$$

The mapping $\alpha \to M\alpha$ is thus an isometric isomorphism of A onto a Banach subalgebra of B(A) and it allows as to identity the algebra A with can create Banach algebra of operation on A

Regular and singular element:

Defn: Regular element:

Let R be a ring with identity if x is any element in R then if there exist $y \in R$ such that $xy = yx = 1$. Then y is called the inverse of x

ie) $y = x^{-1}$

If an element x in R has an inverse that x is said to be regular element which are not regular are called " singular elements".

(9)

Definition :-

Let A be a banach algebra and let G denote the set of all regular elements of A and let S denote the set of all singular elements, here S is nothing but complement of G.

∴ Since G contains the multiplicative identify are and further G is a group under multiplication where an S contains the zero element ie) additive identity.

Theorem :-

Every element x for which $\|x-1\|<1$ is regular and the inverse of such as element is given by the formula, $x^{-1} = 1 + \sum_{n=1}^{\infty} (1-x)^n$

Proof :

Put $\|x-1\| = r$ then we have $r<1$ and

$\|(1-x)^n\| \leq \|(1-x)\|^n = r^n \rightarrow$ (1)

Now, consider the series,

$\sum_{n=1}^{\infty} (1-x)^n$ and let $\{S_n\}$ be a sequence of Partial sum of series. ie) $S_n = \sum_{k=1}^{n} (1-x)^k$

$$\therefore \|S_n - S_m\| = \left\| \sum_{k=1}^{n} (1-x)^k - \sum_{k=1}^{m} (1-x)^k \right\| \quad \forall\, n>m$$

$$= \left\| \sum_{k=1}^{n} (1-x)^k \right\| \leq \sum_{k=m+1}^{n} \|(1-x)\|^k$$

$$\|S_n - S_m\| \leq \sum_{k=m+1}^{n} r^k$$

$< \because r<1$

$\Rightarrow \sum_{k=m+1}^{n} r^k$ is conver (10)

Observe the series $\sum_{k=1}^{\infty} r^k$ is a convergent series of real number of and it sequence of partial sums is a cauchy sequence and

$\therefore$ Given $\varepsilon > 0$ there exist a +ve integer. $n_0 > 0$ such that $\sum_{k=m+1}^{n} r^k < \epsilon \ \forall \ n, m \geq n_0$

$\therefore \| S_n - S_m \| = \sum_{k=m+1}^{n} r^k < \epsilon, \quad \forall n, m \geq n_0$

This shows that $\{S_n\}$ is a cauchy sequence in A. But A being a banach space it is a complete space and hence $\{S_n\}$ it convergent in A

Let the limit to which the sequence $\{S_n\}$ converge in A denoted by $\sum_{n=1}^{k} (1-x)^n$

Then, $\sum_{n=1}^{\infty} (1-x)^n \in A$

Now, consider $y = 1 + \sum_{n=1}^{\infty} (1-x)^n$

For $y - xy = (1-x)y = (1-x)\left[\sum_{n=1}^{\infty} (1-x)^n + 1\right]$

$= (1-x) + \sum_{n=1}^{\infty} (1-x)^{n+1} \Rightarrow (1-x) + \sum_{n=2}^{\infty} (1-x)^n$

$= \sum_{n=1}^{\infty} (1-x)^n$

$y - xy = y^{-1}$

$xy = 1$

$yx = 1 \Rightarrow xy = yx = 1$

$\Rightarrow x$ is regular and $y = x^{-1}$

ie) $x^{-1} = 1 + \sum_{n=1}^{\infty} (1-x)^n$ Hence proved.

Theorem:

The Set of all regular elements is an open Set and set all singular element is closed. (11)

Proof:

Prove that, G is an open set and S is a closed set

For let x_0 be an arbitrary point of G

$x_0 \in G$ and $x_0 \neq 0$

$$\|x^{-1}\| \neq 0$$

Let $x \in A$ such that $\|x - x_0\| < \frac{1}{\|x^{-1}\|} \rightarrow (1)$

consider,

$$\|x_0^{-1}x - 1\| = \|x_0^{-1}x - x_0^{-1}x_0\|$$

$< \because \|xy\| \leq \|x\| \cdot \|y\|$ is a banach space

$$= \|x_0^{-1}(x - x_0)\|$$

$$\leq \|x_0^{-1}\| \|x - x_0\|$$

$$< \|x_0^{-1}\| \cdot \frac{1}{\|x_0^{-1}\|}$$

$$\|x_0^{-1}x - 1\| < 1$$

$\Rightarrow x_0^{-1}x$ is regular by theorem $A\|x^{-1}\|e$,

Then x is algebra

$\Rightarrow x_0^{-1}x \in G$

Now consider, $x = Tx$

$$= (x_0 x_0^{-1})x$$

$< $ where $y = x_0^{-1}x$
$x \in G$

$$= x_0(x_0^{-1}x) = x_0 y$$

$x \in G$, If $S = \{x \in A / \|x - x_0\| < \frac{1}{\|x_0^{-1}\|}\}$

Then $S \subset G$,

Observe that S is an open ball centred at x_0 and radius $\frac{1}{\|x_0^{-1}\|}$ 12

Then $S = B(x_0, r)$ where $r = \frac{1}{\|x^{-1}\|}$ and, hence we have $B(x_0, r) \subset G$ $\angle \therefore x_0 \in B(x_0, r) \subset G$.

$\Rightarrow x_0$ is an interior point of G

This is true for every point in G.

Hence G is open

ie) The set of all regular element is open.

$\therefore G$ is open its complement is closed

ie) S is closed

ie) The set of all singular elements is closed.

Hence proved.

Theorem:

A mapping $x \to x^{-1}$ of G into G is continuous and is therefore homomorphism of G into itself

Proof:

Let $x_0 \in G$ and arbitrary

Let $x \in G$ such that $\|x - x_0\| < \frac{1}{2\|x_0^{-1}\|}$

Consider, $\|x x_0^{-1} - 1\| = \|x_0^{-1} x - x_0^{-1} x_0\|$

$= \|x_0^{-1}(x - x_0)\|$

$\leq \|x_0^{-1}\| \|x - x_0\|$

$< \|x_0^{-1}\| \frac{1}{2\|x_0^{-1}\|}$

$< \frac{1}{2} < 1.$

$\Rightarrow x_0^{-1}x$ is inversible

$\Rightarrow x_0^{-1}x \in G$

$$(x_0^{-1}x)^{-1} = 1 + \sum_{n=1}^{\infty} (1 - x_0^{-1}x)^n$$ 13

$$x_0^{-1}x_0 = 1 + \sum_{n=1}^{\infty} (1 - x_0^{-1}x)^n$$

$$x_0 x^n - 1 = \sum_{n=1}^{\infty} (1 - x_0^{-1}x)^n \to ②$$

$$\|x^{-1} - x_0^{-1}\| = \|x^{-1} \cdot 1 - x_0^{-1}\|$$

$$= \|x^{-1}x_0 x_0^{-1} - x_0^{-1}\|$$

$$= \|(x^{-1} - x_0^{-1}) x_0^{-1}\|$$

$$\leq \|x_0^{-1}\| \, \|x^{-1}x_0 - 1\|$$

$$\leq \|x_0^{-1}\| \cdot \left\| \sum_{n=1}^{\infty} (1 - x_0^{-1}x)^n \right\| \quad \langle \text{by } ②$$

$$\leq \|x_0^{-1}\| \sum_{n=1}^{\infty} \|(1 - x_0^{-1}x)\|^n$$

$$= \|x_0^{-1}\| \, \|1 - x_0^{-1}x\| \sum_{n=0}^{\infty} \|(1 - x_0^{-1}x)^n$$

$$= \|x_0^{-1}\| \cdot \|1 - x_0^{-1}x\| \frac{1}{1 - 1/2}$$

$$\|x^{-1} - x_0^{-1}\| \leq 2\|x_0^{-1}\| \, \|1 - x_0^{-1}x\|$$

$$\langle \therefore \|1 - x_0^{-1}x\| < 1/2, \; -\|1 - x_0^{-1}x\| > -1/2,$$

$$-\|1 - x_0^{-1}x\| < 2 \rangle$$

$$\|x^{-1} - x_0^{-1}\| \leq 2\|x^{-1}\| \cdot \|x_0^{-1}(x_0 - x)\|$$

$$\leq 2\|x_0^{-1}\|^2 \, \|x_0 - x\|$$

$$\|f(x) - f(x_0)\| < 2k^2 \|x_0 - x\|$$

where $k = \|x_0^{-1}\|$

Let $\epsilon > 0$ be given that,

If $||x - x_0|| < \epsilon$ if $||x - x_0|| < \epsilon/2k^2$

choose $\delta = \epsilon/2k^2$ there exist $||f(x) - f(x_0)|| < \epsilon$

where $||x - x_0|| < \delta$ (14)

f is continuous and f^{-1} is also continuous

Next to prove f is one to one

For suppose $x \neq y$ then $x^{-1} \neq y^{-1}$

$\Rightarrow f(x) \neq f(y)$

y is one to one

If $x^{-1} \in G$ then there exist $x \in G$ such that,

$y(x) = x^{-1} \Rightarrow x = y^{-1}(x^{-1})$

$x = y^{-1}(y)$

$\therefore f$ is onto

Hence, the map f is one to one, onto and both f and f^{-1} is continuous.

f is homomorphism of G into itself.

$\Rightarrow f$ is homomorphism of G into itself.

Hence proved.

— Next note.

Definition :

UNIT - IV

Topological division of zero: (15)

An element z is a banach algebra A is called a topological division of zero. If there exist a sequence $\{z_n\} \in A$ such that $\|z_n\| = 1$ and either $z z_n \to 0$ (or) $z_n z \to 0$ (or) $n \to \infty$

The Set of all topological division of zero its denoted by z.

Theorem:

z is a subset of S.

Proof:

Let ξ be an arbitrary element of z Then by definition there exist a sequence $\{z_j n\} \in A$ such that $\|z_j A\| = 1$

$\therefore z_j z_i n \to 0$

$z_j \in S$. Let $\xi \in z$, ξ is topological division of zero.

$\Rightarrow$ There exist $\{\xi_n\} \in A$ such that, $\|z_n\| = 1 \to (1)$

$z, z_n \to 0$ (or) $z_n . z \to 0$ as $n \to \infty$

To prove that, $z \in S$, $A = G \cup S$

$$z \in S \Rightarrow \xi \in G$$

$$z \text{ reg} \Rightarrow z . z^{-1} = 1 = z^{-1} . z$$

$$z_n = z_n . 1 = z_n . z z^{-1} = (z_n z) z^{-1} \to (2)$$

$z_n \to 0$ as $n \to \infty$

$\therefore z \in S$

Theorem :

The boundary of S is a subset of Z

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Proof :

Since, S is a closed its boundary consist of all points in S which are of convergence sequence in G

If S is such a point then it is to be proved that $z \in Z$

So let z be a boundary point of S $\{r_n\}$ in G. Such that $r_n \to z$ as $n \to \infty$

Since $r_n \in G$. r_n is invertible

$\therefore r_n^{-1}$ exists

consider, $r_n^{-1} z - 1 = r_n^{-1} z - r_n r_n^{-1}$

$= r_n^{-1}(z - r_n)$

$\| r_n^{-1} z^{-1} \| \leq \| r_n^{-1} \| \, \| z - r_n \|$

Now $\{r_n^{-1}\}$ is unbounded for otherwise suppose $\{r_n^{-1}\}$ is bounded.

Then we have $\| r_n^{-1} \| \leq k$ and $r_n \to z$ as $n \to \infty$

we have $\| r_n - z \| < \epsilon / k$

$\| r_n^{-1} z^{-1} \| < \epsilon / k \cdot k = \epsilon$

$\| r_n^{-1} z - 1 \| < 1$

$\Rightarrow r_n^{-1} z$ is regular

$\Rightarrow r_n^{-1} z \in G$

Now, $z = (r_n \cdot r_n^{-1}) z$

$\therefore (r_n^{-1} z) \in G$

$\Rightarrow z \in G$

But this contradicts the fact that z is the boundary point of S (17)

$\Rightarrow z \in B(S)$

And our assumption $\{r_n^{-1}\}$ is bounded is wrong and hence $\{r_n^{-1}\}$ is unbounded

$\therefore$ we have, $\|r_n^{-1}\| \to \infty$ as $n \to \infty$

Let, $z_n = \dfrac{r_n^{-1}}{\|r_n^{-1}\|} \ \forall n$

Then $z_n \neq 0$ and $\|z_n\| = \dfrac{\|r_n^{-1}\|}{\|r_n^{-1}\|} = 1$

$\|z_n\| = 1 \ \forall n \to$ (1)

Now consider,

$$z z_n = z \cdot \frac{r_n^{-1}}{\|r_n^{-1}\|}$$

$$I = \frac{z r_n^{-1}}{\|r_n^{-1}\|} = \frac{1 + z r_n^{-1} - 1}{\|r_n^{-1}\|}$$

$$= \frac{1 + r_n^{-1} z - r_n r_n^{-1}}{\|r_n^{-1}\|}$$

$$= \frac{1 + (z - r_n) r_n^{-1}}{\|r_n^{-1}\|}$$

$$z z_n = \frac{1}{\|r_n^{-1}\|} + \frac{(z - r_n) \cdot r_n^{-1}}{\|r_n^{-1}\|}$$

Since, $\|r_n^{-1}\| \to \infty$ as $n \to \infty$

we have $\frac{1}{\ }$ $\to z$ as $n \to \infty$

$(r_n - z) \to 0$ as $n \to \infty$

$$z z_n = \frac{1}{\|r_n^{-1}\|} + \frac{(z - r_n) r_n^{-1}}{\|r_n^{-1}\|} \to 0 \text{ as } n \to \infty$$

$\Rightarrow z z_n \to 0$ as $n \to \infty \to$ ② 18

① and ② we get

$\Rightarrow$ we have for the point I in the boundary of S there exist a $\{z_n\}$ in A.

$\Rightarrow \|z_n\| = 1$ and $z z_n \to 0$ as $n \to \infty$

$\Rightarrow$ By defn. z is the topological divisor of zero. ie) $z \in Z$

$\therefore$ Boundary of S is a subset of Z.

The Spectrum:

An element x in banach space A.

we define " the Spectrum of 'x' be the following subset of the complex plane.

$$\sigma(x) = \{ \lambda : x - \lambda S \text{ is singular} \}$$

" The spectrum of x elements" on A as well as x and so we use the notation $\sigma_A(x)$

$\langle \therefore \lambda \in \sigma(x) \Leftrightarrow x - \lambda I \in$ singular

$\lambda \in \sigma(x) \Leftrightarrow x - \lambda I \in S \rangle$

Properties:

I i) $x - \lambda I$ is a continuous function of λ with value in A.

Proof :-

If $f : \sigma(x) \to A$ defined by $f(\lambda) = x - \lambda I$

So that, $f(\lambda_0) = x - \lambda_0 I$ (19)

$$f(\lambda) - f(\lambda_0) = (x - \lambda I) - (x - \lambda_0 I)$$

$$= \lambda_0 I - \lambda I \; ; \; \lambda \to \lambda_0$$

$$= (\lambda_0 - \lambda)(I \to 0, \; \lambda - \lambda_0 \to 0$$

$f(\lambda) - f(\lambda_0)$ as $\lambda \to \lambda_0$

$\Rightarrow f(\lambda)$ is continuous

$\Rightarrow (x - \lambda I)$ is continuous function of λ.

II, ii) The Spectrum of x is a closed set.

Proof :

For since, the set of all singular elements. S is closed and so from the definition of spectrum $\sigma(x)$ is also closed.

III, iii) $\sigma(x)$ as a subset of the closed disc. $\{z / |z| \leq \|x\|\}$. i.e) $\sigma(x) \subset \{z : |z| \leq \|x\|\}$.

Proof :

For otherwise to prove that $|\lambda| \leq \|x\|$

If $|\lambda| \geq \|x\|$

Then $\frac{\|x\|}{|\lambda|} < 1$

$\Rightarrow \therefore \| 1 - (1 - x/\lambda) \| < 1$

$\Rightarrow$ By theorem A, the element $1 - x/\lambda$ is regular ie) $1 - x/\lambda \in G$

$\Rightarrow x - \lambda I \in G$

ie) $x - \lambda I$ is regular (20)

$x - \lambda I \in G \Rightarrow x - \lambda I \notin S$

$\therefore x - \lambda I$ is regular

$x - \lambda I$ is not singular

$\therefore \lambda \notin \sigma(x)$

But this contradicts the fact that $\lambda \in \sigma(x)$

$\therefore$ we have $|\lambda| \leq \|x\|$

(or) $\lambda \in \{z : |z| \leq \|x\|\}$

ie) $\lambda \in \sigma(x)$

$\Rightarrow \lambda \in \{z : |z| \leq \|x\|\}$

$\Rightarrow \sigma(x) \subset \{z : |z| \leq \|x\|\}$

Definition:

" The Resolvant set of x" is denoted by $\rho(x) = [\sigma(x)]^c$.

The resolvant of x is a function $x(\lambda)$ $f(x)$ defined by. $x(\lambda) = (x - \lambda I)^{-1}$

Properties:

i) Resolvant of x is a continuous function of λ.

Proof:

By property (i) of Spectrum we know that $f(\lambda) = (x - \lambda I)$ is continuous.

$$x(\lambda) = (x - \lambda I)^{-1}$$
$$= f^{-1}(\lambda)$$

$\therefore$ f is homeomorphism f^{-1} is also continuous

ie) $x(\lambda)$ is continuous (2)

$\therefore$ Resolvant of x is continuous.

ii) $\rho(x)$ is an open subset of the complete plane which contains $\{ z / |z| > \|x\| \}$.

Proof:

By properties (2) and (3) of spectrum we know that $\sigma(x)$ is closed.

$\therefore$ $[\sigma(x)]^2$ is open

$\therefore$ $\sigma(x)$ is the subset of closed disc

we have $\rho(x)$ is a subset of open disc

iii) $x(\lambda) \to 0$ as $\lambda \to \infty$

$$x(\lambda) = (x - \lambda I)^{-1}$$
$$= \left(\frac{\lambda x}{\lambda} - \lambda I \right)^{-1}$$
$$= \lambda^{-1} (x/\lambda - I)^{-1}$$
$$= \frac{1}{\lambda} (x/\lambda - I)^{-1} \to 1/\infty (x/\lambda - I)^{-1}$$

$\therefore$ $x(\lambda) \to 0$ as $\lambda \to \infty$

iv) Resolvant sequence is given by

$= x(\lambda) \cdot (x - \lambda I + \lambda I - \mu I) \cdot x(\mu)$

$= x(\lambda) [(x - \lambda I) + (\lambda - \mu) I] \cdot x(\mu)$

$= x(\lambda) [x - \lambda I] \cdot x(\mu) + x(\lambda)(\lambda - \mu) I x(\mu)$

$x(\lambda) = x(\mu) + (\lambda - \mu) x \cdot x(\lambda) \cdot x(\mu)$ (22)

$x(\lambda) - x(\mu) = (\lambda - \mu) x(\lambda) \cdot x(\mu)$

Theorem: A-19 5M

$\sigma(x)$ is non-empty.

Proof:

Let f be a functional on A^* so that f is an element in A^* and define $f(\lambda)$ as $f[x(\lambda)]$

Observe that, $f(\lambda)$ is a complex function define,

$f(\lambda) = p(x) \to C$

Then $f(\lambda)$ is continuous on $f(x)$.

we know that the resolvant eqn. in $p(x)$ is

$x(\lambda) - x(\mu) = (\lambda - \mu) x(\lambda) x(\mu) \quad \forall x(\lambda)$

$f[x(\lambda)] - f[x(\mu)] = (\lambda - \mu) f(x(\lambda)) \cdot f[x(\mu)]$

$f(\lambda) - f(\mu) = (\lambda - \mu) f(\lambda) \cdot f(\mu)$

$$\frac{f(\lambda) - f(\mu)}{(\lambda - \mu)} = f(\lambda) \cdot f(\mu)$$

$$\lim_{\lambda \to \mu} \frac{f(\lambda) - f(\mu)}{\lambda - \mu} = \lim_{\lambda \to \mu} f(\lambda) \cdot f(\mu) \quad \langle \because f' \text{ exist}$$

$f'(\mu) = [f(u)]^2 \neq 0$

$\therefore f$ is analytic

Hence, $f(\lambda)$ was derivative at each point of $f(x)$

(23)

$$\|f(\lambda)\| = \|f(x(\lambda))\|$$
$$\leq \|f\| \cdot \|x(\lambda)\|$$

we know that

$x(\lambda) \to 0$ as $\lambda \to \infty$

$f(x) \to 0$ as $\lambda \to \infty$

Now to prove $\sigma(x)$ is non-empty By contradiction

Suppose $\sigma(x)$ is empty

ie) $\sigma(x) = \phi$

Then $f(x)$ is the entire complex plane

$G = (\sigma(x) \cup \rho(x)$

$\sigma(x) = \phi$

$\therefore x(\lambda) \in \rho(x)$

$\rho[x(\lambda)] \in \rho(x)$

$\therefore$ By liouvilles theorem.

The function of $f(\lambda)$ is defined and continuous on the entire complex plane c and has derivative at each point of c and $f(\lambda) \to 0$ as $\lambda \to \infty$

$\therefore x(\lambda) = (x - \lambda I)^{-1} \to 0$

$\Rightarrow f(\lambda) = 0 \; \forall \lambda$

$f[x(\lambda)] \to 0$

$\therefore f$ is an arbitrary functional to every f in A^* on A^* $f(A) \to 0$. This is the

ie) $f(\lambda) = 0 \; \forall \lambda \in C$ and $f \in A^* \to (*)$

But by the corollary of Hahn Banach theorem

$x(\lambda_0) \neq 0$ for some λ_0.

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Then there exists $f_0 \in A^*$ Such that

$f[x(\lambda_0)] = f_0(\lambda_0)$ ...

$= \|x(\lambda_0)\|$, and (24)

$\|f_0\| = 1$

This is impossible Because $x(\lambda) = (x-\lambda I)$

(*) and this contradict each other is to be discarded ...

Hence $\sigma(x)$ is always non-empty.

i.e. The spectrum of x is a non-empty set

Hence, proved.

Theorem:

If zero is the only topological divisor of zero in A. Then $A = C$

Proof:

Let x be an arbitrary element of A

Then its spectrum $\sigma(x)$ is non-empty

$\therefore$ Let λ is a boundary point of $\sigma(x)$

$\Rightarrow$ By defn. $(x-\lambda I)$ is a boundary point of the Set of all singular elements S.

$\Rightarrow x - \lambda I \in Z$

$\angle$ boundary of S is the subspace of Z)

Hence Z is the set of all topological divisor of zero.

$x - \lambda I \in B(S)$

$\therefore x - \lambda I \in I \Rightarrow x - \lambda I \in T$

$\Rightarrow$ $x - \lambda I$ is the topological divisor of zero

But, its given that is the only topological divisor of zero in A

25

we have, $x - \lambda I = 0$

$$x = \lambda I$$

x is arbitrary element in A

This is true for every $x \in A$

ie, Every element of A can be expressed as scalar of multiple of identity.

$\Rightarrow$ $A = C$

$\therefore A = C$

Hence proved.

Remark:

The basic link between the multiplication in A and the norm is $\|xy\| \leq \|x\| . \|y\|$

But the form the previous theorem we have $A = C$

$\therefore \|xy\| = |xy| = |x| . |y| = \|x\| \, \|y\|$

$\Rightarrow \|xy\| = \|x\| \, \|y\|$ provides $A = C$

Theorem:

If the norm in a satisfies the inequality $\|xy\| \geq k \, \|x\| . \|y\|$ for some +ve constant k then $A = C$.

Proof:

Given that $\|xy\| \geq k \|x\| \, \|y\| \rightarrow (1)$

for some positive constant k.

Now in order to prove. $A = C$ by previous theorem. It is sufficient to show that 0 is only topological divisor of zero in A

If possible. Let $x \neq 0$ in A be a topological divisor of zero (26)

Then by defn. there exist a $\{x_n\} \in A$

Such that, $\|x_n\| = 1 \rightarrow$ (2)

$x x_n \rightarrow 0$ as $n \rightarrow \infty \rightarrow$ (3)

But (1) $\Rightarrow \|x x_n\| \geq k \|x\| \, \|y\|$ $\quad \therefore x \neq 0$

$\geq k \|x\|$ $\quad \Rightarrow \|x\| \neq 0$

$\therefore \|x . x_n\| > 0 \rightarrow$ (4)

$x . x_n \rightarrow 0$

$\Rightarrow \|x . x_n\| \rightarrow 0$ as $n \rightarrow \infty$

But this and eqn. (4) contradic each other our assumption is wrong and hence. '0' is the only topological divisor of zero in A we have $A = C$

Hence proved.

Theorem:

If A is a banach subalgebra of a banach algebra A' Then the Spectrum of the element x in A and A' are related as follows.

i) $\sigma_{A'}(x) \subseteq \sigma_A(x)$

ii) Each boundary point of $\sigma_A(x)$ is also a boundary point of $\sigma_{A'}(x)$

Proof :-

Result : (1)

Suppose $\lambda \in \sigma_{A'}(x)$ (27)

Then we have, $x - \lambda I$ is singular in A'

$\Rightarrow x - \lambda I$ is singular in A

$\sigma_{A'}(x) \subseteq \sigma_A(x)$

Hence the result

Result : (ii)

Let λ be a boundary point of $\sigma_A(x)$

Then we have,

$x - I$ is a boundary point of the set of all singular points S on A. such that, $x - \lambda I$ is the topological divisor of zero in A.

[$\therefore$ Boundary of S is a subset of x]

$\Rightarrow x - I$ is topological divisor of zero in A'

$\Rightarrow x - \lambda I$ is a singular point in A' $(A \subset A')$

$\Rightarrow \lambda \in \sigma_{A'}(x)$

ie) λ is boundary point of $\sigma_{A'}(x)$.

$\therefore$ Boundary point of $\sigma_A(x)$ is a boundary point of $\sigma_{A'}(x)$.

Formula for Spectral radius of x:

Statement :

The spectral radius of x

$$r(x) = \lim_{n \to \infty} (\|x^n\|)^{1/n}.$$

Proof:

Before proving this theorem we need to prove the following lemma.

(28)

Lemma:

$$\sigma(x^n) = [\sigma(x)]^n$$

Proof:

Let λ be a non-zero complex number and let $\lambda_1, \lambda_2 \dots \lambda_n$ be its distinct n-roots.

$$(x - \lambda I) = (x - \lambda_1 I)(x - \lambda_2 I) \dots (x - \lambda m I)$$

Hence, $x^n - \lambda I$ is singular

$\Leftrightarrow x - \lambda_k I$ is singular

For atleast one k, $k = 1, 2 \dots$

$$\Rightarrow \lambda \in \sigma(x^n) \Leftrightarrow \lambda \in [\sigma(x)]^n$$

$$\sigma(x^n) = [\sigma(x)]^n$$

Hence, the lemma.

Proof of the main theorem:

From the above lemma we have

$$r(x^n) = [r(x)]^n \rightarrow (1)$$

and from the defn. of spectral radius of x

$$r(x^n) \leq \|x^n\|$$

$$\Rightarrow [r(x)]^n \leq \|x^n\| \text{ by } (1)$$

$$\Rightarrow r(x) \leq [\|x^n\|]^{1/n} \; \forall n \rightarrow (2)$$

Now in order to prove

$$\lim_{n\to\infty} [\|x\|^n]^{1/n} = r(x)$$

(29)

It is enough to prove that

$$| \|x^n\| - r(x)| < \epsilon$$

ie) To given $\epsilon > 0$ there exist a +ve integer number such that $| \|x^n\|^{1/n} - r(x)| < \epsilon \ \forall n \geq n_0$

This is equation to prove that,

$$[\|x^n\|]^{1/n} < r(x) + \epsilon \quad \forall n \geq n_0$$

For it is sufficient to prove that if a is any real number such that.

$$r(x) + \epsilon \leq a \text{ then } [\|x^n\|^{1/n}] < a \quad \forall n$$

But a finite +ve number of values of n

It follows from the theorem which state every element x for which $\|x^{-1}\| < 1$ is regular.

The inverse of such an element is given by the formula,

$$x^{-1} = 1 + \sum_{n=1}^{\infty} (1-x)^n$$

$\sigma(x) \neq \phi$, $\sigma(x)$ resolvant and further if $|\lambda| \geq \|x\|$. Then $x(\lambda) = (x - \lambda I)^{-1}$

$$x(\lambda) = \lambda^{-1} [x/\lambda^{-1}]^{-1}$$

$$= -\lambda^{-1} \left[1 + \sum_{n=1}^{\infty} (x/\lambda)^n \right] \rightarrow (3)$$

Now if f is any fun. of A. By eqn. (3)

$$\Rightarrow f[x(\lambda)] = -\lambda^{-1} \left[f(1) + \sum_{n=1}^{\infty} f\left(x/\lambda \right)^n \right]$$

$$f(\lambda) = -\lambda^{-1}\left[f(1) + \sum_{n=1}^{\infty} f(x^n)\cdot\lambda^{-n}\right] \rightarrow (4)$$

This true for all, $|\lambda| > ||x||$

$\therefore \sigma(x)$ is non-empty (30)

we have $f[x(\lambda)]$ is an analytic func. in region $|\lambda| > ||x||$ and.

$\therefore$ eqn. (4) is have not expansion for $|\lambda| > ||x||$ and this is convergent.

$\therefore$ This expansion is valid for ($\because \lambda$ scalar)

$$|\lambda| > r(x)$$

$\therefore$ If we consider α be any real number Such that $r(x) < \alpha < a$

Then $\sum_{n=1}^{\infty} f\left(\frac{x^n}{\alpha^n}\right)$ converges and so its form a bounded sequence

$\therefore$ By uniform boundedness theorem for elements x^n/α^n form a bounded sequence in A

Thus $||x^n/\alpha^n|| \leq k$. where k is any +ve real number.

$$\Rightarrow ||x^n|| \leq k\alpha$$

$$\Rightarrow ||x^n||^{1/n} \leq k^{1/n}\alpha = a$$

$$||x^n||^{1/n} \leq a \quad \forall n.$$

But a finite number of r

we have $r(x) = \lim_{n\to\infty} ||x^n||^{1/n}$

The radical and semi simplicity:

A subset I is said to be an ideal in A.

If i) I is a linear subspace of A (3)

ii) $i \in I \Rightarrow xi \in I, \forall x \in A$

iii) $i \in I \Rightarrow ix \in I, \forall x \in A$

If I satisfies the condition (i) and (ii) satisfies condition (i) and (ii) then I is called the "light ideal"

It an ideal satisfies all the 3 conditions then it is called two sides ideal

The properation of the ideal in A are clearly related to the properties of its regular and singular elements.

An element x is said to be "regular" if are exist an element y such that $xy = yx = 1$ are exist an element y such that. So T an elementary, say that x is left regular So T an elementary. Such that $yx = 1$ and if x is not left regular. It is called left singular

similarly, right regular are singular are defined

If x is both left and right regular so that there exist $y \in I$ such that $yx = 1$ and $xz = 1$

Then the relation $y = y.1 = y.xz = yx.z = z$

So that x is regular in the ordinary sence and that $x^{-1} = y = z$.

Definition:

The radicals R of A in the intersection of all its maximal left ideal

(32)

ie) $R = \cap$ maximal left ideal

R is clearly a proper left ideal

Lemma: A-19 5M

If r is an element of R. Then $1-r$ is left regular.

Proof:

Assume that, it is left singular

Let $L = A(1-r)$

$= \{ x(1-r) \mid x \in A \}$

ie) L is a proper left ideal containing $(1-r)$

we next imbedded L in a maximal left ideal M which of course also contains $(1-r)$

$r \in R$, regular also and hence $1 = r + (1-r) \in M$

$1 \in M$, r and $(1-r) \in M$

$\Rightarrow M = A$ (maximal left ideal)

which is contradiction

Then M is maximal left ideal

$\therefore (1-r)$ left singular is wrong.

Hence. $(1-r)$ is left regular in R.

Lemma: 2

If r is an element of R. Then $(1-r)$ is

Proof:

$(1-r)$ is left regular. To prove $(1-r)$ right regular. and there exist $S \in A$. such that $S(1-r)=1$

This gives that S is right regular (33)

$$S(1-r) = 1$$

$$S - Sr = 1$$

$$S = 1-(-Sr) = 1-(-S)r$$

As R is left ideal

$r \in R$, $(-S)r \in R$

$1-(-S)r$ is left regular by lemma (1)

$\therefore (1-(-S)r) = S$ is left regular

S is left regular and right regular (19) which is

It's left and right inverse co

$(1-r)$

ie) $(1-r)$ is also regular.

Theorem:

The radical R of A equals each of the four

Sets in F

i) $\cap MLI = \{ r B = 1 - xr$ is regular $\forall x \}$

ii) $\cap MRI = \{ r D = 1 - rx$ is regular $\forall x \}$

an c is therefore proper two sided ideal

proof:

$R = \cap MLI$

$= \{ r : 1 - xr$ is regular, $\forall x \}$

$\Rightarrow R$ is the intersection of all maximal left

$R = \cap MLI$

$= \{ r : 1 - rx \text{ is regular } \forall x$

ie) R is the intersection of all maximal right ideal

Hence, by defn. R is the proper two sided ideal.

(34)

Theorem:

Every maximal left ideal in A is closed

Proof:

we know that If I is an ideal in A then the closure of I is an ideal of some kind.

Our claim is every element of M is singular for it $x \in M$ is not singular. It's be regular

$\therefore x^{-1}$ exists such that $x^{-1}x = xx^{-1} = 1$

Now, $xx^{-1} \in M$ $\langle \because$ M is left ideal

$\Rightarrow r \in M$ which implies $M = A$.

which is contradiction.

x is singular

$\therefore M$ is a subset of S

ie) $M \subset S \rightarrow \bar{M} \subset \bar{S}$

$\therefore r \in M \Rightarrow r \notin S$

$\bar{M}$ is a proper ideal

$\bar{M} = M$ (M is maximal)

$\Rightarrow$ M is closed.

Theorem :

If I is a proper closed two sided ideal in A. Then the quotient algebra A/I is a banach algebra.

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Proof :

1^{st} we have to prove that A/I is closed. Let us define the norm as,

$\|x + I\| = \inf \{ \|x+i\| / i \in I \}$

i) Clearly $\|x + I\| \geq 0$

ii) $\because I$ is closed

we have $\|x+I\|$ there exist a $\{i_n\} \in I$

Such that $\|x + i_n\| \to 0$ iff $x \in I$

ie) $x + I = I$ = zero element of A/I

iii) considering,

$\|(x+I)+(y+I)\| = \|x+y+I\|$

$= \inf \{ \|(x+y)+i\| / i \in I \}$

$= \inf \{ \|x+y+i+i\| / i, i \in I \}$

$= \inf \{ \|(x+i)+(y+i)\| / i, i \in I \}$

$\leq \inf \{ \|x+i\| / i \in I \} + \inf \{ \|y+i\| / i \in I \}$

$\|(x+I)+(y+I)\| = \|x+I\| + \|y+I\|$

Thus, $\|(x+I)+(y+I)\| = \|x+I\| + \|y+I\|$

iv) consider,

$\|\alpha(x+I)\| = \inf \{ \|\alpha(x+i)\| / i \in I \}$

$= \inf \{ |\alpha| \, \|x+i\| / i \in I \}$

$= |\alpha| \inf \{ \|x+i\| \mid i \in I \}$

$= |\alpha| \, \|x + I\|$ 36

Now, we have to prove that A/I is complete

Assume that, A is complete

we shall show that A/I is complete

Let, $\{x_k + I\}$ be a cauchy sequence in A/I then

we can find a cauchy sequence $\{x_n + I\}$ such that

$\|(x+I) - (x_2+I)\| < 1/2$

$\|(x_2+I) - (x_3+I)\| < 1/2^2$

and in general

$\|(x_n+I) - (x_{n+1}+I)\| < 1/2^n$ convergent in A/I

Let $y_1 \in x_1 + I$

we select y_2 in x_2 such that $\|y_1 - y_2\| < 1/2$

we select a vector $y_3 \in x_3 + I$ such that

$\|y_2 - y_3\| < 1/2^2$

Continuing in this process we get $\{y_n\}$ in such

that $\|y_n - y_{n+1}\| < 1/2^n$

ie) If $m<n$ then,

$\|y_m - y_n\| = \|(y_m - y_{m+1}) + (y_{m+1} - y_{m+2}) + \cdots + (y_{n-1} - y_n)\|$

$= \|y_m - y_{m+1}\| + \|y_{m+1} - y_{m+2}\| + \cdots + \|y_{n-2} - y_n\|$

$< 1/2^m + 1/2^{m+1} + \cdots + 1/2^{n-1}$

$$< \frac{1/2^m}{1-1/2} = \frac{1/2^m}{1/2} = 1/2^{m-1}$$ (37)

which is a Geometric progression with common ratio.

So it is convergent

ie) $\{y_n\}$ is a cauchy sequence in I

By completeness of I. There exist a vector $y \in I$ Such that $y_n \to y$

$$\|(x_n+I)-(y+I)\| = \|(x_n-y)+I\|$$
$$= \inf\{\|(x_n-y)+i\| \mid i \in I\}$$
$$\leq \|x_n+I-y\| \ \forall\ i \in I$$
$$\|(x_n+I)-(y+I)\| \leq \|y_n-y\|$$

where $y_n \in x_n+I$, $\therefore y_n \to y$

$$\|(x_n+I)-(y+I)\| \to 0$$
$$\therefore (x_n+I) \to (y+I) \in A/I$$

Thus every cauchy sequence has a convergent Subsequence in A/I and A/I is complete

Hence, we have proved that A/I is a banach Space

we shall prove that A/I is a banach algebra

A/I is clearly an algebra with identity $1+I$

$$\|1+I\| = \inf\{\|1+i\| \mid i \in I\}$$
$$\leq \|1\| = 1 \to (1)$$

consider,

$$\|(x+I)*(y+I)\| = \|xy+I\|$$ (38)

$$= \inf\{\|xy+i\| / i\in I\}$$

$$= \inf\{\|(x+i_1)+(y+i_2)\| / i_1, i_2 \in I\}$$

$$\le \inf\{\|x+i_1\|.\|y+i_2\| / i_1, i_2\in I\}$$

$$\le \inf\{\|x+i_1\| / i_1\in I\}.$$

$$\inf\{\|y+i_2\| / i_2\in I\}$$

Hence $\|(x+I)*(y*I)\| \le \|x+I\|\ \|y+I\|$

also, $\|1+I\| \doteq 1$ for $\|1+I\|\le 1$

$$\|I+1\| = \|(1+I)^2\| = \|(1+I)(1+I)\|$$

$$\le \|1+I\|\ \|1+I\|$$

$$1 \le \|1+I\| \to (2)$$

From (1) and (2)

$$\|1+I\| = 1$$

A/I is a banach algebra

Theorem:

A/R is a semi-simple banach algebra.

Proof:

By the above theorem.

In order to prove that, "A/R is semi-simple

prove that the radical of A/R is $\{0\}$"

consider the homomorphism.

$x\to x+R$ of A onto A/R

If M is a maximal left ideal in A then.

RCMCA

$\therefore$ M/R is a left ideal in A/R (39)

Also, M/R is a maximal left ideal in A/R.

Thus there is a one to one correspondance bet-ween the maximal left ideal in A and there in A/R

Radial of $A = \cap M \perp T$

There exist a one to one correspondence between the radical of A and radical of A/R.

< provided A is simple. If A is simple the radical of $A/R = \{0\}$ >

ie) Between R and radical of A/R

$\Rightarrow$ Radical of A/R is $\{0\}$

If A is semi simple

A/R is a semi - simple banach algebra

Example:

Let $C(X)$ denote the "set of all bounded continuous complex fun" defined on a topological Space X. Then $C(X)$ is a banach space algebra banach space w.r.to supremum norm and point wise addition and multiplication.

Solution:

Let $f \in C(X)$

$$\|f\| = \sup\{|f(x)| ; x \in X\}$$

$$(f+g)(x) = f(x) + g(x)$$

$(\alpha f)x = \alpha f(x)$

$(fg)x = f(x)\, g(x)$

$\|f\| = \operatorname{Sup}\{|f(x)|, x \in X\}$

i) $x \in X$, $|f(x)| \geq 0$

$\Rightarrow \operatorname{Sup}\{|f(x)|; x \in X\} \geq 0$

$\Rightarrow \|f\| \geq 0$

ii) $\|f\| = 0 \Leftrightarrow \operatorname{Sup}\{|f(x)|, x \in X\} = 0$

$\Leftrightarrow |f(x)| = 0 \ \forall x$

$\Leftrightarrow f(x) = 0$

$\Leftrightarrow f = 0$

$\|f\| = 0 \Leftrightarrow f = 0$

iii) $\|\alpha f\| = \operatorname{Sup}\{|(\alpha f)x|, x \in X\}$

$= \operatorname{Sup}\{|\alpha f(x)|, x \in X\}$

$= \operatorname{Sup}\{|\alpha| \cdot |f(x)|, x \in X\}$

$= |\alpha| \cdot \operatorname{Sup}\{|f(x)|, x \in X\}$

$\|\alpha f\| = |\alpha|\, \|f\|$

iv) $\|f+g\| = \operatorname{Sup}\{|(f+g)x|, x \in X\}$

$= \operatorname{Sup}\{|f(x) + g(x)|; x \in X\}$

$\leq \operatorname{Sup}\{|f(x)|, x \in X\} +$

$\operatorname{Sup}\{|g(x)|, x \in X\}$

$\|f+g\| = \|f\| + \|g\|$

$\therefore$ $C(X)$ is a normal linear Space

Let. $\{f_n\}$ be a cauchy sequence in $C(X)$

Let, $\epsilon > 0$ there exist N such that $\|f_m - f_n\| < \epsilon$, $\forall\ m, n \geq N$

$$\|f_m - f_n\| = \sup\{ |(f_m - f_n)x|, x \in X\}$$ (41)

$$= \sup\{ |f_m x - f_n x|, x \in X\}$$

$$= \sup\{ |f_m(x) - f_n x|, x \in X\} < \epsilon, \ n, m \geq N$$

$|f_m(x) - f_n x| < \epsilon$, $n, m \geq N$ $\forall x \in X$

This is cauchy's cartesian for uniform convergence of bounded continuous scalar valued function.

$\therefore \{f_n\}$ converges in $C(X)$

$C(X)$ is complete

$\therefore C(X)$ is a banach space

$$\|(fg)h\| = \sup\{ \|(fg)h).x\|, x \in X\}$$

$$= \sup\{ |(fg)x . h(x)|, x \in X\}$$

$$= \sup\{ |f(x) . g(x) . h(x)|, x \in X\} =$$

$$= \sup\{ |f(x) . (gh)x|$$

$$= \sup\{ |f(gh).x|, x \in X\}$$

$$\|f(g+h)\| = \sup\{ |f(g+h)x|; x \in X\}$$

$$\therefore (f(g+h))x = f(x)(g+h)x$$

$$= f(x)[g(x) + h(x)]$$

$$= f(x) . g(x) + f(x) . h(x)$$

$$= (fg)x + (fh)x$$

$= (fg + fh)x$

$\| f(g+h) \| = \sup \{ |(fg + fh)x|, x \in X \}$

$= \| fg + fh \|$ (42)

Similarly,

$\| (f+g)h \| = \| fh + gh \|$

$\| \alpha(fg) \| = \sup \{ \| \alpha (fg) x \|, x \in X \}$

$= \sup \{ \| (\alpha f) g, x \in X \}$

$\| \alpha(fg) \| = \| (\alpha f) g \|$

$\| \alpha(fg) \| = \sup \{ |((\alpha f) g) x|, x \in X \}$

$= \sup \{ |(f(\alpha g)) x, x \in X \}$

$= \| f(\alpha g) \|$

$\therefore C(X)$ is an algebra

$\| fg \| = \sup \{ |(fg)x|, x \in X \}$

$= \sup \{ |f(x), g(x), x \in X \}$

$\leq \sup \{ |f(x), x \in X \} . \sup \{ |g(x), x \in X \}$

$\| fg \| \leq \| f \| . \| g \|$

Define a fun. $I : X \to C$ by $Ix = 1 \ \forall x$

$(fI)x = f(x) . Ix$

$= f(x) . 1$

$= f(x)$

I is the identity element of $C(X)$

$\| 1 \| = \| I \|$

$= \sup \{ |Ix|, x \in X \}$

$= \sup \{ |1| \}$

$\therefore C(X)$ is a banach algebra. 43

Definition :-

An algebra is a linear space, whose vectors can be multiplied in such a way that

i) $x(yz) = (xy)z$

ii) $x(y+z) = xy + yz$,

$(x+y)z = xz + yz$

iii) $\alpha(xy) = (\alpha x)y = x(\alpha y)$

A linear space A is called an algebra if it is vectors can be multiplied in such a way that A is a ring in which scalar multiplication is related to the multiplication by the following property. $\alpha(xy) = (\alpha x)y = x(\alpha y)$

Notation :-

Let A be a banach Space and G denote the set of all regular elements (Invertible elements) and S denote the set of all singular element (Non-line invertiable elements)

clearly G and S are complement to each other.

Result :

G is a group w.r.to multiplication

Proof :

Let $x \in G$

$\therefore x^{-1}$ exists

A Banach algebra $\Rightarrow 1 \in A$ and $x^{-1}x = xx^{-1} = 1$

$\therefore$ Inverse of x^{-1} exist and $(x^{-1})^{-1} = x$ 44

$\therefore x^{-1}$ is a regular graph element

$\therefore x^{-1} \in G$

$\therefore$ Inverse exist in G

clearly, associative property hold

Let $x, y \in G$,

$\therefore x^{-1}\ y^{-1} \in G$

To prove, $x, y \in G$

ie) To prove : $(xy)^{-1}$ exists

$(xy)(y^{-1}x^{-1}) = x.1.x^{-1}$

$= x.x^{-1} = 1$

similarly $y^{-1}x^{-1} = 1$

$\therefore y^{-1}x^{-1}$ is the inverse of xy

$\therefore (xy)^{-1}$ exists

$\Rightarrow xy \in G$

$x \in G \Rightarrow x^{-1} \in G$

$\Rightarrow xx^{-1} \in G$

$\Rightarrow 1 \in G$

$\therefore$ G is a graph.

UNIT -V

UNIT - V

Commutative Banach Algebra :- (1)

Theorem : A

If M is a maximal ideal in A, then the Banach Algebra A/M is a division Algebra and therefore equally the Banach Algebra C of complex numbers. the natural homomorphism $x : x+M$ of A complex number $x(M)$ defined by $x(M) = x+M$ and the mapping $x \to x(M)$ has the following properties

i) $(x+y)(M) = x(M) + y(M)$

ii) $(\alpha x)(M) = \alpha x(M)$

iii) $(xy)(M) = x(M)\, y(M)$

iv) $x(M) = 0 \Leftrightarrow x \in M$

v) $1.(M) = 1$

vi) $|x(M)| \leq \|x\|$

Proof :-

M is a maximal ideal

$\therefore M$ is a closed sets

M is ideal $\Rightarrow M$ is linear subspace of A

Hence, M is a closed linear subspace of A

A Banach algebra $\Rightarrow A$ is a Banach Space

$\therefore A/M$ is a Banach Space with norm defined by

$$\|x+M\| = \inf\{\|x+m\| ; m \in M\}$$

M is a maximal ideal

$\Rightarrow$ M is a proper subset of A

$\therefore$ A/M is a Banach Space (2)

" Theorem C, if R is a commutative ring with

velocity then an ideal I in R as maximal $\Leftrightarrow$ R/I

is a field".

Now A is a commutative ring with identity and

M is a maximal ideal in A

$\therefore$ A/M is a field

$\therefore$ Every non-zero element in A/M is regular

$\Rightarrow$ A/M is a division algebra

By theorem. A/M equals C

$$x \rightarrow x+M$$

$$A \rightarrow A/M$$

Since, A/M equals C, the mapping $x \rightarrow x+M$ assigns

to reach $x \in A$ a complex number

Since, $x(M)$ is defined by,

$$x(M) = x+M$$

$x(M)$ is a complex number

i) $(x+y)(M) = x+y+M$

$= x+M+y+M$

$= x(M)+y(M)$

ii) $(\alpha x)(M) = \alpha x+M$

$= \alpha(x+M)$

$= \alpha . x(M)$

iii) $(xy)(M) = xy + M$

$= (x+M)\cdot(y+M)$ (3)

$= x(M)\cdot y(M)$

iv) $x(M) = 0 \Leftrightarrow x+M =$ zero elements of A/M

$\Leftrightarrow x+M = M$

$\Leftrightarrow x \in M$

v) $1\cdot(M) = 1+M$ The identity of A/M

$= 1$ (the identity of C)

vi) $|x(M)| = |x+M| = \|x+M\|$

$= \inf\{\|x+m\|, m \in M\}$

$\leq \|x+m\| \ \forall m \in M$

$|x(M)| \leq \|x\|$ $\because 0 \in M$

$\mathcal{M}$ - Set of all maximal ideal in A For $x \in A$.

Let $\hat{x}$ denote the function of $\mathcal{M}$ by $\hat{x}(M) = x(M)$

$\therefore \hat{x}$ is a function from $\mathcal{M}$ to C

Let $\hat{A} = \{\hat{x}, x \in A\}$

Define a topology on $\mathcal{M}$ such that every function $\hat{x} \in \hat{A}$ is continuous

The mapping $x \to \hat{x}$ is known as "self and mapping".

ie) $A \to \hat{A}$

Theorem : B

The self and mapping $x \to \hat{x}$ is a norm decreasing (and therefore continuous) homomorphism of A into $C_e(\mathcal{M})$ with the following properties

i) The image $\hat{A}$ of A is a subalgebra of $C(M)$ which seperates the points of $C(M)$ and contains the identity of $C(\mu)$ (4)

ii) The radial R of A equals the set of all elements x for which $\hat{x} = 0$ so $x \to \hat{x}$ is an isomor... $\Leftrightarrow$ A its semi simple

iii) An element x in A is regular $\Leftrightarrow$ it doesn... belong to any maximal ideal $\Leftrightarrow$ $\hat{x}(m)$ to $\forall m$

iv) If x is an element of A Then its spectrum equals the range of the function $\hat{x}$ and its spectr... radius equals the norm of $\hat{x}$

ie) $\sigma(x) = \hat{x}(M)$ and $r(x) = \sup |\hat{x}(m)| = \|\hat{x}\|$

Proof:

Define a topology on M such that $\hat{x}$ is continuous $\forall \hat{x} \in \hat{A}$

$$|\hat{x}(m)| = |x(m)| \leq \|x\|$$

$$\|\hat{x}\| = \sup_{x \in M} \{ |\hat{x}(x)| \}$$

$$\therefore \|\hat{x}\| \leq \|x\|$$

$\therefore x \to \hat{x}$ is a norm decreasing function

$\therefore \hat{x}$ is a bounded function

$$\therefore \hat{x} \in C(M) \; \forall \hat{x} \in \hat{A}$$

$$\therefore \hat{A} \subseteq C(M)$$

Let, $\phi : A \to C(M)$ defined by $\phi(x) = \hat{x}$

[illegible]

a) $\widehat{x+y}(M) = (x+y)(M) = x(M) + y(M)$

$= \hat{x}(M) + \hat{y}(M) = (\hat{x}+\hat{y})M$ 5

$\therefore \widehat{x+y} = \hat{x} + \hat{y}$

$\therefore \phi(x+y) = \widehat{x+y} = \hat{x} + \hat{y} = \phi(x) + \phi(y)$

$\Rightarrow \phi(x+y) = \phi(x) + \phi(y)$

b) $\phi(xy) = \widehat{xy}$

$\widehat{xy}(M) = (xy)(M) = x(M)\, y(M)$

$= \hat{x}(M) + \hat{y}(M) = (\hat{x}\hat{y})(M)$

$\widehat{xy} = \hat{x}\hat{y}$

c) $\phi(\alpha x) = \widehat{\alpha x}$

$(\widehat{\alpha x})(M) = (\alpha x)(M) = \alpha\hat{x}(M)$

$\Rightarrow \alpha\hat{x} = \widehat{\alpha x}$

$\therefore \phi(\alpha x) = \widehat{\alpha x} = \alpha . \phi(x)$

$\Rightarrow$ ϕ is a homomorphism of A into $C(M)$

$\Rightarrow$ Image of A under ϕ is $\bar{A}$ and $\bar{A} \subseteq C(M)$

(i) Image of A under ϕ is $\bar{A}$ and $\bar{A} \subseteq C(M)$

Let $\hat{x}, \hat{y} \in \hat{A}$

$\hat{x} + \hat{y} = \widehat{x+y}$

$\hat{x}, \hat{y} \in \hat{A} \Rightarrow x, y \in A$

$\Rightarrow x+y \in A$

$\Rightarrow \widehat{x+y} \in \hat{A}$

$\Rightarrow \hat{x} + \hat{y} \in \hat{A}$

$\therefore \hat{A}$ is closed w.r.to addition

$\hat{x}\,\hat{y} = \widehat{xy} \in \hat{A}$

$\langle \because x, y \in A \Rightarrow xy \in A \rangle$

$\therefore \hat{x}\hat{y} \in \hat{A}$ (6)

$\therefore$ A is closed w.r.to multiplication

Let α be a scalar then $\alpha\hat{x} = \hat{\alpha x}$

$\alpha x \in A \Rightarrow \hat{\alpha x} \in \hat{A}$

$\Rightarrow \alpha\hat{x} \in \hat{A}$

$\therefore \hat{A}$ is closed w.r.to scalar multiplication

Let α be a scalar then αx

$\Rightarrow \hat{A}$ is closed w.r.to addition multiplication scalar multiplication.

$\Rightarrow \hat{A}$ is a subalgebra of $\mathcal{C}(M)$

Let $M_1, M_2 \in \mathcal{C}(M)$ and $M_1 \neq M_2$

There exist $x \in M_1$ and such that $x \notin M_2$

$\Rightarrow x(M_1) = 0$ and $x(M_2) \neq 0$

$\Rightarrow \hat{x}(M_1) = 0$ and $\hat{x}(M_2) \neq 0$

$\Rightarrow \hat{x}(M_1) \neq \hat{x}(M_2)$

$\Rightarrow \hat{x}$ seperates point in $\mathcal{C}(M)$

$1 \in A \Rightarrow \hat{1} \in \hat{A}$

$\hat{1}(M) = 1(M) = 1$.

Let $f \in \mathcal{C}(M)$

$(f(\hat{1})(M) = f(M)\,\hat{1}(M) = f(M)$

$= \hat{1}$ is the identity element of $\mathcal{C}(M)$

Since $\hat{1} \in \hat{A}$, $\hat{A}$ contains the identity element of $\mathcal{C}(M)$

(ii) Let $P = \{x : \hat{x} = 0\}$

$P = \cap MLI = \bigcap_{M \in \mathcal{M}} M$ (7)

$x \in R \Leftrightarrow x \in M \ \forall M \in \mathcal{M}$

$\Leftrightarrow x(M) = 0 \ \forall M \in \mathcal{M}$

$\Leftrightarrow \hat{x}(M) = 0 \ \forall M \in \mathcal{M}$

$\Leftrightarrow \hat{x} = 0$

$\Leftrightarrow x \in P$

$R = P$

i.e. $R = \{x : \hat{x} = 0$

To prove: $x \to \hat{x}$ is an isomorphism iff A is semi simple

Let $\phi : A \to \hat{A}'$, $\phi(x) = \hat{x}$ be an isomorphism

Let $x \in R \Rightarrow \hat{x} = 0$

$\Rightarrow \phi(x) = 0$

$\Rightarrow x = 0$ $\angle \because \phi$ is an isomorphism

$\Rightarrow R = \{0\}$

$\Rightarrow A$ is semi simple

conversely

Suppose A is semi simple

$\therefore R = \{0\}$

Let $\phi : x \to \hat{x}$ be defined by $\phi(x) = \hat{x}$

clearly ϕ is an onto homomorphism

Let $\phi(x) = \phi(y)$

$\phi(x) \cdot \phi(y) = 0 \Rightarrow \phi(x-y) = 0$

$\Rightarrow \phi \ \widehat{x-y} = 0$ $\angle \because R = \{x, \hat{x} = 0\}$

$\Rightarrow x - y \in R$

Again $R = \{0\} \Rightarrow x - y = 0$

$\Rightarrow x = y$

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$\therefore \phi$ is one to one

$\Rightarrow \phi$ is isomorphism

(ii) Let x be regular

$\Rightarrow x \notin$ any maximal ideal in M

$\Rightarrow x \notin M \ \forall M \in \mathcal{M}$

$\Rightarrow x(M) \neq 0 \ \forall M \in \mathcal{M}$

$\Rightarrow \hat{x}(M) \neq 0 \ \forall M \in \mathcal{M}$

Suppose $\hat{x}(M) \neq 0 \ \forall M \in \mathcal{M}$

$\Rightarrow x(M) \neq 0 \ \forall M \in \mathcal{M}$

$\Rightarrow x \notin M \ \forall M \in \mathcal{M}$

$\Rightarrow x \notin$ any maximal ideal in A

To prove x is regular

Suppose x is singular

claim:

Ax is a proper ideal in A

$Ax = \{ax, a \in A\}$

Ax is a linear space

$y \in A$; $y(ax) = (ya)x \in Ax$

$\therefore Ax$ is an ideal

Suppose Ax is not proper

$\therefore xy = yx = 1$

$\Rightarrow x$ is regular

$\Rightarrow$ Contradition to our assumption.

Hence, our assumption that Ax is not proper is wrong.

(9)

$\therefore Ax$ is proper ideal in A

Hence there exist a maximal ideal M such that $Ax \subset M$

$1 \in A \Rightarrow 1 \cdot x \in Ax$

$\therefore x \in Ax \subset M \Rightarrow x \in M$

$\Rightarrow x(M) = 0$

$\Rightarrow \hat{x}(M) = 0$

$\therefore$ contradiction $\hat{x}(M) \neq 0 \ \forall \ M \in \mathcal{M}$

contradiction to hypothesis.

$\Rightarrow$ our assumption that is singular is wrong

$\therefore x$ is regular

iv) $x \in A$, $\sigma(x) = \{\lambda ; x = \lambda\}$ is singular

$\lambda \in \sigma(x) \Leftrightarrow \lambda \in \hat{x}(M)$

$\lambda \in \sigma(x) \Leftrightarrow x - \lambda \cdot 1$ is singular

$\Leftrightarrow \widehat{x - \lambda \cdot 1}(M) = 0$ for some $M \in \mathcal{M}$.

If $\widehat{x - \lambda \cdot 1}(M) \neq 0 \ \forall \ M \in \mathcal{M}$ then by (iii)

$x - \lambda \cdot 1$ is regular

$x - \lambda \cdot 1$ is regular for some $x \in M$

$\Leftrightarrow \hat{x}(M) - \lambda \hat{1}(M) = 0$

$\Leftrightarrow \hat{x}(M) - \lambda \cdot 1 = 0$ $\qquad \angle M \in \mathcal{M} \Rightarrow \hat{x}(M) \in \hat{x}'(M)$

$\Leftrightarrow \hat{x}(M) = \lambda$

$\Leftrightarrow x \in \hat{x}(M)$

$\therefore \lambda \in \sigma(x) \Leftrightarrow \lambda \in \hat{x}(M)$

$\therefore \sigma(x) = \hat{x}(M)$

$\sigma(x) = \hat{x}(M)$

$r(x) = \operatorname{Sup}\{|\lambda|, \lambda \in \sigma(x)\}$

$= \operatorname{Sup}\{|\lambda|, \lambda \in \hat{x} M\}$

$= \operatorname{Sup}\{|\hat{x}(M)|, M \in \mathcal{M}$

$= \|\hat{x}\|$

$\therefore r(x) = \|\hat{x}\| = \operatorname{Sup}_{M \in \mathcal{M}} \{\hat{x}(M)\}$

Definition:

A multiplication functional on A is a functional on A if $f \in A^*$ which is non-zero and satisfies the additional condition.

$f(xy) = f(x) f(y)$

Observation:

To each $M \in \mathcal{M}$ there corresponds a multiplicative functional f_M defined by $f_M(x) = x(M)$ and the null space of f_M is M

Let $M \in \mathcal{M}$ and f_M be defined on A by $f_M(x) = x$

"Clearly f_M is a mapping from A to M

Maximal":

$\Rightarrow$ M is a proper subset of A".

$\therefore$ There exist $z \in M$ such that $z \notin M$

$\Rightarrow f_M(z) = z(M) \neq 0$

ie) $f_M(x) \neq 0$

$\Rightarrow f_M$ is a non-zero mapping

Since, $x(M)$ is unique $\forall x$ and for a given M $\forall x \in A$, $f_M(A)$ is unique (11)

$\therefore f_M$ is a function of A to C

$\Rightarrow f_M$ is a non-zero function

$$f_M(\alpha x+\beta y) = (\alpha x+\beta y)(M)$$
$$= \alpha x(M) + \beta y(M)$$
$$= \alpha\, x(M) + \beta\, y(M)$$
$$= \alpha . f_M(x) + \beta f_M(y)$$

$\therefore f_M$ is a linear transformation

$$|f_M(x)| = |x(M)| \le \|x\|$$

$\therefore f_M$ is bounded $\Rightarrow f_M$ is continuous

Hence, f_M is a continuous linear transformation from $A \to C$

$\therefore f_M$ is functional

$$f_M(xy) = (xy)(M)$$
$$= x(M) \cdot y(M)$$
$$= f_M(x) \cdot f_M(y)$$

$\therefore f_M$ is a multiplicative functional null space

$f_M = \{x, f_M(x) = 0\}$

Now, $f_M(x) = 0 \Leftrightarrow x(M) = 0$

$\Leftrightarrow x \in M$

Null space of $f_M \Leftrightarrow f_M(x) = 0 \Leftrightarrow x \in M$

$\therefore$ Null space of $f_M = M$

Lemma:

If f_1 and f_2 are multiplicative function with the same null space. Then $f_1 = f_2$

(12)

Proof:

1st to prove: $f_1 = \alpha f_2$ for some scalar α.

Let M = Null space of f_1 and f_2

$\because f_1$ is a multiplicative functional

$\Rightarrow f_1 \neq 0$

$\therefore$ There exist x_0 such that $f_1(x_0) \neq 0$

$\Rightarrow x_0 \notin M$

$\Rightarrow M$ is a proper subset of A

Clearly M is an ideal

$\langle \because (\alpha x + \beta y) = \alpha f_1(x) + \beta f_1(y) = \alpha.0 + \beta.0 = 0$

$\therefore M$ is linear

$x \in A,\ m \in M,\ f_1(xm) = f_1(x) f_1(m)$

$= f_1(x) \cdot 0$

$= 0 \quad \forall m \in M$

$\therefore M$ is left ideal

commutative $\Rightarrow$ right ideal

$\therefore M$ is an ideal $\rangle$

$\therefore M$ is a proper ideal of A

Also, $x_0 \notin M \Rightarrow x_0 \neq 0 \quad \langle f_1(0) = 0 \Rightarrow 0 \in M \rangle$

Let, $x \in A$ then x can be uniquely expressed

as $x = m + \beta x_0$, $m \in M$ and β scalar

For let, $m = \alpha - \beta x_0$ where $\beta = \dfrac{f_2(\alpha)}{f_2(x_0)}$ $(f_2(x_0) \neq 0$

$f_2(m) = f_2(\alpha - \beta x_0)$ (13)

$= f_2(\alpha) - \beta f_2(x_0)$

$= f_2(\alpha) - \dfrac{f_2(\alpha)}{f_2(x_0)} f_2(x_0)$

$f_2(m) = 0$

$m \in$ Null space of f_2

$\therefore m \in M$

Since $\alpha - \beta x_0$ is unique

$\therefore \alpha = m + \beta x_0$ is unique $m \in M$, β - scalar

Now,
$f_1(\alpha) = f_1(m + \beta x_0)$

$= f_1(m) + \beta f_1(x_0)$

$f_1(\alpha) = 0 + \dfrac{f_2(\alpha)}{f_2(x_0)} \cdot f_1(x_0)$

$f_1(\alpha) = \dfrac{f_1(x_0)}{f_2(x_0)} f_2(\alpha)$

Put $\alpha = \dfrac{f_1(\alpha)}{f_2(x_0)}$

Then, $f_1(\alpha) = \alpha \cdot f_2(\alpha)$

$= (\alpha f_2)(\alpha)$

$\Rightarrow f_1 = \alpha f_2$

Let $\alpha \in A$ and $\alpha \notin M$, $f_2(\alpha) \neq 0$

$\alpha (f_2(\alpha))^2 = \alpha \cdot f_2(\alpha) f_2(\alpha)$

$= \alpha \cdot f_2(\alpha \cdot \alpha)$

$= \alpha \cdot f_2(\alpha^2)$

$= f_1(\alpha^2)$

$$= f_1(x \cdot x) = f_1(x) \cdot f_1(x)$$

$$= [f_1(x)]^2$$ (14)

$$= \alpha\,[f_2(x)]^2$$

$$= \alpha^2\,[f_2(x)]^2$$

$$(\alpha - \alpha^2)\,[f_2(x)]^2 = 0$$

$$f_2(x) \neq 0 \quad \therefore \alpha - \alpha^2 = 0$$

$$\alpha(1-\alpha) = 0$$

$$\alpha \neq 0, \quad 1 - \alpha = 0$$

$$\alpha = 1$$

$$\therefore F_1 = F_2$$

Theorem : (A-19, 5M)

$M \to F_M$ is a one to one mapping of the set $\mathcal{M}$ of all maximal ideals in A onto the set of its multiplicative functional.

Proof:

Let $\mathcal{F}$ be the set of all multiplicative functionals defined on A

Let $\phi : \mathcal{M} \to f_M$ be defined by

$$\phi(M) = f_M$$

[This mapping is well defined as for each $M \in \mathcal{M}$ there exist a multiplicative functional $f_M \in A^*$ defined by $f_M(x) = x(M)$ $x \in A$]

$$\therefore \forall\ M \in \mathcal{M},\ f_M \in \mathcal{F}$$

Let, $M_1, M_2 \in \mathcal{M}$ and $M_1 \neq M_2$

$\therefore$ There exist $x \in M_1$ and $x \notin M_2$

$x \in M_1 \Rightarrow f_{M_1}(x) = 0$ and $x \notin M_2 \Rightarrow f_{M_2}(x) \neq 0$

$f_{M_1}(x) \neq f_{M_2}(x)$ (15)

$f_{M_1} \neq f_{M_2}$

$\Rightarrow \phi$ is one to one

Let f be an arbitrary multiplicative functional in A^* ($f \in \mathcal{F}$)

Let M be the null space of f multiplicative functional $\Rightarrow f \neq 0$

$\therefore$ There exist $x \in A$ such that $f(x) \neq 0$

$\Rightarrow x \notin M$

$\therefore M$ is a proper subset of A

Clearly, M is a closed subset of A and also M is an ideal

$[f_1(M) = \{0\} \Rightarrow M = f_1^{-1}\{0\}$ and $\{0\}$ closed and continuous $\Rightarrow M$ is closed$]$

Now to prove M is maximal on the contrary

Let M be not maximal

$\therefore$ There exist a maximal proper ideal such that $M \subset I$

I is proper $\Rightarrow f(I)$ is proper subset of A (or) C

$\therefore f(I)$ is not proper

If $f(I) = C$ then $I = A$

which is contradiction

If $f(I) = 0 \Rightarrow I \subset M$

f the

et of all

n $M \in \mathcal{M}$

*

which is contradiction >

Clearly $f(I)$ is a proper ideal in A (16)

By theorem.

" If R is a ring with identity Then R is a field $\Leftrightarrow$ R has no non-traivial ideal"

Now c is a field

$\therefore$ c cannot have proper ideals

Since, $f(I)$ is a proper ideals in c

we arrive at a contradiction

$\therefore$ M is maximal

$\therefore M \in \mathcal{M}$

$\Rightarrow$ There exist a multiplicative functional f_M

Such that null space of $f_M \in M$

Now f and f_M are two - multiplicative functionals having the same null space

Hence by above lemma $f = f_M$

Now, $\phi(M) = f_M = f$

Hence, for each multiplicative functional $f \in \mathcal{F}$ there exist a maximal ideal $M \in \mathcal{M}$ such that $\phi(M) = f$ and ϕ is onto

N Normal linear space and N^* - conjugate Space of N.

(17)

N^{**} - conjugate space of N^* and conjugate space of N for all $x \in N$ gives rise to a functional F_x in N^{**} defined by

$$F_x(f) = f(x) \ \forall \ f \in N^*$$

$x \to F_x$ is a norm preserving function

$$\|x\| = \|F_x\|$$

$$N \subseteq N^{**} \text{ (always)}$$

If $N = N^{**}$ then N is said

i) p and a real number $1 \leq p \leq \infty$

l_p^n - Space of all n-tuples $x = (x_1, x_2 \dots x_n)$ of scalar when the norm

$$\|x\|_p = \left[\sum_{i=1}^{\infty} |x_i|^p\right]^{1/p}$$

ii) l_p is the space of all sequences $x = \{x_n\}$ of scalar such that

$$\sum_1^{\infty} |x_i|^p < \infty \text{ and } \|x\|_p = \left[\sum_1^{\infty} |x_i|^p\right]^{1/p}, \ 1 \leq p < \infty$$

iii) l_∞^n - Space of all n-tuples $x = \{x_1, x_2 \dots x_n\}$ of scalars and $\|x\|_\infty = \max\{|x_1|, \dots |x_n|\}$

iv) ~~l_∞~~ l_∞ - Space of all bounded sequence $\{x_n\}$ of scalars and $\|x\| = \sup|x_n|$

v) L_p - Space of all measurable functional f defined on the measurable space X with p real number such that $1 < p < \infty$ for which $|f(x)|^p$

is integrable

If g is a function in L_q where $1/p + 1/q$

define F_g in L_p by.

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$$F_g(f) = \int f(x)\, g(x)\, dm(x)$$

Then F_g is well defined scalar valued line

function on L_p with $\| F_g \| = \| g \|_q$

and F_g is a functional

$\therefore F_g \in L_p^*$

Also $g \leftrightarrow F_g$ is isometric isomorphism

$L_q = L_p^* \rightarrow$ (1)

Suppose these conditions are to the n-tuple

$l_q^* = l_p$

$l_p^{**} = l_p$

$\therefore l_p$ is a reflexive space

Definition:

Let $X \neq \phi$ and $\{X_i\}$ be a non-empty class of topological spaces and for each i, let f_i be a mapping of X into X_i

The weak topology generated by the f_i's is defined to be the intersection of all topologies on X w.r.to each of which all the f_i's are continuous mapping

In a similar way, weak topology on a No

linear space can be defined. Let N be a Normed linear space the weak topology on N is the weak topology on N generated by functionals in N^*. It is weakest topology on N w.r.to which each functionals N^* remains continuous. 19

weak* topology of N^*:

The weak* topology on N^* is the weak topology on N^* generated by all the iduced functionals in N^*

If f_0 is an arbitrary element in N^* and if $\epsilon > 0$ then the set

$S(x, f_0, \epsilon) = \{ f : f \in N^*, |F(x)(f) - Fx(f_0)| < \epsilon \}$

$= \{ f : f \in N^* | Fx - F(x_0)| < \epsilon \}$

ie) All open set (ie) neighbourhood of f_0) in the weak* topology.

Theorem:

N^* is a H.S w.r.to its weak* topology

proof:

Let $f, g \in N^*$ and $f \neq g$

Then there exist $x \in N$ such that $f(x) \neq g(x)$

put $\epsilon = \dfrac{|f(x) - g(x)|}{3}$

Consider; $S(x, f, \epsilon)$ and $S(x, g, \epsilon)$ nbhd.

of f and g and they are not intresting

$\therefore N^*$ is a Hausdorff Space w.r.to itsweak topology.

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(S^* in N^* is defined by,

$$S^* = \{F ; f \in N^*, \|f\| \leq 1\}$$

closed unit sphere)

Theorem :

If N is a normed linear space then the closed unit sphere S^* in N^* is a compact Hausdorff Space.

Proof :

N^* is a Hausdorff Space

Any subspace of H.S is Hausdorff

S^* is a Hausdorff space with each $x \in N$ associates a compact space C_x

where $C_x = \{-\|x\|, \|x\|\}$ (or)

$$C_x = \{z : \|z\| \leq \|x\|\}$$

According as N is a real normed linear Space (or) a complex normed linear Space

Hence, C_x is Compact $\forall x \in N$ (in both cases)

Let, $C = \prod_{x \in N} C_x$

C is a product of compact spaces

By Tychonoff's theorem C is compact For any $f \in S^*$

$|f(x)| \leq \|f\| \|x\| \leq \|x\|$ $\langle \because \|f\| \leq 1, f \in S^*$

$-\|x\| \leq f(x) \leq \|x\|$

(21)

$\Rightarrow f(x) \in C_x$

Hence for each $x \in N$ the values $f(x)$ of all f in S^* lie in C_x

This enables to imbed S^* in C by regarding each f in S^* as identical with the array of all its values at the vectors x in N.

$\therefore S^* \subseteq C$

Hence, the weak * topology on S^* equals its topology as a subspace of C

In the product topology C consists of all fun. $g : N \to \cup \{C_x : x \in N\}$ such that $g(x) \in C_x$. and these funs. and need not be linear.

A basis element for this product topology for $\varepsilon > 0$ is given by

$u = \{g_0, x_1, x_2 \ldots x_n, \varepsilon\}$

$= \{g \in C, |g(x_i) - g_0(x_i)| < \varepsilon, \text{ for } i = 1, 2 \ldots n\}$

Since $S^* \subset C$, a basis element of S^* as a subspace of C is, $u(g_0, x, \ldots x_n, \varepsilon) \cap S^*$

$= \{g \in S^*, |g(x_i) - g_0(x_i)| < \varepsilon; \; i = 1, 2 \ldots n\}$

These topologies coincide and C is compact

It is enough to prove $\overline{S^*} = S^*$

If g is defined on N, Then $g \in C$ and

$|g(x)| \leq \|x\| \; \forall x$

(22)

Let $g \in \overline{S^*}$

Every basis open set containing g intersect in

Let $f \in S^*$ such that $f \in U(g, x, y, x+y, \epsilon$

$\therefore |g(x) - f(x)| < \epsilon/3$

$|g(y) - f(y)| < \epsilon/3$

$|g(x+y) - f(x+y)| < \epsilon/3$

$$|g(x+y) - g(x) - g(y)| = |g(x+y) - f(x+y) + f(x+y) - g(x) - g(y)|$$

$$= |g(x+y) - f(x+y) + f(x) + f(y) - g(x) - g(y)|$$

$$= |g(x+y) - f(x+y) - |g(x) - f(x) - g(y) - f(y)||$$

$$\leq |g(x+y) - f(x+y)| + |g(x) - f(x)| + |g(y) - f(y)|$$

$$< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$$

This holds $\forall \; \epsilon > 0$

$$\therefore g(x+y) = g(x) + g(y)$$

Similarly, $g(\alpha x) = \alpha g(x)$

$\therefore g(x)$ is linear

For any $x \in N$, there exist $f_x \in N^{**}$ such that

$f_x \in U(g, x, \epsilon) \cap S^*$

$\therefore |g(x) - f_x(x)| < \epsilon$

Now, $|g(x)| = |g(x) - f_x(x) + f_x(x)|$

$\leq |g(x) - f_x(x)| + |f_x(x)|$

$< \epsilon + |f_x(x)|$ (23)

$\|g\| \leq |f_x(x)| = \|f_x\| \cdot \|x\|$

$\|f_x\| = \sup\{|f_x(x)|, \|x\| \leq 1\}$

$f_x \in S^* \Rightarrow \|f_x\| \leq 1$

$\therefore \|g\| \leq 1$ and $g \in S^*$

$\therefore \overline{S^*} \subset S^*$

$\Rightarrow S^*$ is a closed subset of C

$\therefore S^*$ is a compact Hausdorff space

Definition :

A is said to be self adjoint if for each $x \in A$ there is a y in A such that $\hat{y}(M) = \overline{\hat{x}(M)}$ $\forall M$

Theorem : B

If A is self - adjoint then $\hat{A}$ is dense in $C(M)$

proof :

Theorem : 1

If $x \to \hat{x}$ is a self and mapping from $A \to \hat{A}$ then $\hat{A}$ is a subalgebra of $\mathcal{C}(M)$ and $\hat{A}$ separates point in $\mathcal{C}(M)$ and contains the identity of $\mathcal{C}(M)$

Result :: Theorem : 2

Let X be a topological space and A be a subalgebra of $\mathcal{C}(X, R)$ (or) $\mathcal{C}(X, C)$ Then its closure is also a subalgebra of. 'A' is a subalgebra of $\mathcal{C}(X, C)$ which contains the conj of each of its function Then $\bar{A}$ also contains the conjugate of each of its function.

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Theorem: B:

The complex stone Kleiesotrass Theorem

Let X be a compact Hausdorff space and a be a closed subalgebra of $\mathcal{C}(X, C)$ which separates points contains a non-zero constant function and contains the conjugate of each of function then $\hat{A}$ squals $\mathcal{C}(X, C)$

proof:

A is self adjoint

$\therefore \forall x \in A$ there exits $y \in A$ such that

$$\hat{y}(M) = \overline{\hat{x}(M)} \; \forall M$$

Hence $\forall \hat{x} \in \hat{A}$ there is a conjugate $\hat{y} \in \hat{A}$ Satisfy the required condition

Hence $\hat{A}$ contains conjugate of each of its function. Since $\hat{A}$ is a subalgebra its closure $\bar{A}$ is a closed subalgebra

$\hat{A}$ contains conjugate of each of its functions

Further $\hat{A}$ separates points. hence $\hat{A}$ also Separate points

$\mathcal{C}(M)$ - compact Hausdorff space $\hat{A}$ is a closed subalgebra of $\mathcal{C}(M)$ its separates oints and contains conjugate of each of its fun.

$\bar{\hat{A}}$ contains the identity function $\hat{1}$ 25.

is a non-zero functions Also $\hat{1}(M) = 1$

$\therefore$ $\hat{1}$ is a non-zero functions

$\bar{\hat{A}}$ satisfies all the conditions of stone weierstrass

heorem

$$\bar{\hat{A}} = \mathcal{C}(M)$$

$\Rightarrow$ $\hat{A}$ is dense in $\mathcal{C}(M)$.

Theorem : C

If A is self adjoint and if $\|x^2\| = \|x\|^2 \; \forall x$.

hen the Gelfand mapping $x \to \hat{x}$ is an isometric

somorphism of A onto $\mathcal{C}(M)$.

Proof :

$$\|x^2\| = \|x\|^2$$

By theorem A,

$$\|\hat{x}\| = \|x\|$$

$\therefore$ $x \to \hat{x}$ is norm preserving

$$\phi(x) = \hat{x}$$

Then ϕ is clearly, one to one, onto also linear

$\therefore$ $x \to \hat{x}$ is an isometric isomorphism of A onto $\hat{A}$

Since $\hat{A}$ is closed, $\hat{A}$ is a closed subalgebra

of $\mathcal{C}(M)$

Then by theorem B. $\hat{A}$ is dense in $\mathcal{C}(M)$

$$\therefore \bar{\hat{A}} = \mathcal{C}(M)$$

$$\Rightarrow \hat{A} = \mathcal{C}(M)$$

$\therefore x \to \hat{x}$ is an isometric isomorphism of A onto $\mathcal{C}(M)$

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Definition:

A Banach algebra A is called a Banach B^* algebra if it has an involution.

ie) If there exist a mapping $x \to x^*$ of A into itself with the following properties.

i) $(x+y)^* = x^* + y^*$ ii) $(\alpha x)^* = \bar{\alpha} x^*$

iii) $(xy)^* = y^* x^*$ iv) $x^{**} = x$

Deductions:

(1) An involution is a one to one mapping of A onto itself.

Proof:

Let, $\phi : A \to A$ by $\phi(x) = x^*$

$$\phi(x_1) = \phi(x_2)$$

$$\Rightarrow x_1^* = x_2^*$$

$$\Rightarrow x_1^{**} = x_2^{**}$$

$$\Rightarrow x_1 = x_2$$

$\therefore \phi$ is one to one

Let $x \in A$ Then $x^* \in A$

$$\phi(x^*) = x^{**} = x$$

$\therefore \phi$ is onto

(2) $0^* = 0$ and $1^* = 1$

Proof:

$$x = x + 0$$

$$x^* = (x+0)^*$$

$$x^* = x^* + 0^*$$

$$0^* = 0$$

$$\begin{aligned} 1^* &= 1.1^* \\ &= 1^{**}.1^* \\ &= (1.1^*)^* 1^* \end{aligned}$$

$$1^* = (1^*)^*$$

$$1^* = 1$$

Note:

i) The element x^* is called the adjoint of x

ii) A subalgebra of A is said to be self adjoint if it contains the conjugate of each of its element

iii) If A and A^* are Banach B^* algebra and if f is any isomorphism of A to A' Then f is called a $*$- isomorphism if it preserves the involution in the sense that $f(x^*) = f(x)^*$

Note:

Involution is continuous if $\|x^*\| = \|x\|$

Proof:

Let $x_n \to x$

To prove: $x_n^* \to x^*$

$$\begin{aligned} \|x_n^* - x^*\| &= \|(x_n - x)^*\| \\ &= \|x_n - x\| \to 0 \end{aligned}$$

$\therefore$ Involution is continuous.

Definition :

Any Banach B^* - algebra which satisfie[s] the condition $\|x^*x\| = \|x\|^2$ is called B^* - algebr[a]

(28)

Result :

In any B^* - algebra $\|x^*\| = \|x\|$

Proof :

$$\|x\|^2 = \|x^*x\|$$
$$\leq \|x^*\| \|x\|$$
$$\Rightarrow \|x\| \|x\| \leq \|x^*\| \|x\|$$
$$\Rightarrow \|x\| \leq \|x^*\| \rightarrow (1)$$

using (1), we have

$$\|x^*\| \leq \|x^{**}\|$$
$$\Rightarrow \|x^*\| \leq \|x\| \rightarrow (2)$$
$$\therefore \|x^*\| = \|x\|$$

Observation :

In a B^* - algebra $\|x^*x\| = \|x^*\| \|x\|$

$$\|x^*\| \|x\| = \|x\| \|x\|$$
$$= \|x\|^2$$
$$= \|x^*x\|$$

i) Banach B^* - algebra and $\|x^*x\| = \|x\|^2$

$\Rightarrow B^*$ algebra

Definition :

i) An element x is said to be self adjoint if $x = x^*$

ii) An element x is said to be normal if if $xx^* = x^*x$ It is a projection if $x = x^*$ and $x = x^2$

(29)

Theorem : ①

The comp. maximal ideal space M is a compact Hausdorff space.

Proof :

A, is a Banach algebra with identity and A^* functionals on A

$$S^* = \{ f : f \in A^*, \|f\| \leq 1 \}$$

Then S^* is a compact H.S in the weak* topology

Since $M \to f_M$ is one to one mapping of M set of all maximal ideal onto the set of all multiplicative generates in A

For $M \in \mathcal{M}$, $f_M \in A^*$

$f_M(1) = f_M(1^2) = f_M(1.1)$

$f_M(1) \cdot 1 = f_M(1) \cdot f_M(1)$

$\Rightarrow f_M(1) = 1 \quad \forall M \in \mathcal{M}$

$\| f_M \| = \sup \{ | f_M(x) | ; \|x\| \leq 1 \}$

Since $x \in A$ and $\|1\| = 1$.

$| f_M(1) | \in \{ \| f_M(x) \| , \|x\| \leq 1 \}$

$\Rightarrow 1 \in \{ | f_M(x) | ; \|x\| \leq 1 \}$

$\| f_M \| \geq 1 \to (1) \quad \forall M \in \mathcal{M}$

$\| f_M \| = \sup \{ f_M(x) ; \|x\| \leq 1 \}$

$= \text{Sup}\{|x(M)| ; \|x\| \leq 1\}$

$\leq \text{Sup}\{\|x\|, \|x\| \leq 1\}$

$\|f_M\| \leq 1 \rightarrow (2)$

$\therefore$ From (1) and (2)

$\|f_M\| = 1 \quad \forall M \in \mathcal{M}$

Hence, $f_M \in S^* \quad \forall M \in \mathcal{M}$

Hence, $\mathcal{M}$ can be considered as a subset of S^*. The topology on S^* is the weak* topology which is the weak topology generated by all functionals $F_x \in x^{**}$ defined on S^* defined by $F_x(f) = f(x)$ $F(x)$ restricted to elements of $\mathcal{M}$ is $\hat{x}$

For, $F_x(f_M) = f_M(x)$

$= x(M)$

$= \hat{x}(M)$

$\therefore$ The topology M has as a subspace of S^* is same as its topology as the space of maximal ideals.

$\therefore M$ can be regarded as a subspace of S^*

Let $X = \bigcap_{x,y \in A} \{f, f \in S^*, f(xy) = f(x) f(y)\}$

$= \bigcap_{x,y \in A} \{f : f \in S^* \text{ and } f(xy) - f(x) f(y) = 0\}$

$X = \bigcap_{x,y \in A} \{f : f \in S^* \text{ and } F_{xy}(f) - F_x(f) F_y(f) = 0\}$

$X = \bigcap_{x,y \in A} \{f : f \in S^*, (F_{xy} - F_x F_y)(f) = 0\}$

$\{0\}$ is a closed set and $F_{xy} - F_x F_y$ continuous

(31)

Each set in the intersection as a closed set

$\therefore$ X is a closed set in S^*

clearly, X is set of all multiplicative functionals together with zero functionals

Hence, X is M together with 0

$F_1(f_M) = f_M(1) = \hat{1}(M) = 1$

$F_1(f_M) = 1 \quad \forall M \in M$

Also, F_1 is 0 at the zero functional

$F_1(0) = 0(1) = 0$

Hence, $M = F_1^{-1}(\{1\})$

Hence M is a closed set in X as F_1 is cts

Hence, M is a closed set in S^*

$\Rightarrow$ M is compact

$\Rightarrow$ M is a Hausdorff as M a subspace of S^*

Gelfand Representation Theorem : Theorem B + Theorem D

Theorem :

If T is a normal operator on a finite dimensional Hilbert space H Then there exist an orthogonal basis for H relative to which the matrix of T is diagonal.

Proof :-

Let $T = \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_m P_m$ be the Spectral resolution of T. Then $\lambda_1, \lambda_2 \ldots \lambda_m$ are precisely the eigen values of T and $P_1, P_2 \ldots P_m$ are the projections on the eigen spaces $M_1, M_2 \ldots M_m$ corresponding $\lambda_1, \lambda_2 \ldots \lambda_m$ respectively.

Since T is normal $i \neq j$, $M_i \perp M_j$ and $P_i P_j = 0$

Each M_i is finite dimensional and hence has a finite orthogonal basis ($M_i \neq 0$).

Let $B_1, B_2 \ldots B_m$ be the basis for $M_1, M_2 \ldots M_m$ respectively.

Let $B = \bigcup_{i=1}^{m} B_i$

Claim :-

B is an orthogonal basis for H

Since, for any $x \in B$, $x \in B_i$ for some i

$\therefore \|x\| = 1$

Hence, it is enough to prove

$x, y \in B$ then $x \perp y$

Since, $x, y \in B_i \Rightarrow x \perp y$

Suppose $x \in B_i$, $y \in B_j$ $i \neq j$

$x \in B_i \Rightarrow x \in M_i$

$y \in B_j \Rightarrow y \in M_j$

$\therefore M_i \perp M_j \Rightarrow x \perp y$

$\Rightarrow$ B is an orthogonal set in H

$\Rightarrow$ B is linear transformation

Let $x \in H$

$x = Ix = (P_1 + P_2 + \dots + P_m) x$

$= P_1 x + P_2 x + \dots + P_m x$ $\quad \therefore P_i x = x_i, \forall i$

$= x_1 + x_2 + \dots + x_m$

Now, $P_i \in M_i$ = Range of P_i

$\therefore x_i \in M_i \ \forall i$

Now B_i is a basis for $M_i, \forall i$

Hence each x_i is a linear combination of elements of B_i

Hence x is a linear combination of elements of $B_1, B_2, \dots B_m$

$\therefore x$ is a linear combination of elements of B.

Thus B is a basis for H

Let, $B_1 = \{e_{11}, e_{12}, \dots e_{1n_1}\}$

$B_2 = \{e_{21}, e_{22}, \dots e_{2n_2}\}$

$\vdots$

$B_m = \{e_{m1}, e_{m2}, \dots e_{mn_m}\}$

M_i corresponds to λ_i

$Tx = \lambda_1 x \ \forall x \in M_i \ \forall i$

$Te_{11} = \lambda_1 e_{11}$

$Te_{12} = \lambda_1 e_{12}$

$\vdots$

$Te_{1n_1} = \lambda_1 e_{1n_1}$

$Te_{21} = \lambda_2 e_{21}$

$\vdots$

$Te_{2n_2} = \lambda_2 e_{2n_2}$

$\vdots$

$Te_{m1} = \lambda_m e_{m1}$ (34)

$\vdots$

$Te_{mn} = \lambda_m e_{mn}$

$Te_{11} = \lambda_1 e_{11} + 0e_{12} + \dots + 0e_{mn}$

$Te_{12} = 0e_{11} + \lambda_1 e_{12} + \dots + 0e_{mnm}$

$\vdots$

$Te_{1n_1} = 0e_{11} + \dots + \lambda_1 e_{m1} + \dots + 0e_{mnm}$

$Te_{21} = 0e_{11} + \dots + 0e_{1n_1} + \lambda_2 e_{21} + \dots + 0e_{mnm}$

$\vdots$

$Te_{mnm} = 0e_{11} + \dots + \lambda_m e_{mnm}$

$T\alpha = \alpha_1 Te_{11} + \alpha_2 Te_{12} + \dots + \alpha_{ni} Te_{ini}$

$= \alpha_1 \lambda_i e_{i1} + \alpha_2 \lambda_i e_{i2} + \dots + \alpha_{ni} \lambda_i e_{ini}$

$= \lambda_i [\alpha_1 e_{i1} + \alpha_2 e_{i2} + \dots + \alpha_{ni} e_{ini}]$

$T\alpha = \lambda_i \alpha$

Claim : 2

If $T\alpha = \lambda_i \alpha$ then $\alpha \in M_i$, $\alpha \in H$ and

Let $\alpha = \alpha_{11} e_{11} + \alpha_{12} e_{12} + \dots + \alpha_{1n_1} e_{1n_1} + \alpha_{i1} e_{i1} + \dots$
$+ \alpha_{ini} e_{ini} + \dots$
$+ e_{m1} e_{m1} + \dots + \alpha_{mnm} e_{mnm}$

$\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_m$

where $\alpha_i = \alpha_{i1} e_{i1} + \alpha_{i2} e_{i2} + \dots + \alpha_{ini} e_{ini}$

$T\alpha = T\alpha_1 + T\alpha_2 + \dots + T\alpha_m$

$= \lambda_1 \alpha_1 + \lambda_2 \alpha_2 + \dots + \lambda_m \alpha_m$

$\lambda_i \alpha = \lambda_i \alpha_1 + \lambda_i \alpha_2 + \dots + \lambda_i \alpha_i + \dots + \lambda_i \alpha_m$

Given, $T\alpha = \lambda_i \alpha$, $\lambda_i \neq \lambda_j$, $i \neq j$

$\Rightarrow \lambda_1 x_1 = \lambda_i x_1, \lambda_2 x_2 = \lambda_i x_2 \ldots$

$\lambda_m x_m = \lambda_i x_m, \; i = 1 \text{ to } m$ 35

$\lambda_j x_j = \lambda_j x_j \;, j \neq i$

Since $\lambda_i \neq \lambda_j$, ~~[illegible]~~ if $i \neq j$

$\therefore x = x_i \in M_i$

$\Rightarrow x \in M_i$

Let us show that M_i is closed.

Let x be a limit point of M_i

$\therefore$ There exist a sequence $\{x_n\}$ in M_i such that $x_n \to x$.

$\therefore T$ is continuous $\Rightarrow T(x_n) \to T(x)$

$Tx_n = \lambda_i x_n \Rightarrow T(x_n) \to T(x)$

But, $\lambda_i x_n \to \lambda_i x$

$\therefore T(x) = \lambda_i x$

By claim (2); $x \in M_i$

$\therefore M_i$ is closed

Let $P_1, P_2 \ldots P_m$ be the projection on $M_1, M_2 \ldots M_m$ respectively.

$\therefore P_i \neq 0 \; \forall i.$

M_i's span H

For $x \in H$, $x = x_1 + x_2 + \cdots + x_m$. $x_i \in M_i$

M_i's are pairwise orthogonal

For M_i's generated by B_i's where B_i's are orthogonal basis and $B_i \perp B$ and B is a set of

Orthogonal basis

$\therefore M_i \perp M_j \Rightarrow P_i P_j = 0, \; i \neq j$

Let $\alpha \in H$ $\alpha = \alpha_1 + \alpha_2 + \dots + \alpha_m$

$$T\alpha = T\alpha_1 + \dots + T\alpha_m$$

$$= \lambda_1 P_1 \alpha + \lambda_2 P_2 \alpha + \dots + \lambda_m P_m \alpha$$

$$= (\lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_m P_m)\alpha$$

$$T = \lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_m P_m \rightarrow (1)$$

λ_i's distinct

$$I\alpha = \alpha = \alpha_1 + \alpha_2 + \dots + \alpha_m$$

$$= P_1\alpha + P_2\alpha + \dots + P_m\alpha$$

$$I_m = (P_1 + P_2 + \dots + P_m)\alpha$$

$$\therefore I = P_1 + P_2 + \dots + P_m$$

$P_i P_j = 0 \Rightarrow M_i$'s Span H

$\therefore$ eqn (1) is the spectral ~~resolviant~~ a resolution of T

Since, spectral solution is unique $\lambda_1, \lambda_2 \dots \lambda_m$ are eigen values of T M_i's are eigen space corresponding to λ_i's

There exist for H an orthonormal basis relative to which the matrix T is diagonal

[T is normal $\Rightarrow$ IV]

$$\mathrm{I} = \overline{\mathrm{II}} \Rightarrow \overline{\mathrm{III}} \Rightarrow \mathrm{I}$$

$$\overline{\mathrm{III}} \Rightarrow \overline{\mathrm{IV}}$$

$$\overline{\mathrm{IV}} \Rightarrow \overline{\mathrm{II}}$$

<T is diagonal $\Rightarrow$ T is normal>

$\overline{IV} \Rightarrow \overline{II} \Rightarrow \overline{IV}$

The Galfand Newmark Theorem: A-19 2M

If A is a commutative B^* algebra then the Galfand mapping $x \to \hat{x}^*$ is an isomorphic $*$-isomorphism of A onto the commutative B^*-algebra $C(M)$ onto the commutative B^*-algebra $C(M)$

<Banach $*$- algebra>

i) $xx^* = x^*x \Rightarrow x$ is normal

ii) Theorem A: x is maximal then $\|x^2\| = \|x\|^2$

iii) A is self adjoint $x \in A$ there exist y such that.

$$\hat{y}(M) = \overline{\hat{x}(M)}$$

iv) A is self adjoint if A contains adjoint of its each of its element

v) In a B^*-algebra $\|x^*\| = \|x\|$

vi) Banach B^*-algebra is a B^*-algebra if $\|x^*x\| = \|x\|^2$

vii) x^* is the adjoint of x

viii) Theorem C: $x \to \hat{x}$ is an isomorphic isomorphism if A is self adjoint and $\|x^2\| = \|x\|^2$

Proof:

By theorem C: $x \to \hat{x}$ is an isomorphic isomorphism if A is self adjoint and $\|x^2\| = \|x\|^2$

A is commutative $\Rightarrow xx^* = x^*x$

(38)

$\Rightarrow x$ is normal

By theorem B. $\|x\|^2 = \|x^2\|$

It is enough to prove A is self adjoint

ie) To prove: $\forall x \in A$ there exist $y \in A$ such that

$\hat{y}(M) = \overline{\hat{x}(M)}\ \forall x \in M$

ie) To prove; $\hat{x^*}(M) = \overline{\hat{x}(M)}\ \forall M \in \mathcal{M}$

claim:

If x is self adjoint, then $\hat{x}(M)$ is real

Assume $\hat{x}(M)$ is not real

Let $\hat{x}(M) = \alpha + i\beta$ where $\beta \neq 0$

x is self adjoint

$\therefore \dfrac{x - \alpha \cdot 1}{\beta}$ is self adjoint

For let, $y = \dfrac{x - \alpha \cdot 1}{\beta}$

$$y^* = \left(\frac{x - \alpha \cdot 1}{\beta}\right)^* = \frac{x^* - \alpha^* \cdot 1^*}{\beta^*} = \frac{x - \bar{\alpha} \cdot 1}{\beta}$$

$$= \frac{x - \alpha \cdot 1}{\beta} = y$$

$\therefore y$ is self adjoint

$$\hat{y}(M) = \widehat{\left(\frac{x - \alpha \cdot 1}{\beta}\right)}(M) = \frac{1}{\beta}(x - \widehat{\alpha \cdot 1})(M)$$

$$= 1/\beta\,[\hat{x}(M) - \alpha \cdot \hat{1}(M)] = 1/\beta\,[\alpha + i\beta - \alpha \cdot 1]$$

$$= 1/\beta\,[\alpha + i\beta - \alpha]$$

$\hat{y}(M) = i$

$\widehat{y-i}(M) = \hat{y}(M) - i\hat{i}(M) = i - i.1 = i-i = 0$

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$\widehat{y-i}(M) = 0$

$\Rightarrow y - i.1 \in M \quad \because \hat{x}(M) = 0 \Leftrightarrow x \in M$

$\Rightarrow (y-i.1)^* \in M^* \rightarrow (1)$

$(y-i.1)^* = y^* - i^* - 1^* = y - \bar{i} - 1 = y - (-i).1$

$(y-i.1)^* = y + i.1$

$\Rightarrow y + i.1 \in M^* \quad <$ by (1)

$\Rightarrow \widehat{y+i.1}(M)^* = 0$

$\Rightarrow \hat{y}(M^*) + i\,\hat{i}(M^*) = 0$

$\Rightarrow \hat{y}(M^*) + i.1 = 0 \Rightarrow \hat{y}(M^*) = -i.1 = -i$

$\therefore \hat{y}(M^*) = -i$

Suppose k is any +ve number

$(\widehat{y - i.k.1})(M^*) = \hat{y}(M^*) - i.k\,\hat{i}(M^*)$

$= -i - i.k.1 = -i - ik = -i(1+k)$

$= -i(1+k) \rightarrow (2)$

$\widehat{y+ik.1}(M) = \hat{y}(M) + ik\,\hat{i}(M) = i + ik.1$

$= i(1+k) \rightarrow (3)$

$|i(1+k)| = |\widehat{y+ik.1}(M)| \leq \| y + ik.1 \|$

$< k > 0, |1+k| = 1+k$

$[1+k] \leq \| y + ik.1 \|$

Again from (2)

$$|-i(1+k)| = |y - \hat{i}k + (M^*y)| = \|y - i \cdot k \cdot 1\|$$

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$$(1+k) \leq \|y - i \cdot k \cdot 1\|$$

$$(1+k)^2 \leq \|y + i \cdot k \cdot 1\| \; \|y - i - k \cdot 1\|$$

$$(y - i \cdot k \cdot 1)^* = y^* - i^* k^* 1^* = y - \bar{i} \cdot k \cdot 1 = y - (-i)k \cdot 1$$

$$= y + i \cdot k \cdot 1$$

$$(y - i \cdot k \cdot 1) = (y + i \cdot k \cdot 1)^*$$

$$(1+k)^2 \leq \|y + i \cdot k \cdot 1\| \; \|(y + i \cdot k \cdot 1)^*\|$$

$$= \|(y + ik1)(y + ik \cdot 1)^*\|$$

$$= \|(y + ik \cdot 1)(y - ik \cdot 1)\|$$

$$= \|y^2 - iky + iky + k^2 1\|$$

$$= \|y^2 + k^2 - 1\|$$

$$= \|y^2\| + \|k^2 \cdot 1\|$$

$$= \|y^2\| + \|k^2\| \cdot \|1\|$$

$$\leq \|y^2\| + k^2$$

$$1 + 2k + k^2 \leq \|y^2\| + k^2$$

$$1 + 2k = \|y^2\|$$

Since, k is any +ve number the above cannot hold

$\therefore$ our assumption is wrong

$\therefore \sigma(M)$ is real

Define, $y = \dfrac{x + x^*}{2}$, $z = \dfrac{x - x^*}{2i}$

Scanned with CamScanner

$$y^* = \frac{x^* + x^{**}}{2} = \frac{x^* + x}{2} = y$$

$$z^* = \frac{x^* - x^{**}}{2i^*} = \frac{x^* - x}{-2i} = \frac{x - x^*}{2i} = z$$

∴ y and z are self adjoint

$$y + iz = \frac{x + x^*}{2} + i\left(\frac{x - x^*}{2i}\right) = \frac{x^* + x}{2} + \frac{x - x^*}{2}$$

$$y + iz = x$$

$$\hat{x}^*(M) = \widehat{(y+iz)^*}(M)$$

$$= \widehat{(y^* + i^* z^*)}(M)$$

$$= \widehat{(y^* - iz^*)}(M)$$

$$= (y - iz)(M)$$

$$\hat{x}^*(M) = \hat{y}(M) - iz(M)$$

y and z are self adjoint

⇒ $\hat{y}(M)$ and $\hat{z}(M)$ are real

$$\hat{x}^*(M) = \overline{\hat{y}(M)} - \overline{\hat{z}(M)}$$

$$= \overline{\hat{y}(M) + \hat{z}(M)}$$

$$= \overline{\hat{x}(M)}$$

$$\therefore \hat{x}^*(M) = \overline{\hat{x}(M)}$$

∴ A is self adjoint

Hence, proved.