Unit - 1

Space curves :-

Differential Geometry :-

Differential Geometry is a branch of mathematics inwhich the study the curves and surfaces with the help of differential calculus.

Local Differential Geometry :-

This is a study of the properties of curves and surfaces in the neighbourhood of the point.

Global Differential Geometry :-

This is a study of the properties of curves and surfaces as a whole.

Space curve :- 2M

✓ we can represent a space curve in two ways

(i) As the intersection of two surfaces

(ii) parametric representation

As the intersection of two surfaces :-

Let $f_1(x, y, z) = 0$ and $f_2(x, y, z) = 0$ —— ① represent the two surfaces then these two equations together represent the curve which is the intersection of these two surfaces and this curve will be called a plane curve, if it is lies on a plane otherwise it is said to be skew twisted

Example :-

we know that, if $f(x, y, z) = 0$ represents a sphere

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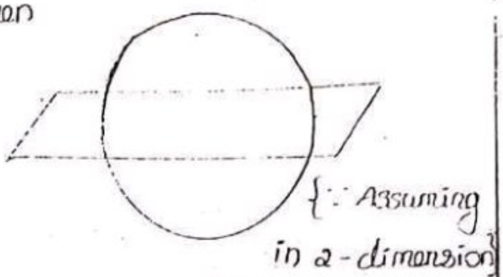
and $f_2(x,y,z) = 0$ represents a plane then these two equations together represents a circle which is the intersection of the given sphere and the given plane. In this case the curve is a plane curve.



{∵ Assuming in 2-dimension}

Parametric representation :-

If the co-ordinates of a point on a space curve be represented by the equations of the form $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$ —— ② where f_1, f_2, f_3 are real valued functions of the single real variable 't', ranging over a set of values $a \le t \le b$. The equations in ② are called parametric equations of the space curve.

Functions of class 'm' :-

Let 'I' be a real interval and 'm' is a positive integer. A real valued function 'f' defined on I is said to be of class 'm' (or) c^m function, if 'f' has an m^{th} derivative at every point of I and if this derivative is continuous on I

Note :-

* when a function is infinitely differentiable then 'f' is said to be of class '∞' (or) c^∞ function

* when a function is analytic, we say it is a class of ω (or) c^ω function

vector valued functions :-

A vector valued function $'R' = (x, y, z)$ defined on I is said to be of class 'm', if it has an m^{th} derivative at every point and if this derivative is continuous on I.

Equivalently, if each of its components x, y, z is of class 'm'

Regular :-

If the derivative $\frac{dR}{du} = \dot{r}$ never vanishes on I, then the function is said to be regular.

{∵ never vanishes means zero ஆகாது }

Equivalently, If $\dot{x}, \dot{y}, \dot{z}$ never vanish simultaneously then the function is said to be regular.

path of class 'm' :- 2m

A regular vector valued function of class 'm' is called a path of class 'm'

Equivalent classes :-

Two paths R_1 and R_2 of the same class 'm' on I_1 and I_2 are called equivalent, if there exist a strictly increasing function φ of class 'm' which maps I_1 onto I_2 such that $R_1 = R_2 \circ \phi$

If we take $R_1 = (x_1, y_1, z_1)$, $R_2 = (x_2, y_2, z_2)$ then the above condition is same as

$$x_1(u) = x_2[\phi(u)]$$

$$y_1(u) = y_2[\phi(u)]$$

$$z_1(u) = z_2[\phi(u)]$$

hange of parameter :-

The function ϕ which relates two quivalent paths is called a change of parameter.

ac length :- 2m

The distance between two points $r_1 = (x_1, y_1, z_1)$, $_2 = (x_2, y_2, z_2)$ in space is the number

$$|r_1 - r_2| = \sqrt{(r_1 - r_2)^2} = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

This distance in space will be used to efine distance along a curve of class $m \geq 1$.

If we are given a path $r = R(u)$ and o numbers a, b $(a < b)$ as the parameters in the range hen the path $r = R(u)$, $(a \leq u \leq b)$ is an arc of the riginal path joining the points corresponding to 'a' nd 'b'

To any subdivision 'Δ' of the interval (a, b) y points $a = u_0 < u_1 < u_2 < \dots < u_n = b$ there are corresponds s the length

$$L_\Delta = \sum_{i=1}^{n} |R(u_i) - R(u_{i-1})|$$ of the polygon nscribed to the arc by joining successive points on it.

Addition to further points of subdivision ncreases the length of the polygon. It is reasonable to efine the length of the arc to be the upperbound of L_Δ taken over all possible subdivisions of (a, b). This upperbound is always finite, because for any

$$L_\Delta = \sum_{i=1}^{n} \left| \int_{u_{i-1}}^{u_i} \dot{R}(u)\, du \right| \leq \sum_{i=1}^{n} \int_{u_{i-1}}^{u_i} |\dot{R}(u)|\, du$$

$$L_\Delta \leq \int_a^b |\dot{R}(u)|\, du \text{ — ①}$$ and R.H.S of equation ① is finite and independent of 'Δ'

{ let $s = s(u)$ denote the arc length from 'a' to any point 'u' then the arc length from u_0 to u is given by $s(u) - s(u_0)$

From equation ①, $$s(u) - s(u_0) \leq \int_{u_0}^{u} |\dot{R}(u)|\, du \text{ — ②}$$

$s(u) - s(u_0) \leq \dot{s}(u)$ ... $\dot{R}(u)$)

and from the definition of arc length,

$$R(u) - R(u_0) \leq s(u) - s(u_0) \text{ — ③}$$

$$\Rightarrow \left| \frac{R(u) - R(u_0)}{u - u_0} \right| \leq \frac{s(u) - s(u_0)}{u - u_0}$$

$$\leq \frac{1}{u - u_0} \int_{u_0}^{u} |\dot{R}(u)|\, du$$

Taking the limit $u \to u_0$ we get

$|\dot{R}(u)| \leq \dot{s}(u)$

$|\dot{R}(u)| \geq \dot{s}(u)$

$$\{ \dot{R}(u) \} \leq \dot{s}(u) \leq \{ \dot{R}(u) \}$$

$$\therefore \dot{s}(u) = \dot{R}(u) \text{ — ④}$$

This is true for any value of u_0 in the range of 'u'

Hence $$\boxed{s = s(u) = \int_a^u |\dot{R}(u)|\, du \text{ — ⑤}}$$

Equation ⑤ is the arc length of the point 'a' to 'u' }

Cartesian equivalent :-

Let $R = x\vec{i} + y\vec{j} + z\vec{k}$, $\dot{R} = \dot{x} i + \dot{y} + \dot{z}$

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$|\dot{R}| = \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}$, we know that $s = \int_a^{u} |\dot{R}(u)|\, du$

$$\dot{s} = \dot{R} \quad , \quad \dot{s}^2 = |\dot{R}|^2 = \dot{x}^2+\dot{y}^2+\dot{z}^2$$

(or) interms of differentials $ds^2 = dx^2+dy^2+dz^2$ where 'ds' is called the <u>linear element</u> of the curve.

Jesus. Praise the lord.

2M

4) obtain the equations of circular helix $r = (a\cos u, a\sin u, bu)$, $-\infty < u < \infty$ where $a > 0$, referred to 's' as parameter and show that the length of one complete turn of the helix is $2\pi c$, where $c = \sqrt{a^2+b^2}$

<u>solution</u> :- Given $R = (a\cos u, a\sin u, bu)$

we know that, $s = \int_0^u |\dot{R}(u)|\, du$

$$\dot{R} = (-a\sin u, a\cos u, b)$$

$$|\dot{R}| = \sqrt{a^2\sin^2 u + a^2\cos^2 u + b^2} = \sqrt{a^2+b^2}$$

$$s = \int_0^u \sqrt{a^2+b^2}\, du = \sqrt{a^2+b^2}\,(u)_0^u$$

$$= \sqrt{a^2+b^2}\,(u) = cu \;,\; c = \sqrt{a^2+b^2}$$

$$s = cu \Rightarrow u = s/c$$

$$\therefore R = \left(a\cos(s/c), a\sin(s/c), b(s/c)\right)$$

The range of 'u' corresponding to one complete turn of the helix is $u_0 \le u \le u_0+2\pi$

$$\therefore s = \int_{u_0}^{u_0+2\pi} |\dot{R}(u)|\, du = \int_{u_0}^{u_0+2\pi} \sqrt{a^2+b^2}\, du$$

$$= \sqrt{a^2+b^2}\,[u]_{u_0}^{u_0+2\pi} = c\,[u_0+2\pi-u_0]$$

$$s = 2\pi c$$

2] Find the length of the curve given as the intersection of the surfaces $x^2/a^2 - y^2/b^2 = 1$, $x = a\cosh(z/a)$ from the point $(a, 0, 0)$ to the point (x, y, z)

solution :-

The equation of the curve in the parametric form may be taken as $x = a\cosh u$, $y = b\sinh u$, $z = au$

The position vector 'r' at any point of the curve is given by $S = \int_0^u |\dot{R}(u)|\, du$

$$R = (a\cosh u,\ b\sinh u,\ au)$$

$$\dot{R} = (a\sinh u,\ b\cosh u,\ a)$$

$$|\dot{R}| = \sqrt{a^2\sinh^2 u + b^2\cosh^2 u + a^2}$$

$$= \sqrt{a^2(1+\sinh^2 u) + b^2\cosh^2 u}$$

$$= \sqrt{a^2\cosh^2 u + b^2\cosh^2 u} = \sqrt{a^2+b^2}\cosh u$$

$$S = \int_0^u \sqrt{a^2+b^2}\cosh u\, du = \sqrt{a^2+b^2}\,(\sinh u)_0^u$$

$$= \sqrt{a^2+b^2}\sinh u = y/b\,\sqrt{a^2+b^2}$$

$$\{\because y = b\sinh u \Rightarrow \sinh u = y/b\}$$

3] Find the length of the arc $x = 3\cosh 2u$, $y = 3\sinh 2u$, $z = 6u$, where 'u' take limits 0 to 2π

solution :-

$$R = (3\cosh 2u,\ 3\sinh 2u,\ 6u)$$

$$\dot{R} = (6\cosh 2u,\ 6\sinh 2u,\ 6).$$

$$|\dot{R}| = \sqrt{36\sinh^2 2u + 36\cosh^2 2u + 36}$$

$$= 6\sqrt{(1+\sinh^2 2u) + \cosh^2 2u} \quad 6\sqrt{\cosh^2 2u + \cosh^2 2u}$$

$$= 6\sqrt{2\cosh^2 2u} \qquad = 6\sqrt{2}\cosh 2u$$

$$s = \int_0^u |\dot{R}(u)|\,du = \int_0^{2\pi} 6\sqrt{2}\cosh 2u\,du$$

$$= 6\sqrt{2}\left[\frac{\sinh 2u}{2}\right]_0^{2\pi} = 3\sqrt{2}\sinh \overset{4\pi}{(2\pi)}$$

Tangent normal and binormal :-

Let γ be a curve represented by the parametric equation $r = r(u)$ and let P and Q be two neighbouring points on the curve and [illegible] have parameters u_0 and u respectively.

Since γ is of class ≥ 1

$$r(u_0+h) = r(u_0) + \frac{h}{1!}\dot{r}(u_0) + \frac{h^2}{2!}\ddot{r}(u_0) + \cdots + o(h^n)$$

where $u - u_0 = h$

$$\therefore \quad r(u) = r(u_0) + (u-u_0)\,\dot{r}(u_0) + o(u-u_0) \text{ — ①}$$

The unit vectors alongs the chord PQ is $\underline{r(u)-r(u_0)}$

$$\lim_{u\to u_0} \frac{r(u)-r(u_0)}{|r(u)-r(u_0)|} = \lim_{u\to u_0} \frac{r(u)-r(u_0)}{|u-u_0|}\;\frac{\substack{|r(u)-r(u_0)| \\ |u-u_0|}}{|r(u)-r(u_0)|}$$

$$= \frac{\dot{r}(u_0)}{|\dot{r}(u_0)|} \text{ —— ②}$$

(ie) The unit vector along the chord PQ tends to unit vector at P as $Q \to P$. This is called unit ~~length~~ tangent vector to γ at 'P' and is denoted by 't'

$$\therefore \quad t = \frac{\dot{r}(u_0)}{|\dot{r}(u_0)|}$$

Since $\dot{s} = |\dot{r}(u_0)|$

$$t = \dot{r}/\dot{s} = \frac{dr/du}{ds/du} = dr/ds$$

The line through 'p' parallel to 't' is called the tangent line to γ at 'p'

If R is any point on this line, the vector from the point of contact 'p' to R is called a tangent vector to γ at 'p'

The unit tangent vector becomes $t = \dot{r}$ where $\dot{r} = dr/ds$

① Osculating plane :- 2m

2m

Let γ be a curve of class ≥ 2 and let P, Q be two neighbouring points on γ. Then the limiting position as Q → P of that plane which contains the tangent line at 'p' and the point 'Q' is called the osculating plane of γ at 'p'

Note :-

when γ is a straight line the osculating plane is indeterminate at each point. the equation of the osculating plane is $[R-r(0), r'(0), r''(0)] = 0$ where $r'' \neq 0$ provided the vectors $r'(0)$, $r''(0)$ are linearly independent.

Inflexion :-

The point 'p' on the curve for which $r'' = 0$ is called a point of Inflexion and the tangent line at 'p' is called Inflexional.

Result :-

If a curve is given interms of a general parameter 'u' then the equation of osculating plane corresponding to

$[R-r . \dot{r}, \ddot{r}] = 0$

Cartesian equation of the osculating plane :-

Cartesian eqn/- of the Osculating plane : If $R = (x, y, z)$ and $r = (x, y, z)$ then the equation of osculating plane is given by the scalar triple product takes the form

$$\begin{vmatrix} X - x & Y - y & Z - z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0$$

Examples :-

1] Find the osculating plane at the point 't' on the helix $r = (a\cos t, a\sin t, ct)$

Solution :- Given $r = (a\cos t, a\sin t, ct)$

$$\dot{r} = (-a\sin t, a\cos t, c)$$

$$\ddot{r} = (-a\cos t, -a\sin t, 0)$$

The osculating plane equation is

$$\begin{vmatrix} x - a\cos t & y - a\sin t & z - ct \\ -a\sin t & a\cos t & c \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = 0$$

$$(x - a\cos t)[0 + ac\sin t] - (y - a\sin t)[0 + ac\cos t] + (z - ct)[a^2\sin^2 t + a^2\cos^2 t] = 0$$

$$ac\sin t\, x - a^2c\sin t\cos t - ac\cos t\, y + a^2c\sin t.\cos t + za^2 - a^2ct = 0$$

$$ac\sin t\, x - ac\cos t\, y + za^2 - a^2ct = 0$$

Divided by 'a', $c\sin t\, x - c\cos t\, y + a(z - ct) = 0$

2] For the helix $x = a\cos u$, $y = a\sin u$, $z = au\tan\alpha$ prove that $ds/du = a\sec\alpha$ and find the length of the curve measure from the point $u = 0$

solution :- Given $r = (a\cos u, a\sin u, au\tan\alpha)$

$$\dot{r} = (-a\sin u, a\cos u, a\tan\alpha)$$

$$|\dot{r}| = \sqrt{a^2\sin^2u + a^2\cos^2u + a^2\tan^2\alpha}$$

$$= \sqrt{a^2 + a^2\tan^2\alpha} = a\sqrt{1+\tan^2\alpha} = a\sqrt{\sec^2\alpha} = a\sec\alpha$$

$$dr/du = |\dot{r}| = a\sec\alpha$$

$$s = \int_0^u |\dot{R}(u)|\,du = \int_0^u a\sec\alpha\,du = a\sec\alpha\,[u]_0^u$$

$$\therefore \quad s = au\sec\alpha$$

Normal plane :- 2m

Let 'p' be a point on the curve γ. The plane through 'p' orthogonal to tangent at 'p' is called the normal plane at 'p'

principal Normal :-

The line of intersection of normal plane and osculating plane is called the principal normal at 'p'

The unit vector along the principal normal is denoted by 'n'

Example :- 10m

The following examples shows that at the point of inflexion over a curve of class '∞' need not possess an osculating plane

let γ be the curve define by $r(u) = (u, 0, e^{-1/u^2})$ where $u < 0$ and where $u > 0$, $r(u) = (u, e^{-1/u^2}, 0)$

$r(u) = (0, 0, 0)$ where $u = 0$

Now we have to show that the given curve is of class '∞' with $u = 0$ as an inflexion point

Take $f(u) = e^{-1/u^2}$

To prove : $f^{k}(0) = 0 \;,\; \forall\; k \geq 2$

$$f'(0) = \lim_{u \to 0} \frac{f(u) - f(0)}{u - 0} = \lim_{u \to 0} \frac{e^{-1/u^2}}{u} = \frac{0}{0}$$

$$f''(0) = \lim_{u \to 0} \frac{f'(u) - f(0)}{u - 0} = \lim_{u \to 0} \frac{2/u^3 \; e^{-1/u^2} - 0}{u - 0} = 0$$

Similarly $f'''(0) = 0$

$$\left\{ \because f'''(0) = \lim_{u \to 0} \frac{f''(u) - f(0)}{u - 0} = \lim_{u \to 0} \frac{1}{u} \left\{ \frac{4e^{-1/u^2}}{u^6} - \frac{6e^{-1/u^2}}{u^4} \right\} \right.$$

and hence, when $u \to 0$

$r''(u) = (0, 0, 0)$

$\therefore u = 0$ is the inflexion point

Hence r is the curve of class ∞ with $u = 0$ is an inflexion point

Now we have to find the equation of osculating plane when $u > 0$

$$r(u) = \left(u \,,\, 0 \,,\, e^{-1/u^2} \right)$$

$$\dot{r}(u) = \left(1 \,,\, 0 \,,\, 2/u^3 \; e^{-1/u^2} \right)$$

$$\ddot{r}(u) = \left(0 \,,\, 0 \,,\, \frac{4e^{-1/u^2}}{u^6} - \frac{6e^{-1/u^2}}{u^4} \right)$$

$$\begin{vmatrix} X - x & Y - y & Z - z \\ \dot{x} & \dot{y} & \dot{z} \\ \ddot{x} & \ddot{y} & \ddot{z} \end{vmatrix} = 0$$

$$\begin{vmatrix} x-u & y-0 & z-e^{-1/u^2} \\ 1 & 0 & 2/u^3 \; e^{-1/u^2} \\ 0 & 0 & \frac{4e^{-1/u^2}}{u^6} - \frac{6e^{-1/u^2}}{u^4} \end{vmatrix} = 0$$

$$(x-u)(0-0) - (y-0)\left[\frac{4e^{-1/u^2}}{u^6} - \frac{6e^{-1/u^2}}{u^4}\right] +$$

$$(z-e^{-1/u^2})(0-0) = 0$$

$$\left[\frac{4e^{-1/u^2}}{u^6} - \frac{6e^{-1/u^2}}{u^4}\right] y = 0$$

since $u > 0 \Rightarrow y = 0$

The equation of the osculating plane is $y=0$ when $u>0$

Similarly, the equation of the osculating plane is $z=0$ when $u<0$ and also the osculating plane at $u=0$ is indeterminant.

The above process shows that at the point of inflexion even a curve of class '∞' need not possess an osculating plane.

Formula :-

* The equation of the normal plane is $(R-r)t=0$ where 'R' is the position vector of any point of the plane and $r=r(u)$ be a point on the curve.

* The equation of the normal line at 'p' is $R = r + \lambda n$ where λ is scalar

Binormal :-

The normal at 'p' orthogonal to the osculating plane is called the binormal at 'p'. The unit vector along the binormal is denoted by 'b'

Note :-

The behaviour of t, n, b [t = tangent, n = normal and b = binormal] at a point 'p' on the curve is same as the unit vectors i, j, k along the co-ordinate axis, also we have

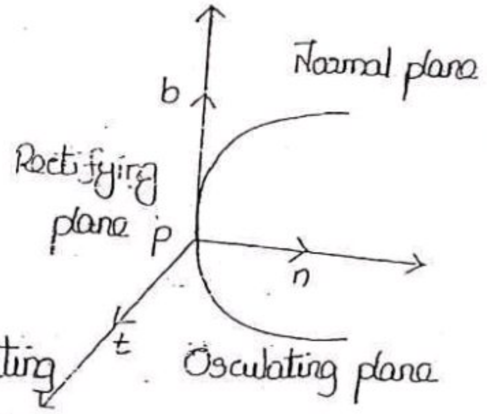
$$b = t \times n \,, \quad t = n \times b \,, \quad n = t \times b$$

$$t \cdot n = n \cdot b = t \cdot b = 0 \quad \text{and}$$

$$t \cdot t = n \cdot n = b \cdot b = 1$$

Rectifying plane :- 2m Uni

The plane containing the tangent and binormal lines is called the rectifying plane.



Result :-

* The equation of the osculating plane is contains 't' and 'n' and its is normal to 'b'. Its equation is given by $(R-r)b=0$

* The equation of the Rectifying plane contains 't' and 'b' and is normal to 'n'. Its equation is $(R-r)n=0$

* The equation of the binormal line at 'p' is $R = r + \mu b$

* The equation of the tangent ~~plane~~ line at 'p' is $R = r + \lambda t$

Find the directions and equations of the tangent, normal, binormal lines and also obtain the normal, rectifying and osculating planes at a point on the circular helix $r = (a\cos s/c\,,\ a\sin s/c\,,\ bs/c)$.

Solution:- Given $r = (a\cos s/c\,,\ a\sin s/c\,,\ bs/c)$

$$\dot{r} = \left(-a/c\ \sin s/c\,,\ a/c\ \cos s/c\,,\ b/c\right) \text{ —— ①}$$

$$\therefore \ddot{r} = \left(-a/c^2\ \cos s/c\,,\ -a/c^2\ \sin s/c\,,\ 0\right) \text{ —— ②}$$

$$\dot{r} \times \ddot{r} = \begin{vmatrix} t & n & b \\ -a/c\ \sin s/c & a/c\ \cos s/c & b/c \\ -a/c^2\ \cos s/c & -a/c^2\ \sin s/c & 0 \end{vmatrix}$$

$$= t\left[0 + ab/c^3\ \sin s/c\right] - n\left[0 + ab/c^3\ \cos s/c\right] + b\left[a^2/c^3\ \sin^2 s/c + a^2/c^3\ \cos^2 s/c\right]$$

$$= \frac{ab}{c^3}\sin s/c\ t - \frac{ab}{c^3}\cos s/c\ n + \frac{a^2}{c^3}\,b \text{ —— ③}$$

Equations ①, ② and ③ gives the directions of the tangent, normal and binormal lines.

The equation of the tangent line is $R = r + \lambda t$ which is given by.

$$\frac{x - a\cos s/c}{-a/c\ \sin s/c} = \frac{y - a\sin s/c}{a/c\ \cos s/c} = \frac{z - bs/c}{b/c}$$

$$\Rightarrow \frac{x - a\cos s/c}{-a\sin s/c} = \frac{y - a\sin s/c}{a\cos s/c} = \frac{z - bs/c}{b}$$

The equation of the normal line is $R = r + \lambda n$

which is given by

$$\frac{x - a\cos s/c}{-a/c^2 \cos s/c} = \frac{y - a\sin s/c}{-a/c^2 \sin s/c} = \frac{z - bs/c}{0} .$$

$$\frac{x - a\cos s/c}{-a \cos s/c} = \frac{y - a\sin s/c}{-a \sin s/c} = \frac{z - bs/c}{0}$$

The equation of the binormal line is $R = r + \lambda b$ ich is given by

$$\frac{x - a\cos s/c}{ab/c^3 \sin s/c} = \frac{y - a\sin s/c}{-ab/c^3 \cos s/c} = \frac{z - bs/c}{a^2/c^3}$$

$$\frac{x - a\cos s/c}{b \sin s/c} = \frac{y - a\sin s/c}{-b \cos s/c} = \frac{z - bs/c}{a}$$

The equation of the normal plane is $(R-r)t = 0$

$x - a\cos s/c\ ,\ y - a\sin s/c\ ,\ z - bs/c \Big)\Big(-a/c \sin s/c\ ,\ a/c \cos s/c\ ,\ b/c \Big) = 0$

$x - a\cos s/c \big) \big(-a/c \sin s/c \big) + \big(y - a\sin s/c \big)$

$\big(a/c \cos s/c \big) + \big(z - bs/c \big) \big(b/c \big) = 0$

$-a/c \sin s/c\ x + a^2/c \sin s/c \cos s/c + a/c \cos s/c\ y -$

$a^2/c \sin s/c \cos s/c + b/c\ z - b^2 s/c^2 = 0$

$-a/c \sin s/c\ x + a/c \cos s/c\ y + b/c\ z - b^2 s/c^2 = 0$

$-ac \sin s/c\ x + ac \cos s/c\ y + bcz - b^2 s = 0$

The equation of the rectifying plane is $(R-r)n = 0$

$x - a\cos s/c\ ,\ y - a\sin s/c\ ,\ z - bs/c \Big)\Big(-a/c^2 \cos s/c\ ,\ -a/c^2 \sin s/c\ ,\ 0 \Big) = 0$

$x - a\cos s/c \big) \big(-a/c^2 \cos s/c \big) +$

$\big(y - a\sin s/c \big) \big(-a/c^2 \sin s/c \big) + \big(z - bs/c \big) (0) = 0$

$$-\frac{a}{c^2}\cos\frac{s}{c}\cdot x+\frac{a^2}{c^2}\cos^2\frac{s}{c}-\frac{a}{c^2}\sin\frac{s}{c}\cdot y+\frac{a^2}{c^2}$$

$$\left[\frac{a^2}{c^2}-\frac{a}{c^2}\cos\frac{s}{c}\cdot x-\frac{a}{c^2}\sin\frac{s}{c}\cdot y=0 \quad \sin^2\frac{s}{c}=0\right.$$

$$-\cos\frac{s}{c}\,x-\sin\frac{s}{c}\,y+a=0$$

The equation of the osculating plane is $[R-r,\dot{r},\ddot{r}]=0$

$$\begin{vmatrix} x-a\cos s/c & y-a\sin s/c & z-bs/c \\ -a/c\sin s/c & a/c\cos s/c & b/c \\ -a/c^2\cos s/c & -a/c^2\sin s/c & 0 \end{vmatrix}=0$$

$$(x-a\cos s/c)(0+ab/c^3\sin s/c)-(y-a\sin s/c)$$

$$(0+ab/c^3\cos s/c)+(z-bs/c)(a^2/c^3\sin^2 s/c+a^2/c^3\cos^2 s/c)=$$

$$\frac{ab}{c^3}\sin s/c\;x-\frac{a^2b}{c^3}\sin s/c\cos s/c$$

$$-\frac{ab}{c^3}\cos s/c\;y+\frac{a^2b}{c^3}\sin s/c\cos s/c+a^2/c^3\,z-\frac{a^2bs}{c^4}=0$$

$$\frac{ab}{c^3}\sin s/c\;x-\frac{ab}{c^3}\cos s/c\;y+a^2/c^3\,z-a^2bs/c^4=0$$

$$b\sin s/c\;x-b\cos s/c\;y+az-abs/c=0$$

(or) $(R-r)\cdot b=0$ என்ற formula -ஐயும் பயன்

பருக்கனம்

Curvature and Torsion :-

Curvature :- • Uni // (Imp)

The arc rate at which the tangent changes direction as 'p' moves along the curve is the curvature of the curve and is denoted by 'k'. where $k=\frac{dt}{ds}=t'$

$$\Rightarrow |k|=|t'|$$

(18) (18)

radius

~~m~~ of curvature :- (P) (2m)

The reciprocal of the kappa [(ie) $1/K$]

led radius of curvature and is denoted by ρ, where $\rho = \frac{1}{K}$.

on :- Torsion :- A-19 2m

) As 'p' moves along a curve the arc rate at which osculating plane turns about the tangent is called Torsion of the curve and is denoted by 'τ'

$$(ie)\ \tau = \left|\frac{db}{ds}\right|$$

(P) 6 N-2009

) 5m $\sigma = \left|\frac{1}{\tau}\right|$ is called a radius of Torsion.

et - Frenet Formula :- M (X) 5m (10m) (2) 10m

10m (X) If (t, n, b) is moving in orthogonal of a unit vectors at a point 'p' on a space curve 'r'

(i) $\frac{dt}{ds} = \kappa n$ [(ie) $t' = \kappa n$]

(ii) $\frac{dn}{ds} = \tau b - \kappa t$ [(ie) $n' = \tau b - \kappa t$] ✓

(iii) $\frac{db}{ds} = -\tau n$ [(ie) $b' = -\tau n$]

of :-

Let us first prove the results (i) and (iii) and ive the second result from them.

(i) To prove : $t' = \kappa n$

We know that, $t \cdot t = 1$

Differentiating w.r.to 's' at a point 'p' of the curve

$$t t' + t' t = 0 \Rightarrow 2 t t' = 0$$

$$\Rightarrow t t' = 0$$

$\therefore$ t and t' are perpendicular [$\perp$]

Since $r' = t \Rightarrow r'' = t'$

As r'' lies in the osculating plane, t' lies in the osculating plane.

$\therefore$ Both t and t' are lies in the osculating plane

since t and t' are perpendicular and lies in the osculating plane, t' is parallel to principal normal.

$\therefore$ t' is proportional to 'n' [(ie) $t' \alpha n$]

By the definition of curvature $|t'| = |k|$

$\therefore t' = \pm nk$

since curvature is positive, $t' = nk$

(iii) To prove : $b' = -\tau n$

We know that $b.b = 1$

differentiating w.r.to 's' at a point 'p' of the curve

$bb' + b'b = 0 \Rightarrow 2bb' = 0 \Rightarrow bb' = 0$

$\therefore$ b and b' are perpendicular [$\perp$]

$\therefore$ b' must lie in the osculating plane

Also $b.t = 0$, differentiating w.r.to 's' at a point 'p' of the curve

$bt' + b't = 0$

$b(n.k) + b't = 0$ {$\because$ By (i)}

$\Rightarrow k(n.b) + b't = 0$ [$\because a(b.c) = (a.b)c$]

$0 + b't = 0$ {$\because n.b = 0$}

$b't = 0$

$\therefore$ b' and t are perpendicular [$\perp$]

since b' lies on the osculating plane and it is perpendicular to 't', ~~and~~ b' must be parallel to the principal normal at 'p' and b' is proportional to 'n' [(ie) $b' \alpha n$]

By the definition of Torsion $|b'| = \tau$

$b' = \pm n\tau$

since Torsion may be positive (or) negative

$b' = -n\tau$

ii) To prove : $n' = \tau b - \kappa t$

We know that, $b \times t = n$

Differentiating w.r.to 's' at a point 'p' of the curve,

$$b \times \frac{dt}{ds} + \frac{db}{ds} \times t = \frac{dn}{ds}$$

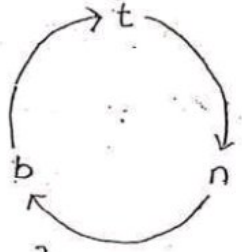
$$b \times t' + b' \times t = n'$$

$$b \times (n \cdot \kappa) + (-n \cdot \tau) \times t = n' \quad \{\because \text{By (i), (iii)}$$

$$(b \times n)\kappa - \tau(n \times t) = n'$$

$$\kappa(-t) - \tau(-b) = n'$$

$$n' = \tau b - \kappa t$$



{∵ $t \times n = b$, $n \times b = t$, $b \times t = n$

But $n \times t = -b$, $b \times n = -t$, $t \times b = -n$ }

Note :-

The vector $t' = r''$ is some times called as a curvature vector.

A-19 2M 5m 5M (3) ✓

Theorem :-

A necessary and sufficient condition that a curve be a straight line is that $\kappa = 0$ at all the points.

Proof :- Necessary part :- Assume that the given curve is a straight line.

To prove that $\kappa = 0$

The equation of the straight line is $r(s) = as + b$, where 'a' and 'b' are vector constants.

Differentiating the above equation w.r.to 's', we get

$$r' = a$$

since $t = r'$ then $t = r' = a$

Again differentiating the above equation w.r.to 's'

$$t' = r'' = 0$$

we know that $t' = kn \Rightarrow t' = 0 \Rightarrow kn = 0$

$$\Rightarrow k = 0$$

sufficient part :- Assume that $k = 0$

To prove that the given curve is a straight line

$$k = 0 \Rightarrow t' = 0 \quad \{\because t' = kn\}$$

Integrating w.r.to 's' we get $t = a$, where 'a' is constant

we know that $t = r' \Rightarrow r' = a$

Integrating w.r.to 's' we get $r = as + b$, where 'a' and 'b' are constant

This is the equation of a straight line.

Theorem :- 5m Q

Let γ be a curve for which 'b' varies differentiably with arc length. Then a necessary and sufficient condition that γ be a plane curve is that $\underline{\tau = 0}$ at all points.

proof :-

Necessary part :- Assume that the curve is a plane curve.

To prove that $\tau = 0$

since the given curve is a plane curve, It must lie in a plane.

since 'b' is normal to the osculating plane and the curve lies in a plane, the curve must lie in all the osculating plane.

Also we know that 't' and 'n' are also lie is that plane Hence $b = t \times n$ must be a constant.

(ie) $b = a$, where a is constant

$\therefore b' = 0$

We know that, $b' = -\tau n \Rightarrow -\tau n = 0$

$\Rightarrow \tau = 0$

Sufficient part :-

Assume that $\tau = 0$, To prove that the curve is a plane curve

We know that $b' = -\tau n \Rightarrow b' = 0$ {By our assumption}

Integrating, $b =$ constant

The curve equation is $r = r(s)$

we have $(rb)' = rb' + r'b = rb' + t b$

$= 0$ $\{\because r' = t\}$

Integrating $rb = a$, where 'a' is constant

$\therefore$ The given curve is a plane curve $\{\because b' = 0$ and $t \cdot b = 0\}$

Theorem :-

unit 5m 5 ✓

Imp 5. If $r = r(s)$ is the position vector of the point 'p' with arc length as a parameter on a curve then ~~true~~ prove the following results (i) $k^2 = r'' \cdot r''$

(ii) $\tau = \dfrac{[r', r'', r''']}{r'' \cdot r''}$

(iii) $k^2 \tau = [r', r'', r''']$

Proof :-

(i) Let $t = r' \Rightarrow r'' = t' = kn$

$r'' \cdot r'' = kn \cdot kn = k^2\, n \cdot n = k^2$ —— ①

(ii) $r' \times r'' = t \times (k.n) = k.(t \times n)$ $\{\because n.n = 1\}$

$= kb$

differentiate w.r.to 's',

$$r' \times r''' + r'' \times r'' = kb' + k'b$$

$$r' \times r''' + (0) = k(-\tau n) + k'b \quad \{\because a \times a = 0$$

$$r' \times r''' = k'b - k\tau n$$

Taking dot product by r'' on bothsides.

$$(r' \times r''') \cdot r'' = (k'b - k\tau n) \cdot r''$$

$$r' \cdot (r''' \times r'') = (k'b - k\tau n) \cdot kn \quad \{\because a \times b = -b \times a$$

$$-r' \cdot (r'' \times r''') = kk' b \cdot n - k^2 \tau \, n \cdot n \quad b \cdot n = 0, \; n \cdot n = 1\}$$

$$-[r', r'', r'''] = 0 - k^2 \tau$$

$$\tau = \frac{[r', r'', r''']}{k^2} \text{ —— } ②$$

$$\tau = \frac{[r', r'', r''']}{r'' \, r''} \text{ —— } ③ \quad \{\because \text{By } ①$$

(iii) From ②, $k^2 \tau = [r', r'', r''']$

Theorem :-

If $r = r(u)$ is the equation of a curve with parameter 'u' then (i) $k = \frac{|\dot r \times \ddot r|}{|\dot r|^3}$

(ii) $\tau = \frac{[\dot r, \ddot r, \dddot r]}{|\dot r \times \ddot r|^2}$

proof :- (i) $\dot r = \frac{dr}{du} = \frac{dr}{ds} \frac{ds}{du} = dr/ds \; \dot s$

$$\dot r = r' \dot s = t \dot s$$

$$|\dot r| = |t \dot s| = \dot s \text{ —— } ① \quad \{\because |t| = 1$$

$\ddot r = d^2r/du^2 = \frac{d}{du}(dr/du)$ $\frac{d}{du}$ [illegible]

$$= \frac{dt}{du}\dot{s} + t\frac{d\dot{s}}{du} = \frac{dt}{ds}\frac{ds}{du}\dot{s} + t\ddot{s}$$

$$= t'\dot{s}\dot{s} + t\ddot{s} = kn\dot{s}^2 + t\ddot{s} \quad \{\because kn = t'\} \quad \text{└②}$$

$$\dot{r} \times \ddot{r} = t\dot{s} \times (kn\dot{s}^2 + t\ddot{s}) = (t\dot{s} \times kn\dot{s}^2) + (t\dot{s} \times t\ddot{s})$$

$$= k\dot{s}^3(t \times n) + \dot{s}\ddot{s}(t \times t)$$

$$= k\dot{s}^3(b) + 0 = k\dot{s}^3 b \quad \{\because t \times t = 0\}$$

$$|\dot{r} \times \ddot{r}| = |k\dot{s}^3 b| = k\dot{s}^3 \quad \text{└③} \quad \{\because |b| = 1\}$$

$$k = |\dot{r} \times \ddot{r}| / \dot{s}^3$$

$$k = \frac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3} \quad \text{using ①}$$

(ii) To prove $T = \dfrac{[\dot{r}, \ddot{r}, \dddot{r}]}{|\dot{r} \times \ddot{r}|^2}$

using ③, $\dot{r} \times \ddot{r} = \dot{s}^3 kb$

differentiate w.r.to 'u' we get

$$\dot{r} \times \dddot{r} + \ddot{r} \times \ddot{r} = b\frac{d}{du}(\dot{s}^3 k) + k\dot{s}^3\frac{db}{du}$$

$$\dot{r} \times \dddot{r} + (0) = b\frac{d}{du}(k\dot{s}^3) + k\dot{s}^3\frac{db}{ds}\frac{ds}{du}$$

$$\dot{r} \times \dddot{r} = b\frac{d}{du}(k\dot{s}^3) + k\dot{s}^3(-\tau n)\cdot\dot{s}$$

$$\dot{r} \times \dddot{r} = b\frac{d}{du}(k\dot{s}^3) - k\dot{s}^4\tau n \quad \left\{\because \frac{db}{ds} = b' = -\tau n\right\}$$

Taking dot product with $\ddot{r}$ on bothsides

$$(\dot{r} \times \ddot{r}) \cdot \ddot{r} = \left(b\frac{d}{du}(k\dot{s}^3) - k\dot{s}^4\tau n\right)\cdot\ddot{r}$$

$$= \left(b\frac{d}{du}(k\dot{s}^3) - k\dot{s}^4\tau n\right)(kn\dot{s}^2 + \ddot{s}t)$$

using ③

$$= b\frac{d}{du}(k\dot{s}^3)(kn\dot{s}^2) + b\frac{d}{du}(k\dot{s}^3)(\ddot{s}t)$$

$$- (k\dot{s}^4 \tau n)(kn\dot{s}^2) - (k\dot{s}^4 \tau n)(\ddot{s}t)$$

$$= \frac{d}{du}(k\dot{s}^3)(k\dot{s}^2)(b\cdot n) + \frac{d}{du}(k\dot{s}^3)\ddot{s}(b\cdot t) -$$

$$k^2\dot{s}^6\tau(n.n) - k\dot{s}^4\ddot{s}\tau(n.t)$$

$(\dot{r} \times \ddot{r})\cdot\dddot{r} = -k^2\dot{s}^6\tau$ $\{\because n.n = 1$ and $b.n = b.t = n.t = 0\}$

$$\dot{r}\cdot(\dddot{r} \times \ddot{r}) = -k^2\dot{s}^6\tau$$

$-[\dot{r}, \ddot{r}, \dddot{r}] = -k^2\dot{s}^6\tau$ $\{\because$ order மாற்று செய்வதால் '-' வரும்$\}$

$$\tau = [\dot{r}, \ddot{r}, \dddot{r}] / k^2\dot{s}^6$$

$\tau = \dfrac{[\dot{r}, \ddot{r}, \dddot{r}]}{|\dot{r} \times \ddot{r}|^2}$ $\{\because |\dot{r} \times \ddot{r}| = k\dot{s}^3$ by above case$\}$

2) Calculate the curvature and Torsion of the curbe given by $r = (u, u^2, u^3)$ A-19 5M ⑥

Solution :- we know that, $k = \dfrac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3}$, $\tau = \dfrac{[\dot{r}. \ddot{r}. \dddot{r}]}{|\dot{r} \times \ddot{r}|^2}$

Given $r = (u, u^2, u^3)$

$$\dot{r} = (1, 2u, 3u^2)$$

$$\ddot{r} = (0, 2, 6u)$$

$$\dddot{r} = (0, 0, 6)$$

$$\dot{r} \times \ddot{r} = \begin{vmatrix} t & n & b \\ 1 & 2u & 3u^2 \\ 0 & 2 & 6u \end{vmatrix} = t[12u^2 - 6u^2] - n[6u - 0] + b[2 - 0]$$

$$= 6u^2t - 6un + 2b$$

$|\dot{r} \times \ddot{r}| = \sqrt{36u^4 + 36u^2 + 4}$ $\{\because |t| = |b| = |n| = 1\}$

$$= 2\sqrt{9u^4 + 9u^2 + 1}$$

$$[\dot{r}, \ddot{r}, \dddot{r}] = \begin{vmatrix} 1 & 2u & 3u^2 \\ 0 & 2 & 6u \\ 0 & 0 & 6 \end{vmatrix} = 12$$

$$|\dot{r}| = \sqrt{1+4u^2+9u^4}$$

$$k = \frac{|\dot{r} \times \ddot{r}|}{|\dot{r}|^3} = \frac{2\sqrt{9u^4+9u^2+1}}{\left(\sqrt{1+4u^2+9u^4}\right)^3} = \frac{2\left(9u^4+9u^2+1\right)^{1/2}}{\left(1+4u^2+9u^4\right)^{3/2}}$$

$$\tau = \frac{[\dot{r}, \ddot{r}, \dddot{r}]}{|\dot{r} \times \ddot{r}|^2} = \frac{12}{4(9u^4+9u^2+1)} = \frac{3}{9u^4+9u^2+1}$$

2) Calculate the curvature and Torsion of the cube given by $r = \left(a\cos s/c, \ a\sin s/c, \ bs/c\right)$ where $c^2 = a^2+b^2$

Solution :- We know that, $k^2 = r'' \cdot r''$, $\tau = \dfrac{[r', r'', r''']}{r'' \cdot r''}$

{∵ parameter 's' ல் இருந்தால் இந்த formula use பண்ணனும், 'u' - வில் இருந்தால் previous உள்ள formula use பண்ணனும் }

Given, $r = \left(a\cos s/c, \ a\sin s/c, \ bs/c\right)$ where $c^2 = a^2+b^2$

$$r' = \left(-a/c \sin s/c, \ a/c \cos s/c, \ b/c\right)$$

$$r'' = \left(-a/c^2 \cos s/c, \ -a/c^2 \sin s/c, \ 0\right)$$

$$r''' = \left(a/c^3 \sin s/c, \ -a/c^3 \cos s/c, \ 0\right)$$

$$r'' \cdot r'' = \left(-a/c^2 \cos s/c, \ -a/c^2 \sin s/c, \ 0\right) \cdot \left(-a/c^2 \cos s/c, \ -a/c^2 \sin s/c, \ 0\right)$$

$$= a^2/c^4 \cos^2 s/c + a^2/c^4 \sin^2 s/c + 0$$

$$= a^2/c^4$$

$$k^2 = r'' \cdot r'' = a^2/c^4 \Rightarrow k = a/c^2$$

$$[r', r'', r'''] = \begin{vmatrix} -a/_c \sin s/_c & a/_c \cos s/_c & b/_c \\ -a/_{c^2} \cos s/_c & -a/_{c^2} \sin s/_c & 0 \\ a/_{c^3} \sin s/_c & -a/_{c^3} \cos s/_c & 0 \end{vmatrix}$$

$$= -a/_c \sin s/_c [0-0] - a/_c \cos s/_c [0-0] + b/_c \left[a^2/_{c^5} \cos^2 s/_c + a^2/_{c^5} \sin^2 s/_c \right]$$

$$= b/_c \left[a^2/_{c^5} \right] = a^2 b/_{c^6}$$

$$\tau = \frac{[r', r'', r''']}{r'' \cdot r''} = \frac{a^2 b/c^6}{a^2/c^4} = \frac{a^2 b}{c^6} \frac{c^4}{a^2} = \frac{b}{c^2}$$

Theorem :- ⑦ ✓

⑧ A necessary and sufficient condition that the curve be a plane is $[\dot{r}, \ddot{r}, \dddot{r}] = 0$

proof :- Necessary part :- Assume that a curve is a plane curve

To prove that $[\dot{r}, \ddot{r}, \dddot{r}] = 0$

since the curve is a plane curve, $\tau = 0$ at all the points [By a theorem]

Also we know that, $[\dot{r}, \ddot{r}, \dddot{r}] = k^2 \dot{s}^6 \tau$

$\therefore [\dot{r}, \ddot{r}, \dddot{r}] = 0 \quad \{\because \text{By } \tau = 0\}$

Sufficient part :- Assume that $[\dot{r}, \ddot{r}, \dddot{r}] = 0$

To prove that the curve is a plane curve

since $[\dot{r}, \ddot{r}, \dddot{r}] = 0$ and $[\dot{r}, \ddot{r}, \dddot{r}] = k^2 \dot{s}^6 \tau$

$\Rightarrow k^2 \dot{s}^6 \tau = 0$

Suppose $\tau = 0$ at all points of the curve Then by a theorem that curve must be a plane curve

Suppose $\tau \neq 0$, at some points of the curve at which $k = 0$

Since $k = 0$ in arc of the curve, that should be a straight line, also it may be a plane curve

This is a contradiction to our assumption that $\tau \neq 0$.

$\therefore$ In this case also $\tau = 0$

$\therefore$ $\tau = 0$ at all the points of the curve, Hence it is a plane curve.

Result :-

If a curve is given interms of a general parameter 'u' then the equation of osculating plane corresponding to $[R-r, \dot{r}, \ddot{r}] = 0$

proof :-

Given $r = r(s)$, differentiate w.r.to 's'

$$r' = \frac{dr}{ds} = \frac{dr}{du} \cdot \frac{du}{ds} = \dot{r}\,\frac{1}{\dot{s}} = \dot{r}/\dot{s}$$

$$r'' = \frac{d}{ds}\left(\frac{dr}{ds}\right) = \frac{d}{ds}\left(\dot{r}/\dot{s}\right) = \frac{d}{du}\left(\dot{r}/\dot{s}\right)\frac{du}{ds}$$

$$= \frac{\ddot{r}\dot{s} - \dot{r}\ddot{s}}{\dot{s}^2} \times \frac{1}{\dot{s}} = \frac{\ddot{r}\dot{s} - \dot{r}\ddot{s}}{\dot{s}^3}$$

we know that, $[R-r, r', r'''] = 0$

Substitute the values of r', r'' in the above equation,

$$\left[R-r,\ \dot{r}/\dot{s},\ \frac{\ddot{r}\dot{s} - \dot{r}\ddot{s}}{\dot{s}^3}\right] = 0$$

$$(R-r)\cdot\left[\frac{\dot{r}}{\dot{s}} \times \left(\frac{\ddot{r}\dot{s} - \dot{r}\ddot{s}}{\dot{s}^3}\right)\right] = 0$$

$$(R-r)\cdot\left[\frac{\dot{r}}{\dot{s}}\times\frac{\ddot{r}\dot{s}}{\dot{s}^3}-\frac{\dot{r}}{\dot{s}}\times\frac{\dot{r}\ddot{s}}{\dot{s}^3}\right]=0$$

$$(R-r)\cdot\left[\dot{r}/\dot{s}\times\ddot{r}/\dot{s}^2\right]=0 \quad \left\{\because \frac{\dot{r}}{\dot{s}}\times\frac{\dot{r}\ddot{s}}{\dot{s}^3}=0\right\}$$

$$\left[R-r,\frac{\dot{r}}{\dot{s}},\frac{\ddot{r}}{\dot{s}^2}\right]=0$$

$$\frac{1}{\dot{s}^3}\left[R-r,\dot{r},\ddot{r}\right]=0$$

$$\Rightarrow \left[R-r,\dot{r},\ddot{r}\right]=0$$

sm

Theorem :- sm

The length of the common perpendicular 'd' between the tangents at two neighbouring points with angular distance 's' between them approximately $d = KTs^3/_{12}$

proof :-

Let p and Q be the two neighbouring points of the curve with parameters '0' and 's' respectively. The unit tangent vectors at 'p' and 'Q' are $r'(0)$ and $r'(s)$ respectively.

So the unit vector of the common perpendicular is along $r'(0)\times r'(s)$

The projection of the vector $r(s)-r(0)$ in this direction is equal to 'd'

$$d=\frac{\left[r(s)-r(0),\ r'(s),\ r'(0)\right]}{|r'(s)\times r'(0)|}$$

since $r(0)=0$,

$$d=\frac{\left[r(s),\ r'(s),\ r'(0)\right]}{|r'(s)\times r'(0)|} \qquad \frac{r(s)\cdot(r'(s)\times r'(0))}{|r'(s)\times r'(0)|}$$

By Taylor's theorem

$$\lambda\, kk'(n.b) - \lambda^{8}k^{2}T\,(n.n) + \lambda^{3}\lambda' k'(t.b) - \lambda^{5}\lambda' k T\,(t.n)$$

$$\Delta h\ \Delta A = -\lambda^{6}k^{2}T$$

$$\boxed{T = \frac{-\Delta h . \Delta A}{\lambda^{6}k^{2}}} \text{ — (11)}$$

$\{\because n.n = 1$

$n.b = t.b = t.n = 0$

Equations (9) and (11) gives the require curvature and torsion of the given curve.

10M

(X) Imp Uni

Obtain the curvature and torsion of the curve given as the intersection of two surfaces $ax^2+by^2+cz^2=1$ and $a'x^2+b'y^2+c'z^2=1$.

Solution :- Let $f = ax^2+by^2+cz^2-1$

$g = a'x^2+b'y^2+c'z^2-1$ } —— (*)

$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$

$\nabla f = (2ax, 2by, 2cz)$

$\nabla g = (2a'x, 2b'y, 2c'z)$ $\nabla g = \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z}\right)$

∇f and ∇g both are normal to the surfaces $f=0$ and $g=0$ respectively, then

$$\nabla f \times \nabla g = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2ax & 2by & 2cz \\ 2a'x & 2b'y & 2c'z \end{vmatrix}$$

$= \vec{i}\,(4bc'yz - 4b'cyz) - \vec{j}\,(4ac'xz - 4a'cxz) + \vec{k}\,(4ab'xy - 4a'bxy)$

$= \vec{i}\ 4yz\,(bc' - cb') - \vec{j}\ 4xz\,(ac' - ca') + \vec{k}\ 4xy\,(ab' - ba')$

$\nabla f \times \nabla g = xyz\left(\frac{A}{x}, \frac{B}{y}, \frac{C}{z}\right)$ where $A = 4(bc' - cb')$,

$B = -4(ac' - ca') = 4(ca' - ac')$, $C = 4(ab' - ba')$ —(*)(*)

$\lambda t = \nabla f \times \nabla g = \left(\frac{A}{x}, \frac{B}{y}, \frac{C}{z}\right)$ — (1) $\{\because$ consider the triplet only (ie) omit $xyz\}$

$|\lambda t|^2 = \frac{A^2}{x^2} + \frac{B^2}{y^2} + \frac{C^2}{z^2}$ —— (2)

$\left(\frac{A}{x}, \frac{B}{y}, \frac{C}{z}\right) = \lambda r'$

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$$\left(\frac{A}{x}, \frac{B}{y}, \frac{C}{z}\right) = \lambda\left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)$$

$$\lambda \frac{dx}{ds} = \frac{A}{x}, \quad \lambda \frac{dy}{ds} = \frac{B}{y}, \quad \lambda \frac{dz}{ds} = \frac{C}{z} \quad \text{—— ③}$$

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds}$$

$$\lambda \frac{df}{ds} = \lambda \frac{\partial f}{\partial x}\frac{dx}{ds} + \lambda \frac{\partial f}{\partial y}\frac{dy}{ds} + \lambda \frac{\partial f}{\partial z}\frac{dz}{ds}$$

$$\lambda \frac{df}{ds} = \frac{A}{x}\frac{\partial f}{\partial x} + \frac{B}{y}\frac{\partial f}{\partial y} + \frac{C}{z}\frac{\partial f}{\partial z}, \text{ using ③}$$

$$\text{and} \quad \lambda \frac{d}{ds} = \frac{A}{x}\frac{\partial}{\partial x} + \frac{B}{y}\frac{\partial}{\partial y} + \frac{C}{z}\frac{\partial}{\partial z} \quad \text{—— ④}$$

$$\lambda \frac{d}{ds}(\lambda t) = \left(\frac{A}{x}\frac{\partial}{\partial x} + \frac{B}{y}\frac{\partial}{\partial y} + \frac{C}{z}\frac{\partial}{\partial z}\right)\cdot\left(\frac{A}{x}, \frac{B}{y}, \frac{C}{z}\right)$$

$$\lambda[\lambda t' + \lambda' t] = \left(\frac{A}{x}\frac{\partial}{\partial x}\left(\frac{A}{x}\right), \frac{B}{y}\frac{\partial}{\partial y}\left(\frac{B}{y}\right), \frac{C}{z}\frac{\partial}{\partial z}\left(\frac{C}{z}\right)\right)$$

$$= \left(\frac{A}{x}\left(\frac{-A}{x^2}\right), \frac{B}{y}\left(\frac{-B}{y^2}\right), \frac{C}{z}\left(\frac{-C}{z^2}\right)\right)$$

$$\lambda^2 t' + \lambda\lambda' t = \left(-A^2/x^3, -B^2/y^3, -C^2/z^3\right) \quad \text{—— ⑤}$$

Taking cross product of equation ① with equation ⑤

$$\lambda t \times (\lambda^2 \underline{\underline{t'}} + \lambda\lambda' t) = \left(\frac{A}{x}, \frac{B}{y}, \frac{C}{z}\right) \times \left(\frac{-A^2}{x^3}, \frac{-B^2}{y^3}, \frac{-C^2}{z^3}\right)$$

$t' = kn$

$$\lambda t \times [\lambda^2 kn + \lambda\lambda' t] = \left(\frac{A}{x}, \frac{B}{y}, \frac{C}{z}\right) \times \left(\frac{-A^2}{x^3}, \frac{-B^2}{y^3}, \frac{-C^2}{z^3}\right)$$

$$\lambda^3 k(t \times n) + \lambda^2\lambda'(t \times t) = \left[\frac{BC}{y^3 z^3}(Bz^2 - Cy^2), \frac{CA}{z^3 x^3}(Cx^2 - Az^2), \frac{AB}{x^3 y^3}(Ay^2 - Bx^2)\right]$$

∴ Expand R.H.S of above equation by
$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A/x & B/y & C/z \\ -\frac{A^2}{x^3} & -\frac{B^2}{y^3} & -\frac{C^2}{z^3} \end{vmatrix}$$

$$\lambda^3 k\, b = \left[\frac{BC}{y^3 z^3}(Bz^2 - Cy^2), \frac{CA}{z^3 x^3}(Cx^2 - Az^2), \frac{AB}{x^3 y^3} (Ay^2 - Bx^2) \right]$$ $\left\{ \because t \times t = 0,\ t \times n = b \right.$

But $Bz^2 - Cy^2 = (ca' - c'a)z^2 - (ab' - ba')y^2$ $\left[\because \right.$ using ⑦ and ④⑤ omit constant

$= a'(cz^2 + by^2) - a(c'z^2 + b'y^2)$

$= a'(1 - ax^2) - a(1 - a'x^2)$

$= a' - aa'x^2 - a + aa'x^2 = a' - a$

similarly, $Cx^2 - Az^2 = b' - b$, $Ay^2 - Bx^2 = c' - c$

$$\lambda^3 kb = \left(\frac{BC}{y^3 z^3}(a' - a), \frac{CA}{z^3 x^3}(b' - b), \frac{AB}{x^3 y^3}(c' - c) \right)$$

$$= \frac{ABC}{x^3 y^3 z^3} \left[\frac{x^3}{A}(a' - a), \frac{y^3}{B}(b' - b), \frac{z^3}{C}(c' - c) \right] \quad ⑥$$

squaring on bothsides,

$$\lambda^6 k^2 = \frac{A^2 B^2 C^2}{x^6 y^6 z^6} \sum \frac{x^6}{A^2}(a' - a)^2 \quad — ⑦$$ $\left\{ \because b^2 = b \cdot b = 1 \right.$

$$k^2 = \frac{A^2 B^2 C^2}{x^6 y^6 z^6} \; \frac{\sum x^6/A^2 \, (a' - a)^2}{\left[\sum A^2/x^2 \right]^3} \quad — ⑧$$ $\left\{ \because \right.$ using ② $\lambda^6 = (\lambda^2)^3$

⑥ $\Rightarrow$ $$\frac{\lambda^3 kb\, x^3 y^3 z^3}{ABC} = \left(\frac{x^3}{A}(a' - a), \frac{y^3}{B}(b' - b), \frac{z^3}{C}(c' - c) \right)$$

$$\mu \cdot b = \left(\frac{x^3}{A}(a' - a), \frac{y^3}{B}(b' - b), \frac{z^3}{C}(c' - c) \right)$$

where $\mu = \dfrac{\lambda^3 k x^3 y^3 z^3}{ABC}$ —— ⑨

using (4) and ⑨ we get.

$$\therefore \frac{d}{dn}(\mu b) = \left(\frac{A}{x}\frac{\partial}{\partial x} + \frac{B}{y}\frac{\partial}{\partial y} + \frac{C}{z}\frac{\partial}{\partial z} \right) \left(\frac{x^3}{A}(a' - a), \right.$$

$$, \frac{y^3}{B}(b'-b), \frac{z^3}{C}(c'-c)\Big)$$

$$\lambda[\mu' b + b'\mu] = \Big(\frac{A}{x}\frac{(a'-a)}{A}\frac{\partial}{\partial x}(x^3), \frac{B}{y}\frac{(b'-b)}{B}\frac{\partial}{\partial y}(y^3)$$

$$, \frac{C}{z}\frac{(c'-c)}{C}\frac{\partial}{\partial z}(z^3)\Big)$$

$$\lambda\mu' b + \lambda\mu(-\tau n) = \Big(\frac{a'-a}{x}3x^2, \frac{b'-b}{y}3y^2, \frac{c'-c}{z}3z^2\Big)$$

$$\lambda\mu' b - \lambda\mu\tau n = \big(3x(a'-a), 3y(b'-b), 3z(c'-c)\big) \quad ⑩$$

Taking dot product between ⑤ and ⑩

$$(\lambda^2 t' + \lambda\lambda' t)\cdot(\lambda\mu' b - \lambda\mu\tau n) = \Big(\frac{-A^2}{x^3}, \frac{-B^2}{y^3}, \frac{-C^2}{z^3}\Big)$$

$$\cdot\big(3x(a'-a), 3y(b'-b), 3z(c'-c)\big)$$

$$(\lambda^2 \kappa n + \lambda\lambda' t)\cdot(\lambda\mu' b - \lambda\mu\tau n) = \Big(\frac{-A^2}{x^3}, \frac{-B^2}{y^3}, \frac{-C^2}{z^3}\Big)$$

$$\cdot\big(3x(a'-a), 3y(b'-b), 3z(c'-c)\big)$$

$$\lambda^3\kappa\mu'(n\cdot b) - \lambda^3\mu\tau\kappa(n\cdot n) + \lambda^2\lambda'\mu'(t\cdot b) - \lambda^2\lambda'\mu\tau(t\cdot n)$$

$$= \Big(\frac{-3A^2x(a'-a)}{x^3} + \frac{-3B^2y(b'-b)}{y^3} + \frac{-3C^2z(c'-c)}{z^3}\Big)$$

$$= -\Big[\frac{3A^2}{x^2}(a'-a) + \frac{3B^2}{y^2}(b'-b) + \frac{3C^2}{z^2}(c'-c)\Big]$$

$$-\lambda^3\mu\tau\kappa = -3\sum A^2/x^2\,(a'-a) \qquad \Big\{\because n\cdot b = t\cdot b = t\cdot n = 0,\ n\cdot n = 1$$

$$\mu = \frac{3}{\lambda^3\tau\kappa}\sum\frac{A^2}{x^2}(a'-a) \longrightarrow ⑪$$

From ⑨ and ⑪

$$\frac{\lambda^3\kappa x^3y^3z^3}{ABC} = \frac{3}{\lambda^3\tau\kappa}\sum\frac{A^2}{x^2}(a'-a)$$

$$\lambda^6\kappa^2\tau = \frac{3ABC}{x^3y^3z^3}\sum A^2/x^2\,(a'-a)$$

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using (1), $\frac{1}{\left[\frac{ABC}{x^6y^6z^6}\sum\frac{x}{A^2}(a'-a)^2\right]} = \frac{3\,ABC}{x^3y^3z^3}\sum\frac{A}{x^2}(a'-a)$

$$T = \frac{3x^3y^3z^3}{ABC}\;\frac{\sum A^2/x^2\,(a'-a)}{\sum x^6/A^2\,(a'-a)^2} \quad \text{——} ⑫$$

From equations ⑧ and ⑫ is the curvature and torsion of the given surfaces.

Contact between curves and surfaces :-

Let γ be a curve given by the equation $r = \{f(u), g(u), h(u)\}$ and let 's' be a surface given by $F(x,y,z) = 0$, where the function 'F' has a sufficiently high class. Then the parameters of the points of 'γ' which also lie on 's' are zeros of the function
$F(u) = F\{f(u), g(u), h(u)\}$

If u_0 is a zero, then the function $F(u)$ may be expressed by Taylor's theorem in the form

$$F(u) = \xi F'(u_0) + \frac{\xi^2}{2!}F''(u_0) + \cdots + \frac{\xi^n}{n!}F^{(n)}(u_0) + O(\xi^{n+1}) \text{——} ①$$

where $\xi = u - u_0$

If $F'(u_0) \neq 0$ then u_0 is a simple zero of $F(u)$ and in this case γ and s have a simple intersection at $r(u_0)$

If $F'(u_0) = 0$ but $F''(u_0) \neq 0$ then γ and s have two point contact. If $F'(u_0) = F''(u_0) = 0$ but $F'''(u_0) \neq 0$ then γ and s have three point contact

In general, If $F'(u_0) = F''(u_0) = \cdots = F^{(n-1)}(u_0) = 0$ and $F^{(n)}(u_0) \neq 0$ then γ and s have 'n' point of contact at $r(u_0)$

Remark :-

These conditions remain invariant over a change of parameter.

Example :- 2M, 5M

Show that the osculating plane at 'p' has three point contact in general, with the curve at 'p'

proof :- let 'p' be any point on the curve and let 's' be the arc length of the curve.

We know that the equation of the osculating plane is $[r(s)-r(0), r'(0), r''(0)] = 0$ where $r''(0) \neq 0$

$$\text{let } F(s) = [r(s)-r(0), r'(0), r''(0)]$$

Using Taylor's series,

$$r(s) = r(0) + \frac{s}{1!} r'(0) + \frac{s^2}{2!} r''(0) + \frac{s^3}{3!} r'''(0) + O(s^4)$$

Neglecting the powers of 's' more than 3

$$F(s) = \left[r(0) + \frac{s}{1!} r'(0) + \frac{s^2}{2!} r''(0) + \frac{s^3}{3!} r'''(0) - r(0), r'(0), r''(0)\right]$$

$$= \left[\frac{s^3}{6} r'''(0), r'(0), r''(0)\right]$$

{∵ Take only higher power (ie) $\frac{s^3}{3!} r'''(0)$}

$$= \frac{s^3}{6} [r'''(0), r'(0), r''(0)]$$

$$= s^3/6 \, [r'(0), r''(0), r'''(0)]$$

{∵ $[r''', r', r'']$
$= (-)[r', r''', r'']$
$= (-)(-)[r', r'', r''']$
order மாற்றும் போது '-' வரும் }

{∵ By known result $[r'(0), r''(0), r'''(0)] = K^2 T$ }

$$F(s) = K^2 T \, s^3/6$$

$$F'(s) = K^2 T \, 3s^2/6 = K^2 T \, s^2/2$$

$$F''(s) = K^2 T \, 2s/2 = K^2 T s$$

$$F'''(0) = K^2 T$$

As $s \to 0$, $f(0), f'(0) = f''(0) = 0$ & $f'''(0) \neq 0$.
Hence the curve and osculating plane ~~is~~ has three point contact

Note :-

If $\kappa = 0$ (or) $\tau = 0$ at 'p'. then the plane has atleast four point contact with the curve

Osculating circle :-

The osculating circle at a point 'p' on a curve is the circle which has three point contact with the curve at 'p' (ie) $c - r = \rho n$

Radius of curvature :-

The radius of the osculating circle is $|\rho| = |\kappa^{-1}|$, ρ is called the radius of curvature of the curve at 'p'. Note that ρ may be negative

Centre of curvature :-

The centre of curvature is the centre of the osculating circle, and its position vector is given by $c = r + \rho n$

Radius of Torsion :-

σ is called the radius of torsion it has no simple geometrical significance analogous to the radius of curvature [where $\sigma = \tau^{-1}$]

Osculating sphere :-

The osculating sphere at a point 'p' on a curve is the sphere which has four point contact with the curve at 'p' If 'c' is its centre and 'R' its radius. the equation of the sphere is $(c - r)^2 = R^2$

uni

centre of spherical curvature :-

The centre of the osculating sphere is called the centre of spherical curvature, & its position vector is given by $c = r + \rho n + \sigma \rho' b$

The radius of spherical curvature is given by $R = (\rho^2 + \sigma^2 \rho'^2)^{1/2}$

To find the centre of the osculating sphere :-

If 'c' is its centre and 'R' is its radius then the equation of the sphere is $(r-c)^2 = R^2$ —①

$$F(u) = (r-c)^2 - R^2$$

The condition for four point of contact are $F = F' = F'' = F''' = 0$. These conditions provides $(r-c)t = 0$ —②

Differentiate equation ③ we get,

$$(r-c)t' + t(r') = 0 \qquad \{\because t' = kn$$

$$(r-c)kn + t \cdot t = 0 \qquad r' = t$$

$$(r-c)kn + 1 = 0 \qquad t \cdot t = 1$$

$$(r-c)\frac{n}{\rho} = -1 \qquad k = \frac{1}{\rho}$$

$$(r-c)n = -\rho \text{ —③}$$

Again differentiate equation ③ we get,

$$(r-c)n' + nr' = -\rho' \qquad \{\because n' = \tau b - kt$$

$$(r-c)(\tau b - kt) + n \cdot t = -\rho' \qquad r' = t$$

$$(r-c)\tau b - kt(r-c) + n \cdot t = -\rho' \qquad n \cdot t = 0$$

By ②, $t(r-c) = 0$

$$(r-c)\tau b = -\rho'$$

$$(r-c)b = -\frac{\rho'}{\tau} = -\rho'\sigma \text{ —④} \qquad \sigma = \frac{1}{\tau}$$

eqn ② shows that $(r-c)$ is perpendicu

t' (ie) $(r-c)$ lies in the normal plane at 'p'

$\therefore$ We can express $(r-c)$ as a linear combinat

of n and b.

(ie) $r-c = \lambda n + \mu b$ where λ and μ are scalars

so let $\lambda = -\rho$, $\mu = -\sigma\rho'$

$$r-c = -\rho n - \sigma\rho' b$$

$$-c = -\rho n - \sigma\rho' b - r$$

$$c = r + \rho n + \sigma\rho' b \quad \text{——} ⑤$$

The centre of the osculating sphere is called the centre of spherical curvature.

Again we expressed $(r-c)$ $\overset{as}{\cancel{to}}$ a linear combination of 'n' and 'b'

(ie) $r-c = \lambda n + \mu b$ where λ and μ are scalars

$$(r-c)^2 = (\lambda n + \mu b)^2$$

$$R^2 = \lambda^2 n^2 + \mu^2 b^2 + 2\lambda\mu(n.b)$$

$$R^2 = \lambda^2 + \mu^2 \qquad \left\{\because n^2 = n.n = 1,\ b^2 = b.b = 1,\ n.b = 0\right\}$$

Hence $\lambda = -\rho$, $\mu = -\sigma\rho'$

$$R^2 = (-\rho)^2 + (-\sigma\rho')^2$$

$$R^2 = \rho^2 + \sigma^2\rho'^2$$

$$R = \sqrt{\rho^2 + \sigma^2\rho'^2} \quad \text{——} ⑥$$

Equation ⑥ $\overset{is}{\cancel{gs}}$ the radius of the osculating sphere.

Equivalently, if κ is constant then $R = \rho$ and the two radius of curvatures coincide with each other.

The centre of curvature is the centre of the osculating sphere is $c = r + \rho n + \sigma \rho' b$

since the curve has constant curvature

ρ = constant $\Rightarrow \rho' = 0 \Rightarrow \rho'' = 0$

$\Rightarrow c = r + \rho n$

Locus of the centre of spherical curvature :5m

If the point 'p' traces out a curve 'c', the corresponding centre of spherical curvature traces out another curve 'c_1' whose curvature and torsion are simply related to the curvature and torsion of the original curve 'c' also the product of the torsions is equal to the product to the curvatures.

Proof :- The position vector r_1 of the centre of spherical curvature is given by $r_1 = r + \rho n + \sigma \rho' b$ —— ①

's_1' will denote the arc length of the curve c_1

Differentiate equation ① w.r.to 's'

$$\frac{dr_1}{ds} = \frac{dr}{ds} + (\rho' n + n' \rho) + (\sigma' \rho' b + \sigma \rho'' b + \sigma \rho' b')$$

$$\frac{dr_1}{ds_1} \cdot \frac{ds_1}{ds} = \dot{r} + \rho' n + n' \rho + (\sigma' \rho' + \sigma \rho'') b + \sigma \rho' b'$$

$$= r + \rho' n + (\tau b - kt) \rho + (\sigma' \rho' + \sigma \rho'') b + \sigma \rho' (-\tau n)$$

$\{\because n' = \tau b - kt,\ b' = -\tau n,\ t = r',\ k = 1/\rho,\ \tau = 1/\sigma\}$

$$= t + \rho' n + \tau b \rho - t/\rho\, \rho + \sigma' \rho' b + \sigma \rho'' b - \sigma \rho' n\, 1/\sigma$$

$$= \tau b \rho + \sigma' \rho' b + \sigma \rho'' b$$

$r_1' \cdot s_1' = (\tau \rho + \sigma' \rho' + \sigma \rho'') b$ $\quad$ $\{\because r' = t$

$t_1 \cdot s_1' = (\tau \rho + \sigma' \rho' + \sigma \rho'') b$ —— ② $\quad$ $r_1' = t_1$

(44).

∴ curve C_1 is parametrized by s_1 and s_1 is an increasing function of 's' so that s_1' is non negative

$\therefore t_1 = e.b$ — ③ , where $e = \pm 1$

Then we have $s_1' = (\rho/\sigma + \sigma'\rho' + \sigma\rho'')\, e^{-1}$

differentiate equation ③ w.r.to s'

$$\frac{dt_1}{ds} = e'b + b'e$$

$$\frac{dt_1}{ds_1}\cdot\frac{ds_1}{ds} = e'b + b'e \qquad \{\because e'b = 0\}$$

$$t_1' \cdot s_1' = e'b + b'e = b'e$$

$$t_1' s_1' = -e\tau n \qquad \{\because b' = -\tau n\}$$

$$\kappa_1 n_1 s_1' = -e\tau n \text{ — ④} \qquad \{\because t' = \kappa n,\ t_1' = \kappa_1 n_1\}$$

so n_1 is parallel to 'n' then

we take $n_1 = e_1 n$ —— ⑤ where $e_1 = \pm 1$

substitute ⑥ in ④ we get, $\kappa_1(e_1 n)s_1' = -e\tau n$

$$\kappa_1 e_1 n s_1' = -e\tau n$$

we know that, $b_1 = t_1 \times n_1$ $\{\because t\times n = b,\ n\times b = t,\ b\times n = -t\}$

$$b_1 = eb \times e_1 n$$

$$= ee_1(b\times n) = -ee_1 t$$

$$db_1/ds = -ee_1 t'$$

$$\frac{db_1}{ds_1}\cdot\frac{ds_1}{ds} = -ee_1(\kappa n)$$

$$b_1' s_1' = -ee_1\kappa n \qquad \{\because b' = -\tau n,\ b_1' = -\tau_1 n_1\}$$

$$-\tau_1 n_1 s_1' = -ee_1\kappa n$$

$$-\tau_1(e_1 n)s_1' = -ee_1\kappa n$$

$$\tau_1 s_1' = e\kappa \text{ —— ⑦}$$

multiply bothsides on '$e\tau$' we have

$$\therefore e\tau\tau_1 s_1' = e_\tau e_\kappa$$

$$e\tau\tau_1 s_1' = \frac{-\kappa_1 n_1 s_1'}{n} e_\kappa \quad \{\because \text{using ④ put } e_\tau \text{ value}\}$$

$$\tau\tau_1 = \frac{-\kappa\kappa_1 n_1}{n} = -e_1\kappa\kappa_1, \text{ using ⑥}$$

$$\tau\tau_1 = \kappa\kappa_1 \quad \{\because e_1 = \pm 1\}$$

Hence the result

5M

Theorem :-

If a curve lies on a sphere, then ρ and σ are related by $\frac{d}{ds}(\sigma\rho') + \rho/\sigma = 0$.

proof:- Let the curve lies in a sphere.

Now the sphere will be the osculating sphere for every point of the curve.

The radius 'R' of the osculating sphere is given by $R^2 = \rho^2 + \sigma^2\rho'^2$ ——①

Differentiate equation ①, we get

$$0 = 2\rho\rho' + 2\sigma\sigma'\rho'^2 + \sigma^2 2\rho'\rho''$$

$$0 = \rho\rho' + \sigma\sigma'\rho'^2 + \sigma^2\rho'\rho''$$

divided by $\sigma\rho'$, we get

$$0 = \rho/\sigma + \sigma'\rho' + \sigma\rho''$$

$$\rho/\sigma + d/ds(\sigma\rho') = 0$$

$$\Rightarrow \frac{d}{ds}(\sigma\rho') + \rho/\sigma = 0$$

Theorem :- 10M

The radius of curvature of the locus of the centre of curvature of a curve is given by.

(4B)

$$\rho_1 = \left[\left\{ \frac{\rho^2\sigma}{R^3} \cdot \frac{d}{ds}\left(\frac{\sigma\rho'}{\rho}\right) - \frac{1}{R} \right\}^2 + \frac{\rho'^2\sigma^4}{\rho^2 R^4} \right]^{-1/2}$$

Proof :- consider $r_1 = r + \rho n$ —— ①

Differentiate w.r.to 's' we get

$$\frac{dr_1}{ds} = r' + \rho' n + n'\rho \qquad \{\because r' = t$$

$$\frac{dr_1}{ds_1} \cdot \frac{ds_1}{ds} = t + n\rho' + \rho(\tau b - \kappa t) \qquad n' = \tau b - \kappa t$$

$$r_1' s_1' = t + n\rho' + \rho\tau b - \rho\kappa t$$

$$t_1 s_1' = t[1 - \kappa\rho] + \rho\tau b + n\rho' \qquad \{\because \kappa = 1/\rho$$

$$= t\left[1 - \frac{1}{\rho}\rho\right] + \rho\tau b + n\rho'$$

$$t_1 s_1' = 0 + \rho\tau b + n\rho' \text{ —— ②}$$

Multiply by σ/ρ, we get

$$\sigma/\rho \; t_1 s_1' = \sigma/\rho \; \frac{1}{\sigma} b\rho + \sigma/\rho \; n\rho' \qquad \{\because \tau = 1/\sigma$$

$$\frac{\sigma}{\rho} t_1 s_1' = b + \frac{\sigma}{\rho} n\rho' \text{ —— ③}$$

squaring on bothsides, we get

$$\frac{\sigma^2}{\rho^2} t_1^2 s_1'^2 = b^2 + \frac{\sigma^2 n^2 \rho'^2}{\rho^2} + \frac{2(b \cdot n)\sigma\rho'}{\rho}$$

$$\frac{\sigma^2}{\rho^2} s_1'^2 = 1 + \frac{\sigma^2\rho'^2}{\rho^2} \qquad \{\because t_1^2 = t_1 \cdot t_1 = 1, \; b \cdot n = 0$$

$$\frac{\sigma^2 s_1'^2}{\rho^2} = \frac{\rho^2 + \sigma^2\rho'^2}{\rho^2} \qquad b^2 = b \cdot b = 1 \qquad n^2 = n \cdot n = 1\}$$

$$\sigma^2 s_1'^2 = \rho^2 + \sigma^2\rho'^2 = R^2 \qquad \{\because \rho^2 + \sigma^2\rho'^2 = R^2$$

$$\sigma^2 s_1'^2 = R^2 \Rightarrow s_1' = R/\sigma \text{ ------ ④}$$

Differentiate ③ w.r.to 's',

A. Boopalan.

$\{\because b \cdot b = t \cdot t = n \cdot n = 1\}$

$$\frac{\sigma^4}{\rho^4} \kappa_1^2 \frac{R^8}{\sigma^6} = \left[\frac{R^2}{\sigma\rho^2} - \frac{d}{ds}\left(\frac{\sigma\rho'}{\rho}\right)\right]^2 + \frac{\sigma^4\rho'^4 + \sigma^2\rho'^2\rho^2}{\rho^6}$$

$\{\because$ using ① $s_1' = R/\sigma\}$

$$\Rightarrow = \left[\frac{R^2}{\sigma\rho^2} - \frac{d}{ds}\left(\frac{\sigma\rho'}{\rho}\right)\right]^2 + \frac{\sigma^2\rho'^2(\sigma^2\rho'^2 + \rho^2)}{\rho^6}$$

$$\kappa_1^2 \frac{R^6}{\rho^4\sigma^2} = \left[\frac{R^2}{\sigma\rho^2} - \frac{d}{ds}\left(\frac{\sigma\rho'}{\rho}\right)\right]^2 + \frac{\sigma^2\rho'^2 R^2}{\rho^6}$$

$$\kappa_1^2 = \frac{\rho^4\sigma^2}{R^6}\left[\frac{R^2}{\sigma\rho^2} - \frac{d}{ds}\left(\frac{\sigma\rho'}{\rho}\right)\right]^2 + \frac{\sigma^4\rho'^2 R^2}{\rho^2 R^{4+2}}$$

$$= \left[\frac{\rho^2\sigma}{R^3} \cdot \frac{R^2}{\sigma\rho^2} - \frac{\rho^2\sigma}{R^3} \cdot \frac{d}{ds}\left(\frac{\sigma\rho'}{\rho}\right)\right]^2 + \frac{\sigma^4\rho'^2}{\rho^2 R^4}$$

{∵ square உள்ளே போனால் வெளிய power [illegible] குறையும் }

$$\kappa_1^2 = \left[\frac{1}{R} - \frac{\rho^2\sigma}{R^3}\frac{d}{ds}\left(\frac{\sigma\rho'}{\rho}\right)\right]^2 + \frac{\sigma^4\rho'^2}{\rho^2 R^4}$$

$$\kappa_1 = \left\{\left[\frac{1}{R} - \frac{\rho^2\sigma}{R^3}\frac{d}{ds}\left(\frac{\sigma\rho'}{\rho}\right)\right]^2 + \frac{\sigma^4\rho'^2}{\rho^2 R^4}\right\}^{1/2}$$

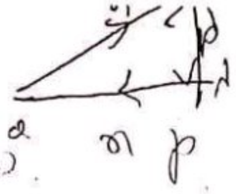
since $\rho_1 = 1/\kappa_1$

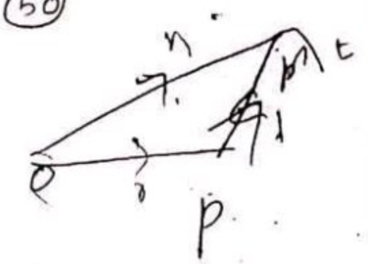
$$\rho_1 = \left\{\left[\frac{\rho^2\sigma}{R^3}\frac{d}{ds}\left(\frac{\sigma\rho'}{\rho}\right) - \frac{1}{R}\right]^2 + \frac{\sigma^4\rho'^2}{\rho^2 R^4}\right\}^{-1/2}$$

Tangent surface, Involutes and Evolutes :-

Tangent surface :-

The surface generated by the tangent lines to the given curve 'c' is called the tangent surface to 'c'. The position vector of the point 'P' on the tangent surface R





$= r(s) + \lambda(s)\, t(s)$.

volute :-

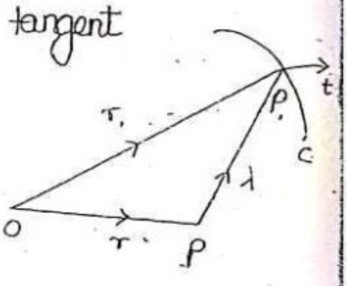
3) A curve which lies on the tangent surface of 'c' & intersects the generators of the tangent surface orthogonally is called the involute of 'c' denoted by $\tilde{c}$)

quation of an Involute :- (4)

If 'r_1' is the position vector of a point P_1 on the curve $\tilde{c}$ of c then $r_1 = r + (c - s)t$, where 'c' is a arbitrary constant and 'r' is the position vector of 'p' on 'c'.

proof:-

Since the involute lies on the tangent surfaces, the position vectors r_1 of a point P_1 on the involute is



$$r_1 = r + \lambda(s)\, t \quad \text{—— ①}$$

differentiate w.r.to 's'

$$\frac{dr_1}{ds} = \frac{dr}{ds} + \lambda'(s)\, t + \lambda(s)\, t'$$

$$\frac{dr_1}{ds_1} \cdot \frac{ds_1}{ds} = r' + \lambda'(s)t + \lambda(s)(kn). \quad \{\because t' = kn\}$$

$$t_1\, s_1' = t + \lambda'(s)t + \lambda(s)\, kn \quad \text{—— ②} \quad \{\because \frac{dr_1}{ds_1} = r_1' = t_1$$

Since the tangent to the involute cuts the generator orthogonally $\Rightarrow t \cdot t_1 = 0$

Taking dot product with t on bothsides of ②.

$$s_1'\, t_1 \cdot t = t \cdot t + \lambda'(s)\, t \cdot t + \lambda(s)\, k(n \cdot t)$$

$$0 = 1 + \lambda'(s) \Rightarrow \lambda'(s) = -1 \quad \{\because n \cdot t = 0,\ t \cdot t = 1$$

$$d\lambda / ds = -1 \Rightarrow d\lambda = -ds$$

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Integrating w.r.to 's'

$\lambda = -s+c = c-s$

Hence the equation of involute is $r_1 = r + (c-s)t$

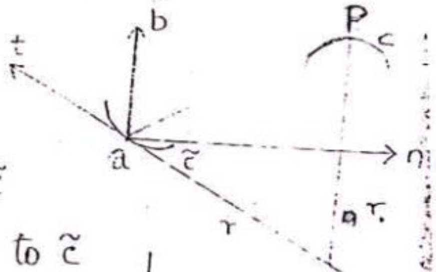
Evolute :-

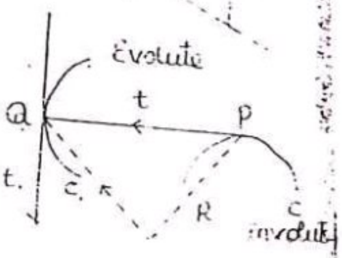
If $\tilde{c}$ is an involute of a given curve 'c' then c is defined to be evolute of $\tilde{c}$

Equation of a Evolute :-

If $r = r(s)$ is the equation of an involute $\tilde{c}$ of a curve 'c' and (t, n, b) is a moving orbit at any point of $r = r(s)$ then the position vector r_1 of a point on 'c' is $r_1 = r + \rho n + \rho \cot(\psi + c) b$ where $\psi = \int \tau ds$

proof :- Let p be a point on 'c' corresponding to the point Q on $\tilde{c}$ PQ is a tangent at p orthogonal to $\tilde{c}$ Hence PQ is perpendicular to the tangent at Q to $\tilde{c}$





Since the tangent at Q to the involute is at the right angles to the tangent PQ to the curve c, PQ lies in the normal plane at Q to $\tilde{c}$

Let $\bar{QP} = \lambda n + \mu b$

The co-efficients λ, μ change from point to point on $\tilde{c}$ so that λ, μ are functions of s on c. Using this position vector of any point [illegible] on c, is

[illegible]

$$r_1 = r + \lambda n + \mu b \quad \text{—— ①}$$

differentiate w.r.to 's' we get

$$\frac{dr_1}{ds} = \frac{dr}{ds} + \lambda' n + \lambda n' + \mu' b + \mu b'$$

$$\frac{dr_1}{ds_1} \cdot \frac{ds_1}{ds} = r' + \lambda' n + \lambda[\tau b - \kappa t] + \mu' b + \mu[-\tau n]$$

$$\{\because n' = \tau b - \kappa t,\ b' = -\tau n\}$$

$$r_1' \, s_1' = t + \lambda' n + \lambda \tau b - \lambda \kappa t + \mu' b - \mu \tau n \qquad \{\because r' = t$$

$$t_1 \, s_1' = (1 - \lambda\kappa) t + (\lambda' - \mu\tau) n + (\lambda\tau + \mu') b \quad \text{—— ②}$$

It must be parallel to $\lambda n + \mu b$ —— ③

since ② and ③ are parallel then,

$$1 - \lambda\kappa = 0$$

$$1 = \lambda\kappa$$

$$\lambda = 1/\kappa = \rho$$

$$\lambda = \rho$$

$$\{\because \kappa = 1/\rho\}$$

$$\frac{\lambda' - \mu\tau}{\lambda} = \frac{\mu' + \lambda\tau}{\mu} \quad \text{—— ④}$$

$$\mu\lambda' - \mu^2\tau = \lambda^2\tau + \lambda\mu'$$

$$\tau\lambda^2 + \mu^2\tau = \mu\lambda' - \lambda\mu'$$

$$\tau = \frac{\mu\lambda' - \lambda\mu'}{\lambda^2 + \mu^2}$$

We know that,

$$\frac{d}{ds}\tan^{-1}(u/v) = \frac{vu' - uv'}{u^2 + v^2}$$

$$\tau = \frac{d}{ds}\tan^{-1}(\lambda/\mu)$$

Integrating w.r.to 's', we get

$$\int \tau\, ds = \int \frac{d}{ds}\tan^{-1}(\lambda/\mu)\, ds = \tan^{-1}(\lambda/\mu) + c$$

$$\tan \int \tau\, ds = \lambda/\mu + c$$

$$\tan\left[\int \tau ds + c\right] = \lambda/\mu$$

since $\int \tau ds = \chi$, $\lambda/\mu = \tan(\chi + c)$

---

or $\mu = \frac{\lambda}{\tan(\chi + c)}$ =

$\mu = \rho$ cot

Hence the equation

$r_1 = r + \rho$

Space curve :-

A curve

-ons is the locus of

with respect to the

parameter 't' the ca

point 'p' are called

of the parameter 't'

∴ We

interms of the single

Thus r

simply r = (

Intrinsic Equation :

functions of arc

(or) Natural equ

Intrinsic equation

$\tau = g(s)$

Congruent space

or. $\mu = \dfrac{\lambda}{\tan(\chi+c)} = \lambda \cot(\chi+c)$

$\mu = \rho \cot(\chi+c) \qquad \{\because \lambda = \rho$

Hence the equation of evolute of $\tilde{c}$ is

$r_1 = r + \rho n + \rho \cot(\chi+c)\, b$ where $\chi = \int \tau\, ds$

pace curve :-

A curve in Euclidean space of three dimensions is the locus of a point whose position vector 'r' with respect to the origin say 'o' is a function of a single parameter 't' the cartesian co-ordinates (x, y, z) of point 'p' are called components of 'r' and are functions of the parameter 't'

$\therefore$ We can express the equation of curve interms of the single parameter 't'

Thus $r(t) = x(t)\vec{i} + y(t)\vec{j} + z(t)\vec{k}$

simply $r = (x, y, z)$ represents a curve in space

Intrinsic Equation :-

The equations expressing κ and τ are the functions of arc length 's' are called Intrinsic equation (or) Natural equation of a curve. Hence we can write the Intrinsic equation of the curve are of the form $\kappa = f(s)$, $\tau = g(s)$

Congruent space curves :-

Two space curves are said to be congruent if one curve is brought into coincidence with

(7) Find the intrinsic equations of the curve $r = (ae^u\cos u, ae^u\sin u, be^u)$

Solution :- Given $r = (ae^u\cos u, ae^u\sin u, be^u)$ — ①

differentiate w.r.to 'u',

$$\frac{dr}{du} = \left(a\left[e^u(-\sin u) + e^u\cos u\right], a\left[e^u\cos u + e^u\sin u\right], be^u\right)$$

$$\frac{dr}{ds}\frac{ds}{du} = \left(ae^u(\cos u - \sin u), ae^u(\cos u + \sin u), be^u\right)$$

$$r'\dot{s} = \left(ae^u(\cos u - \sin u), ae^u(\cos u + \sin u), be^u\right)$$

$$t\dot{s} = \left(ae^u(\cos u - \sin u), ae^u(\cos u + \sin u), be^u\right) \text{ — ②}$$

Taking dot product of equation ② with itself $\{\because r' = t$

$$(t\dot{s})\cdot(t\dot{s}) = a^2e^{2u}(\cos u - \sin u)^2 + a^2e^{2u}(\sin u + \cos u)^2 + b^2e^{2u}$$

$$\dot{s}^2(t\cdot t) = e^{2u}\left[a^2\cos^2 u + a^2\sin^2 u - 2a^2\sin u\cos u + a^2\sin^2 u + a^2\cos^2 u + 2a^2\cos u\sin u + b^2\right]$$

$$\dot{s}^2 = e^{2u}(2a^2 + b^2)$$

$$\dot{s} = e^u(2a^2+b^2)^{1/2} \text{ —— ③}$$

$$\frac{ds}{du} = e^u(2a^2+b^2)^{1/2} \Rightarrow ds = (2a^2+b^2)^{1/2}e^u\,du$$

Integrating $\int ds = \int (2a^2+b^2)^{1/2}e^u\,du$

$$s = (2a^2+b^2)^{1/2}e^u = \dot{s}, \text{ using ③}$$

$$\therefore s = \dot{s} \text{ —— ④}$$

[illegible] $e^u(2a^2+b^2)^{1/2} = \left(ae^u(\cos u - \sin u), ae^u(\cos u + \sin u), be^u\right)$

$$t = \frac{e^u}{e^u(2a^2+b^2)^{1/2}}\left[a(\cos u\right.$$

$$= \frac{1}{(2a^2+b^2)^{1/2}}\left[a(\cos u\right.$$

differentiate with respect

$$\frac{dt}{du} = \frac{1}{(2a^2+b^2)^{1/2}}\left[a(-\right.$$

$$\frac{dt}{ds}\frac{ds}{du} = \frac{1}{(2a^2+b^2)^{1/2}}\left[a(-\right.$$

$$t'\cdot\dot{s} = \frac{1}{(2a^2+b^2)^{1/2}}\left[a(\right.$$

$$\kappa n\cdot\dot{s} = \frac{1}{(2a^2+b^2)^{1/2}}\left[a(\right.$$

Taking dot product of

with itself

$$\kappa^2\dot{s}^2(n\cdot n) = \frac{1}{(2a^2+b^2)}$$

$$\kappa^2\dot{s}^2 = \frac{1}{(2a^2+b^2)}\left[a^2\right.$$

$$+a^2\cos^2 u +$$

$$\kappa^2 = \frac{1}{\dot{s}^2}\,\frac{1}{(2a^2+}$$

$$\kappa = \frac{1}{\dot{s}}$$

Substitute the value

$$\therefore t = \frac{e^{u}}{e^{u}(2a^2+b^2)^{1/2}}\Big[a(\cos u - \sin u),\ a(\cos u + \sin u),\ b\Big]$$

$$= \frac{1}{(2a^2+b^2)^{1/2}}\Big[a(\cos u - \sin u),\ a(\cos u + \sin u),\ b\Big] \quad \llcorner (*)$$

Differentiate with respect to 'u',

$$\frac{dt}{du} = \frac{1}{(2a^2+b^2)^{1/2}}\Big[a(-\sin u - \cos u),\ a(-\sin u + \cos u),0\Big]$$

$$\frac{dt}{ds}\frac{ds}{du} = \frac{1}{(2a^2+b^2)^{1/2}}\Big[a(-\sin u - \cos u),\ a(\cos u - \sin u),\ 0\Big]$$

$$t'.\dot{s} = \frac{1}{(2a^2+b^2)^{1/2}}\Big[a(-\sin u - \cos u),a(\cos u - \sin u),0\Big]$$

$$\kappa n.\dot{s} = \frac{1}{(2a^2+b^2)^{1/2}}\Big[a(-\sin u - \cos u),\ a(\cos u - \sin u),0\Big]$$

$\llcorner$ ⑤ $\{\because t' = \kappa n$

Taking dot product of equation ⑤ with itself

$$\kappa^2 \dot{s}^2 (n.n) = \frac{1}{(2a^2+b^2)}\Big[a^2(-\sin u - \cos u)^2 + a^2(\cos u - \sin u)^2\Big]$$

$$\kappa^2 \dot{s}^2 = \frac{1}{(2a^2+b^2)}\Big[a^2\sin^2 u + a^2\cos^2 u + 2a^2\cos u \sin u$$

$$+ a^2\cos^2 u + a^2\sin^2 u - 2a^2\cos u \sin u\Big]$$

$$\kappa^2 = \frac{1}{\dot{s}^2}\frac{1}{(2a^2+b^2)}\Big[2a^2\Big] = \frac{2a^2}{\dot{s}^2(2a^2+b^2)}$$

$$\kappa = \frac{1}{\dot{s}}\frac{\sqrt{2}a}{\sqrt{2a^2+b^2}} = \frac{\sqrt{2}a}{\sqrt{2a^2+b^2}\ s}, \text{ using } ④ \quad \llcorner ⑥$$

Substitute the value of κ and ṡ in equation ⑤ $\{\because \dot{s} = s\}$

[illegible] $n = \frac{1}{[illegible]}\Big[a(-\sin u - \cos u),\ a(\cos u\ \sin u)$ [illegible]

$$n = \frac{(2a^2+b^2)^{1/2}}{\sqrt{2}\,a\,(2a^2+b^2)^{1/2}}\left[a(-\sin u - \cos u),\ a(\cos u - \sin u),\ 0\right]$$

$$= \frac{1}{\sqrt{2}\,a} \cdot a\left[-\sin u - \cos u,\ \cos u - \sin u,\ 0\right]$$

$$n = \frac{1}{\sqrt{2}}\left[-\sin u - \cos u,\ \cos u - \sin u,\ 0\right] \text{ --- } ⑦$$

Taking ~~dot~~ cross product of equation ④ with ⑦

$$t \times n = \frac{1}{\sqrt{2a^2+b^2}\,\sqrt{2}}\Big[\{a(\cos u - \sin u),\ a(\cos u + \sin u),\ b\}$$
$$\times \{-\sin u - \cos u,\ \cos u - \sin u,\ 0\}\Big]$$

$$= \frac{1}{\sqrt{2}\sqrt{2a^2+b^2}}\begin{vmatrix} t & n & b \\ a\cos u - a\sin u & a\cos u + a\sin u & b \\ -\sin u - \cos u & \cos u - \sin u & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{2}\sqrt{2a^2+b^2}}\Big[t\{-b(\cos u - \sin u)\} - n\{-b(-\sin u - \cos u)\} + b\{(a\cos u - a\sin u)(\cos u - \sin u) - (-\sin u - \cos u)(a\cos u + a\sin u)\}\Big]$$

$$= \frac{1}{\sqrt{2}\sqrt{2a^2+b^2}}\Big[b(\sin u - \cos u)t - b(\sin u + \cos u)n + b\{a(\cos u - \sin u)^2 + a(\sin u + \cos u)^2\}\Big]$$

$$= \frac{1}{\sqrt{2}\sqrt{2a^2+b^2}}\Big[b(\sin u - \cos u)t - b(\sin u + \cos u)n + b\{a\cos^2 u + a\sin^2 u - 2a\sin u\cos u + a\sin^2 u + a\cos^2 u + 2a\sin u\cos u\}\Big]$$

$$= \frac{1}{\sqrt{2a^2+b^2}}\Big[b(\sin u - \cos u)t - b(\sin u + \cos u)n + 2ab\Big]$$

$$b = \frac{1}{\sqrt{2}\sqrt{2a^2+b^2}}\left[b(\sin u-\cos u),\ -b(\sin u+\cos u),\ 2a\right] \quad \text{⑧}$$

Differentiate w.r.to 'u'

$$\frac{db}{du} = \frac{1}{\sqrt{2}\sqrt{2a^2+b^2}}\left[b(\cos u+\sin u),\ -b(\cos u-\sin u),\ 0\right]$$

$$\frac{db}{ds}\cdot\frac{ds}{du} = \frac{1}{\sqrt{2}\sqrt{2a^2+b^2}}\left[b(\cos u+\sin u),\ b(\sin u-\cos u),\ 0\right]$$

$$b'\cdot\dot{s} = \frac{1}{\sqrt{2}\sqrt{2a^2+b^2}}\left[b(\cos u+\sin u),\ b(\sin u-\cos u),\ 0\right]$$

$$-\tau n\cdot\dot{s} = \frac{1}{\sqrt{2}\sqrt{2a^2+b^2}}\left[b(\cos u+\sin u),\ b(\sin u-\cos u),\ 0\right]$$

$$\{\because b' = -\tau n\} \quad \text{⑨}$$

Taking dot product of equation ⑨ with itself

$$(-\tau n\cdot\dot{s})(-\tau n\cdot\dot{s}) = \frac{1}{2(2a^2+b^2)}\left[b^2(\cos u+\sin u)^2 + b^2(\sin u-\cos u)^2 + 0\right]$$

$$\tau^2\dot{s}^2(n\cdot n) = \frac{1}{2(2a^2+b^2)}\left[b^2\cos^2 u + b^2\sin^2 u + 2b^2\cos u\sin u + b^2\sin^2 u + b^2\cos^2 u - 2b^2\sin u\cos u\right]$$

$$\tau^2\dot{s}^2 = \frac{1}{2(2a^2+b^2)}\left[2b^2\right] = \frac{b^2}{2a^2+b^2}$$

$$\tau^2 = \frac{b^2}{(2a^2+b^2)\cdot\dot{s}^2} = \frac{b^2}{(2a^2+b^2)s^2}, \text{ using ④}$$

$$\tau = \frac{b}{s(2a^2+b^2)^{1/2}} \quad \text{---- ⑩}$$

Equations ⑥ and ⑩ are the intrinsic equation given curve

A-19
5M

Theorem: I Fundamental Existence theorem for space curves

If $K(s)$, $T(s)$ are continuous functions of the real riable 's', where $s \geq 0$, then there exist a space curve for hich K is the curvature, T is the torsion and 's' is the arc ngth measured from some suitable base point

roof :- consider the differential equation of the first order in α, β and γ

$$\frac{d\alpha}{ds} = K\beta \;,\; \frac{d\beta}{ds} = T\gamma - K\alpha \;,\; \frac{d\gamma}{ds} = -T\beta \quad \text{——} ①$$

where α, β, γ are unknown functions of 's' and 'K' and 'T' are the given functions $K(s)$ and $T(s)$

The set of equation ① admits a unique set of solution which is $(\alpha_0, \beta_0, \gamma_0)$ when $s = 0$

let $(\alpha_1, \beta_1, \gamma_1)$ be one set solution taking the values $\alpha_1(0) = 1$, $\beta_1(0) = 0$, $\gamma_1(0) = 0$ when $s = 0$

similarly there is a unique set $(\alpha_2, \beta_2, \gamma_2)$ which assume the value $(0,1,0)$ when $s=0$ and also a unique set $(\alpha_3, \beta_3, \gamma_3)$ which assumes the value $(0,0,1)$ when $s=0$

step (i) :- we establish the following properties of the above three solutions

$$\left.\begin{aligned} \alpha_1^2 + \beta_1^2 + \gamma_1^2 &= 1 \;, & \alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 &= 0 \\ \alpha_2^2 + \beta_2^2 + \gamma_2^2 &= 1 \;, & \alpha_1\alpha_3 + \beta_1\beta_3 + \gamma_1\gamma_3 &= 0 \\ \alpha_3^2 + \beta_3^2 + \gamma_3^2 &= 1 \;, & \alpha_2\alpha_3 + \beta_2\beta_3 + \gamma_2\gamma_3 &= 0 \end{aligned}\right\} \text{——} ②$$

for all values of 's'

when $s = 0$, the curves C and C_1 coincides at $A = n$.
then $t = t_1$, $n = n_1$, $b = b_1$

(ie) The angle between the unit normal vectors is zero, then $r' = r_1'$

$$r' - r_1' = 0$$

$$\frac{d}{ds}(r - r_1) = 0$$

Integrating, $r - r_1 = e$, where e is any constant

At $s = 0$, $r(0) - r_1(0) = e$

$$0 - 0 = e \Rightarrow e = 0$$

$$r - r_1 = 0 \Rightarrow r = r_1 \text{ identically}$$

Hence the proof.

Helices :-

cylindrical helix :-

A cylindrical helix is a space curve which lies on a cylinder and cuts the generators at a constant angle. Its tangent makes a constant angle 'α' with a fixed line is known as the axis of the helix

Theorem :- 5M

sm A necessary and sufficient condition for a curve to be helix is that the ratio of the curvature to the torsion is constant at all the points.

proof :-

Assume that the curve is a cylindrical helix.

To prove that $\frac{k}{\tau}$ = constant

let $\vec{a}$ be a unit vector along the direction of the axis Since the helix cuts the generators at a constant angle α $\quad \therefore t \cdot \vec{a} = \cos\alpha \rightarrow$ ①

$t \cdot \vec{a} = \cos\alpha$

Differentiate w.r.to 's'

$$t' \cdot a + a' \cdot t = 0$$

since 'a' is a constant vector and $t' = kn$

we get $\quad kn \cdot a + t \cdot 0 = 0 \qquad \{\because$ 'a' is unit vector

$kna = 0 \qquad a = 1 =$ constant

If $k = 0$ then the curve is a straight $\quad \therefore a' = 0$

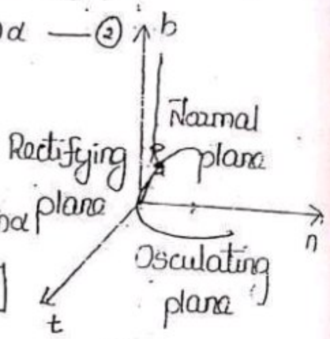
line and hence the theorem.

If $n \cdot a = 0$ then 'a' is perpendicular to normal at 'p'

since 'a' passes through 'p' and making an angle 'α' which tangent at 'p' and it is perpendicular to the normal at 'p'. It lies in the rectifying plane at 'p'

$$\therefore a = t\cos\alpha + b\sin\alpha \text{ — ②}$$

Differentiate w.r.to 's'



$$0 = t'\cos\alpha + b'\sin\alpha$$

$\{\because t' = kn$

$b' = -\tau n\}$ $\quad 0 = (kn)\cos\alpha + (-\tau n)\sin\alpha$

$$0 = n\left[k\cos\alpha - \tau\sin\alpha\right]$$

$$k\cos\alpha - \tau\sin\alpha = 0 \qquad \{\because n \neq 0\}$$

$$k\cos\alpha = \tau\sin\alpha$$

$$k/\tau = \sin\alpha/\cos\alpha = \tan\alpha \text{ (constant)}$$

$$k/\tau = \text{constant}$$

Conversely, let $k/\tau =$ constant

We have to prove that the curve is helix

Given $k/\tau =$ constant $= \lambda$ (say)

For any constant λ we can always find the smallest angle 'α' suchthat $k/\tau = \tan\alpha = \sin\alpha/\cos\alpha$

$$k\cos\alpha - \tau\sin\alpha = 0$$

$$n[k\cos\alpha - \tau\sin\alpha] = 0 \quad , n \neq 0$$

$$kn\cos\alpha - \tau n\sin\alpha = 0$$

$$t'\cos\alpha + b'\sin\alpha = 0$$

$$\frac{d}{ds}[t\cos\alpha + b\sin\alpha] = 0$$

Integrating $t\cos\alpha + b\sin\alpha = a$, where 'a' is constant

Taking dot product with 't' on bothsides

$$t\cdot t\cos\alpha + t\cdot b\sin\alpha = t\cdot a$$

$$\cos\alpha = at \quad \{\because t\cdot t = 1, t\cdot b = 0\}$$

Hence the curve is helix $\{\because$ a is constant so $t \cdot a = at\}$

Hence the proof.

circular helices :-

A circular helix is one which lies on the surface of a circular cylinder, the axis of the helix being that of the cylinder.

Theorem :-

If the z axis is the axis of the cylinder as well as that of the helix, the parametric equation of the helix is of the form $x = a\cos u$, $y = a\sin u$, $z = bu$ where the base circle is $x^2+y^2=a^2$, $z=0$ and 'b' is any constant

Note :-

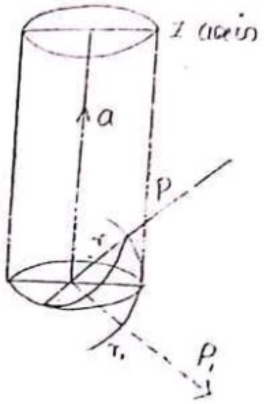
* If $b > 0$ then the helix is right handed and if $b < 0$ then the helix is left handed.

* The pitch of the helix is equal to $2\pi b$. The pitch gives the displacement along the axis corresponding to a complete turn round the axis

Theorem :-

The projection 'c_1' of the general helix 'c' on a ne perpendicular to its axis has its principal normal allel to the corresponding principal normal of the helix d its corresponding curvature is given by $\kappa = \kappa_1 \sin^2\alpha$

oof :-



Let 'p' be any point on the helix ith the position vector 'r' and 'P_1' s its projection on xoy plane with the osition vector 'r_1' Let 'a' be the unit ector in the direction of the axis of he helix.

By our choice of axis of the helix 'a' is parallel o the z-axis. so 'P_1' be parallel to 'a' and hence

$$P_1 P = r \cdot a$$

$$\overrightarrow{P_1 P} = (r \cdot a) \cdot a$$

$$\overrightarrow{op} = \overrightarrow{op_1} + \overrightarrow{P_1 P} \Rightarrow \overrightarrow{op_1} = \overrightarrow{op} - \overrightarrow{P_1 P}$$

$$r_1 = r - (r \cdot a) \cdot a = r - (a \cdot r) \cdot a$$

$$r = r_1 + (a \cdot r) \cdot a \text{ —— ①}$$

Differentiate w.r.to 's', $\frac{dr}{ds} = \frac{dr_1}{ds} + a \frac{dr}{ds} \cdot a$

$$r' = r_1' + a\, r'\, a$$

$$t = \frac{dr_1}{ds_1} \cdot \frac{ds_1}{ds} + (a \cdot t) \cdot a \quad \{\because t = r'\}$$

$$= t_1 \frac{ds_1}{ds} + (a \cdot t)\, a \quad \{\because \frac{dr_1}{ds_1} = r_1' = t_1\}$$

By [illegible] theorem, $t \cdot a = \cos\alpha$, $\frac{ds_1}{ds} = \sin\alpha$ —— (+)

$$t = t_1 \sin\alpha + a\cos\alpha \quad \text{---} ④$$

Differentiate w.r.to 's' $\quad \frac{dt}{ds} = \frac{dt_1}{ds}\sin\alpha + 0$

$$\frac{dt}{ds} = \frac{dt_1}{ds_1}\frac{ds_1}{ds}\sin\alpha$$

$$t' = t_1' \sin\alpha \sin\alpha = t_1' \sin^2\alpha \qquad \{\because \text{using } ④$$

$$kn = k_1 n_1 \sin^2\alpha \quad \text{---} ⑤ \quad \{\because t' = kn,\ t_1' = k_1 n_1\}$$

Equation ⑤ proves that the normal n_1 to c_1 is parallel to the principal normal 'n' of 'c' of the helix.

Taking dot product of equation ⑤ with itself

$$(kn)\cdot(kn) = (k_1 n_1 \sin^2\alpha)\cdot(k_1 n_1 \sin^2\alpha)$$

$$k^2 (n\cdot n) = k_1^2 \sin^4\alpha\,(n_1\cdot n_1)$$

$$k^2 = k_1^2 \sin^4\alpha \qquad \{\because n\cdot n = n_1\cdot n_1 = 1$$

$$k = k_1^{\cancel{2}} \sin^2\alpha$$

<u>Definition</u> :-

If a curve on sphere (or) cone is a helix then the curve is called spherical (or) conical helix

1] Find the involute and Evolute of the circular helix $r = (a\cos\theta, a\sin\theta, b\theta)$

<u>Solution</u> :-

Given $r = (a\cos\theta, a\sin\theta, b\theta)$ --- ①

The equation of involute is $r_1 = r + (\lambda - s)t$ --- ②

We have to find the values of 's' and 't'

$$\frac{dr}{d\theta} = (-a\sin\theta, a\cos\theta, b)$$

$$\frac{dr}{ds}\cdot\frac{ds}{d\theta} = (-a\sin\theta, a\cos\theta, b)$$

$$t\,\frac{ds}{d\theta} = (a\sin\theta, a\cos\theta, b)$$

$$t \cdot \frac{ds}{d\theta} = (-a\sin\theta, a\cos\theta, b) \quad \{\because r' = t\} \; \text{③}$$

Taking dot product of equation ③ with itself

$$\left(t\frac{ds}{d\theta}\right)\cdot\left(t\frac{ds}{d\theta}\right) = (-a\sin\theta, a\cos\theta, b)\cdot(-a\sin\theta, a\cos\theta, b)$$

$$(t\cdot t)\left(\frac{ds}{d\theta}\right)^2 = a^2\sin^2\theta + a^2\cos^2\theta + b^2$$

$$\left(\frac{ds}{d\theta}\right)^2 = a^2+b^2 = c^2 \text{ (say)}$$

$$\frac{ds}{d\theta} = c \Rightarrow ds = c\,d\theta$$

Integrating, $s = c\theta$ — ④

$$\text{③} \Rightarrow t = \frac{d\theta}{ds}(-a\sin\theta, a\cos\theta, b)$$

$$= \frac{1}{ds/d\theta}(-a\sin\theta, a\cos\theta, b)$$

$$= \frac{1}{c}(-a\sin\theta, a\cos\theta, b) \text{ — ⑤}$$

using ④ and ⑤ in ③,

$$r_1 = (a\cos\theta, a\sin\theta, b\theta) + \frac{1}{c}(\lambda - c\theta)(-a\sin\theta, a\cos\theta, b) \text{ — ⑥}$$

The equation of the evolute is,

$$r_1 = r + \rho n + \rho\cot(\psi+\lambda)b \text{ — ✱}$$

where $\psi = \int \tau\,ds$

we have to find the values of n, b, ρ, τ and ψ

we know that, $\kappa = a/c^2$ and $\tau = b/c^2$ — ⑦

{∵ Curvature and Torsion section-ல் [illegible] }

$$\therefore \psi = \int \tau\,ds = \int b/c^2\,ds = \frac{b}{c^2}s$$

$$\psi = b/c^2(c\theta) = b\theta/c \text{ — (8)}, \text{ using ④}$$

$$\kappa = \frac{1}{\rho} \Rightarrow \rho = \frac{1}{\kappa}$$

differentiate ⑤ w.r.to 's'

$$\frac{dt}{ds} = \frac{1}{c}(-a$$

{∵ Here R.H.S -ல் θ [illegible]

[illegible]

$$\kappa n = \frac{1}{c}\cdot\frac{1}{c}$$

$$n = \frac{a}{\kappa c^2}$$

$$n = \frac{a}{a/c^2}$$

$$n = \frac{ac}{ac}$$

$$n = (-$$

$$\therefore b = t \times n =$$

$$b = t\,[$$

$$b/c$$

$$b\,(b/c$$

$k = \frac{1}{\rho} \Rightarrow \rho = \frac{1}{k} = \frac{1}{a/c^2} = \frac{c^2}{a}$ —— ⑨, using ⑦

differentiate ⑧ w.r.to 's',

$$\frac{dt}{ds} = \frac{1}{c}(-a\cos\theta, -a\sin\theta, 0)\frac{d\theta}{ds}$$

{∵ Here R.H.S -ல் θ மட்டும் உள்ளதால் differentiate பண்ணும் போது θ-வை பொறுத்து பண்ணலாம். பின் θ-வை 's' பொறுத்து பண்ணலாம் }

$$kn = \frac{1}{c}(a)(-\cos\theta, -\sin\theta, 0)\frac{d\theta}{ds}$$

$$kn = \frac{1}{c}\cdot\frac{1}{c}(a)(-\cos\theta, -\sin\theta, 0) \quad \left\{\because t' = \frac{dt}{ds} = kn\right.$$

$$n = \frac{a}{kc^2}(-\cos\theta, -\sin\theta, 0) \quad \left\{\because ④ \Rightarrow \frac{ds}{d\theta} = \right.$$

$$n = \frac{a}{a/c^2 \cdot c^2}(-\cos\theta, -\sin\theta, 0), \text{ using } ⑦$$

$$n = \frac{ac^2}{ac^2}(-\cos\theta, -\sin\theta, 0)$$

$$n = (-\cos\theta, -\sin\theta, 0) \text{ —— } ⑩$$

$$\therefore b = t \times n = \begin{vmatrix} t & n & b \\ -a/c\sin\theta & a/c\cos\theta & b/c \\ -\cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$b = t\left[0 + b/c\sin\theta\right] - n\left[0 + b/c\cos\theta\right] + b\left[a/c\sin^2\theta + a/c\cos^2\theta\right]$$

$$= b/c\sin\theta\, t - b/c\cos\theta\, n + ab/c$$

$$b = (b/c\sin\theta, -b/c\cos\theta, a/c) \text{ —— } ⑪$$

$$r_1 = (a\cos\theta, a\sin\theta, b\theta) + \frac{c}{a}(-\cos\theta, -\sin\theta, 0) + \frac{c^2}{a}\cot\left(\frac{b\theta}{c} + \lambda\right)\left(\frac{b}{c}\sin\theta, -\frac{b}{c}\cos\theta, \frac{a}{c}\right)$$

Unit - 2

Intrinsic properties of a surface :- ①

Surface :-

A surface is a locus of a point $p(x, y, z)$ satisfying some restriction on (x, y, z) which is expressed by a relation of the form $F(x, y, z) = 0$. This is called the implicit (or) constraint equation of the surface.

Parametric (or) Freedom :-

A parametric (or) freedom equation of a surface are of the form $x = f(u, v)$, $y = g(u, v)$, $z = h(u, v)$, where u and v are parameters which take real values and vary freely in some domain D

Surface of class 'r' :-

A surface is said to be of class 'r' if the functions f, g and h are single valued continuous and possess continuous partial derivatives of the r^{th} order

Example :-

The parametric equation of the surface is not unique

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Proof :-

consider the following two set of equations

$x = u+v \quad , \; y = u-v \quad , \; z = 4uv$ —— ①

$x = u \quad , \; y = v \quad , \; z = u^2 - v^2$ —— ②

① => $x^2 - y^2 = (u+v)^2 - (u-v)^2$

$= u^2 + v^2 + 2uv - u^2 - v^2 + 2uv$

$= 4uv = z$

$x^2 - y^2 = z$ —— ③

② => $x^2 - y^2 = u^2 - v^2 = z \quad => \quad x^2 - y^2 = z$

Example :-

sometimes the constraint equations obtained by eliminating the parameters represents more than the given surface, so that parametric equations and constraint equations are not equivalent.

proof :- consider the parametric equations

$x = u\cosh v \;, \; y = u\sinh v \;, \; z = u^2$, where u and v take all real values. —— ④

$x^2 - y^2 = u^2\cosh^2 v - u^2\sinh^2 v$

$= u^2 [\cosh^2 v - \sinh^2 v] = u^2 = z$

$x^2 - y^2 = z$

The constraint equation in ③ [above example] which represents the whole of the paraboloid but the parametric equations ④ represents only the part of the surface for which $z \geq 0$. since 'u' takes only real values

(81)

⑫

$$\therefore N = \frac{(-f'g\cos v, -f'g\sin v, gg')}{g\sqrt{f'^2+g'^2}}$$

$$N = \frac{(-f'\cos v, -f'\sin v, g')}{\sqrt{f'^2+g'^2}}$$

Result :-

The right circular cone of semi vertical angle 'α' is given by $g(u) = u$, $f(u) = u\cot\alpha$ then the representation of the point of the cone is $r = (u\cos v, u\sin v, u\cot\alpha)$

An anchor ring :- 2m A-19 2m

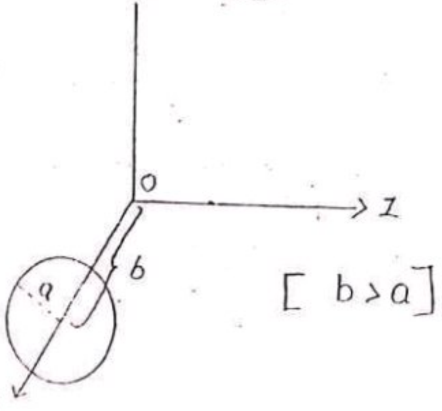
The anchor ring is obtained by rotating a circle of radius 'a' about a line in its plane and at a distance 'b' [b > a] from its centre. Here $g(u) = b + a\cos u$, $f(u) = a\sin u$. Then the position vector of a point in the anchor ring is given by

$$r = (g(u)\cos v, g(u)\sin v . f(u))$$

$$r = ((b+a\cos u)\cos v, (b+a\cos u)\sin v, a\sin u)$$

where x axis is the axis of the revolution generating a circle in the xoz plane with centre (b.0.0) on x axis and the domain of u, v is $0 < u < 2\pi$,

2π



Helicoids :-

Screw motion :-

We consider ~~d~~ the surfaces obtain only by rotation about an axis in its plane such as sphere, one and anchor ring, but there are surfaces which are not only generated by the rotation alone, but the rotation followed by a translation. such a motion is called a screw motion.

Helicoids :- A-19 2M

" A helicoid is a surface generated by the screw motion of a curve about a fixed line, the axis"

The various positions of the generating curve are obtained by translating it through a distance 'λ' parallel to the axis and then rotating it through an angle 'υ' about the axis, where λ/υ has a constant value 'a'

The constant 2πa is the pitch of the helicoid being the distance translated in one complete revolution.

it is positive (or) negative according as the helicoid is right (or) left handed and is zero for the surface of revolution.

Right helicoid :-

5M

This is the helicoid generated by a straight line which meets the axis at right angles. Taking the axis to be the z axis, the position vector of a general point on right helicoid is $r = (u\cos v, u\sin v, av)$ where 'u' is the distance from the axis, and 'v' is the angle of rotation. The generator being assumed to be the x axis when $v = 0$, Here u and v take all real values. The curves $v =$ constant are the generators and $u =$ constant are, circular helices.

$$\text{Now } r_1 = \frac{\partial r}{\partial u} = (\cos v, \sin v, 0)$$

$$r_2 = \frac{\partial r}{\partial v} = (-u\sin v, u\cos v; a)$$

$$r_1 \cdot r_2 = -u\cos v\sin v + u\cos v\sin v + 0$$

$$= 0$$

Hence the parametric curves of the right helicoid are orthogonal.

Metric :- 5M

Let $r = r(u, v)$ be the given surface then the parameters u and v are the functions of a single variable 't'. then $r = r(u(t), v(t))$ is a function y a single variable t along the curve $u = u(t)$, $v = v(t$

The arc length 's' is related to the parameter 't' is given by

$$\left(\frac{ds}{dt}\right)^2 = \frac{dr}{dt} \cdot \frac{dr}{dt} = \left(\frac{dr}{dt}\right)^2$$

But
$$\frac{dr}{dt} = \frac{\partial r}{\partial u}\frac{du}{dt} + \frac{\partial r}{\partial v}\frac{dv}{dt}$$
$$= r_1 \frac{du}{dt} + r_2 \frac{dv}{dt}.$$

$$\left(\frac{ds}{dt}\right)^2 = \left(\frac{dr}{dt}\right)^2 = \left(r_1 \frac{du}{dt} + r_2 \frac{dv}{dt}\right)^2$$
$$= r_1 \cdot r_1 \left(\frac{du}{dt}\right)^2 + r_2 \cdot r_2 \left(\frac{dv}{dt}\right)^2 + 2\, r_1 \cdot r_2 \, \frac{du}{dt} \cdot \frac{dv}{dt}$$
$$= E\left(\frac{du}{dt}\right)^2 + 2F\left(\frac{du}{dt}\right)\left(\frac{dv}{dt}\right) + G\left(\frac{dv}{dt}\right)^2$$

2M

where $E = r_1 \cdot r_1$, $F = r_1 \cdot r_2$, $G = r_2 \cdot r_2$

2'm

The above differential equation can be expressed as

$$(ds)^2 = E(du)^2 + 2F(du)(dv) + G(dv)^2 \quad \text{—①}$$

The R.H.S of equation ① doesnot involve the parameter 't' except in the case that 'u' and 'v' depends on 't'. The differential quadratic form in equation ① is called the first fundamental form (or) the metric of the surface.

Geometrically 'ds' can be interpreted as the "infinitesimal distance" from the point

(u, v) to the point $(u+du, v+dv)$

We know that, the identity $(r_1 \times r_2)^2 = r_1^2 r_2^2 - (r_1 \cdot r_2)^2$

The coefficients of equation ① satisfies the following conditions. $E > 0,\ G > 0,\ H^2 = EG - F^2 > 0$

These inequalities show that the metric is a positive definite quadratic form in du, dv

Example :- 2m

2m Find E, F, G and H for the paraboloid $x = u$, $y = v$, $z = u^2 - v^2$

solution :- Given $x = u, y = v, z = u^2 - v^2$

$$r = (u, v, u^2 - v^2)$$

$$r_1 = \frac{\partial r}{\partial u} = (1, 0, 2u)$$

$$r_2 = \frac{\partial r}{\partial v} = (0, 1, -2v)$$

$$E = r_1 \cdot r_1 = (1, 0, 2u) \cdot (1, 0, 2u)$$

$$= 1 + 4u^2$$

$$F = r_1 \cdot r_2 = (1, 0, 2u) \cdot (0, 1, -2v)$$

$$= -4uv$$

$$G = r_2 \cdot r_2 = (0, 1, -2v) \cdot (0, 1, -2v)$$

$$= 1 + 4v^2$$

$$H^2 = EG - F^2 = (1+4u^2)(1+4v^2) - (-4uv)^2$$

$$= 1 + 4v^2 + 4u^2 + 16u^2v^2 - 16u^2v^2$$

$$= 1 + 4u^2 + 4v^2$$

(88) (4) Angle b/w parametric

Angle between parametric curves :- (17)

5m

The parametric directions re given by r_1 and r_2. The angle 'w' ($0 < w < \pi$) between rem is given by,

$$\cos w = \frac{r_1 \cdot r_2}{|r_1|\,|r_2|} = \frac{F}{\sqrt{EG}}.$$

$$\sin w = \frac{|r_1 \times r_2|}{|r_1|\,|r_2|} = \frac{H}{\sqrt{EG}}$$

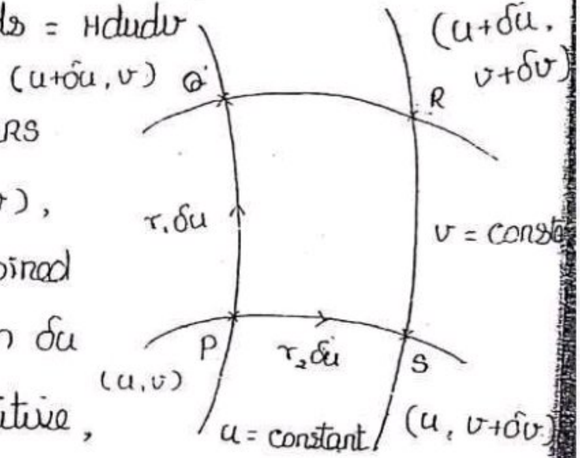
8m

Element of area :-

If 'ds' represents the element of area PQRS on the surface then ds = Hdudv

roof :-

consider the figure PQRS with vertices (u,v), $(u+\delta u, v)$, $u, v+\delta v)$, $(u+\delta u, v+\delta v)$ joined by parametric curves. when δu and δv are small and positive, this figure is approximately a parallelogram with adjacent sides given by the vectors $r_1\delta u$, $r_2\delta v$ and the area is $|r_1\delta u \times r_2\delta v| = H\,\delta u\,\delta v$



Taking du, dv in the place of $\delta u, \delta v$ then the element of area 'ds' for the surface is given by ds = Hdudv

5m

1] Calculate ~~that~~ 5m first fundamental coefficients and area of
(i) 4m Anchor ring corresponding to the domain $0 \le u \le 2\pi$.
$0 \le v \le 2\pi$

solution :-

The position vector of any point on the anchor ring is $r = ((b+a\cos u)\cos v, (b+a\cos u)\sin v, a\sin u)$

we know that, $E = r_1 \cdot r_1$, $F = r_1 \cdot r_2$, $G = r_2 \cdot r_2$

$$r_1 = \frac{\partial r}{\partial u} = (-a\sin u\cos v, -a\sin u\sin v, a\cos u)$$

$$r_2 = \frac{\partial r}{\partial v} = (-(b+a\cos u)\sin v, (b+a\cos u)\cos v, 0)$$

$$E = a^2\sin^2 u\cos^2 v + a^2\sin^2 u\sin^2 v + a^2\cos^2 u$$

$$= a^2\sin^2 u(\cos^2 v + \sin^2 v) + a^2\cos^2 u$$

$$= a^2(\sin^2 u + \cos^2 u) = a^2$$

$$F = a\sin u\cos v(b\sin v + a\cos u\sin v) - a\sin u \sin v(b\cos v + a\cos u\cos v) + 0$$

$$= ab\sin u\sin v\cos v + a^2\sin u\sin v\cos u\cos v - ab\sin u\sin v\cos v - a^2\sin u\sin v\cos u\cos v$$

$$= 0$$

$$G = (b+a\cos u)^2\sin^2 v + (b+a\cos u)^2\cos^2 v + 0$$

$$= (b+a\cos u)^2[\sin^2 v + \cos^2 v]$$

$$= (b+a\cos u)^2$$

we know that, $ds = H\,du\,dv$

$$H^2 = EG - F^2 = a^2(b+a\cos u)^2 - 0$$

$$H = a(b+a\cos u)$$

{∵ R.H.S - a^2
two variable du.
dv change to
[illegible]

$$ds = H\,du\,dv$$

integrating, $\int ds = \int_0^{2\pi}\int_0^{2\pi} H\,du\,dv$

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$$\therefore \int_0 \int_0 a(b + a\cos u)\, du\, dv$$

$$= \int_0^{2\pi} \int_0^{2\pi} (ab + a^2 \cos u)\, du\, dv$$

$$= \int_0^{2\pi} \left[abu + a^2 \sin u\right]_0^{2\pi} dv$$

$$= \int_0^{2\pi} \left[(ab2\pi + a^2 \sin 2\pi) - (0)\right] dv$$

$$= \int_0^{2\pi} 2ab\pi\, dv \quad \{\because \sin n\pi = 0\}$$

$$= 2\pi ab\, [v]_0^{2\pi} = 2\pi ab\, [2\pi - 0]$$

$$S = 4\pi^2 ab$$

Theorem :- Uni/ (5m) (4)

The metric is invariant under parametric transformation

Proof :-

Let $r = r(u, v)$ be the equation of the surface with parameter u and v

Let us transform the parameters u and v into the parameter u' and v' by the relationship $u' = \phi(u, v)$ and $v' = \varphi(u, v)$

$$r_1' = \frac{\partial r}{\partial u'} = \frac{\partial r}{\partial u}\frac{\partial u}{\partial u'} + \frac{\partial r}{\partial v}\frac{\partial v}{\partial u'}$$

$$r_1' = r_1 \frac{\partial u}{\partial u'} + r_2 \frac{\partial v}{\partial u'} \quad \text{——} ①$$

similarly, $$r_2' = r_1 \frac{\partial u}{\partial v'} + r_2 \frac{\partial v}{\partial v'} \quad \text{——} ②$$

Further, $$du = \frac{\partial u}{\partial u'} du' + \frac{\partial u}{\partial v'} dv' \quad \text{——} ③$$

$$dv = \frac{\partial v}{\partial u'} du' + \frac{\partial v}{\partial v'} dv' \quad \text{——} ④$$

If E', F' and G' are the first fundamental coefficients in the new parametric system then we have

$$E'du'^2 + 2F'du'dv' + G'dv'^2 = (r_1' . r_1')du'^2 + 2(r_1' . r_2')du'dv' + (r_2' . r_2')dv'^2$$

$$= (r_1'du' + r_2'dv')^2$$

using ① and ②,

$$= \left[\left(r_1 \frac{\partial u}{\partial u'} + r_2 \frac{\partial v}{\partial u'}\right)du' + \left(r_1 \frac{\partial u}{\partial v'} + r_2 \frac{\partial v}{\partial v'}\right)dv'\right]^2$$

$$= \left[r_1\left(\frac{\partial u}{\partial u'}du' + \frac{\partial u}{\partial v'}dv'\right) + r_2\left(\frac{\partial v}{\partial u'}du' + \frac{\partial v}{\partial v'}dv'\right)\right]^2$$

$$= [r_1 du + r_2 dv]^2, \text{ using ③ and ④}$$

$$= r_1 . r_1 du^2 + 2r_1 . r_2 dudv + r_2 . r_2 dv^2$$

$$= Edu^2 + 2Fdudv + Gdv^2$$

∴ The metric is invariant

2m

Direction coefficients :- 2m

Consider a surface $r = r(u,v)$ and 'p' be any point on a surface. Then we know that the vectors r_1 and r_2 are tangents to the parametric curves v = constant and u = constant passing through 'p'.

Let 'N' be the surface normal at 'p'.

Since $r_1 \times r_2 \neq 0$. r_1 and r_2 are linearly independent. Thus for any point 'p' on the surface there are three linearly independent vectors N, r_1, r_2

Hence every vector a through p can be expressed in the form $a = a_n N + \lambda r_1 + \mu r_2$ — ①

where the scalars a_n, λ and μ are define uniquely by this relation. Thus equation ① expresses any vector through 'p' as the sum of two vectors $a_n N$ normal to the surface and $\lambda r_1 + \mu r_2$ lying in the tangent plane to the surface at 'p'.

Taking dot product with 'N' on bothsides of equation ①, we get

$$a \cdot N = (a_n N + \lambda r_1 + \mu r_2) \cdot N$$

$$a \cdot N = a_n N \cdot N \quad \{\because (N, r_1, r_2) \text{ are triplet so } r_1 \cdot N = r_2 \cdot N = 0 \text{ and } N \cdot N = 1\}$$

$$\therefore aN = a_n$$

(i) The scalar 'a_n' is called normal component of 'a'

(ii) The vector 'a' lies in the tangent plane at 'p' iff $a_n =$

(iii) The vector $\lambda r_1 + \mu r_2$ is called the tangential part of 'a' and λ, μ are called the tangential components of 'a'

directional on the surface :- 2m

The direction of any tangent line to a surface at a point 'p' is called the directional on the surface at the point 'p'

If 'a' is a vector (λ, μ) then ~~|a|~~ = $a = \lambda r_1$

$|a| = |\lambda r_1 + \mu r_2|$

$$\sqrt{(a)^2} = \sqrt{(\lambda r_1 + \mu r_2)^2} \Rightarrow a^2 = (\lambda r_1 + \mu r_2)^2$$

$$a^2 = \lambda^2 r_1 \cdot r_1 + 2\lambda\mu\, r_1 \cdot r_2 + \mu^2 r \quad r$$

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$$|a|^2 = E\lambda^2 + 2F\lambda\mu + G\mu^2$$

$$\therefore \quad a = \sqrt{E\lambda^2 + 2F\lambda\mu + G\mu^2}$$

which is the magnitude of the tangential vector interms of the components and first fundamental coefficients E, F and G.

irectional Coefficients :-

(P) Let 'b' be the unit vector along the tangential vector 'a' at 'p'. Let the components of 'b' be (l, m) then 'b' can be written as $b = lr_1 + mr_2$. The components of unit vector 'b' at 'p' along the direction 'a' are called directional coefficients of 'a'

since $b = lr_1 + mr_2$ and $|b| = 1$ —(*)

$$|lr_1 + mr_2|^2 = (lr_1 + mr_2)^2$$

$$|b|^2 = l^2 r_1 \cdot r_1 + 2lm\, r_1 \cdot r_2 + m^2 r_2 \cdot r_2$$

$$1 = El^2 + 2Flm + Gm^2, \text{ using } (*)$$

Hence directional coefficients satisfies the above identity.

Note :-

* The directional coefficients are $(\cos\varphi, \sin\varphi)$ where φ is the angle between the plane and the x axis then the metric becomes dx^2+dy^2 and the above identity becomes $\cos^2\varphi + \sin^2\varphi = 1$

* The directional coefficients (l, m) are analogous to the direction cosines (l, m, n) satisfying the identity [illegible] solid geometry [illegible] $l^2 + m^2 + n^2 = 1$ [illegible]

5m

Theorem:

If (l, m) and (l', m') are directional coeffici… of two directions at a point 'p' on a surface and 'θ' is an angle between the two directions at 'p', then

(i) $\cos\theta = Ell' + F(lm' + ml') + Gmm'$

(ii) $\sin\theta = H(lm' - l'm)$

Proof:-

If (l, m) and (l', m') are the directional coeffici-nts of the two directions at the same point 'p' on a surface $r = r(u, v)$, then the corresponding unit vectors along 'p' are $a = lr_1 + mr_2$, $a' = l'r_1 + m'r_2$ —①

let 'θ' be the angle between the two directions then we have $a \cdot a' = \cos\theta$, $a \times a' = N \sin\theta$ —②
[for assumption]

(i) consider $a \cdot a' = (lr_1 + mr_2) \cdot (l'r_1 + m'r_2)$

$$\cos\theta = ll'(r_1 \cdot r_1) + lm'(r_1 \cdot r_2) + ml'(r_2 \cdot r_1) + mm'(r_2 \cdot r_2)$$

using ②, $\cos\theta = ll'E + (lm' + ml')F + mmG$

(ii) consider, $a \times a' = (lr_1 + mr_2) \times (l'r_1 + m'r_2)$

using ③, $N\sin\theta = ll'(r_1 \times r_1) + lm'(r_1 \times r_2) + ml'(r_2 \times r_1) + mm'(r_2 \times r_2)$

$N\sin\theta = 0 + (r_1 \times r_2) lm' - (r_1 \times r_2) ml' + 0$

∴ ஏன் [illegible] '-' வரும் ?

$N\sin\theta = (r_1 \times r_2)(lm' - ml')$

$$\sin\theta = \frac{(r_1 \times r_2)}{N}(lm' - ml') = H(lm' - m$$

$$\left\{\because N = \frac{r_1 \times r_2}{H} \Rightarrow H = \frac{(r_1 \times r_2)}{N},\ H = |r_1 \times r_2| \neq 0\right\}$$

ectional ratio's of the direction :-

If (l, m) are the direction al coefficients of the direction at a point 'p' on the surface then the scalar (λ, μ) which are proportional to (l, m) are called differential ratio's of the direction.

The relation between (l, m) and (λ, μ) :-

Suppose (l, m) are the directional coefficients and (λ, μ) are the directional ratio's of a surface at a point 'p' then (l, m) are proportional to (λ, μ)

$$\text{let } l/\lambda = m/\mu = K \text{ (say)}$$

$$\text{Then } l = K\lambda,\ m = K\mu \text{ —①}$$

We know that, the directional coefficients must satisfy the identity $El^2 + 2Flm + Gm^2 = 1$ —②

using ① in ②. $EK^2\lambda^2 + 2FK^2\lambda\mu + GK^2\mu^2 = 1$

$$K^2\left[E\lambda^2 + 2F\lambda\mu + G\mu^2\right] = 1$$

$$K^2 = 1 \Big/ \left[E\lambda^2 + 2F\lambda\mu + G\mu^2\right] \text{ —③}$$

$$K = 1 \Big/ \sqrt{E\lambda^2 + 2F\lambda\mu + G\mu^2} \text{ and}$$

$$1 \Big/ \sqrt{E\lambda^2 + 2F\lambda\mu + G\mu^2}$$

using ③ in ①. l

Note :

If $r_1 = (1, 0)$ and $r_2 = (0, 1)$ are the directions ratio's for the parametric directions then the directions coefficients are $\left(\frac{1}{\sqrt{E}}, 0\right)$ and $\left(0, \frac{1}{\sqrt{G}}\right)$.

(7) 2m A-19 10M 2M

(Q) Find the coefficients of the directions which makes an angle $\pi/2$, with the direction whose coefficients are (l, m)

Solution :-

The angle between the directional coefficient is $\theta = \pi/2$, one of the directional coefficients is (l, m)

we have to find the another directional coefficients (l', m')

We know that, $\cos\theta = Ell' + F(lm' + ml') + Gmm'$ — ①

and $\sin\theta = H(lm' - ml')$ — ②

when $\theta = \pi/2$, ① => $\cos \pi/2 = Ell' + F(lm' + ml') + Gmm'$

$$0 = Ell' + F(lm' + ml') + Gmm'$$

$$l'[El + Fm] + m'[Fl + Gm] = 0$$

$$l'[El + Fm] = -m'[Fl + Gm]$$

$$\frac{l'}{-[Fl + Gm]} = \frac{m'}{[El + Fm]} = \alpha \text{ (say)}$$

$$\left.\begin{aligned} l' &= -\alpha(Fl + Gm) \\ m' &= \alpha(El + Fm) \end{aligned}\right\} \text{ — ③}$$

when $\theta = \pi/2$, ② => $\sin \pi/2 = H(lm' - ml')$

$$1 = Hlm' - Hml'$$

$$1 = Hl[\alpha(El + Fm)] - Hm[-\alpha(Fl + Gm)]$$

$\therefore \alpha \left[HEl^2 + HFlm + HFlm + HGm^2 \right] = 1$

$\alpha H \left[El^2 + Flm + Flm + Gm^2 \right] = 1$

$$\alpha = 1 \Big/ H\left[El^2 + 2Flm + Gm^2 \right] = 1/H$$

{$\because$ The directional coefficients must satisfy the identity $El^2 + 2Flm + Gm^2 = 1$}

From ⑤, $l' = \frac{-1}{H}(Fl+Gm)$

$$m' = \frac{1}{H}(El+Fm)$$

Families of curves :- 2m

A family of curves on a surface is a system given by an implicit equation of the form $\phi(u,v) = c$, where ϕ is single valued and has continuous derivatives ϕ_1, ϕ_2 which do not vanish together, and 'c' is a real parameter.

From the above definition we have the following properties.

(i) There is just one curve of the family passing through every point (u,v) of this surface;

(ii) At any point (u,v) the tangent to the curve through the point has direction ratio's $(-\phi_2, \phi_1)$

Theorem :-

The curves of a family $\phi(u,v)$ = constant are the solutions of the differential equations $\phi_1 du + \phi_2 dv = 0$ or the differential equation of the [illegible] first order differential [illegible] form $p(u,v)\,du + Q(u,v)\,dv = 0$ —— ② where p and Q are differential functions which donot vanish simultaneously, always define a family of curves

proof:-

since $\phi_1 = \frac{\partial \phi}{\partial u}$, $\phi_2 = \frac{\partial \phi}{\partial v}$ then the equation ① becomes $\frac{\partial \phi}{\partial u}\,du + \frac{\partial \phi}{\partial v}\,dv = 0$

$$\Rightarrow d\phi = 0$$

Hence we conclude that $\phi(u,v) = \text{constant} = c$

conversely, let us consider ②, unless this equation is exact, it is not (in general) possible to find the single functions $\phi(u,v)$ such that $\phi_1 = P$ and $\phi_2 = Q$

However, we can find integrating factor $\lambda(u,v)$ such that $\lambda P = \phi_1$ and $\lambda Q = \phi_2$

substitute the value in ①,

$$\lambda P\,du + \lambda Q\,dv = 0$$

$$\phi_1\,du + \phi_2\,dv = 0$$

which shows that the solution of the equation is $\phi(u,v) = c$

Note:-

The condition for the orthogonal direction is $\cos\theta = 0$ interms of the direction ratio's, this becomes

(A) 2m $$E\lambda\lambda' + F(\lambda\mu' + \mu\lambda') + G\mu\mu' = 0$$

Orthogonal Trajectories:- 2m

let $\phi(u,v) = c$ be the equation of the given family of curves on the surface $r = r(u,v)$.

If there exist second (another) family of curves $\psi(u,v) = k$ lying on the surface suchthat at every poin of the surface of two curves, one from each family are orthogonal. Then the second family of curves are called the orthogonal trajectories of the first family of curves

1) On the paraboloid $x^2 - y^2 = z^2$, find the orthogonal trajectories in the sections by the planes z = constant A-19 5M

solution :-

The parametric representation of the given surface can be taken as $x = u$, $y = v$, $z = u^2 - v^2$

Hence the position vector of any point on the surface is $r(u,v) = (u, v, u^2 - v^2)$

Since the given family of curves are the sections by the planes z = constant, the first family of curves is $\phi(u,v) = u^2 - v^2$ = constant —①

The tangential directions of any point on the surface is given by $(-\phi_2, \phi_1) = (2v, 2u)$

(ie) $(-\phi_2, \phi_1) = (v, u)$

The condition for orthogonal direction is $E\lambda\lambda' + F(\lambda\mu' + \mu\lambda') + G\mu\mu' = 0$. If (du, dv) is orthogonal to the direction (u, v) then the above condition can be written as. $+ Gudv = 0$ —②

$E v u du + F(v dv + u du$

$)$

$r_1 = \frac{\partial r}{\partial u} = (1, 0, 2u)$

$r_2 = \frac{\partial r}{\partial v} = (0, 1, -2v)$

$$\left.\begin{array}{l} E = r_1 \cdot r_1 = 1+4u^2 \\ F = r_1 \cdot r_2 = -4uv \\ G = r_2 \cdot r_2 = 1+4v^2 \end{array}\right\} \text{——} ③$$

using ③ in ②,

$$(1+4u^2)\, v\,du + (-4uv)(v\,dv + u\,du) + (1+4v^2)\, u\,dv = 0$$

$$v\,du + 4u^2 v\,du - 4uv^2 dv - 4u^2 v\,du + u\,dv + 4uv^2 dv = 0$$

$$u\,dv + v\,du = 0$$

$$d(uv) = 0 \text{ —— } ④$$

Integrating, $uv = \text{constant}$

∴ The orthogonal trajectories are given by $uv = \text{constant}$.

They are the section of the paraboloid by the hyperbolic cylinders $xy = \text{constant}$

Double family of curves :-

If P, Q and R are continuous functions of u and v which do not vanish together, the quadratic differential equation $Pdu^2 + 2Qdudv + Rdv^2 = 0$ represents two families of curves provided that $Q^2 - PR > 0$

The differential equations for the separate families are found by solving the quadratic for the ratio $du : dv$

Theorem :-

The two families of curves given by $p\,du^2 + 2Q\,du\,dv + R\,dv^2 = 0$ — ① are orthogonal on the surface iff $ER - 2QF + Gp = 0$

proof :-

If (l, m) and (l', m') are the direction coefficients of the two families of the curves given by the equation ①, At any point 'p' on the surface then l/m and l'/m' are the roots of the quadratic equations ①

Hence the sum of roots is $\frac{l}{m} + \frac{l'}{m'} = \frac{-2Q}{P}$

and the product of roots is $\frac{ll'}{mm'} = \frac{R}{P}$ — ②

The given directional coefficients are orthogonal iff $E\lambda\lambda' + F(\lambda\mu' + \mu\lambda') + G\mu\mu' = 0$

$$\Leftrightarrow \quad Ell' + F(lm' + ml') + Gmm' = 0$$

$$\Leftrightarrow \quad E\frac{ll'}{mm'} + F\left(\frac{l}{m} + \frac{l'}{m'}\right) + G = 0$$

$$\Leftrightarrow \quad E\left(R/p\right) + F\left(-2Q/p\right) + G = 0 \text{, using ②}$$

$$\Leftrightarrow \quad ER - 2QF + Gp = 0$$

5M

1) prove that. If 'θ' is an angle at the point (u, v) between the two directions given by $p\,du^2 + 2Q\,du\,dv + R\,dv^2 = 0$ then $\tan\theta = \frac{2H(Q^2 - PR)^{1/2}}{ER - 2QF + Gp}$

└①

Solution :

If (l, m) and (l', m') are the directional coefficients of the tangent directions at a point of the double families of curves given by equation ①, then the sum of roots is $l/m + l'/m' = -2Q/P$ and the product of the roots is $ll'/mm' = R/P$ —②

If 'θ' is an angle between the two directions

then $\tan\theta = \sin\theta / \cos\theta$

$$\tan\theta = \frac{H(lm' - l'm)}{Ell' + F(lm' + ml') + Gmm'}$$, using previous results

$$= \frac{H\left(l/m - l'/m'\right)}{E\frac{ll'}{mm'} + F\left(\frac{l}{m} + \frac{l'}{m'}\right) + G}$$, {∵ divided by mm' on up and down}

$$= \frac{H\left[\left(\frac{l}{m} + \frac{l'}{m'}\right)^2 - \frac{4ll'}{mm'}\right]^{1/2}}{E\frac{ll'}{mm'} + F\left(\frac{l}{m} + \frac{l'}{m'}\right) + G}$$, {∵ இதனை சுருக்கினால் previous term வரும்}

$$= \frac{H\left[\left(-2Q/P\right)^2 - 4\left(R/P\right)\right]^{1/2}}{E\left(R/P\right) + F\left(-2Q/P\right) + G}$$

$$= \frac{H\left[\frac{4Q^2}{P^2} - \frac{4R}{P}\right]^{1/2}}{\frac{ER - 2FQ + GP}{P}} = \frac{H\left[4/P^2\left(Q^2 - RP\right)\right]^{1/2}}{\frac{ER - 2FQ + GP}{P}}$$

$$\therefore \tan\theta = \frac{2H/P\left[Q^2-RP\right]^{1/2}}{\frac{ER-2FQ+GP}{P}} = \frac{2H(Q^2-PR)^{1/2}}{ER-2FQ+GP}$$

Isometric correspondence :-

In this section, we are going to discuss the correspondence between surfaces S and S'. We shall consider only correspondence between parts of the surfaces. Each part will be assumed to carry a parametric system, so that if the point (u', v') on S' corresponds to the point (u, v) on S then u', v' are single valued functions of u and v say $u' = \phi(u,v)$, $v' = \psi(u,v)$

If S and S' are of class r and r' respectively, we assume that ϕ and ψ are functions of class minimum $\{r, r'\}$ with Jacobian $\frac{\partial(\phi,\psi)}{\partial(u,v)} \neq 0$ in the domain u, v

We also assume that the mapping is one-to-one through out this domain. Thus we have the maps between a part of S and a part of S' to be differentiable homeomorphism of sufficiently high class, regular at each point of the domain of u.v.

Isometric :-

Two surfaces S and S' are said to be isometric (or) applicable, if there exist a correspondence

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$= \phi(u,v)$, $v' = \psi(u,v)$ between their parameters, are ϕ and ψ are single value and $\partial(\phi,\psi)/\partial(u,v) \neq 0$ ch that the metric of 's' is transformed into a metric of Then the correspondence itself is called an isometric.

e :-

Two surfaces s and s' are said to be isometric) applicable, if there is a correspondence between the oints of s and s' such that correspondings arcs of curves are the same length. The correspondence is called an ometry."

Repeated.

Find the surface of revolution which is isometric with region of right helicoid.

lution :- let 's' be the surface of revolution given by

$= (g(u)\cos v, g(u)\sin v, f(u))$ —— ①

The surface s' which is isometric with 's' is right helicoid and is given by $r' = (u'\cos v', u'\sin v', av')$ —— ②

The metric of 's' is given by

$$S = Edu^2 + 2Fdudv + Gdv^2 \text{ —— ③}$$

here $E = r_1 \cdot r_1$, $F = r_1 \cdot r_2$, $G = r_2 \cdot r_2$

$$r_1 = \frac{\partial r}{\partial u} = (\dot{g}(u)\cos v, \dot{g}(u)\sin v, f(u))$$

$$r_2 = \frac{\partial r}{\partial v} = (-g(u)\sin v, g(u)\cos v, 0)$$

$\therefore E = (g'(u))^2 \cos^2 v + (g'(u))^2 \sin^2 v + (f'(u))^2$

$= (g'(u))^2 [\cos^2 v + \sin^2 v] + (f'(u))^2$

$E = g'^2 + f'^2$

$F = -g(u) g'(u) \sin v \cos v + g(u) g'(u) \sin v \cos v + 0$

$F = 0$.

$G = (g(u))^2 \sin^2 v + (g(u))^2 \cos^2 v + 0$

$= (g(u))^2 [\sin^2 v + \cos^2 v]$

$G = g^2$

Using E, F and G in equation in ③, the metric of

$S = (g'^2 + f'^2) du^2 + 2(0) du dv + g^2 dv^2$

$= (g'^2 + f'^2) du^2 + g^2 dv^2 \rightarrow$ ④

From ②, $r_1' = \frac{\partial r'}{\partial u'} = (\cos v', \sin v', 0)$

$r_2' = \frac{\partial r'}{\partial v'} = (-u' \sin v', u' \cos v', a)$

The metric of S' is given by $E' du'^2 + 2 F' du dv + G' dv^2$ ⑤

where $E' = r_1' \cdot r_1'$, $F' = r_1' \cdot r_2'$, $G' = r_2' \cdot r_2'$

$E' = \cos^2 v' + \sin^2 v' + 0 = 1$

$F' = -u' \sin v' \cos v' + u' \sin v' \cos v' + 0 = 0$

$G' = u'^2 \sin^2 v' + u'^2 \cos^2 v' + a^2 = u'^2 + a^2$

using E', F' and G' in equation ⑤, the metric of

$S' = du'^2 + (u'^2 + a^2) dv'^2$ —— ⑥

To find the transformation $(u, v) \rightarrow (u', v')$ from S to S' make these metric are identical

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taking $u' = \phi(u)$, $v' = v$ $\{\because \phi_1 = \frac{\partial \phi}{\partial u}\} = \phi_1$

$\therefore du' = \phi_1 du$, $dv' = dv$ — ⑦

Using equation ⑦ in equation ⑥ we got

$s' = \phi_1^2 du^2 + (\phi^2 + a^2) dv^2$ — ⑧

Equation ⑧ is a metric transformation. Hence equations ④ and ⑧ are identical. So that we have

$g'^2 + f'^2 = \phi_1^2$ and $\phi^2 + a^2 = g^2$ — ⑨

From equation ⑨ we have to obtain f and g by eliminating 'ϕ'. Let us assume that $\phi(u) = a \sinh u$, $g(u) = a \cosh u$ — ⑩

Using equation ⑩ in equation ⑨ we get

$$g'^2 + f'^2 = \phi_1^2$$

$$a^2 \sinh^2 u + f'^2 = a^2 \cosh^2 u$$

$$f'^2 = a^2 [\cosh^2 u - \sinh^2 u]$$

$$f'^2 = a^2 \Rightarrow f(u) = a$$

$\{\because g = a \cosh u$

$g' = a \sinh u$

$\phi = a \sinh u$

$\phi' = a \cosh u$

Integrating, $f(u) = au$

Hence the surface of the revolution is generated by $x = a \cosh u$, $y = 0$, $z = au$ about the z axis

B) Intrinsic properties :-

Thm. Let E, F and G are any real single valued continuous functions of u and v satisfying $E > 0$ and $EG - F^2 > 0$ in some domain R of (u, v) then every point of R has a neighbourhood R' of R in which

$E du^2 + 2F du dv + G dv^2$ is the metric of the surface referred to u and v as parameters. The vector function $r(u, v)$ satisfying the partial differential equation $E = r_1^2$, $F = r_1 \cdot r_2$, $G = r_2^2$ in some domain D'. This kind of properties are called intrinsic properties.

2m Unit - 3

Geodesics :- 2m

①

Let A and B be any two points on a surface 's' and let these points be joined by curves lying on 's' then any curve possessing the shortest distance over S is called Geodesics.

Note :-

Let A and B be two points on the surface $r = r(u, v)$, and the arcs ~~with~~ which join A and B are given by the equation of the form $u = u(t)$, $v = v(t)$, where $u(t)$ and $v(t)$ are of class two.

Let us assume that for every arc 'α', $t = 0$, at A and $t = 1$ at B: so that 'α' is given by $0 \leq t \leq 1$. Let $s(\alpha)$ be the length of the arc. We know that $ds^2 = E du^2 + 2F du dv + G dv^2$ (ie) $\dot{s}^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$

$$\therefore S(\alpha) = \int_0^1 \dot{s}\, dt = \int_0^1 \sqrt{E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2}\, dt$$

where u(t) and v(t) are substituted for u and v in E, F and G. Suppose that an arc α' is obtained

By deforming 'α' slightly, keeping its ends A and B fixed, then α' is given by the equation of the form

$$u = u'(t) = u(t) + \xi\lambda(t)$$
$$v = v'(t) = v(t) + \xi\mu(t)$$

where $\xi > 0$ is small and λ, μ are arbitrary function of 't' of class two in $0 \le t \le 1$ and satisfying $\lambda = \mu = 0$ at $t = 0$ and $t = 1$. Let us denote the length of the arc α' as $S(\alpha')$

The variation in $S(\alpha)$ is given by $S(\alpha') - S(\alpha)$.

Definition :-

If α is such that the variation in $S(\alpha)$ is atmost of order ξ^2 for all small variation in α for different $\lambda(t)$ and $\mu(t)$, then $S(\alpha)$ is said to be stationary and 'α' is the geodesic.

geodesic eqn

Theorem :- 5M 5M

Derive d.E SD

(Th) 5M A necessary and sufficient condition for a curve $u = u(t)$, $v = v(t)$ on the surface $r = r(u,v)$ to be geodesic is that $U\frac{\partial T}{\partial \dot v} - V\frac{\partial T}{\partial \dot u} = 0$ —① where

$U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot u}\right) - \frac{\partial T}{\partial u} = \frac{1}{2T}\frac{dT}{dt}\frac{\partial T}{\partial \dot u}$, $V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot v}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T}\frac{dT}{dt}\frac{\partial T}{\partial \dot v}$ —② and $T = \frac{1}{2}\left[E\dot u^2 + 2F\dot u\dot v + G\dot v^2\right]$

[∴ The equations ② are called geodesic equations] which is the contradiction to the hypothesis $\int_0^1 v(t)g(t)\,dt = 0$ for all admissible function $v(t)$

Hence our assumption $g(t_0) \neq 0$ is wrong. consequently $g(t) = 0$ for all 't' in $(0,1)$

proof of the main theorem :-

Let $f(u, v, \dot{u}, \dot{v}) = \sqrt{2T}$ where

$$2T(u, v, \dot{u}, \dot{v}) = \dot{s}^2 = E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2$$

The arc length $s(\alpha)$ is given by $s(\alpha) = \int_0^1 \dot{s}\,dt$

$$s(\alpha) = \int_0^1 \sqrt{2T}\,dt = \int_0^1 f(u, v, \dot{u}, \dot{v})\,dt$$

After a slight deformation of the arc 'α' we get a new arc α' whose length is given by $s(\alpha')$ and

$$S(\alpha') = \int_0^1 f(u+\xi\lambda,\ v+\xi\mu,\ \dot{u}+\xi\dot{\lambda},+\ \dot{v}+\xi\dot{\mu})\,dt$$

$$S(\alpha') - S(\alpha) = \int_0^1 \left[f(u+\xi\lambda,\ v+\xi\mu,\ \dot{u}+\xi\dot{\lambda},\ \dot{v}+\xi\dot{\mu}) - f(u, v, \dot{u}, \dot{v}) \right] dt \quad \text{—⑥}$$

Using Taylor's theorem we have

$$f(u+\xi\lambda,\ v+\xi\mu,\ \dot{u}+\xi\dot{\lambda},\ \dot{v}+\xi\dot{\mu}) = f(u, v, \dot{u}, \dot{v}) + \xi\lambda\frac{\partial f}{\partial u} + \xi\mu\frac{\partial f}{\partial v} + \xi\dot{\lambda}\frac{\partial f}{\partial \dot{u}} + \xi\dot{\mu}\frac{\partial f}{\partial \dot{v}} + o(\xi^2) \quad \text{—⑥}$$

Using ⑥ in ⑤,

$$S(\alpha') - S(\alpha) = \xi\int_0^1 \left(\lambda\frac{\partial f}{\partial u} + \mu\frac{\partial f}{\partial v} + \dot{\lambda}\frac{\partial f}{\partial \dot{u}} + \dot{\mu}\frac{\partial f}{\partial \dot{v}}\right) + o(\xi^2) \quad \text{—⑦}$$

Now consider,

Unit M 1/2 ⑤m Repeat ⑧

1. Prove that the curves of the family $\frac{v^3}{u^2}$ = constant

are geodesics in the surface with a metric $v^2du^2 - 2uv\,du\,dv + 2u^2dv^2$, $(u>0, v>0)$ used Trans 5m

solution :-

let $u = ct^3$, $v = ct^2$

$$\dot u = 3ct^2, \quad \dot v = 2ct$$

$$2T = v^2du^2 - 2uv\,du\,dv + 2u^2dv^2$$

$$T = \frac{1}{2}\left[v^2\dot u^2 - 2uv\dot u\dot v + 2u^2\dot v^2\right]$$

We know that, $U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot u}\right) - \frac{\partial T}{\partial u}$

$$V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot v}\right) - \frac{\partial T}{\partial v}$$

$$\frac{\partial T}{\partial u} = \frac{1}{2}\left[0 - 2v\dot u\dot v + 4u\dot v^2\right]$$

$$= \frac{1}{2}\left[-2(ct^2)(3ct^2)(2ct) + 4(ct^3)(2ct)^2\right]$$

$$= -6c^3t^5 + 8c^3t^5 = 2c^3t^5.$$

$$\frac{\partial T}{\partial v} = \frac{1}{2}\left[2v\dot u^2 - 2u\dot u\dot v + 0\right] = v\dot u^2 - u\dot u\dot v$$

$$= (ct^2)(3ct^2)^2 - (ct^3)(3ct^2)(2ct)$$

$$= 9c^3t^6 - 6c^3t^6 = 3c^3t^6$$

$$\frac{\partial T}{\partial \dot u} = \frac{1}{2}\left[v^2 2\dot u - 2uv\dot v\right] = \dot u v^2 - uv\dot v$$

$$= (3ct^2)(ct^2)^2 - (ct^3)(ct^2)(2ct)$$

$$= 3c^3t^6 - 2c^3t^6 = c^3t^6$$

$$\frac{\partial T}{\partial \dot v} = \frac{1}{2}\left[-2uv\dot u + 4u^2\dot v\right] = -uv\dot u + 2u^2\dot v$$

⑨

$$= -.(ct^3)(ct^2)(3ct^2) + 2(ct^3)^3(2ct)$$

$$= -3c^3t^7 + 4c^3t^7 = c^3t^7$$

$$U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = \frac{d}{dt}(c^3t^6) - 2c^3t^5$$

$$= 6c^3t^5 - 2c^3t^5 = 4c^3t^5$$

$$V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = \frac{d}{dt}(c^3t^7) - 3c^3t^6$$

$$= c^3 7t^6 - 3c^3t^6 = 4c^3t^6.$$

If the given family of curves v^3/u^2 is geodesic then $U\frac{\partial T}{\partial \dot{v}} - V\frac{\partial T}{\partial \dot{u}} = 0$

$$U\frac{\partial T}{\partial \dot{v}} - V\frac{\partial T}{\partial \dot{u}} = (4c^3t^5)(c^3t^7) - (4c^3t^6)(c^3t^6)$$

$$= 4c^6t^{12} - 4c^6t^{12} = 0$$

The given family of curves v^3/u^2 = constant is geodesic

5m cycle test

ⓐ 2] Prove that on the general surface, a necessary and sufficient condition that the curve $v = c$ be a geodesic is $EE_2 + FE_1 - 2EF_1 = 0$

$U\frac{\partial T}{\partial \dot{v}} - V\frac{\partial T}{\partial \dot{u}} = 0$

Solution :- On the curve $v = c$, 'u' can be taken as parameter

(ie) $u = t$, $v = c$

$\dot{u} = 1$, $\dot{v} = 0$ ---- (1)

We know that, if the given curve is geodesic then

[illegible] ---- (i)

where $U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u}$, $V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v}$ ---- ②

$$T = \frac{1}{2}\left[E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2\right]$$

$$\frac{\partial T}{\partial u} = \frac{1}{2}\left[E_1\dot{u}^2 + 2F_1\dot{u}\dot{v} + G_1\dot{v}^2\right] = \frac{1}{2}E_1$$

$$\frac{\partial T}{\partial v} = \frac{1}{2}\left[E_2\dot{u}^2 + 2F_2\dot{u}\dot{v} + G_2\dot{v}^2\right] = \frac{1}{2}E_2$$

{∵ E, F and G are functions of u and v }

$$\frac{\partial T}{\partial \dot{u}} = \frac{1}{2}\left[2E\dot{u} + 2F\dot{v}\right] = E\dot{u} + F\dot{v} = E$$

$$\frac{\partial T}{\partial \dot{v}} = \frac{1}{2}\left[2F\dot{u} + 2G\dot{v}\right] = F\dot{u} + G\dot{v} = F$$

② ⇒ $U = \frac{d}{dt}(E) - \frac{E_1}{2} = \frac{\partial E}{\partial u}\frac{du}{dt} + \frac{\partial E}{\partial v}\cdot\frac{dv}{dt} - \frac{E_1}{2}$

$= E_1\dot{u} + E_2\dot{v} - E_1/2$, using ⊛

$= E_1 - E_1/2 = E_1/2$

$V = \frac{d}{dt}(F) - \frac{E_2}{2} = \frac{\partial F}{\partial u}\frac{du}{dt} + \frac{\partial F}{\partial v}\frac{dv}{dt} - \frac{E_2}{2}$

$= F_1\dot{u} + F_2\dot{v} - E_2/2 = F_1 - E_2/2$, using ⊛

① ⇒ $U\frac{\partial T}{\partial \dot{v}} - V\frac{\partial T}{\partial \dot{u}} = 0$

$$E_1/2\,(F) - (F_1 - E_2/2)\,E = 0$$

$$FE_1/2 - EF_1 + EE_2/2 = 0$$

$$FE_1 - 2EF_1 + EE_2 = 0$$

5] Prove that on the general surface, a necessary and sufficient condition that the curve $u = c$ be a geodesic is $GG_1 + FG_2 - 2GF_2 = 0$

⑪

solution :- on the curve $u = c$, v can be taken as the parameter 't' (ie) $u = c$, $v = t$

$$\dot{u} = 0 , \dot{v} = 1 \text{ ——— } ⊛$$

we know that, If the given curve is geodesic then

$$U \frac{\partial T}{\partial \dot{v}} - V \frac{\partial T}{\partial \dot{u}} = 0 \text{ ——— } ①$$

where $U = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u}$, $V = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v}$ ——— ②

$$T = \frac{1}{2}\left[E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2\right]$$

$$\frac{\partial T}{\partial u} = \frac{1}{2}\left[E_1\dot{u}^2 + 2F_1\dot{u}\dot{v} + G_1\dot{v}^2\right] = \frac{1}{2}G_1$$

$$\frac{\partial T}{\partial v} = \frac{1}{2}\left[E_2\dot{u}^2 + 2F_2\dot{u}\dot{v} + G_2\dot{v}^2\right] = \frac{1}{2}G_2$$

{∵ E, F and G are functions of u and v}

$$\frac{\partial T}{\partial \dot{u}} = \frac{1}{2}\left[2E\dot{u} + 2F\dot{v}\right] = E\dot{u} + F\dot{v} = F$$

$$\frac{\partial T}{\partial \dot{v}} = \frac{1}{2}\left[2F\dot{u} + 2G\dot{v}\right] = F\dot{u} + G\dot{v} = G$$

② ⇒ $U = \frac{d}{dt}(F) - \frac{1}{2}G_1 = \frac{\partial F}{\partial u}\frac{du}{dt} + \frac{\partial F}{\partial v}\frac{dv}{dt} - \frac{1}{2}G_1$

$= F_1\dot{u} + F_2\dot{v} - \frac{1}{2}G_1 = F_2 - \frac{1}{2}G_1$, using ⊛

$V = \frac{d}{dt}(G) - \frac{1}{2}G_2 = \frac{\partial G}{\partial u}\frac{du}{dt} + \frac{\partial G}{\partial v}\frac{dv}{dt} - \frac{1}{2}G_2$

$= G_1\dot{u} + G_2\dot{v} - \frac{1}{2}G_2 = G_2 - \frac{1}{2}G_2 = \frac{1}{2}G_2$

① ⇒ $U \frac{\partial T}{\partial \dot{v}} - V \frac{\partial T}{\partial \dot{u}} = 0$

$(F_2 - \frac{1}{2}G_1)(G) - (\frac{1}{2}G_2)(F) = 0$

$GF_2 - \frac{1}{2}GG_1 - \frac{1}{2}FG_2 = 0 \Rightarrow 2GF_2 - GG_1 - FG_2 = 0$

$\therefore GG_1 + FG_2 - 2GF_2 = 0$

Canonical geodesic equations :-

Theorem :-

2M [If the arc length 's' is the parameter of the curve then the geodesic equations are $U = \frac{d}{ds}\left(\frac{\partial T}{\partial \dot{u}}\right) - \frac{\partial T}{\partial u} = 0$, $V = \frac{d}{ds}\left(\frac{\partial T}{\partial \dot{v}}\right) - \frac{\partial T}{\partial v} = 0$ —— ① the equations ① are called canonical geodesic equations.] 2M

proof :- since the geodesic equations are true for any arbitrary parameter 't'. It is true for the parameter 's' also

(ie)

$$U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = \frac{1}{2T}\frac{dT}{ds}\frac{\partial T}{\partial u'}$$

$$V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = \frac{1}{2T}\frac{dT}{ds}\frac{\partial T}{\partial v'} \quad \text{——} ②$$

where $T = \frac{1}{2}\left[E\dot{u}^2 + 2F\dot{u}\dot{v} + G\dot{v}^2\right]$

since $u' = du/ds = l$, $v' = \frac{dv}{ds} = m$ are the differential coefficients at any point on the curve and $El^2 + 2Flm + Gm^2 = 1$, we have

$$T = \frac{1}{2}\left[El^2 + 2Flm + Gm^2\right] = \frac{1}{2}$$

$$\therefore T = \frac{1}{2}$$

$$dT/ds = 0 \quad \text{——} ③$$

Using ③ in ② we get, $U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = 0$

$$V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = 0.$$

Hence the proof

Theorem :-

(i) If the curves on the surface are not parametric curves then the sufficient conditions for the curve to be geodesic is either $U = 0$ (or) $V = 0$

uni ② m (ii) For the parametric curve u = constant to be a geodesic, a sufficient condition is $V = 0$; similarly $U = 0$ is a sufficient condition for a curve v = constant to be a geodesic

proof :-

(i) If 's' is used as a parameter then from the previous theorem, we have $Uu' + Vv' = \frac{dT}{ds}$;

$$\text{since } \frac{dT}{ds} = 0 \quad (\text{By previous theorem})$$

$$Uu' + Vv' = 0 \quad \text{—①}$$

If the curves are not parametric curves then $u' \neq 0$, $v' \neq 0$

$$\therefore ① \Rightarrow U \text{ and } V \text{ are not independent}$$

Hence U is a scalar multiple of V and by vice versa. Hence either $U = 0$ (or) $V = 0$ is a sufficient condition for a curve to be a geodesic.

(ii) For a curve to be geodesic on a surface, it should satisfies the canonical equations

$$U = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u} = 0$$

$$V = \frac{d}{ds}\left(\frac{\partial T}{\partial v'}\right) - \frac{\partial T}{\partial v} = 0$$

[illegible] take [illegible] constant then we have

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$$r_1 \frac{dr}{dt} + \dot{r} \frac{dr_1}{dt} - \dot{r}(r_{11}\dot{u} + r_{21}\dot{v})$$

$$= r_1 \ddot{r} + \dot{r}(r_{11}\dot{u} + r_{12}\dot{v}) - \dot{r}(r_{11}\dot{u} + r_{21}\dot{v})$$

$$\left\{\because \frac{dr_1}{dt} = \frac{dr_1}{du}\frac{du}{dt} + \frac{dr_1}{dv}\frac{dv}{dt} = r_{11}\dot{u} + r_{12}\dot{v}\right\}$$

{∵ $r_{12} = r_{21}$, differentiate [illegible] [illegible] change ஆகாது}

$$U(T) = r_1 \ddot{r} \quad \text{—} ⑦$$

similarly $V(T) = r_2 \ddot{r}$ — ⑧

If we replace 't' by 's' then we have

$$U(S) = r_1 r''$$

$$V(S) = r_2 r''$$

The canonical geodesic equations are $U(S) = 0$, $V(S) = 0$. From the above equation we get

$$r'' r_1 = 0, \quad r'' r_2 = 0$$

This shows that r'' is perpendicular to r_1 and r_2 at 'p' (ie) The principle normal is perpendicular to the tangential directions at any point 'p' (ie) The principle normal is orthogonal (or) normal to the surface.

Theorem: 2M 5M

⑥ Every helix on a cylinder is geodesic

proof:-

To prove this theorem, It is enough to prove that the surface normal to the cylinder is parallel to the principle normal to the helix (ie) 'n' is parallel to 'N'

let $\vec{a}$ be the unit vector in the direction of the cylinder and let 'c' be any helix on the cylinder suppose that $\vec{t}, \vec{n}$ be the unit vectors at any point on 'c', along the tangent and principle normal to 'c'

Let 'N' be the unit vector along the normal to the surface of the cylinder at any point 'p'. then we have $\vec{n}\cdot\vec{t} = 0$ —— ① also. $\vec{a}\cdot\vec{t}$ = constant

differentiate w.r.to 's' we get

$$\vec{a}\cdot\frac{d\vec{t}}{ds} + \vec{t}\cdot\frac{d\vec{a}}{ds} = 0 \qquad \left\{\because \vec{a} \text{ is unit vector } \frac{d\vec{a}}{dt} = 0\right\}$$

$$\vec{a}\cdot\frac{d\vec{t}}{ds} = 0$$

$$\vec{a}\cdot t' = 0, \text{ where } t' = kn \text{ and } k \neq 0$$

$$\vec{a}\cdot kn = 0$$

$$\vec{a}\cdot n = 0 \quad \text{—— ②}$$

$\therefore$ 'n' is parallel to 'a×t' {∵ using ① and ②}

since 'a' and 't' lies on the tangent plane at 'p', 'a×t' is parallel to 'N'

=> 'n' is parallel to 'N'

Hence every helix on a cylinder is geodesic

Theorem :-

For any given family of geodesics on a surface parametric system can be choosen, ~~show that~~ the metric takes the form $ds^2 = du^2 + G(u,v)dv^2$

proof :

Given a family of geodesic curves

let us choose a system of parametric equations of the family of geodesic curves [illegible] constant and

~~that orthogonal trajectories which~~ their orthogonal trajectories with are given by u = constant;

We know that, v = constant is a geodesic iff

$EE_2 + FE_1 - 2EF_1 = 0$

Since $E \neq 0$ and $F = 0$, the above condition reduces to $EE_2 = 0$

$\Rightarrow E_2 = 0$ $\{\because E \neq 0\}$

$\{\because E_1$ means 'u' ஐ பொறுத்து differentiate பண்ணது -அது E_2 means 'v' ஐ பொறுத்து differentiate பண்ணுவது $\}$

∴ E is independent of 'v' and it is a function of 'u' only.

∴ The metric becomes $ds^2 = E(u)du^2 + G(u,v)dv^2$ — ①

Now let us consider the orthogonal trajectories u = constant, consider a distance between any two of the orthogonal trajectories, say $u = u_1$ and $u = u_2$ measured along the geodesic $v = c$

Since v = constant, $dv = 0$

$\therefore$ ① $\Rightarrow$ $ds^2 = E(u)du^2$

$ds = \sqrt{E(u)}\, du$ — ③

Integrating, we get $s = \int_{u_1}^{u_2} \sqrt{E(u)}\, du$ — ③

From ③, the distance between any two orthogonal trajectories are independent of v = constant.

The distance is the same along whichever geodesic v = constant, it is measured.

Hence the orthogonal trajectories are parallel.

let the curve be a geodesic then the normal property at geodesic $n = N$ —— ③

using ③ in ④ we got, $kN = k_n N + \lambda r_1 + \mu r_2$

Equating the co-efficients r_1 and r_2 on bothsides

$\lambda = \mu = 0$

so that $k_g = 0$

∴ Geodesic curvature vector is zero.

conversely, let geodesic curvature vector is zero

let $k_g = 0$, $\lambda = \mu = 0$

we have the form ③, $kn = k_n N$

Thus the principle normal to the curve is parallel to the surface normal.

∴ The curve is geodesic by the normal property.

Hence the proof.

Theorem :-

The geodesic curvature vector of any curve is orthogonal to the curve

proof :- If λ and μ are the curvature vector of $r = r(s)$ at 'P'. then we know that $r'' = k_n N + \lambda r_1 + \mu r_2$ —— ①

let us take 't' is the tangent vector to the curve as well as to the surface

multiplying equation ① by r' on bothsides we get

$r' \cdot r'' = r' \cdot k_n N$ [illegible] $(\lambda r_1 + \mu r_2)$ [illegible]

[illegible]

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Corollary:

$$K_g = H^{-1}\dot{s}^{-3}\left[(r_1\cdot\dot{r})(r_2\cdot\ddot{r}) - (r_2\cdot\dot{r})(r_1\cdot\ddot{r})\right]$$

Theorem:

If U and V are the intrinsic quantities of the surface at a point (u, v) then (i) $K_g = \frac{1}{H}\frac{V(s)}{u'}$ and (ii) $K_g = \frac{-1}{H}\frac{U(s)}{v'}$

Proof: We know that $T = \frac{1}{2}\dot{r}^2$ —①

$$\frac{\partial T}{\partial \dot{u}} = \frac{1}{2}\,2\dot{r}\frac{\partial \dot{r}}{\partial \dot{u}} = \dot{r}\frac{\partial \dot{r}}{\partial \dot{u}}$$

$$\frac{\partial T}{\partial \dot{v}} = \frac{1}{2}\,2\dot{r}\frac{\partial \dot{r}}{\partial \dot{v}} = \dot{r}\frac{\partial \dot{r}}{\partial \dot{v}} \Bigg\} \text{—②}$$

Now $\dot{r} = r_1\dot{u} + r_2\dot{v}$ $\Big\{\because \dot{r} = \frac{\partial r}{\partial t} = \frac{\partial r}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial r}{\partial v}\frac{\partial v}{\partial t}$, $\dot{r} = r_1\dot{u} + r_2\dot{v}\Big\}$

$$\left.\begin{aligned}\frac{\partial \dot{r}}{\partial \dot{u}} &= r_1\\ \frac{\partial \dot{r}}{\partial \dot{v}} &= r_2\end{aligned}\right\} \text{—③}$$

using ③ in ②, we get $\frac{\partial T}{\partial \dot{u}} = \dot{r}r_1$, $\frac{\partial T}{\partial \dot{v}} = \dot{r}r_2$ —④

Also $U(t) = r_1\cdot\ddot{r}$, $V(t) = r_2\cdot\ddot{r}$, By Normal property

By the corollary, $K_g = \frac{\dot{s}^{-3}}{H}\left[(r_1\cdot\dot{r})(r_2\cdot\ddot{r}) - (r_2\cdot\dot{r})(r_1\cdot\ddot{r})\right]$

$$K_g = \frac{\dot{s}^{-3}}{H}\left[\frac{\partial T}{\partial \dot{u}}V(t) - \frac{\partial T}{\partial \dot{v}}U(t)\right]$$

Choosing the parameter t as s such that $\dot{s}^{-3} = 1$

we have $K_g = \frac{1}{H}\left[V(s)\frac{\partial T}{\partial \dot{u}} - U(s)\frac{\partial T}{\partial \dot{v}}\right]$ —⑥

(i) By theorem $Uu' + Vv' = 0$

$$U = \frac{-Vv'}{u'} \quad \text{———} ⑦$$

using ⑦ in ⑥, $K_g = \frac{1}{H}\left[V(s)\frac{\partial T}{\partial \dot{u}} - \frac{\partial T}{\partial \dot{v}}\left(-\frac{V(s)u'}{u'}\right)\right]$

$$K_g = \frac{1}{H}\,\frac{V(s)}{u'}\left[\frac{\partial T}{\partial \dot{u}}u' + \frac{\partial T}{\partial \dot{v}}v'\right]$$

By Euler's theorem for homogeneous functions

$$\left[\frac{\partial T}{\partial \dot{u}}u' + \frac{\partial T}{\partial \dot{v}}v'\right] = 2T \quad \text{———} (*)$$

$$\therefore K_g = \frac{1}{H}\,\frac{V(s)}{u'}(2T)$$

since 's' is the parameter, $T = \frac{1}{2}\gamma'^2$

choosing $\gamma'^2 = 1 \Rightarrow T = \frac{1}{2} \Rightarrow 2T = 1$

$$\therefore K_g = \frac{1}{H}\,\frac{V(s)}{u'}$$

(ii) By theorem, $Uu' + Vv' = 0$

$$V = \frac{-Uu'}{v'} \quad \text{———} ⑧$$

using ⑧ in ⑥, $K_g = \frac{1}{H}\left[-\frac{U(s)u'}{v'}\frac{\partial T}{\partial \dot{u}} - U(s)\frac{\partial T}{\partial \dot{v}}\right]$

$$K_g = \frac{1}{H}\,\frac{U(s)}{v'}\left[-u'\frac{\partial T}{\partial \dot{u}} - v'\frac{\partial T}{\partial \dot{v}}\right]$$

$$= -\frac{1}{H}\,\frac{U(s)}{v'}\left[\frac{\partial T}{\partial \dot{u}}u' + \frac{\partial T}{\partial \dot{v}}v'\right]$$

$$= -\frac{1}{H}\,\frac{U(s)}{v'}[2T], \text{ using } (*)$$

[illegible] parameter $T = \frac{1}{2}\gamma'^2$

choosing $r^{2} = 1$ we get $r = \frac{1}{2} \Rightarrow 2r = 1$

$$\therefore K_g = \frac{-1}{H} \frac{U(s)}{v'}$$

Hence the proof

<u>Result</u> :-

prove that, If (λ, μ) is the geodesic curvature vector then $K_g = \frac{-H\lambda}{Fu' + Gv'} = \frac{H\mu}{Eu' + Fv'}$

<u>proof</u> :- By theorem $\lambda = \frac{1}{H^2}(GU - FV)$

$$= \frac{U}{H^2}\left(G - \frac{FV}{U}\right)$$

$$= \frac{U}{H^2}\left(G + \frac{Fu'}{v'}\right)$$

$$\left\{\because Uu' + Vv' = 0 \Rightarrow Uu' = -Vv' \Rightarrow \frac{V}{U} = \frac{-u'}{v'}\right\}$$

$$= \frac{U}{H^2}\left(Gv' + Fu'\right)\frac{1}{v'} \quad \longrightarrow (*)$$

$\therefore$ By above theorem, $K_g = \frac{1}{H}\frac{V(s)}{u'} = \frac{-1}{H}\frac{U(s)}{v'}$

$$-HK_g = \frac{U(s)}{v'} \quad \longrightarrow ①$$

$$\therefore \lambda = \frac{1}{H^2}(-HK_g)\left[Gv' + Fu'\right]$$

$$= \frac{-K_g}{H}\left[Gv' + Fu'\right]$$

$$\therefore K_g = \frac{-\lambda H}{\left[Gv' + Fu'\right]}$$

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similarly, by theorem $\mu = \frac{1}{H^2}\left[EV - FU\right]$

$$\mu = \frac{V}{H^2}\left[E - F\,U/V\right]$$

$$= \frac{V}{H^2}\left[E + F\frac{v'}{u'}\right] \text{, using } (*)$$

$$= \frac{V}{H^2}\,\frac{1}{u'}\left[Eu' + Fv'\right]$$

$$= \frac{K_g}{H}\left[Eu' + Fv'\right] \text{, using } ①$$

$$\therefore K_g = \frac{\mu H}{\left[Eu' + Fv'\right]}$$

Corollary :-

Under certain conditions for the geodesic curves we have $u' = \frac{1}{\sqrt{E}}$, $v' = \frac{1}{\sqrt{G}}$, $\cos\theta = Ell' + F(lm' + ml')$ where 'θ' is an angle $+Gmm'$ and $\sin\theta = H(lm' - ml')$ Using these formula's we between the two directions. Using can obtain $\sin\theta = \frac{1}{K_g} H(u'\mu - v'\lambda)$ cyclo test Repeated

(X) 5M Liouville's Formula :- 5M A-19 2M.o 5M

If 'θ' is an angle of the curve 'c' with its parametric curve v = constant then $K_g = \theta' + Pu' + Qv'$ where $P = \frac{1}{2HE}\left[2EF_1 - FE_1 - EE_2\right]$ and $Q = \frac{1}{2HE}\left[EG_1 - FE_2\right]$

Proof :- The directional coefficients of the curve 'c' at constant are (u', v') and the unit let θ be the angle between the two directions $(\overset{l}{1/\sqrt{E}}, \overset{m}{0})$ and $(\overset{l'}{u'}, \overset{m'}{v'})$ then we have from the formula

$$\cos\theta = Ell' + F(lm' + ml') + Gmm'$$

$$= E\cdot\frac{1}{\sqrt{E}}\cdot u' + F\left[\frac{1}{\sqrt{E}}\cdot v' + 0\cdot u'\right] + G\cdot 0\cdot v'$$

$$= \frac{Eu'}{\sqrt{E}} + \frac{Fv'}{\sqrt{E}} = \frac{1}{\sqrt{E}}\left[Eu' + Fv'\right] \text{ --- ①}$$

$$\sin\theta = H(lm' - ml')$$

$$= H\left[\frac{1}{\sqrt{E}}\cdot v' - 0\cdot u'\right] = \frac{Hv'}{\sqrt{E}} \text{ --- ②}$$

We know that, $T = \frac{1}{2}\left(Eu'^2 + 2Fu'v' + Gv'^2\right)$

$$\frac{\partial T}{\partial u'} = \frac{1}{2}\left[2Eu' + 2Fv'\right] = Eu' + Fv' \text{ --- ③}$$

using ③ in ①, $\cos\theta = \frac{1}{\sqrt{E}}\frac{\partial T}{\partial u'}$ --- ④

$$\frac{\partial T}{\partial u} = \frac{1}{2}\left[E_1 u'^2 + 2F_1 u'v' + G_1 v'^2\right] \text{ --- ⑤}$$

differentiate equation ④ w.r.to 's'

$$\frac{d}{ds}(\cos\theta) = \frac{d}{ds}\left(\frac{1}{\sqrt{E}}\frac{\partial T}{\partial u'}\right)$$

$$-\sin\theta\frac{d\theta}{ds} = \frac{1}{\sqrt{E}}\frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) + \frac{\partial T}{\partial u'}\frac{d}{ds}\left(\frac{1}{\sqrt{E}}\right)$$

$$= \frac{1}{\sqrt{E}}\frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) + \frac{\partial T}{\partial u'}\frac{-1}{2E^{3/2}}\frac{dE}{ds}$$

$\{\because E^{3/2} = E^{1+1/2} = E\,E^{1/2}\}$

$$= \frac{1}{\sqrt{E}}\left[\frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u'}\frac{1}{2E}\frac{dE}{ds}\right]$$

$$-\sqrt{E}\sin\theta\,\theta' = \frac{d}{ds}\left(\frac{\partial T}{\partial u'}\right) - \frac{\partial T}{\partial u'}\frac{1}{2E}\frac{dE}{ds} \text{ --- ⑥}$$

From Liouville's formula, $K_g = \theta' + Pu' + Qv'$

(i.e) $K_g = d\theta + Pdu + Qdv$

Then $\int_C K_g \, ds = \int_C \theta' + \int_C Pu' + \int_C Qv'$

$$= \int_C d\theta + \int_C Pdu + \int_C Qdv \quad \text{——} ①$$

where 'θ' is the angle between the curve 'c' on the parametric curve $v =$ constant and 'P' and 'Q' are differentiable functions of u and v. Since the curves $v =$ constant from a family in the region 'R' bounded by 'c', the tangent to 'c' turns through 2π relative to these curves.

$$\int_C d\theta + \sum_{r=1}^{n} \alpha_r = 2\pi \quad \text{——} ②$$

Also we know that, $ex(C) = 2\pi - \sum_{r=1}^{n} \alpha_r - \int_C K_g \, ds$ ——③

using ①, ② in ③ we get

$$ex(C) = 2\pi - \left[2\pi - \int_C d\theta\right] - \left[\int_C d\theta + \int_C Pdu + \int_C Qdv\right]$$

$$= 2\pi - 2\pi + \int_C d\theta - \int_C d\theta - \int_C Pdu - \int_C Qdv$$

$$= -\int_C (Pdu + Qdv) \quad \text{——} ④$$

Since R is a simply connected region and P and Q are differentiable functions of u and v

By Green's Theorem $\int Pdu + Qdv = \iint_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) du\, dv$

Since the surface element [illegible] ——⑤

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$$du\,dv = \frac{ds}{H} \quad \text{—— ⑥}$$

From ④, ⑤ and ⑥ $ex(c) = -\iint_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right)\frac{ds}{H}$

$$ex(c) = \frac{-1}{H}\iint_R \left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right) ds \quad \text{—— ⑦}$$

If we take $K = \frac{-1}{H}\left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right)$ in ⑦ then

$$ex(c) = \iint_R K\,ds$$

where 'K' is a function of 'u' and 'v' and it is independent of the curve 'c' and defined over the region 'R' of the surface. If K is not unique then

$$ex(c) = \iint_R \bar{K}\,ds \quad \text{—— ⑧}$$

using ⑦ and ⑧ we get, $\iint_R (\bar{K} - K)\,ds = 0$ —— ⑨

For every region 'R', If $K \neq \bar{K}$ at the some point 'A' where $\bar{K} > K$ (or) $\bar{K} < K$ at 'A'

let us first consider $\bar{K} > K$, since the given surface is of class 3

$\frac{\partial \theta}{\partial u}$ and $\frac{\partial \theta}{\partial v}$ are continuous in R, so that there exist a small region R_1 of R contain the point 'A' such that $\bar{K} - K > 0$ at every point of R_1, ~~for this~~ the region R contain R_1

$$\therefore \iint_{R_1} (\bar{K} - K)\,ds > 0$$

which is contradiction to ⑨

=>

and v

Corollary

= A+B

proof:-

Gaussic

$\left(\frac{\partial Q}{\partial u}\right.$

surfa

curval

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constan

Similarly we can prove for $\bar{K} < K$

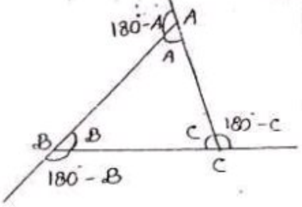
=> $\bar{K} = K$ at every point of R

(ie) K is uniquely determined ~~of the~~ function of u and v

∴ ex(C) = Total curvature

Hence the proof

Corollary :- For a geodesic triangle, prove that ex(C) = A+B+C − π



proof:-

We know that,

$$ex(C) = 2\pi - \sum_{i=1}^{n} \alpha_i - \int_C K_g ds$$

$$= 2\pi - [(\pi - A) + (\pi - B) + (\pi - C)] - 0$$

$$= 2\pi - [3\pi - (A+B+C)]$$

$\{\because A+B+C = 180°$, $\int_C K_g ds = 0\}$

2m of cycle test 6/11

$$= 2\pi - 3\pi + (A+B+C)$$

2m ex(C) = A+B+C − π

Gaussian curvature :- 2m (2m)

The invariant 'K' defined as $K = \frac{-1}{H}\left(\frac{\partial Q}{\partial u} - \frac{\partial P}{\partial v}\right)$ is called the Gaussian curvature of the surface. $\iint_R K ds$ is called the total curvature (or) integral curvature of R where R is any region whether simply connected (or) not.

Constant curvature .

has the same value K at every

corresponding position $\bar{s}$ in the neighbourhood of P whose $\bar{s}$ is a surface of revolution and is given by

$x = g(\bar{u})$, $y = 0$, $z = f(\bar{u})$

$$x = a\cosh\bar{u}, \; y = 0, \; z = a\int_{0}^{\pi}\sqrt{1-\sinh^2\bar{u}}\,d\bar{u}$$

$\{\because$ using ⑥ and ⑦

This surface of revolution is called pseudo sphere.

Unit - 4

①

The second fundamental form :-

⑤ Theorem :- state only 2m state $(K_n)^*$ 2m 5M

5m

{If K_n is a normal curvature of a curve at a point 'p' on a surface then $K_n = \dfrac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}$ $(K_n)^*$

where $L = N \cdot r_{11}$, $M = N \cdot r_{12}$, $N = N \cdot r_{22}$ and E, F, G are first fundamental coefficients.}

proof :-

let 'r' be the position vector of any point on the curve. If K_n is a normal curvature of a curve at 'p' on a surface then we know that,

$$r'' = K_n \bar{N} + \lambda r_1 + \mu r_2 \quad \text{——} ①$$

Also $N \cdot r_1 = 0 = N \cdot r_2$ —— ⊛

Taking dot product of equation ① with 'N' on bothsides, we get

$$r'' \cdot N = (K_n N + \lambda r_1 + \mu r_2) \cdot N$$

$$= K_n (N \cdot N) + \lambda (r_1 \cdot N) + \mu (r_2 \cdot N)$$

$$= K_n (1) + \lambda (0) + \mu (0) \quad \{\because \text{using } ⊛ \text{ and } N \cdot N = 1\}$$

$$= K_n \text{ ——— } ②$$

$$r' = \frac{dr}{ds} = \frac{\partial r}{\partial u}\frac{du}{ds} + \frac{\partial r}{\partial v}\frac{dv}{ds}$$

$$r' = r_1 u' + r_2 v' \text{ ——— } ③$$

differentiate w.r.to 's' we get

$$r'' = r_1 u'' + r_1' u' + r_2 v'' + r_2' v' \text{ ——— } ④$$

$$r_1' = \frac{dr_1}{ds} = \frac{\partial r_1}{\partial u}\frac{du}{ds} + \frac{\partial r_1}{\partial v}\frac{dv}{ds}$$

$$r_1' = r_{11} u' + r_{12} v' \text{ ——— } ⑤$$

$$r_2' = \frac{dr_2}{ds} = \frac{\partial r_2}{\partial u}\frac{du}{ds} + \frac{\partial r_2}{\partial v}\frac{dv}{ds}$$

$$r_2' = r_{21} u' + r_{22} v' \text{ ——— } ⑥$$

using ⑤ and ⑥ in ④ we get,

$$r'' = r_1 u'' + (r_{11} u' + r_{12} v') u' + r_2 v'' + (r_{21} u' + r_{22} v') v'$$

$$= r_1 u'' + r_{11} u'^2 + r_{12} u' v' + r_2 v'' + r_{21} u' v' + r_{22} v'^2 \text{ ——— } ⑦$$

using ⑦ in ③ we get $K_n = r'' \cdot N$

$$K_n = [r_1 u'' + r_{11} u'^2 + r_{12} u' v' + r_2 v'' + r_{21} u' v' + r_{22} v'^2] \cdot N$$

$$= u'' (r_1 \cdot N) + u'^2 (r_{11} \cdot N) + u' v' (r_{12} \cdot \cancel{N}) + v'' (r_2 \cdot N) + u' v' (r_{21} \cdot N) + v'^2 (r_{22} \cdot N)$$

$$= u'^2 (r_{11} \cdot N) + u' v' (r_{12} \cdot N) + u' v' (r_{21} \cdot N) + v'^2 (r_{22} \cdot N) \quad \{\because \text{using } ⊛\}$$

let $L = \bar{N}\cdot\bar{r}_{11}$, $m = N\cdot r_{12}$, $N = N\cdot r_{22}$

$$\therefore K_n = Lu'^2 + mu'v' + mu'v' + Nv'^2$$

$$= Lu'^2 + 2mu'v' + Nv'^2$$

$$= L\left(\frac{du}{ds}\right)^2 + 2m\frac{du}{ds}\cdot\frac{dv}{ds} + N\left(\frac{dv}{ds}\right)^2$$

$$= \frac{Ldu^2 + 2m\,du\,dv + Ndv^2}{ds^2}$$

$$K_n = \frac{Ldu^2 + 2m\,du\,dv + Ndv^2}{Edu^2 + 2F\,du\,dv + Gdv^2} \quad \{\because ds^2 = Edu^2 + 2F\,du\,dv + Gdv^2\}$$

Definition :- 2m 2nd cycle test - II

The quadratic form $Ldu^2 + 2m\,du\,dv + Ndv^2$ is called the second fundamental form of the surface and L, m, N which are functions of u and v are called second fundamental coefficients.)

classification of point on a surface :- 2m

The second fundamental form $Ldu^2 + 2m\,du\,dv + Ndv^2$ is a quadratic form in du and dv. This form can be written as

$$Ldu^2 + 2m\,du\,dv + Ndv^2 = \frac{1}{L}\left[(Ldu + mdv)^2 + (LN - m^2)dv^2\right]$$

This is the discriminant $(LN - m^2)$ of the quadratic form

case (i) :- If $(LN - m^2) > 0$

since the discriminant is positive, the quadratic form is positive at any point 'p' on the surface. Hence K_n has the same sign for all directions at 'p'. In this case the point 'p' is called an Elliptic point.

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using ⑤ and ⑥, $\begin{vmatrix} L-kE & M-kF \\ M-kF & N-kG \end{vmatrix} = 0$

$$(L-kE)(N-kG) - (M-kF)(M-kF) = 0$$

$$LN - LkG - kEN + k^2EG - [M^2 - MkF - MkF + k^2F^2] = 0$$

$$LN - LkG - kEN + k^2EG - M^2 + 2MkF - k^2F^2 = 0$$

$$k^2[EG - F^2] - k[LG + NE - 2MF] + LN - M^2 = 0$$

The roots of the above equation gives the principal curvature at 'p' and let it be k_a and k_b inwhich one must be maximum and other must be minimum.

Definitions 2m 2m cycle test II A-19 2m

(46) * If k_a and k_b are principal curvature at a point 'p' then the mean curvature 'μ' is defined as

$$\mu = \tfrac{1}{2}(k_a + k_b) = \frac{EN + GL - 2MF}{2(EG - F^2)}$$

* If k_a and k_b are principal curvature then the Gaussian curvature 'k' is defined as

$$K = k_a \cdot k_b = \frac{LN - M^2}{EG - F^2}$$

* A point on a surface is called an umbilic if $\frac{L}{E} = \frac{M}{F} = \frac{N}{G}$ is true in that point.

④ **Theorem** :- 5m 10m

The principal directions are given by $(EM - FL)l^2 + (EN - GL)lm + (FN - GM)m^2 = 0$

proof: we know that, $Ll + Mm - \lambda El - \lambda Fm = 0$ —①

$Ml + Nm - \lambda Fl - \lambda Gm = 0$ —②

By previous theorem, equations ① and ②

⑦

① ⇒ $-Ll + Mm - \lambda(El + Fm) = 0$

② ⇒ $Ml + Nm - \lambda(Fl + Gm) = 0$

Eliminating λ from the above equations we get

$$\begin{vmatrix} Ll + Mm & -(El + Fm) \\ Ml + Nm & -(Fl + Gm) \end{vmatrix} = 0$$

$$-(Ll + Mm)(Fl + Gm) + (Ml + Nm)(El + Fm) = 0$$

$$-LFl^2 - LGlm \mp MFlm \mp MGm^2 + MEl^2 + MFlm + NElm + NFm^2 = 0$$

$$[EM - FL]l^2 + [EN - GL]lm + [FN - GM]m^2 = 0$$

Further the discriminant of the above equation is

$$(EN - GL)^2 - 4(EM - FL)(FN - GM) = 0 \quad — ③ \quad \{\because b^2 - 4ac = 0\}$$

consider $FN - GM = \frac{F}{E}(EN - GL) - \frac{G}{E}(EM - FL)$ — ④

using ④ in ③ we get.

$$\therefore (EN - GL)^2 - 4(EM - FL)\left[\frac{F}{E}(EN - GL) - \frac{G}{E}(EM - FL)\right] = 0$$

$$(EN - GL)^2 - \frac{4F}{E}(EM - FL)(EN - GL) + \frac{4G}{E}(EM - FL)^2 = 0$$

$$\left[(EN - GL) - \frac{2F}{E}(EM - FL)\right]^2 + \frac{4(EM - FL)^2}{E^2}(EG - F^2) = 0$$

since $EG - F^2 > 0$, The discriminant is always positive. ∴ The roots of the equations are real and distinct provided that the coefficients E, F, G and L, M, N are not proportional, when the values of these coefficients are proportional. The principal directions are indeterminant and the normal curvature is same in all directions.

0

quations we get

$= 0$

$+ Nm)(El + Fm) = 0$

$+ mEl^2 + mFlm +$

$NElm + NFm^2 = 0$

$+ [FN - Gm] m^2 = 0$

above equation is

$) = 0$ — ③ $\{\because b^2 - 4ac = 0\}$

$(Em - FL)$ — ④

$(EN - GL) - \frac{G}{E}(Em - FL)] = 0$

$+ \frac{4G}{E}(Em - FL)^2 = 0$

$\frac{(Em - FL)^2}{E^2}(EG - F^2) = 0$

iscriminant is always

s are real and distinct

d L, m, N are not

coefficients are proportio

eterminant and the

irections.

Note :-

(45) (44) we know that $L = N \cdot r_{11}$, $m = N \cdot r_{12}$ and $N = N \cdot r_{22}$ — ①

we may obtain an alternative expression for L, m, N as follows. For that consider the equations $N \cdot r_1 = 0$

(4B) and $N \cdot r_2 = 0$

Differentiate the above equations, we get

$$\left.\begin{array}{l} N_1 \cdot r_1 + N \cdot r_{11} = 0 \\ N_2 \cdot r_1 + N \cdot r_{12} = 0 \end{array}\right\} — ②$$

$$\left.\begin{array}{l} N_1 \cdot r_2 + N \cdot r_{21} = 0 \\ N_2 \cdot r_2 + N \cdot r_{22} = 0 \end{array}\right\} — ③$$

② => $N_1 \cdot r_1 + L = 0 \Rightarrow L = -N_1 \cdot r_1$, using ①

②, ③ => $N_2 \cdot r_1 + m = 0 \Rightarrow m = -N_2 \cdot r_1$, using ①

$N_1 \cdot r_2 + m = 0 \Rightarrow m = -N_1 \cdot r_2$, using ①

$\therefore m = -N_2 \cdot r_1 = -N_1 \cdot r_2$

2m ③ => $N_2 \cdot r_2 + N = 0 \Rightarrow N = -N_2 \cdot r_2$, using ①

(1) **Lines of curvature** :- 2m

A curve on a surface whose tangent at each point is along a principal direction is called a line of curvature.

The principal directions are given by

$$\left.\begin{array}{l} Ldu + mdv - k(Edu + Fdv) = 0 \\ mdu + Ndv - k(Fdu + Gdv) = 0 \end{array}\right\} —①$$

where k is one of the principal curvature

the above equation can be written as

$$\left.\begin{array}{l} (L - kE)du + (m - kF)dv = 0 \\ (m - kF)du + (N - kG)dv = 0 \end{array}\right\} — (5)$$

⑨

the above equations are lines of curvature, Eliminating 'K' from equation ① we get the two families of lines of curvature.

$$(Em - FL)du^2 + (EN - GL)dudv + (FN - Gm)dv^2 = 0$$

$$\left\{\because \begin{vmatrix} Ldu + mdv & -(Edu + Fdv) \\ mdu + Ndv & -(Fdu + Gdv) \end{vmatrix} = 0.\right.$$

$$-(Ldu + mdv)(Fdu + Gdv) + (mdu + Ndv)(Edu + Fdv) = 0$$

$$-LFdu^2 - LGdudv - mFdudv - mGdv^2 + mEdu^2 + mF\,dudv + NEdudv + NFdv^2 = 0$$

② $(Em - FL)du^2 + (EN - GL)dudv + (FN - Gm)dv^2 = 0\}$

Theorem :- [Rodrique's Formula] 5M (or) $dN + Kdr =$

A-19 5M

A necessary and sufficient condition for a curve on a surface to be a line curvature is $\underline{Kdr + dN = 0}$ at each point on the line curvature where K is the normal curvature in the direction 'dr' of the lines of curvature

proof :-

let us assume that the curve on a surface be a line curvature.

To prove : $Kdr + dN = 0$

The direction (du, dv) at any point 'p' on the curve is the principal direction at (u, v) on the surface is given by

$$\left.\begin{aligned} Ldu + mdv - K(Edu + Fdv) &= 0 \\ mdu + Ndv - K(Fdu + Gdv) &= 0 \end{aligned}\right\} \text{—①}$$

substitute $L = -N_1 \cdot r_1$, $m = -N_2 \cdot r_1$, $E = r_1 \cdot r_1$ and $F = r_1 \cdot r_2$

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in equation ① we get

$$(-N_1 \cdot r_1)\,du + (-N_2 \cdot r_1)\,dv - k\left[(r_1 \cdot r_1)\,du + (r_1 \cdot r_2)\,dv\right] = 0$$

$$(N_1 \cdot r_1)\,du + (N_2 \cdot r_1)\,dv + k\left[(r_1 \cdot r_1)\,du + (r_1 \cdot r_2)\,dv\right] = 0$$

$$\left[N_1\,du + N_2\,dv + k(r_1\,du + r_2\,dv)\right] \cdot r_1 = 0 \quad —②$$

consider $dr = \dfrac{\partial r}{\partial u}\,du + \dfrac{\partial r}{\partial v}\,dv$

$$= r_1\,du + r_2\,dv \quad —③$$

$$dN = \frac{\partial N}{\partial u}\,du + \frac{\partial N}{\partial v}\,dv$$

$$= N_1\,du + N_2\,dv \quad —④$$

using ③ and ④ in ② we get

$$(dN + k\,dr) \cdot r_1 = 0$$

$$(k\,dr + dN) \cdot r_1 = 0 \quad —⑤$$

$$\therefore\ k\,dr + dN = 0 \qquad \{\because r_1 \neq 0\}$$

similarly substitute $M = -N_1 \cdot r_2$, $N = -N_2 \cdot r_2$, $F = r_1 \cdot r_2$ and $G = r_2 \cdot r_2$ in equation ① we get

$$(k\,dr + dN) \cdot r_2 = 0 \quad —⑥$$

$$\therefore\ k\,dr + dN = 0 \qquad \{\because r_2 \neq 0\}$$

consider $N^2 = 1$

differentiating the above, we get

$$2N\,dN = 0$$

$$\Rightarrow N \cdot dN = 0$$

(ie) 'dN' is normal to 'N' {∵ N means tangential vector}

Also dr is a tangential vector, $k\,dr + dN$ is a tangential vector to the surface

⇒ it lies in the plane vectors r_1 and r_2

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From equation (5) and (6) we conclude that

$dN + kdr$ is perpendicular to r_1 and r_2

$\Rightarrow$ $dN + kdr$) is parallel to $r_1 \times r_2$, which is the direction of the surface normal.

$\therefore$ $dN + kdr$ is parallel to surface normal

which is the contradiction to the fact that $dN + kdr$ is a tangential vector to the surface.

$$\Rightarrow \quad kdr + dN = 0$$

<u>sufficient part</u>: let us assume that there exist a curve on a surface for which $kdr + dN = 0$

<u>To prove</u>: The curve is a line of curvature

(ie) That curve having the normal curvature 'k' at 'p' whose direction coincides with the principal directions

since $kdr + dN = 0$ ---- ⊛ we have

$$(kdr + dN) \cdot r_1 = 0$$

$$(kdr + dN) \cdot r_2 = 0$$

By reversing the steps, we get

$$(L - kE)\,du + (M - kF)\,dv = 0$$

$$(M - kF)\,du + (N - kG)\,dv = 0$$

⊛ $\Rightarrow$ $kdr = -dN$

$$k(r_1 du + r_2 dv) = -(N_1 du + N_2 dv)\text{, using above results}$$

Multiply by $r_1 du + r_2 dv$,

$$\therefore k(r_1 du + r_2 dv)(r_1 du + r_2 dv) = -(N_1 du + N_2 dv)(r_1 du + r_2 dv)$$

$$k[r_1 r_1 du^2 + 2r_1 r_2 du dv + r_2 r_2 dv^2] = -[N_1 r_1 du^2 + (N_2 r_1 +$$

Note
$- N_1 r_1$
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7

$N_1 T_2) du dv + N_2 T_2 dv^2]$; using previous

Note $E = T_1 T_1$, $F = T_1 T_2$, $G = T_2 T_2$, $L = -N_1 T_1$, $M = -N_2 T_1 = -N_1 T_2$ and $N = -N_2 T_2$

$$K[Edu^2 + 2F du dv + G dv^2] = L du^2 + 2M du dv + N dv^2$$

$$K = \frac{L du^2 + 2M du dv + N dv^2}{E du^2 + 2F du dv + G dv^2}$$

=> K is a normal curvature at the point 'p' in the direction (du, dv)

Hence the normal curvature at each point of the curve is coincides with the principal directions (du, dv) { since p is arbitrary }

∴ The curve must be a line of curvature

7 5m cycle test 2 Hence the proof

⑩ The Dupin Indicatrix :- 5m

② 5m ③

suppose at a point 'o' on a given surface a set of rectangular cartesian co-ordinates are choosen, so that ox, oy are along the principal directions at 'o' and 'oz' is along the normal at 'o'. Then the equation of the surface near 'o' is

$z = ax^2 + by^2$

The plane $z = 2h$ intersects the surface near 'o' in the conic $z = 2h$, $2h = ax^2 + by^2$ The normal curvature at 'o' in the direction 'ox' is $\lim_{x \to 0} \frac{2h}{x^2} = a = k_a$

similarly k_n b if we defined R_a, R_b be the reciprocals of k_a, k_b the [illegible] radius in the conic

6

(15)

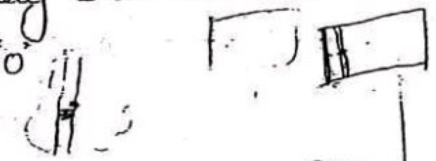
(6)

~~developables~~ :- Developable :-

2m (Q) A developables is a surface enveloped by a one parameter family of planes. such a family is given by the equation $r \cdot a = \rho$, where 'a' and 'ρ' are functions of a real parameter 'u'

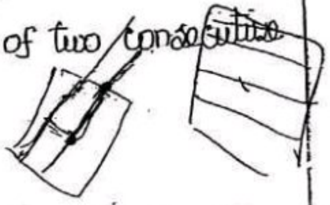
Dupin Indicatrix :- 2m

let 'o' be any point on a given surface than the section of the surface divided by a plane parallel to the tangent plane at 'o' and a very small distance from it. is called dupin Indicatrix at 'o'



characteristic Line :- 2m

(Q) The line of intersection of two consecutive planes is called the characteristic line.



Theorem :-

The characteristic line corresponding to the plane 'μ' are given by the intersection of the planes $r \cdot a = p$ and $r \cdot \dot{a} = \dot{p}$

proof :-

If the planes u and v $(u < v)$ are two neighbouring planes, than the lines of intersection of the planes is given by $f(u) = 0$ and $f(v) = 0$

Here $f(u) = [r \cdot a(u)] - p(u)$

Hence by the Rolle's theorem, there exist u_1, $u < u_1 < v$ suchthat $f(u_1) = 0$

In the limiting case, when $v \to u$, $u_1 \to u$

and fi

$r \cdot \dot{a} =$

Theorem

determine

$r \cdot \ddot{a} =$

proof :-

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we obtain the equation of the characteristic line as $f(u)=0$ and $f(v)=0$ which is equivalent to $r.a = p$ and $r.\dot{a} = \dot{p}$

Theorem :-

⑨ The characteristic point on the plane 'μ' is determined by the equation $r.a = p$; $r.\dot{a} = \dot{p}$ and $r.\ddot{a} = \ddot{p}$

proof :- let u, v, w be the three neighbouring points such that $f(u)=0$, $f(v)=0$, $f(w)=0$

Then by Rollo's theorem, $u < u_1 < v$, $v < u_2 < w$

such that $f(u_1)=0$ and $f(u_2)=0$

Again by Rolle's theorem, there exist u_3 such that $u_1 < u_3 < u_2$, $f(u_3)=0$

when $u_1, u_2, u_3 \to u$ we get

$$\dot{f}(u)=0 ,\ \dot{f}(u)=0 ,\ \ddot{f}(u)=0$$

$$r.a = p ,\ r.\dot{a} = \dot{p} ,\ r.\ddot{a} = \ddot{p}$$

Hence the proof

⑩ **Edge of Regression :-**

The characteristic points corresponding to planes of the family determine a curve on the developable called the edge of regression

a v
... of regression on the

theorem: ⑫, ⑬ The tangent to the edge ...

Minimal surface :- ~~34~~

The mean curvature $\mu = \frac{1}{2}(K_a + K_b)$ is zero at all points of the surface then the surface is called the "minimal surface"

If K_a and K_b are the principal curvature at a point 'p' on the surface then the mean curvature is denoted by 'μ' and it is also defined by

$$\mu = \frac{1}{2}(K_a + K_b) = \frac{EN + GL - 2FM}{2(EG - F^2)}$$

Since $EG - F^2 \neq 0$ then the condition for the minimal surface is $EN + GL - 2FM = 0$

Note :-

The direction (l_1, m_1) and (l_2, m_2) will be conjugate if $L l_1 l_2 + M(l_1 m_2 + l_2 m_1) + N m_1 m_2 = 0$

$= \frac{1}{2}(K_a + K_b)$

the surface is

principal curvatu

mean curvature

d by

$-2FM$

$-F^2)$

n the condition

$2FM = 0$

$m_2)$ will be

$) + N m_1 m_2 = 0$

---

(34)

(35) (36) (37)

Unit - $\overline{IV}$

5 m (*) Repeated.

Theorem :-

(i) If there is a surface of minimum area passing through a closed space curve, it is necessarily a minimal surface. (i.e) a surface of zero mean curvature

Proof :- Let S be the surface $r = r(u,v)$ bounded by a closed curve 'c'. Let us give a small displacement 'ϵ' in the direction of the surface normal and let S' be the new surface obtained ϵ is a function of u and v and its derivatives w.r.to u and v are denoted by ϵ_1 and ϵ_2. Both ϵ_1 and ϵ_2 are small and tend to zero as $\epsilon \to 0$

$$(ie)\ \epsilon_1 = O(\epsilon) \text{ and } \epsilon_2 = O(\epsilon) \text{ as } \epsilon \to 0$$

Let R be the position vector of the displaced surface S'

Then $R = r + \epsilon N$

$$R_1 = r_1 + \epsilon_1 N + \epsilon N_1$$

and $$R_2 = r_2 + \epsilon_2 N + \epsilon N_2$$

Let E', F', G' be the first fundamental coefficients of S'

Then $$E' = R_1 \cdot R_1 = (r_1 + \epsilon_1 N + \epsilon N_1) \cdot (r_1 + \epsilon_1 N + \epsilon N_1)$$

$$= r_1^2 + \epsilon_1 r_1 N + \epsilon r_1 N_1 + \epsilon_1 r_1 N + \epsilon_1^2 N^2 + \epsilon\epsilon_1 N N_1$$
$$+ \epsilon r_1 N_1 + \epsilon\epsilon_1 N N_1 + \epsilon^2 N_1^2$$

38

uled surface :- 2M

2m (37) A surface generated by the motion of one paramete family of straight lines is called a ruled surface and the straight lines of the family are called its generators (ie) the various positions of the line being called generators

Base curve (or) Directories :-

A curve 'c' on a ruled surface with the property that it meets each generator precisely once is called a base curve (or) directories

Equation of a ruled surface :-

Let $r = r(u)$ be the position vector of the point 'p' on the base curve of a ruled surface. Let $g(u)$ be the unit vector along the generator at 'p'. Let 'R' be the position vector of any point on the surface. since the generators passes through 'Q' we have $R(u,v) = r(u) + v\,g(u)$ where v is the distance of 'Q' from 'p' in the direction of 'g'

Note :- On the ruled surface Gaussian curvature is given by

$$K = -\frac{[\dot{r}, g, \dot{g}]^2}{H^4}$$

the motion of one param
uled surface and the
its generators (ie) the
ed generators.

on a ruled surface with
er precisely one is

se the position vector
a ruled surface. let
er at 'P'. let 'R' be
surface. since the
(u, v) = r(u)+ v g(u)
the direction of 'g'

ature is given by

---

ition :- The function $p(u) = \frac{[\dot{r} . g . \dot{g}]}{\dot{g}^2}$ is called the paramet distribution of the ruled surface.

perty associated with p(u) :-

- * p(u) has constant value at each point on the generator.
- * p(u) is independent of the base curve
- * p(u) is independent of choice of parameter chosen.

Note :-

Gaussian curvature $K = \frac{-[\dot{r} . g . \dot{g}]^2}{H^4}$.

$$K = \frac{-p^2(u) \dot{g}^4}{H^4}$$

ntral point :-

Let P and Q be two points on a base curve of a ruled surface. let P_1 and Q_1 be the common perpendicular (ie) shortest distance between the generators through P and Q as Q → P the point 'P'' → a definite point on the generator through 'P' and this is called the central point of the generator through 'P'

ntral plane :- The tangent at any central point of generator is called the central plane of generator.

1 For a ruled surface $R(u,v) = r(\bar{u}) + v\,g(u)$, where $r = r(u)$ is a point on the base curve and $g(u)$ is an unit vector along the generator then the Gaussian curvature $K = \frac{-[\dot{r}, g, \dot{g}]^2}{H^4}$

<u>Solution</u> :-

$$\text{Given } R(u,v) = r(u) + v\,g(u)$$

$$R_1 = \dot{r} + v\dot{g} \quad , \quad R_{11} = \ddot{r} + v\ddot{g}$$

$$R_2 = g \quad , \quad R_{22} = 0 \quad \text{and} \quad R_{12} = R_{21} = \dot{g}$$

$$E = R_1 \cdot R_1 = (\dot{r} + v\dot{g})(\dot{r} + v\dot{g})$$

$$= \dot{r}^2 + 2\dot{r}v\dot{g} + v^2\dot{g}^2$$

$$F = R_1 \cdot R_2 = (\dot{r} + v\dot{g})(g) = \dot{r}g + v\dot{g}g = \dot{r}g \quad \{\because v\dot{g}g = 0\}$$

$$\{\because \text{since } g^2 = g \cdot g = 1$$

$$\text{Differentiate } \dot{g}g + g\dot{g} = 0 \Rightarrow 2g\dot{g} = 0 \Rightarrow g\dot{g} = 0\}$$

$$G = R_2 \cdot R_2 = (g)(g) = 1$$

$$HL = [R_{11}, R_1, R_2]$$

$$= [\ddot{r} + v\ddot{g}, \dot{r} + v\dot{g}, g]$$

$$= [\ddot{r}, \dot{r} + v\dot{g}, g] + [v\ddot{g}, \dot{r} + v\dot{g}, g]$$

$$= [\ddot{r}, \dot{r}, g] + [\ddot{r}, v\dot{g}, g] + [v\ddot{g}, \dot{r}, g] + [v\ddot{g}, v\dot{g}, g]$$

$$= [\ddot{r}, \dot{r}, g] + v[\ddot{r}, \dot{g}, g] + v[\ddot{g}, \dot{r}, g] + v^2[\ddot{g}, \dot{g}, g]$$

$$\{\because L = R_{11} \cdot N \quad \text{where } N = \frac{R_1 \times R_2}{H}$$

$$L = R_{11} \cdot \left(\frac{R_1 \times R_2}{H}\right) = \frac{1}{H}\{\dot{R}_{11} \cdot (R_1 \times R_2)\} = \frac{1}{H}[R_{11}, R_1, R_2]\}$$

$+ v\,g(u)$, where $r = r(u$

$)$ is an unit vector along

ve $K = -\frac{[\dot{r}, g, \dot{g}]^2}{H^4}$

$g(u)$

$v\ddot{g}$

and $R_{12} = R_{21} = \dot{g}$

$v\dot{g}g = \dot{r}g$ $\{\because v\dot{g}g = 0$

$\Rightarrow gg\dot{g} = 0 \Rightarrow g\dot{g} = 0\}$

$+ v\dot{g}, g]$

$v\ddot{g}, \dot{r}, g] + [v\ddot{g}, v\dot{g}, g]$

$\ddot{g}, \dot{r}, g] + v^2[\ddot{g}, \dot{g}, g]$

$R_1 \times R_2)\} = \frac{1}{H}[R_{11}, R_1, R_2]$

---

similarly $HM = [R_{12}, R_1, R_2]$, $HN = [R_{22}, R_1, R_2]\}$

$$HM = [R_{12}, R_1, R_2] = [\dot{g}, \dot{r} + v\dot{g}, g]$$
$$= [\dot{g}, \dot{r}, g] + [\dot{g}, v\dot{g}, g]$$
$$= [\dot{g}, \dot{r}, g] + v[\dot{g}, \dot{g}, g]$$
$$= [\dot{g}, \dot{r}, g] + 0 = [\dot{g}, \dot{r}, g]$$

$\{\because$ Two rows are same then $[\dot{g}, \dot{g}, g] = 0\}$

$$HN = [R_{22}, R_1, R_2] = [0, \dot{r} + v\dot{g}, g] = 0$$

$HN = 0 \Rightarrow N = 0$ $\{\because H \neq 0$ because $H = \frac{R_1 \times R_2}{L}$ Here R_1 and $R_2 \neq 0$ so $H \neq 0\}$

$\{\because$ In matrix any row is zero then determinant value is also zero$\}$

we know that $K = \frac{LN - M^2}{EG - F^2}$

$$K = \frac{(0) - \frac{1}{H^2}[\dot{g}, \dot{r}, g]^2}{H^2}$$

$\{\because HM = [\dot{g}, \dot{r}, g]$, $M = \frac{1}{H}[\dot{g}, \dot{r}, g]\}$

$$K = \frac{-[\dot{g}, \dot{r}, g]^2}{H^4}$$

$$K = \frac{-[\dot{r}, g, \dot{g}]^2}{H^4}$$

$\{\because [\dot{g}, \dot{r}, g] = -[\dot{g}, g, \dot{r}] = (-)(-)[\dot{r}, g, \dot{g}]\}$

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$$\therefore \quad L_2 = \frac{1}{2} E_2 \left(\frac{L}{E} + \frac{N}{G} \right) \quad ②$$

$$N_1 = \frac{1}{2} G_1 \left(\frac{L}{E} + \frac{N}{G} \right)$$

From which we find

$$\frac{\partial (Ka)}{\partial v} = \frac{EL_2 - LE_2}{E^2}$$

$$= \frac{L_2}{E} - \frac{LE_2}{E^2}$$

$$= \frac{1}{2} \frac{E_2}{E} \left(\frac{L}{E} + \frac{N}{G} \right) - \frac{LE_2}{E^2}$$

$$= \frac{E_2 L}{2E^2} + \frac{NE_2}{2EG} - \frac{LE_2}{E^2}$$

$$= \frac{NE_2}{2EG} - \frac{LE_2}{2E^2}$$

$$= \frac{E_2}{2E} \left(\frac{N}{G} - \frac{L}{E} \right)$$

$$\frac{\partial Ka}{\partial v} = \frac{1}{2} \cdot \frac{E_2}{E} (Kb - Ka)$$

$$\frac{\partial Kb}{\partial u} = \frac{GN_1 - NG_1}{G^2}$$

$$= \frac{N_1}{G} - \frac{NG_1}{G^2}$$

$$= \frac{1}{G} \left[\left(\frac{G_1}{2} \right) \left(\frac{L}{E} + \frac{N}{G} \right) \right] - \frac{NG_1}{G^2}$$

$$= \frac{G_1 L}{2EG} - \frac{NG_1}{2G^2}$$

$$= \frac{G_1}{2EG} - \frac{NG_1}{2G^2}$$

$$= \frac{G_1}{2G}\left(\frac{L}{E} - \frac{N}{G}\right) \quad ③$$

$$= \frac{G_1}{2G}\left(Ka - Kb\right)$$

$$\left.\begin{aligned} \frac{\partial Ka}{\partial v} &= \frac{1}{2}\cdot\frac{E_2}{E}\left(Kb - Ka\right) \\ \frac{\partial Kb}{\partial u} &= \frac{1}{2}\cdot\frac{G_1}{G}\left(Ka - Kb\right) \end{aligned}\right\} \longrightarrow ②$$

since the principle curvatures have extreme values then the left hand members vanish at P_0. It follows that,

$E_2 = G_1 = 0$ and

Hence the

$$\left.\begin{aligned} \frac{\partial^2 Ka}{\partial v^2} &= \frac{1}{2}\cdot\frac{E_{22}}{E}\left(Ka - Kb\right) \\ \frac{\partial^2 Kb}{\partial u^2} &= \frac{1}{2}\,\frac{G_{11}}{G}\left(Ka - Kb\right) \end{aligned}\right\} \longrightarrow ③$$

There are now two probabilities either.

i). Ka has maximum has in this case $Ka - Kb > 0$

$$\left.\begin{aligned} \frac{\partial^2 Kb}{\partial v^2} &\leq 0 \\ \frac{\partial^2 Kb}{\partial u^2} &\geq 0 \end{aligned}\right\} \longrightarrow ④$$

$$\left.\begin{aligned} \frac{\partial^2 Ka}{\partial v^2} &\geq 0 \\ \frac{\partial^2 Kb}{\partial u^2} &\leq 0 \end{aligned}\right\} \longrightarrow ⑤$$

In either case we see that,

$\therefore E_{22} \geq 0$ & $G_{11} \geq 0$

But this contradicts the fact that.

$$K' = \frac{-1}{2H}\left(\frac{\partial G_1}{H\partial u} + \frac{\partial E_2}{H\partial v}\right) \quad \text{where} \quad H = \sqrt{EG}$$

That the Gaussian curvature K satisfies

$$K = \frac{-1}{2H^2}\left\{\left(\left(\frac{\partial G_1}{\partial u} + \frac{\partial E_2}{\partial v}\right)\right)\right\}$$

$$E = \frac{-1}{2EG}\left(G_{11} + E_{22}\right)$$

since the R.H.S is -ve (or zero.

while K assumes strictly +ve.

Hence the proof.

A-19 2M OD

Strum's theorem:-

consider the two distinct differential equation $\frac{d^2v}{dx^2} = Hv$, $\frac{d^2v}{dx^2} = H'v$ where for all values of x is the range where $H'(x) \geq H(x)$ Then if $\theta(x)$ is a considered solution of the 1st equation having two consective zeros at x_0 and x_1 a solution of the 2nd equation which has a zero at x_0 cannot have another zero in $[x_0, x_1]$

A compact surface cannot have constant zero curvature.

Theorem:- 2

2M Nov 2014

The only compact surface of a class ≥ 2 for which every point is an umbiles are sphere

proof:-

Let S be a compact Surface of class ≥ 2. for which point is an umbilic.

Let p be an any point on S.

$+\frac{\partial E_2}{H\partial v}\Big)$ ④ where $H = \sqrt{EG}$

n curvature k satisfies

$+\frac{\partial E_2}{\partial v}\Big)\Big]\Big\}$

$+E_{22})$

-ve (or) zero.

strictly +ve.

e proof.

---

two distinct differential

$\frac{d^2v}{dx^2} = H^1 v$

ues of x is the range

(x) Then if $\theta(x)$ is a

quation having two consta

solution of the 2nd equa

t α_0 cannot have an

Surface cannot have consta

zero curvature.

ct surface of a class

is an umbilics are

surface of class

component

n S.

---

⑤

Let V be a ... [illegible]

S containing P, in which part of S; is

resent parametrically by $V = r(u,v)$ 108

since every point of V is an umblic it

follows that every curve lying in V must be a

ne of curvature.

Hence from Rodique's formula at all

point of V

$$dN + K\,dr = 0. \longrightarrow ①$$

where K is a normal curvature of S in the

direction dr.

From ① we get

$N_1 = -Kr_1$

$N_2 = -Kr_2$

which identity $N_{12} = N_{21}$

$K_2 r_1 - K_1 r_2 = 0$

$\because r_1$ & r_2 are Linear independent $K_1, K_2 = 0$

so that K is constant.

$\{\because$ From ① we get,

$\Rightarrow (N_1 du + N_2 dv) + K(r_1 du + r_2 dv)$

$\Rightarrow (N_1 + Kr_1) + (N_2 + Kr_2) dv$

$\Rightarrow N_1 + Kr_1 = 0, \quad N_2 + Kr_2 = 0$

$\Rightarrow N_1 = -Kr_1, \quad N_2 = -Kr_2$

Diff w.r to v: $N_{12} = -[Kr_{12} + K_2 r_1]$ Diff w.r to u: $N_{21} = -[Kr_{21} + K_1 r_2]$

$N_{12} = N_{21}\}$

$\int$ing ① we get

for $K \neq 0$

$$r = a - K^{-1}N \longrightarrow ②$$

so that r lies on the surface

of a sphere of centre 'a' and

radius of a sphere of $a^{-1}K^{-1}$

$\{\because \int dN + \int K\,dr = 0$

$N + Kr = c$

$Kr = c - N$

$r = K^{-1}c - K^{-1}N$

$r = a - K^{-}N$

$\because K^{-1}c = a\}$

when $K = 0$ from ① gives,

$$\boxed{N = b}$$

showing that V lies on a plane.

This completes the local part of the

theorem. Next we have to prove that in the

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(b)

neighbourhood of any point the surface is sphere (OD plane associate at each point P on the surface a neighbourhood v having the above property.

The set of all neighbourhoods v, covers and from the compactness, we deduce at S in covered by a finite sub-cover formed by $V_i (i = 1, 2, --N)$

Consider two overlapping neighbourhoods $\boxed{V_i, V_j}$ from the previous local argument it follows that K is constant in V_i and also in V_j.

By considering the value of K at points in $\boxed{V_i \cap V_j}$ it follow that K takes the same value over the whole of the surface.

more over this value cannot be zero. otherwise the surface would contain a straight line and would not be compact.

Hence the surface must be a sphere.

Theorem is proved.

Compact surface of constant Gaussian or Mean curvature :-

(5M)

UQ **Theorem :- 3**

(5M) Nov-2014

The only compact surfaces with constant Gaussian curvature are spheres.

proof :-

Let S be a compact surface with constant Gaussian curvature K.

∴ S is compact, there is a point P_0 at the maximum value of the principal

---

ture is attained

∴ The product

the Gaussian

It follows t

respectively, a

at P_0 with

minimum.

From Hilb

two principal

i.e) at no

vature exceed

∴ Hence every

theorem r

orem :- 4 [conju

If P & Q

which can be

hen the arc

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ies entirely

overed by th

proof :-

u = consta

The metric

$ds^2 = du^2 +$

Let P ⇒ C

Q ⇒ C

Let C be

theorem

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rvature is attained.

∴ The product of the principal curvature (i.e) the Gaussian curvature) is constant 110

It follows that the principal curvature are respectively, a maximum and a minimum values at P_0 with the maximum not less than the minimum.

From Hilbert's Lemma it follows that the two principal curvature must be equal.

i.e) at no point does either principal curvature exceed $\sqrt{K}$.

∴ Hence every point of S is an umbolic and the theorem now follows theorem (1).

Theorem :- 4 [Conjugate point on geodesics]

If P & Q are two points of a geodesic which can be embadded in a field of geodesic then the arc PQ of the geodesic is shorter than any other arc which joins P to Q and lies entirely in that region of the surface covered by the field.

Proof :-

$u =$ constant , $v =$ constant

The metric reduces to the form

$$ds^2 = du^2 + \lambda^2 dv^2$$

Let $P \Rightarrow (u_1, v_0)$, $r = \phi(u)$

$Q \Rightarrow (u_2, v_0)$, where $\phi(u_1) = \phi(u_2) = v_0$

with $u_2 > u_1$

Let C be an arbitrary curve passing :-

theorem P & Q given by,

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$$\frac{ds^2}{du^2} = 1+\lambda^2 \frac{dv^2}{du^2} \quad ⑧$$

$$\left(\frac{ds^2}{du}\right)^2 = 1+\lambda^2 \frac{d\phi^2}{du^2}$$

$$s = \frac{ds}{du} = \sqrt{1+\lambda^2\left(\frac{d\phi}{du}\right)^2}$$

length of $l = \int_{u_1}^{u_2} \left\{ 1+\lambda^2 \left(\frac{d\phi}{du}\right)^2 \right\}^{1/2} du$ exceed

$u_2 - u_1$ unless $\frac{d\phi}{du} = 0$.

$$\Rightarrow l = \int_{u_1}^{u_2} du \Rightarrow [u]_{u_1}^{u_2} = u_2 - u_1$$

where c is the given geodesic.

Theorem :- 5

U.Q 5M 10M when the surface S have -ve curvature every where a length of geodesic which going any two points A, B is always less than the length of neighbouring curves through A & B.

Proof :-

Let one system of parametric curve be the geodesic normal to the gives geodesic AB and the other system to the orthogonal trafactories.

Let u denote the length of the geodesic normal PQ from P to AB.

Let v denote the length AQ the line element of the surface becomes.

$$ds^2 = du^2 ! \lambda^2 dv^2$$

where $\lambda(0, v) = 1$

$\lambda(0, v) = 0$.

∴ The Gaussian curv

$K = \frac{-\lambda_{11}}{\lambda} \Rightarrow \lambda_{11}$

The function λ must

power series in u is

$\lambda = 1 - \frac{Ku^2}{2} \frac{-Ku^3}{b} +$

$\lambda. = \lambda(u) - \frac{\lambda'(u)}{1!}$

$\lambda'' = -K\lambda$

$\lambda''' = -(K\lambda_1 + K_1\lambda$

$= -(0 + K_1(1$

$= -K_1$

$\lambda = 1+0 - K$

$ds^2 = du^2 + \lambda^2$

$u = \phi(v)$

$\left(\frac{ds}{dv}\right)^2 = \left(\frac{d}{d}\right.$

$l^2 = \lambda^2$

$l = ($

$l =$

$l = \int$

∴ The Gaussian curvature is, ⑨

1.12

$$K = \frac{-\lambda_{11}}{\lambda} \Rightarrow \lambda_{11} = -\lambda K$$

The function λ must be expended as a power series in u is the form.

$$\lambda = 1 - \frac{Ku^2}{2} \frac{-Ku^3}{b} + o(u4)$$

$$\lambda_1 = \lambda(u) - \frac{\lambda'(u)}{1!} u + \frac{\lambda''(u)}{2!} u^2 + \frac{\lambda'''(u) \cdot u^3}{3!} + o(u4)$$

$$\lambda'' = -K\lambda$$

$$\lambda''' = -(K\lambda_1 + K_1\lambda)$$

$$= -(0 + K_1(1))$$

$$= -K_1$$

$$\lambda = 1 + 0 - \frac{Ku^2}{2!} + \frac{K_1 u^3}{3!} + o(u4)$$

$$ds^2 = du^2 + \lambda^2 dv^2$$

$$u = \phi(v)$$

$$\left(\frac{ds}{dv}\right)^2 = \left(\frac{du}{dv}\right)^2 + \lambda^2$$

$$l^2 = \lambda^2 + \phi'^2$$

$$l = (\lambda^2 + \phi'^2)^{1/2}$$

$$l = \int_B^B (\phi'^2 + \lambda^2)^{1/2} dv$$

$$l = \int_A^B \left\{ \phi'^2 + \left(1 - \frac{Ku^2}{2} + \frac{K_1 u^3}{6}\right)^2 \right\}^{1/2} dv$$

$$= \int_A^B \left[\left\{ \phi'^2 + 1 + \frac{K^2 u^2}{4} + \frac{K_1{}^2 u^6}{36} - \frac{2Ku^2}{2} + \frac{2K_1 K u^5}{12} + \frac{2 K_1 u^3}{6} \right\}^{1/2} \right]^2 dv$$

$$= \int_A^B \left\{ \phi'^2 - Ku^2 - \frac{K_1 u^3}{3} + 1 \right\}^{1/2} dv$$

$$l - s = \frac{1}{2}\int_A (\phi'^2 - K\phi^2)\, dv$$ {∵ neglecting the higher power}

If K is always negative the integer and is always +ve & so $l > s$.

Theorem :- 6

In order that geodesic distance AB should be the shortest distance it is necessary and sufficient that B lies b/w A and it is conjugate point A_1.

proof :-

Given B lies b/w A & A_1

prove that : geodesic distance AB should be
test distance.

$$ds^2 = du^2 + \lambda^2 dv^2$$

Taken b/w the geodesic v is $v + \delta v$.

$$p = \lambda \delta v, \quad \lambda_{11} = -K\lambda, \quad P_{11} = -Kp.$$

$$\frac{d^2p}{du^2} = -Kp \quad \& \quad \frac{d^2p}{du^2} + Kp = 0.$$

B lies A & A_1.

Geodesic distance AB should be shortest distance,

conversely,

AB should be shortest distance.

Given AB

To prove : B lies between L and S

$$\delta^2(S) = \frac{1}{2}\int_A^B (u'^2 - Ku^2)\, du$$

$$\left\{ \because L - S = \frac{1}{2}\int_A^B (\phi'^2 - K\phi^2) dv \right\}$$

$$\delta^2(S) \geq 0$$

$\Rightarrow \delta^2(S)$ gives a

$$u'' + Ku = 0$$

$$u = \phi(v)$$

$\rightarrow A = 0, \quad u \rightarrow$

$$\delta^2(S) = \int_A^B (u'^2 - K$$

prove :- $\delta^2(S)$ has ex

$$\int_A^{A_1} uu'' dv$$

$u = u$

$du = du$

$dv = u'' dv$

$v = u'$

$$= [uu']_A^{A_1} -$$

$$= 0 - \int_A^{A_1} u'$$

$$\int_A^{A_1} uu'' dv = -\int_A^{A_1}$$

$$\int_A^B (u'^2 - Ku^2)\, dv$$

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$=0 \Rightarrow \delta^2(S)$ gives a maximum value,

$$u'' + Ku = 0$$

$$u = \phi(V)$$

$$u \rightarrow A = 0, \quad u \longrightarrow A_1 = 0$$

$$\delta^2(S) = \int_A^B (u'^2 - Ku^2)\, dv.$$

prove: $\delta^2(S)$ has extreme value.

$$\int_A^{A_1} uu''\, dv$$

$$\begin{array}{l|l|l} u = u & dv = u''\, dv & u = \phi(v) \\ du = du & v = u' & u' = \phi'(v)\, dv \\ & & u' = u'\, dv. \end{array}$$

$$= [uu']_A^{A_1} - \int_A^{A_1} u'u'\, dv$$

$$= 0 - \int_A^{A_1} u'^2\, dv$$

$$\int^{A_1} uu''\, dv = -\int_A^{A_1} u'^2\, dv$$

$$\int_A^B (u'^2 - Ku^2)\, dv = \int_A^{A_1} (u'^2 - Ku^2)\, dv$$

$$= -\int_A^{A_1} (Ku^2 - u'^2)\, dv$$

$$= -\int_A^{A_1} Ku^2\, dv + \int_A^{A_1} u'^2\, dv$$

$$\Rightarrow -\int_A^{A_1} Ku^2\, dv - \int_A^{A_1} uu''\, dv$$

$$= -\int_A^{A_1} (Ku^2 + uu'')\, dv$$