Mathematical Modelling Through Differential Equation :-

Mathematical Modelling interms of O.D.E aries when the situation Model involves some continuous variable varying with respect to some other continuous variable and we have some reasonable hypothesis about the rate of change of dependent variable with respect to independent variable.

When we have on dependent variable x (say population sides) depending on independent variable on O.D.E 1st order. If the hypothesis about the rate of change dx/dt. The Model with be interm of an O.D.E of 2nd order if the hypothesi involve the rate of change of d^2x/dt^2.

If there are a number of dependent continuous variable and only one dependent variable hypothesis may given a Mathematica Modelling interms of a system of 1st (or)

Higher order O.D.E.

If there is one dependent continuous variable (say velocity "u") and a number of Independent continuous variable (say space co-ordinates x, y, z and time "t"). we get a M.M interms of P.D.E If there are a no. of independent continuous variable. we can get a M.M. interms of system of P.D.E.

Linear Growth and Decay Model :-

i) Population Growth Model :-

let $x(t)$ be the population side of times and "b" & "d" be the birth and death rate per unit time. Then in time interval $(t, t+\Delta t)$ no. of birth and death would be $bx(t)\Delta t + o(\Delta t)$ and $dx(t)\Delta t + o(\Delta t)$. where $o(\Delta t)$ is an infinite symbol with approaches zero as Δt approaches zero. So that which is main equation is,

$$x(t+\Delta t) - x(t) = [bx(t) - dx(t)].\Delta t +$$

$$o(\Delta t) \quad -(1)$$

Taking limit on both sides and divided by Δt. we get

$$\lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{[bx(t) - dx(t)]\Delta t + O(\Delta t)}{\Delta t}$$

$$x'(t) = \lim_{\Delta t \to 0} \frac{bx(t) - dx(t)}{\cancel{\Delta t}} \cdot \cancel{\Delta t} + \lim_{\Delta t \to 0} \frac{O(\Delta t)}{\Delta t}$$

$$x'(t) = \frac{dx}{dt} = (b-d) \cdot x(t) \quad [\because b-d = a,\ x(t) = x]$$

$$\frac{dx}{dt} = ax \longrightarrow (2)$$

$\int$ing on both sides. we get $[\because \frac{dy}{dx} + Py = Q]$

$$(2) \Rightarrow \frac{dx}{dt} - ax = 0 \quad (\because P = (-a),\ y = x,\ Q = 0)$$

Integrating factor $= e^{\int P dx} = e^{\int -a dt} = e^{-at}$

The solution is,

$$ye^{\int P dx} = \int Q e^{\int P dx} \cdot dx + c$$

$$x e^{\int -a dt} = 0 + c$$

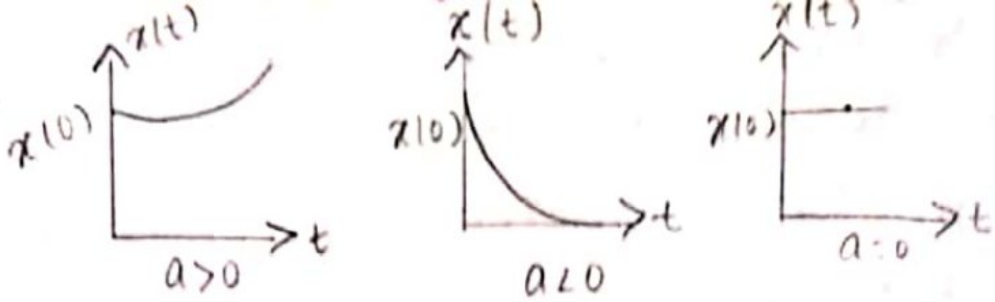
$$\boxed{x e^{-at} = c} \rightarrow (3)$$

put, $t = 0$

$$\boxed{x(0) = c} \longrightarrow (4)$$

Sub (4) in (3). we get, so that the population growth exponentially. If $a > 0$ decays exponentially if $a < 0$ and remains constant if $a = 0$

Scanned with CamScanner



case i) :-

If $a > 0$ the population will become double its present size at time "T". where,

$$2x(0) = x(0)e^{aT} \qquad [\text{time } t = T]$$

$$2 = e^{aT}$$

X value $\boxed{\ln 2 = 0.69314718}$

ln (or) log

$$\ln 2 = aT$$

$$\frac{1}{a} \cdot \ln 2 = T$$

$$T = (0.69314718)a^{-1} \dashrightarrow (0)$$

T is called the doubling period of the population and this doubling period is such that greater the values of "a". The smaller is the doubling period.

case ii)

If $a < 0$ the population will becomes half its present size at time T'. when,

$$\frac{1}{2} \cdot x(0) = x(0)e^{aT'}$$

$$\frac{1}{2} = e^{aT'}$$

$$\ln\left(\frac{1}{2}\right) = aT'.$$

Scanned with CamScanner

$$T' = \frac{1}{a} \cdot \ln(1/2)$$

$$T' = a^{-1} \cdot \ln(2^{-1})$$

$$T' = a^{-1}(-1)\ln 2.$$

$$T' = (0.69314718)a^{-1} \rightarrow ⓐ$$

T' is also independent of $x(0)$ & since $a < 0$, $T' < 0$, T' may be called the half life of the population and it decrease as the excess of death rate over birth date increase.

ii) Growth of science and scientists :-

let $S(t)$ denote the number of scientists at time "t" $b \cdot S(t) \cdot \Delta t + O(\Delta t)$ be the number of new scientist trained in time interval $(t, ~~t+s~~\; t + \Delta t)$. and let $d \cdot S(t) \Delta t + O(\Delta t)$ be the number of scientists refined from the science in the same period. Then the above model applies and the number of scientist should growth exponentially.

The same model applies to the growth of science mathematics and technology. Thus if, $M(t)$ is the amount of mathematics at times "t". Then the

Scanned with CamScanner

rate at growth of mathematics is proportional to the amount of mathematics so that,

6

$$\frac{dM}{dt} = aM \rightarrow \frac{dm}{dt} \propto M(t) = m$$

$$m(t) = M(a)e^{at} \cdot \frac{dM}{dt} \propto M$$

Thus according to this models mathematics science and technology at an exponential ratio and double themselves in a certain period of time.

iii) Effects of Immigration and Emigration population :-

If there is immigration into the population from outside at the rate proportional to the population size.

The effect is equivalent to increasin the birth rate.

Similarly, if their emigration is form the population rate proportional to the population size.

The effect is the same as that of increase in the death rate.

Scanned with CamScanner

If however immigration and emigration take place at a constant rate i & e respectively, 7

$$\frac{dx}{dt} = (b-d)x \quad \text{is modified.}$$

$$\frac{dx}{dt} = ax + i - e \qquad \boxed{\because b-d=a}$$

$$\frac{dx}{dt} = ax + K$$

$$\frac{dx}{dt} - ax = K$$

since, $\boxed{\frac{dy}{dx} + Py = Q}$.

$\boxed{P = -a}$ $\boxed{Q = K}$ $\boxed{y = x}$

The integral function,

$$e^{\int P dx} = e^{\int -a dt}$$

The solution is,

$$ye^{\int P dx} = \int Q e^{\int P dx} . dx + c$$

$$xe^{-at} = \int K e^{-at} . dt + c$$

$$= K . \frac{e^{-at}}{-a} + c$$

$$xe^{-at} = \frac{K}{a} . e^{-at} + c \longrightarrow (1)$$

Scanned with CamScanner

put $t=0$ in ①. we get

$$x \cdot e^0 = \frac{k}{-a} \cdot e^0 + c$$

$$x(t)e^0 = \frac{-k}{a} \cdot e^0 + c$$

$$x(0) = -\frac{k}{a} + c$$

$$c = x(0) + \frac{k}{a} \rightarrow ②$$

sub ② in ① we get

$$x(t)e^{-at} = \frac{-k}{a} \cdot e^{-at} + x(0) + \frac{k}{a}$$

$$x(t)e^{-at} + \frac{k}{a} \cdot e^{-at} = x(0) + \frac{k}{a}$$

$$x(t) + \frac{k}{a} = \left[x(0) + \frac{k}{a}\right] e^{-at} \rightarrow ③$$

The model also applies to growth of population of bacteria and micro-orga to the growth of malignent cells to the increase value of timber in forest.

In the case of forest planting of now plants will corresponds to immigrating and the cutting of will corresponding to emigration.

iv) Interest compounded continuously :-

let the amount at time "t" be x(t.) and let Interest at rate "r" per unit amount per unit time be compound continuously.

$x(t+\Delta t) - x(t)$ 9

Then, $\boxed{\Delta x(t) = r\, x(t) \cdot \Delta t + O(\Delta t)}$

Taking limit on both sides and divided by Δt. we get

$$\lim_{\Delta t \to 0} \frac{\Delta x(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{r\, x(t) \cdot \Delta t + O(\Delta t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{r\, x(t)}{\cancel{\Delta t}} \cdot \cancel{\Delta t} + \lim_{\Delta t \to 0} \frac{O(\Delta t)}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\Delta x(t)}{\Delta t} = r\,[x(t)] + 0$$

$$\Rightarrow \quad \frac{dx(t)}{dt} = r \cdot x(t)$$

$$\frac{d \cdot x(t)}{x(t)} = r \cdot dt$$

$\int$ing on both sides. we get

$$\int \frac{dx(t)}{x(t)} = \int r\, dt$$

$$\log x(t) = rt + c \longrightarrow (1)$$

put $t=0$ in (1). we get

$$\log x(0) = c \longrightarrow (2)$$

$$\log x(t) = rt + \log x(0)$$

Sub (2) in (1) we get

$$\log x(t) - \log x(0) = rt$$

$$\therefore \log \left[\frac{x(t)}{x(0)} \right] = rt$$

Taking exponential on both sides. we get

$$\frac{x(t)}{x(0)} = e^{rt}$$

$$x(t) = x(0).e^{rt} \longrightarrow (3)$$

Remark :-

1) we have another formula for compound interest.

$$x(t) = x(0) \left[1 + \frac{r}{n} \right]^{nt}$$

using, $A = P \left[1 + \frac{r}{100} \right]$

Assume that interest is payable "n" times per unit time by taking the limit as $n \to \infty$.

(i.e) let $\boxed{x(0) = 1}$, $\boxed{r = 1}$, $\boxed{t = 1}$

Sub those values in (3) & (4) we get

(3) $\Rightarrow$ $x(t) = x(0) e^{rt}$

$$x(1) = 1.e^{(1).1}$$

$$x(1) = e$$

(4) $\Rightarrow$ $x(t) = x(0) \left[1 + \frac{r}{n} \right]^{nt}$

Scanned with CamScanner

$$x(t) = \lim_{n\to\infty} (1) \left[1 + \frac{1}{n}\right]^{n(1)}$$

$$\therefore e = \lim_{n\to\infty} \left[1 + \frac{1}{n}\right]^{n} \quad [\because x(1) = e]$$

2) let $x(t) = 1$

$$③ \Rightarrow x(t) = x(0)e^{rt}$$

$$1 = x(0)e^{rt}$$

$$x(0) = e^{-rt}$$

$$x(0) = e^{-rt}$$

So that e^{-rt} is present value of a unit amount due to the one period.

Hence, when interest at the rate "r" per unit amount is compounded continuously.

V) Radio-Active Decay :-

n a) ;) Nov-2018

Many substances undergoes Radio-Active Decay at the rate proportional to the amount of the Radio-Active ~~Decay~~ substance present at any time and each of then has a half life period.

For uranium (U^{238}), it is 4.55 billion years

For Potassium , " " 1.3 " "

For Rubidium , " " 50 " "

For Thorium , " " 13.9 " "

while for carbon 14, for white led it is only 22 years.

In Radiogeology these results are used for radio-active dating. Thus the radio of radio-carbon to ordinary carbon [carbon 12] in decay plants and animals enables us to estimate there time of death. 12

Radio active dating has also been used to estimate the age to the solar system and of earth of 4.5 billion years.

vi) Decrease of Temperature :-

According to [Newton's Law of cooling] the rate of change of temperature of a body is proportional to the difference between the temperature "T" of the body and temperature T_S of surrounding

Scanned with CamScanner

$$T(t)-T_s = (T(0)-T_s)e^{kt}$$

At $t=0$ we get $\left[\because x(t) = x(0)e^{at}\right]$

$$\log(T(t)-T(s)) = k(0)+c$$

$$c = \log(T(0)-T_s) \rightarrow ②$$

Sub ② in ① we get

$$\log(T(t)-T_s) = kt + \log(T(0)-T_s)$$

$$\log(T(t)-T_s) - \log(T(0)-T_s) = kt$$

$$\log\left(\frac{T(t)-T_s}{T(0)-T_s}\right) = kt$$

Taking exponential on both sides. we get

$$\frac{T(t)-T(s)}{T(0)-T_s} = e^{kt}$$

$$T(t)-T_s = e^{kt}[T(0)-T_s]$$

$$\therefore T(t)-T_s = e^{kt}[T(0)-T_s]$$

vii) Diffusion :-

According to [Fick's law of diffusion] the time rate of movement of solute accross a thin membrane is proportional if the area of the membrane and so the difference in concentration of two solute on the two sides of the membrane.

If the area of the membrane is constant and the concentration of solute on one side is kept fixed at "a" and concentration of the solution on the other side initially is $C_0 < a$.

14

The fick's law gives,

$$\boxed{\frac{dc(t)}{dt} \propto a - c} \qquad [\because c(0) = c_0]$$

$$\frac{dc(t)}{dt} = k(a-c)$$

$$\frac{dc(t)}{a-c} = k.dt \Rightarrow \frac{dc(t)}{a-c(t)} = k.dt$$

$\int^{ing}$ on both sides. we get

$$\int \frac{dc(t)}{a-c(t)} = k\int dt$$

$$-\log(a-c(t)) = kt + c \rightarrow (1)$$

At $\boxed{t=0}$ we get

$$-\log(a-c(0)) = c \longrightarrow (2)$$

sub (2) in (1), we get

$$-\log(a-c(t)) = kt - \log(a-c(0))$$

$$\Rightarrow -kt = \log(a-c(t)) - \log(a-c(0))$$

$$\Rightarrow -kt = \log\left(\frac{a-c(t)}{\right.$$

$$\Rightarrow e^{-kt} = \frac{a - c(t)}{a - c(0)}$$

$\Rightarrow e^{-kt}(a - c(0)) = a - c(t)$ and

$c(t) \rightarrow a$ as $t \rightarrow \infty$ whatever be the value of c_0.

viii) change of price of a commodity:-

Let $p(t)$ be the price of a commodity and time "t" then its rate of change is proportional to the difference between the demand $d(t)$ and the supply $s(t)$ of the common in the market.

(i.e) $\boxed{\frac{dp(t)}{dt} \propto d(t) - s(t)}$

Scanned with CamScanner

$$= K(\alpha - \beta p(t))$$

[where, $\alpha = d_1 - s_1$ position $\beta = d_2 - s_2 p$ price $\beta = s_2 - d_2 p_1$ $\because \alpha/\beta = pe$]

$$\frac{dp(t)}{dt} = K\beta\left(\alpha/\beta - p(t)\right)$$

$$\cancel{\frac{dp(t)}{dt} = K\beta (}$$

16

$$\therefore pe = \alpha/\beta = \text{Equilibrium.}$$

Here, $\boxed{\beta > 0}$, so $\boxed{\beta = 1}$

$$\frac{dp(t)}{dt} = k\left[pe - p(t)\right]$$

$$\Rightarrow \frac{dp(t)}{pe - p(t)} = k.dt$$

$\int^{ing}$ on both sides. we get

$$\int \frac{dp(t)}{pe - p(t)} = k\int dt$$

$$-\log\left[pe - p(t)\right] = kt + C \rightarrow ①$$

At $\boxed{t = 0}$ we get

$$-\log\left[pe - p(0)\right] = C \longrightarrow ②$$

Sub ② in ① we get

$$-\log\left[pe - p(t)\right] = kt - \log\left[pe - p(0)\right]$$

$$-kt = \log\left[pe - p(t)\right] - \log\left[pe - p(0)\right]$$

$$-kt = \log\left[pe - p(t)\right]$$

Scanned with CamScanner

Taking exponential on both sides.

$$e^{-kt} = \frac{pe - p(t)}{pe - p(0)}$$

$$\Rightarrow [pe - p(0)]\, e^{-kt} = pe - p(t)$$

$$\therefore \boxed{pe - p(t) = e^{-kt}\,[pe - p(0)]}$$

and

$$\boxed{p(t) - pe = 0} \quad \text{as } t \to \infty$$

problems :-

1. suppose the population of the world new 4 billion and its doubling period is 35 years what well be the population of the world of her 350 years 700 years, 1050 years? If the surface area of the earth is 18,60,000 billion square fees. How much space would's each person get of her 1050 years.

Soln:

Given :

$x(0) = 4$ billion

$T = 35$ years

W.K.T $\quad T = \frac{\ln e^2}{a}$

$$35 = \frac{\ln e^2}{a} \qquad [\log \text{ (or) } \ln]$$

$$T = \frac{1}{a}.\ln x$$

$a = \log e^2$

Scanned with CamScanner

$$= \frac{0.69314718}{35}$$

$$\boxed{a = 0.0198}$$

W.K.T $x(t) = x(0)e^{at}$ //

when, $t = 350$ years.

$x(350) = 4e^{(0.0198)(350)}$

$= 4e^{6.93}$

$\Rightarrow x(350) = 4090$ billion (4089.97591

//

when, $t = 700$ years

$x(700) = 4e^{(0.0198)(700)}$

$= 4e^{13.86}$

$\Rightarrow x(700) = 41,01,917$ billion (4181975.735

//

when, $t = 1050$ years.

$x(1050) = 4e^{(0.0198)(1050)}$

$= 4e^{20.79}$

$= 4427,60,45,03,1$ billions

$\Rightarrow x(1050) = 4427{\cdot}6045,03,1$ billions

//

Given that surface area of earth is 18,60,000 billion square feat.

square would each person art after 1050 years.

$= 4276045031 \times 10^{9}$

2. Find the relation b/w doubling, tripling, quadrupling times for a population.

Soln: let the doubling period be T.

let the tripling period be T'.

let the quadrupling period be T''.

Initial population be x(0).

$$x(t) = x(0)e^{aT}.$$

If the population is doubling. Then,

$$2x(0) = x(0)e^{aT}$$

$$2 = e^{aT}$$

$$\log 2 = aT$$

$$\therefore \boxed{a = \frac{\log 2}{T}} \longrightarrow (1)$$

If the population is tripling. Then,

$$3x(0) = x(0)e^{aT'}$$

$$3 = e^{aT'}$$

$$\log 3 = aT'$$

$$\therefore \boxed{a = \frac{\log 3}{T'}} \longrightarrow (2)$$

If the population is quadrupling. Then,

$$4x(0) = x(0)e^{aT''}$$

$$4 = e^{aT''}$$

$$\log 4 = aT''$$

$$\therefore \boxed{a = \frac{\log 4}{T''}} \longrightarrow (3)$$

Scanned with CamScanner

From ①, ② & ③ we get

$$a = \frac{\log 2}{T} = \frac{\log 3}{T'} = \frac{\log 4}{T''}$$ //

Non-Linear growth and Decay Models :-

i) Logistic law of population Growth :-

As population increases due to over growing and limitation of resource. The birth rate "b" be decreases and death rate "d" be increases with population size

The simplest assumption is to take,

$$\left.\begin{aligned} b &= b_1 - b_2 x \\ d &= d_1 + d_2 x \end{aligned}\right\} \rightarrow ①$$

where, $b_1, b_2, d_1, d_2 > 0$.

W.K.T $x'(t) = (b-d)x \rightarrow ②$

Sub Equ ① in Equ ② we get

$$x'(t) = [(b_1 - b_2 x) - (d_1 + d_2 x)]x$$

$$= [(b_1 - d_1) - (b_2 + d_2)x]x$$

$$x'(t) = [a - bx]x$$

where, $a > 0$, $b > 0$

$a = b_1 - d_1$ $\qquad$ $b = b_2 + d_2$

$x'(t) = \frac{dx}{dt}$ $\Rightarrow$ $\frac{dx(t)}{dt} = [a - bx]x$

$$\frac{dx(t)}{(a-bx)x} = dt \longrightarrow ③$$

Scanned with CamScanner

using the partial fraction. we get

$$\frac{1}{(a-bx)x} = \frac{A}{(a-bx)} + \frac{B}{x}$$ 21

$$\frac{1}{\cancel{(a-bx)x}} = \frac{A(x) + B(a-bx)}{\cancel{(a-bx)x}}$$

$$1 = A(x) + B(a-bx)$$

put $\boxed{x=0}$ $\quad 1 = A(0) + B(a-b(0))$

$$1 = 0 + B(a)$$

$$\therefore \boxed{B = \frac{1}{a}}$$

put $\boxed{x = \frac{a}{b}}$ $\quad 1 = A\left(\frac{a}{b}\right) + B\left(a - \cancel{b}\left(\frac{a}{\cancel{b}}\right)\right)$

$$1 = A\left(\frac{a}{b}\right) + 0$$

$$\therefore \boxed{A = \frac{b}{a}}$$

So, $\dfrac{1}{(a-bx)x} = \dfrac{b}{a(a-bx)} + \dfrac{1}{ax} \rightarrow$ 4

sub equ 4 in 3 we get

$$\frac{b}{a(a-bx)} \cdot dx(t) + \frac{dx(t)}{ax} = dt$$

$\int$ing on both sides. we get

$$\frac{b}{a}\int \frac{dx(t)}{[a-bx(t)]} + \frac{1}{a}\int \frac{dx(t)}{x} = \int dt$$

$$\frac{\cancel{b}}{a}\left[-\log\left(\frac{a-bx(t)}{\cancel{b}}\right)\right] + \frac{1}{a}.\log x(t) = t + c$$

$$\frac{-1}{a} \log[a-bx(t)] + \frac{1}{a}\cdot\log x(t) = t + c$$

$$\frac{1}{a}\log x(t) - \frac{1}{a}\log[a - bx(t)] = t + c$$

$$\frac{1}{a}[\log x(t) - \log(a - bx(t))] = t + c$$

$$\frac{1}{a}\log\left(\frac{x(t)}{a - bx(t)}\right) = t + c \quad // \rightarrow (5)$$

put $\boxed{t=0}$

$$\boxed{\frac{1}{a}\cdot\log\left(\frac{x(0)}{a - bx(0)}\right) = c}$$

sub c in (5) we get

$$\frac{1}{a}\left[\log\frac{x(t)}{a - bx(t)}\right] = t + \frac{1}{a}\left[\log\frac{x(0)}{a - bx(0)}\right]$$

$$\frac{1}{a}\left[\log\frac{x(t)}{a - bx(t)}\right] - \frac{1}{a}\left[\log\frac{x(0)}{a - bx(0)}\right] = t$$

$$\frac{1}{a}\left[\log\left(\frac{x(t)/a - bx(t)}{x(0)/a - bx(0)}\right)\right] = t$$

$$\log\left(\frac{x(t)/a - bx(t)}{x(0)/a - bx(0)}\right) = at$$

Taking exponential on both sides. we get

$$\left(\frac{x(t)}{a - bx(t)}\right) = e^{at}$$

Scanned with CamScanner

$$\frac{x(t)}{a-bx(t)} = e^{at} \frac{x(0)}{a-bx(0)}$$ 23

case i) :-

Assume $x(0) < a/b$

(ie) $x(0) - a/b > 0$

$bx(0) - a < 0$

Multiply (−) on both sides.

$a - bx(0) > 0$

$\therefore a > bx(0)$

$\frac{x(0)}{a-bx(0)} . e^{at}$ becomes greater than zero. (i

$\Rightarrow a - bx(t) > 0$

$a > bx(t)$

$x(t) < a/b$

$\frac{dx}{dt} = (a-bx)x > 0$ (ii

(ie) $x(t)$ is a monotonic increasing function of "t" which approaches $\frac{a}{b}$ as $t \to \infty$.

case ii) :- (iii

Assume, $x(0) > a/b$.

Then, $x(t) > a/b$

$\Rightarrow \frac{dx}{dt} < 0$.

Scanned with CamScanner

$\Rightarrow x(t)$ is a monotonic decreasing function of "t" which approaches a/b as $t \to \infty$

case iii) :-

Assume, $x(0) = a/b$ //

Then, $x(t) = a/b \quad \forall\, t$

$$\Rightarrow \frac{dx(t)}{dt} = 0 \quad //$$

But we have, $\dfrac{dx(t)}{dt} = (a-bx)x$

$$\frac{d^2x(t)}{dt} = (a-bx) + x(-b)$$

$$= a - 2bx$$

i) $\dfrac{d^2x}{dt^2} = 0 \quad \Rightarrow a - 2bx = 0$

$$x = \frac{a}{2b} \, //$$

$\therefore$ The curve has a point of inflection at $x = a/ab$.

ii) $\dfrac{d^2x}{dt^2} > 0 \quad \Rightarrow a - 2bx > 0$

$a > 2bx$

$\Rightarrow x < a/2b$ //

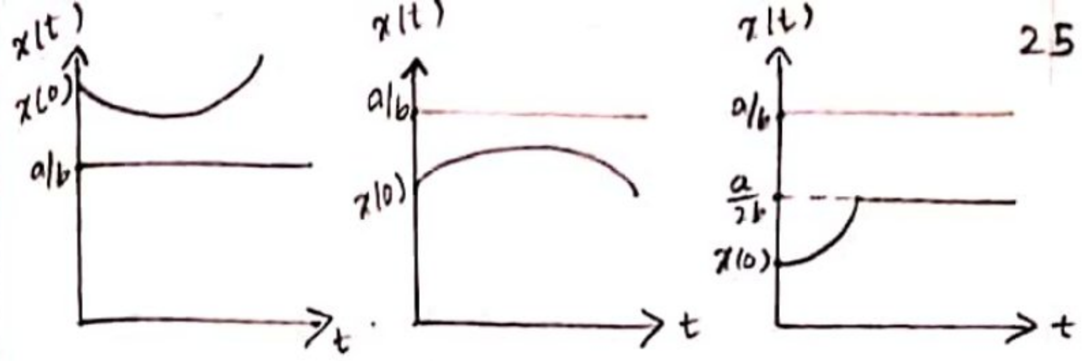
$\therefore$ The curve is convex.

iii) $\dfrac{d^2x}{dt^2} < 0 \quad \Rightarrow a - 2bx < 0$

$a < 2bx$

$a/2b < x$

$\Rightarrow x > a/2b$ // $\therefore$ The curve is concave



$x(0) > a/b$ $\qquad$ $\frac{a}{2b} < x(0) < a/b$ $\qquad$ $x(0) < \frac{a}{2b}$

ii) Spread of Technological innovation and infections Disease :-

Let $N(t)$ be the no. of compainer, which have adopted a technological innovation till time "t". Then the rate of change of the no. of these companies which have adopted this innovation and in the no. of there which have not yet adopted it, so that if R is the total no. of companies in the region.

$$\frac{dN(t)}{dt} \propto N(t)\,[R - N(t)]$$

$$\Rightarrow \frac{dN(t)}{dt} = K\,N(t)\,[R - N(t)]$$

Similarly, let $N(t)$ be the no. of persons in the region affected by on infliction disease the rate of increase of the disease varies with the no. of affected persons and

Scanned with CamScanner

on affected persons.

(ie) $\frac{dN(t)}{dt} \propto N(t)\,[R - N(t)]$

$$\Rightarrow \frac{dN(t)}{dt} = K\,(N(t))\,[R - N(t)].$$

where, R is the total no. of. person in a region.

iii) Rate of Disolutions :-

let "$x(t)$" be the amount of disolved solute is a solvent at time "t" and let "C_0" be the maximum concentration.

(ie) The maximum amount of the solute that can be disolved in a unit volume of the solvent.

let V be the volume of the solvent, it is found that rate at which the solute is dissolved is proportional to the amount of undisolved is ~~proport~~ solute and to the different b/w the concentration of the solute at time "t" and the maximum possible concentration. So that,

Concentration at time (t) } = Amount of the solute that has formed solutions

Scanned with CamScanner

(ie) $$\frac{dx(t)}{dt} \propto x(t)\left[\frac{x(0)-x(t)}{V}\right]-C_0$$

$$\frac{dx(t)}{dt} = K.x(t)\left[\frac{x(0)-x(t)}{V}\right]-C_0$$

$$= \frac{K.x(t)}{V}\left[x(0)-x(t)-VC_0\right]$$

$$= \frac{K.x(t)}{V}\left[x(0)-VC_0-x(t)\right]$$

$$\frac{dx}{dt} = \frac{Kx}{V}\left[a-x\right] \qquad [\because x(0)-VC_0=a,\ x(t)=x]$$

$$\frac{dx}{x(a-x)} = \frac{K}{V}$$

Using partial fraction. we get

$$\frac{1}{x(a-x)} = \frac{A}{x} + \frac{B}{a-x} \longrightarrow (1)$$

$$\frac{1}{x(a-x)} = \frac{A(a-x)+B(x)}{x(a-x)}$$

$$1 = A(a-x)+B(x)$$

Put $x=a$ $\quad 1 = A(a-a)+B(a)$

$$1 = A(0)+Ba$$

$$\boxed{B = 1/a}$$

put $x=0$ $\quad 1 = A(a-0)+B(0)$

$$1 = Aa$$

$$\boxed{A = 1/a}$$

Sub A & B value in (1) we get

$$\frac{1}{x(a-x)} = \frac{1}{ax} + \frac{1}{a(a-x)}$$

Sub ② in ✱ we get

$$\frac{dx}{ax} + \frac{dx}{a(a-x)} = \frac{k}{V}.dt$$

$\int$ing on both sides. we get

$$\int \frac{dx}{ax} + \int \frac{dx}{a(a-x)} = \frac{k}{V}\int dt$$

$$\frac{1}{a}\int \frac{dx}{x} + \frac{1}{a}\int \frac{dx}{a-x} = \frac{kt}{V} + c$$

$$\frac{1}{a}\log x - \frac{1}{a}\log(a-x) = \frac{kt}{V} + c$$

$$\frac{1}{a}\left[\log x - \log(a-x)\right] = \frac{kt}{V} + c$$

$$\frac{1}{a}\log\left(\frac{x}{a-x}\right) = \frac{kt}{V} + c \longrightarrow ③$$

At $t=0$ in ③, we get

$$\frac{1}{a}\left[\log \frac{x(0)}{a-x(0)}\right] = c \longrightarrow ④$$

Sub ④ in ③ we get

$$\frac{1}{a}\left[\log \frac{x(t)}{a-x(t)}\right] = \left[\log \frac{x(0)}{a-x(0)}\right]\frac{1}{a} + \frac{k}{V}.t$$

$$\frac{1}{a}\left[\log \frac{x(t)}{a-x(t)} - \log \frac{x(0)}{a-x(0)}\right] = \frac{k}{V}.t$$

$$\frac{1}{a}.\log\left[\frac{x(t)/a-x(t)}{x(0)/a-x(0)}\right] = \frac{k}{V}.t$$

Scanned with CamScanner

Taking ~~log~~ $x(t)$ exponential on both sides. 29

$$\frac{x(t)}{a-x(t)} \times \frac{a-x(0)}{x(0)} = e^{\frac{k}{v}.at}$$

$$\frac{x(t)}{a-x(t)} = e^{\frac{k}{v}.at} \times \frac{x(0)}{a-x(0)}$$

put "a" value. we get

$$\left[\frac{x(t)}{x(0)-VC_0-x(t)}\right] = e^{k/v}$$ //

Law of mass action Chemical Reaction :-

Two chemical substance combined in the ratio $a:b$ to form a 3^{rd} substance z. if $z(t)$

$$\frac{dz}{dt} = k\left(A - \frac{az}{a+b}\right)\left(B - \frac{bz}{a+b}\right)$$

$$= k(A - A_1 z)(B - B_1 z)$$

$$\frac{dz}{(A-A_1 z)(B-B_1 z)} = k.dt \longrightarrow (1)$$

where, $A_1 = \frac{a}{a+b}$, $B_1 = \frac{b}{a+b}$

using partial fraction,

$$\frac{1}{(A-A_1 z)(B-B_1 z)} = \frac{A}{(A-A_1 z)} + \frac{B}{(B-B_1 z)}$$

$$\frac{1}{\cancel{(A-A_1 z)(B-B_1 z)}} = \frac{A(B-B_1 z) + B(A-A_1 z)}{\cancel{(A-A_1 z)(B-B_1 z)}}$$

$$1 = A(B-B_1 z) + B(A-A_1 z)$$

put $\boxed{z = \frac{A}{A_1}}$ $1 = A\left[B - B_1\left(\frac{A}{A_1}\right)\right] + B\left[A - A_1\left(\frac{A}{A_1}\right)\right]$

$$1 = A\left[\frac{BA_1 - B_1 A}{A_1}\right] + B\left[\frac{AA_1 - A_1 A}{A_1}\right]$$

$$1 = A\left[\frac{BA_1 - B_1 A}{A_1}\right] + 0$$

$$\frac{1}{A} = \left[\frac{BA_1 - B_1 A}{A_1}\right]$$

$$\therefore \boxed{A = \frac{A_1}{A_1 B - B_1 A}}$$

Scanned with CamScanner

Scanned with CamScanner

put $\boxed{z = \frac{B}{B_1}}$ $\quad 1 = A\left(B - \frac{B_1 B}{B_1}\right) + B\left(A - \frac{A_1 B}{B_1}\right)$

$$1 = A\left(\frac{BB_1 - B_1 B}{B_1}\right) + B\left(\frac{AB_1 - A_1 B}{B_1}\right)$$

$$1 = 0 + B\left(\frac{AB_1 - A_1 B}{B_1}\right)$$

$$1 = B\left(\frac{AB_1 - A_1 B}{B_1}\right)$$

$$\frac{1}{B} = \frac{AB_1 - A_1 B}{B_1}$$

$$\therefore \boxed{B = \frac{B_1}{AB_1 - A_1 B}}$$

$$\frac{1}{(A - A_1 z)(B - B_1 z)} = \frac{A_1}{(A - A_1 z)(A_1 B - B_1 A)} + \frac{B_1}{(B - B_1 z)(AB_1 - A_1 B)} \quad (2)$$

Sub (2) in (1), we get

$$\frac{dz}{(A - A_1 z)(B - B_1 z)} = \frac{A_1 dz}{(A_1 B - B_1 A)(A - A_1 z)} - \frac{B_1 dz}{(AB_1 - A_1 B)(B - B_1 z)} = k\,dt$$

$$\frac{1}{A_1 B - B_1 A}\left[\frac{A_1 dz}{A - A_1 z} - \frac{B_1 dz}{B - B_1 z}\right] = k.dt$$

$\int$ing on both sides. we get

$1 \quad -A_1(\log(A - A_1 z)) \quad -B_1(\log(B - B_1 z))$

Scanned with CamScanner

$$\frac{1}{A_1B - B_1A}\left[\log(B - B_1Z) - \log(A - A_1Z)\right] = Kt + c$$

$$\frac{1}{A_1B - B_1A}\left[\log\left(\frac{B - B_1Z(t)}{A - A_1Z(t)}\right)\right] = Kt + C \longrightarrow (3)$$

At $\boxed{t=0}$ we get

$$\frac{1}{A_1B - B_1A}\left[\log\left(\frac{B - B_1(0)Z}{A - A_1(0)Z}\right)\right] = C \;//$$

Sub C value. we get

$$\frac{1}{A_1B - B_1A}\left[\log\left(\frac{B - B_1Z(t)}{A - A_1Z(t)}\right)\right] = Kt + \frac{1}{A_1B - B_1A}\left[\log\left(\frac{B - B_1(0)}{A - A_1(0)}\right.\right.$$

$$\frac{1}{A_1B - B_1A}\left[\log\left(\frac{B - B_1Z(t)}{A - A_1Z(t)}\right) - \log\left(\frac{B - B_1Z(0)}{A - A_1Z(0)}\right)\right] = Kt$$

$$\log\left[\frac{B - B_1Z(t) / A - A_1Z(t)}{B - B_1Z(0) / A - A_1Z(0)}\right] = Kt\,[A_1B - B_1A$$

Taking exponential on both sides. we get

(ie) $$\frac{B - b/a+b \cdot (Z(t))}{A - \frac{a}{a+b}\cdot Z(t)} = e^{Kt\left(\frac{a}{a+b}\cdot B - \frac{b}{a+b}\cdot A\right)}\cdot$$

$$\left(B - \frac{b}{a+b}\cdot Z(0)\right)$$

In spread of technological innovation and inflections $K = 0.007$, $R = 1000$, $N(0) = 50$. Find $N(10)$ and find "t" when $N(t) = 500$. 33

Soln: Given: $R = 1000$, $K = 0.007$

W.K.T $$\frac{dN(t)}{dt} = K \cdot N(t)\,[R - N(t)]$$

$$\Rightarrow \frac{dN(t)}{dt} = (0.007)\,N(t)\,[1000 - N(t)]$$

$$\frac{dN(t)}{N(t)\,[1000 - N(t)]} = 0.007 \cdot dt \rightarrow (1)$$

using partial fraction,

$$\frac{1}{N(t)\,[1000 - N(t)]} = \frac{A}{N(t)} + \frac{B}{[1000 - N(t)]}$$

$$1 = A\,[1000 - N(t)] + B \cdot N(t)$$

At $\boxed{N(t) = 1000}$

$$1 = A(0) + B(1000)$$

$$\therefore \boxed{\frac{1}{1000} = B}$$

At, $N(t) = 0$

$$1 = A(1000) + B(0)$$

$$1 = A(1000)$$

$$\therefore \boxed{A = \frac{1}{1000}}$$

Scanned with CamScanner

Equ ①,

$$\frac{1}{1000}\left[\frac{dN(t)}{N(t)}+\frac{dN(t).}{1000-N(t)}\right]=0.007\,dt$$

$\int$ing on both sides. we get

$$\frac{1}{1000}\left[\log N(t)-\log(1000-N(t))\right]=0.007t+c$$

$$\frac{1}{1000}\left[\log\left(\frac{N(t)}{1000-N(t)}\right)\right]=0.007t+C \rightarrow ②$$

put $t=0$

$$\frac{1}{1000}\left[\log\left(\frac{N(0)}{1000-N(0)}\right)\right]=c \rightarrow ③$$

sub ③ in ② we get

$$\frac{1}{1000}\left[\log\left(\frac{N(t)}{1000-N(t)}\right)\right]=0.007t+\frac{1}{1000}\left[\log\left(\frac{N(0)}{1000-N(0)}\right)\right]$$

$$\frac{1}{1000}\left[\log\left(\frac{N(t)}{1000-N(t)}\right)-\log\left(\frac{N(0)}{1000-N(0)}\right)\right]=0.007t$$

$$\frac{1}{1000}\left[\log\frac{N(t)/1000-N(t)}{N(0)/1000-N(0)}\right]=0.007t.$$

$$\log\left[\frac{N(t)/1000-N(t)}{\quad}\right]=0.007t\,(1000)$$

Taking exponential on both sides. we get

$$\frac{N(t)/1000-N(t)}{N(0)/1000-N(0)} = e^{7t}$$

$$\frac{N(t)}{1000-N(t)} = e^{7t}\left(\frac{N(0)}{1000-N(0)}\right) \rightarrow (4)$$

When, $t=10$, $N(0)=50$ in (4) we get

$$\frac{N(10)}{1000-N(10)} = e^{7t}\,\frac{50}{1000-50}$$

$$\frac{N(10)}{1000-N(10)} = e^{7(10)}\left(\frac{\cancel{50}^{1}}{\cancel{950}_{19}}\right)$$

$$\frac{N(10)}{1000-N(10)} = e^{70}\left(\frac{1}{19}\right)$$

$$19.N(10) = e^{70}[1000-N(10)]$$

$$19N(10) = e^{70}(1000) - e^{70}N(10)$$

$$19N(10) + e^{70}N(10) = e^{70}(1000)$$

$$(19+e^{70})N(10) = e^{70}(1000)$$

$$N(10) = \frac{e^{70}(1000)}{19+e^{70}}$$

$$= \frac{\cancel{e^{70}}(1000)}{\cancel{e^{70}}(19/e^{70+1})}$$

$$= \frac{1000}{19/e^{70+1}}$$

$$= \frac{1000}{19/2.5154\times10^{30}} + 1$$ 36

$\therefore \boxed{N(10) = 1000}$

· when, $N(t) = 500$. sub in ④ we get

$$\frac{500}{1000-500} = e^{7t}\left(\frac{50}{1000-50}\right)$$

$$\frac{500}{500} = e^{7t}\left(\frac{50}{950}\right)$$

$$1 = e^{7t}\left(\frac{1}{19}\right)$$

$$1 = \frac{e^{7t}}{19} \Rightarrow 19 = e^{7t}$$

$\therefore \boxed{e^{7t} = 19}$

Taking log on both sides. we get

$$7t = \text{In}\, e^{19}$$

$$t = \cancel{\text{In}\, e} \frac{\log e^{19}}{7} \quad (\because \log = \text{In})$$

$\therefore \boxed{t = 0.4203}$

Scanned with CamScanner

ii) How most time we will it take 0 to get 70 grams if $t = ?$ 37

Soln: By using the substance formula,

$$\frac{1}{A_1B - B_1A}\left[\log\left(\frac{B - B_1 Z}{A - A_1 Z}\right)\right] = kt + C$$

Here, $A_1 = \frac{a}{a+b}$, $B_1 = \frac{b}{a+b}$

Given: $a : b = 2 : 3$

$A = 45$ grams

$B = 60$ grams

$$\frac{1}{\frac{2}{5}(60) - \frac{3}{5}(45)}\left[\log\left(\frac{60 - \frac{3}{5}Z}{45 - \frac{2}{5}\cdot Z}\right)\right] = kt + C$$

$$\frac{1}{24 - 27}\left[\log\left(\frac{\frac{300 - 3Z}{5}}{\frac{225 - 2Z}{5}}\right)\right] = kt + C$$

$$\frac{-1}{3}\left[\log\left(\frac{300 - 3Z}{225 - 2Z}\right)\right] = kt + C$$

$$\log\left(\frac{300 - 3Z}{225 - 2Z}\right) = kt + C \cdot \frac{3kt + 3c}{-3kt + C_1}$$

Taking exponential on both sides.

$$\frac{300 - 3Z}{225 - 2Z} = e^{-3kt} \cdot e^{C_1}$$

Given: $\boxed{Z(t) = 50}$ at time 5 minutes.

$$\frac{300 - 3(50)}{225 - 2(50)} = \frac{4}{3} \cdot e^{-3k(5)}$$

Scanned with CamScanner

$$\frac{300-150}{225-100} = \frac{4}{3} \cdot e^{-15K}$$

$$\frac{150}{125} \times \frac{3}{4} = e^{-15K}$$

Taking exponential on both sides.

$$0.9 = e^{-15K}$$

$$\log 0.9 = -15K.$$

$$K = \log \frac{0.9}{-15}$$

$$\therefore \boxed{K = -0.00702}$$

sub K value in ② we get

$$\frac{300-3Z}{225-2Z} = e^{-3(0.00702)t} \left(4/3\right) \cancel{\left(4/3\right)}$$

To find Z:

when, $t = 210$ minutes.

$$\frac{300-3Z}{225-2Z} = \frac{4}{3} \cdot e^{(0.02106)(210)}$$

$$\frac{300-3Z}{225-2Z} = \frac{4}{3} \cdot e^{4.4206}$$

$$\frac{300-3Z}{225-2Z} = 111.0835$$

$$300-3Z = 111.0835(225-2Z)$$

$$300-3Z = 24993.7875 - \cancel{222}\ 222.167Z.$$

$$222.167Z - 3Z = 24993.7875 - 300$$

$$219.167Z = 24693.7875$$

$$Z = \frac{24693.7875}{219.167}$$

$\therefore \boxed{z = 112.679 \text{ grams}}$ //

To find t :-

when $z = 70$ grams

$$\frac{300 - 3(z)}{225 - 2(z)} = \frac{4}{3} e^{(0.02106)t}$$

$$\frac{300 - 3(70)}{225 - 2(70)} = \frac{4}{3} \cdot e^{(0.02106)t}$$

$$\frac{300 - 210}{225 - 140} = \frac{4}{3} \cdot e^{(0.02106)t}$$

$$\frac{90}{85} = \frac{4}{3} \cdot e^{0.02106t}$$

$$\frac{19}{17} \times \frac{3}{4} = e^{0.02106t}$$

$$\frac{27}{34} = e^{0.02106t}$$

$$0.7941 = e^{0.02106t}$$

Taking log on both sides. we get

$$\log 0.7941 = 0.02106t$$

$$t = \log \frac{e^{0.7941}}{0.02106}$$

$$= -10.946$$

$\therefore \boxed{t = 10.95}$ //

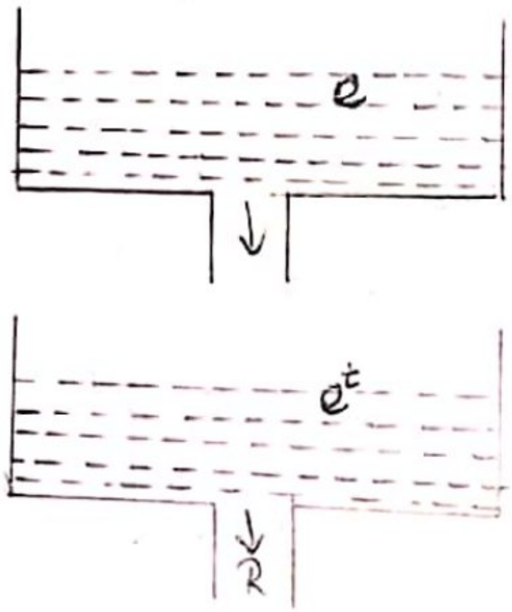
Compartment Model :-

Models interms of Linear differential equations of the 1st order.

Principle of Continuity :.

The gain in amount of Substances in a medium in any times is equal to the excess of the amount has external the times over the amount that the left medium in this time.

A Simple compartment Model :-



let, a vessel contains a volume "V" of a Solution with a concentration $C(t)$ of a substance at time "t". let a solution with constant concentration "c" in an over heat tank enter the vessel at a constant rate R and after mixing through with

Scanned with CamScanner

solution in the vessel. let the mixture with the concentration $c(t)$ leave the vessel at the same rate "R". So that the volume of the solution in the vessel remains v. 41

Using the principle of continuously, we get,

$$v\left[c(t+\Delta t)-c(t)\right] = RC.\Delta t - RC(t) + O(\Delta t)$$

Dividing by Δt and take $\lim \Delta t \to 0$.

$$v\, d \lim_{\Delta t \to 0}\left(\frac{c(t+\Delta t)-c(t)}{\Delta t}\right) = RC - RC(t).$$

$$\lim_{\Delta t \to 0} \frac{\Delta(t)}{\Delta t}.$$

$$v.\frac{dc(t)}{dt} \,\&\, c(t)$$

$$\Rightarrow v.\frac{dc(t)}{dt} = RC - RC(t)$$

$$v.\frac{dc(t)}{dt} + Rc(t) = Rc$$

Divided by v, we get

$$\frac{dc(t)}{dt} + \frac{Rc(t)}{v} = \frac{Rc}{V}$$

This is an 1st order diff' equ' of the form,

$$\boxed{\frac{dy}{dx} + Py = Q}.$$

The integrating fraction,

$$e^{\int p\,dx} = e^{\int R/v \cdot dt} = e^{\frac{R}{V}\cdot t}$$ 42

The general solution is,

$$\boxed{y e^{\int p\,dx} = \int Q e^{\int p\,dx} \cdot dx + k}$$

$$c(t) e^{R/v \cdot t} = \int \frac{Rc}{V} \cdot e^{\frac{R}{V}\cdot t} \cdot dt + k$$

$$= \frac{Rc}{V} \cdot \frac{V}{R} \cdot e^{\frac{R}{V}\cdot t} + c$$

$$c(t) e^{\frac{R}{V}\cdot t} = c e^{\frac{R}{V}\cdot t} + k \longrightarrow (1)$$

At $t=0$, $c(0) e^{\frac{R}{V}\cdot(0)} = c e^{0} \cdot k$

$(\because e^0 = 1)$

$$c(0) = c + k$$

$$k = c(0) - c \longrightarrow (2)$$

Sub (2) in (1) we get

$$c(t) e^{\frac{R}{V}\cdot t} = c \cdot e^{\frac{R}{V}\cdot t} + [c(0) - c]$$

$$c(t) = \frac{c e^{\frac{R}{V}\cdot t}}{e^{\frac{R}{V}\cdot t}} + \frac{c(0) - c}{e^{\frac{R}{V}\cdot t}}$$

$$\therefore \boxed{c(t) = c - [c - c(0)]\, e^{-R/v \cdot t}}$$

At $t \to 0$, $c(t) \to c$

If $c > c_0$. Then the concentration in the vessel (ie) c_0 increase to c.

If $c < c_0$. Then the concentration in

the vessel. (ie) C_0 decrease to C, volume of solution is the vessel at time.

$$t = V_0 + R(t) - R'(t)$$ 43

where, V_0 is the volume of the solution in the vessel initially and the rate R' at which the solution leave the vessel is less than R. Hence in this the corresponding diff' eq' is,

$$\frac{d}{dt}\left\{\left[V_0 + R(t) - R'(t)\right] c(t)\right\} = Rc - R'c(t).$$

Diffusion of Glucose (or) A Medicine in a Blood Stream :-

let the volume of blood in the human body be "V" and let the initial concentration of glucose in the blood stream be $c(0)$.

let Glucose be introduced in the blood stream at a constant rate "I".

Glucose is also removed from the blood stream due to the physiological needs of the human body at a rate

Scanned with CamScanner

Scanned with CamScanner

$$V \cdot \frac{dc}{dt} = I - Kc$$

$$\frac{dc}{I - Kc} = \frac{dt}{V}$$

$\int^{ing}$ on both sides. we get

$$\frac{-\log(I - Kc)}{K} = \frac{t}{V} + a.$$

Taking exponential on both sides.

$$(I - Kc) = e^{-Kt/V} \cdot e^{-aK} \longrightarrow ②$$

put $t=0$.

$$I - Kc = e^{0} \cdot e^{-aK}$$

$$I - Kc = e^{-aK} \longrightarrow ③$$

Sub ③ in ② we get

$$I - Kc = e^{-KV/t}\,(I - K(0))$$

let the dose "D" of a medicine to give to the patient as regular interval of duration T each the medicine also disappears from the system at a rate "proportional" to C(t), the concentration of the medicine in the blood system the differential equation given by the continuity principle.

$$V \cdot \frac{dc(t)}{dt} \propto C(t)$$

$$V \cdot \frac{dc(t)}{dt} = K \cdot C(t)$$

where, K is constant.

$$\frac{dC(t)}{C(t)} = \frac{-k}{V} \cdot dt$$

$\int$ing on both sides. we get

$$\log C(t) = \frac{-k}{V} \cdot t + a \longrightarrow (4)$$

put $\boxed{t=0}$, $\log C(0) = a$

$$(4) \Rightarrow \log C(t) = \frac{-k}{V} \cdot t + \log C(0)$$

$$\log C(t) - \log C(0) = \frac{-k}{V} \cdot t$$

$$\log\left(\frac{C(t)}{C(0)}\right) = \frac{-k}{V} \cdot t$$

Taking exponential on both sides.

$$\frac{C(t)}{C(0)} = e^{-k/v \cdot t}$$

$$C(t) = e^{-k/v \cdot t} \cdot C(0)$$

1st does, $C(t) = De^{-k/v \cdot t}$ $\quad 0 < t < T$.

At the time T, the residue of the 1st does is $De^{-k/v \cdot t}$ and now another does D is given, so that we get,

$$C(t) = \left[De^{-k/v \cdot t} + D\right] e^{-k/v \cdot (t-T)}$$

$$= De^{-k/v \cdot t} + De^{-k/v (t-T)} \quad T \le t \le 2T$$

II nd - Does, The 1st terms gives the residue of the 1st does and the 2nd term gives residual of the 2nd does proceeding in the same way. we get after "n" does have been

Scanned with CamScanner

$$c(t) = De^{-K/v \cdot t} + De^{-K/v(t-T)} + De^{-K/v(t-2T)} + \dots$$

$$\dots + De^{-K/v(t-(n-1))}$$

$$= De^{-K/v \cdot t}\left[1+e^{K/v \cdot t} + e^{K/v \cdot 2T} + \dots + e^{K/v(n-1)T}\right] \to (A)$$

$$1+x+x^2+\dots+x^{n-1} = (1-x)^{-1} - [x^n + x^{n+1} + \dots]$$

$$= (1-x)^{-1} - x^n[1+x+x^2+\dots]$$

$$= (1-x)^{-1} - x^n(1-x)^{-1}$$

46

$$= (1-x^n)(1-x^{-1})$$

$$= \frac{\cancel{-}(x^n-1)}{\cancel{-}(x-1)}$$

$$1+x+x^2+\dots+x^{n-1} = \frac{1-x^n}{1-x} = \frac{x^n-1}{x-1}$$

$$C(t) = De^{-K/v \cdot t}\left[\frac{e^{n \cdot \frac{K}{V} \cdot T} - 1}{e^{\frac{K}{V} \cdot T} - 1}\right] \to (5)$$

$$(n-1)T \le t \le nT.$$

(A) $$C(t) = De^{-\frac{K}{V} \cdot Tn}\left[1 + e^{K/v \cdot T} + e^{K/v \cdot 2T} + \dots + e^{n\frac{K}{V} \cdot T}\right].$$

$$nT \le t \le \frac{(n+1)}{T}$$

$$C(nt) = De^{-K/v \cdot nT}\left[\frac{e^{(n+1)K/v \cdot T} - 1}{e^{K/v \cdot T} - 1}\right]$$

Put $t = nT$ in (5)

$$C(nT-0) = De^{-\frac{K}{V} \cdot nT}\left[\frac{e^{n\frac{K}{V} \cdot T} - 1}{e^{K \cdot \frac{T}{V}} \dots}\right.$$

47

Put t

C(n

(6) ⇒ C(

[Put

(ie) C

T

D

$$= De^{\dots} \cdot e^{\dots}$$

put $t = nT$ in (5)

$$= De^{-\frac{K}{V}\cdot nT} \cdot e^{nKT} \left[\frac{1 - e^{-\frac{nK}{V}\cdot T}}{e^{\frac{KT}{V}} - 1}\right]$$

$$C(nT-0) = D\left[\frac{1 - e^{-nK\cdot\frac{T}{V}}}{e^{\frac{KT}{V}} - 1}\right] \longrightarrow (7)$$

$$(6) \Rightarrow C(nT+0) = De^{-\frac{KnT}{V}} \cdot e^{nK\cdot\frac{T}{V}} \left[\frac{e^{TK/V} - e^{-Kn\cdot T/V}}{e^{KT/V} - 1}\right]$$

[put $nT = t$ in (5) & split to
$(n+1)K\cdot\frac{T}{V} = nK\cdot\frac{T}{V}$ & $\frac{KT}{V}$]

(ie) $$C(nT+0) = De^{-\frac{K}{V}\cdot nT}\left[\frac{e^{(n+1)K\frac{T}{V}} - 1}{e^{\frac{KT}{V}} - 1}\right]$$

$$= De^{-\frac{K}{V}\cdot nT}\left[\frac{e^{nK\frac{T}{V}} - e^{-KnT/V}}{e^{KT/V} - 1}\right]$$

$$= D\cancel{e^{-KT/V}}\left[\frac{1 - e^{-K(n+1)T/V}}{\cancel{e^{KT/V}}(1 - e^{-KT/V})}\right]$$

$$= D\left[1 - e^{-K(n+1)T/V} \dots\right]$$

This is each interval concentration decrease in any interval the concentration is maximum at the beginning this interval and thus maximum concentration at the begining of this interval goes on increasing as the no. of intervals increase but the maximum value in an interval occurs at the end of each intervals.

The concentration curve is piecewise continuous and has points of discontinuity of $T, 2T, 3T \ldots\ldots$

The case of a succession of compactments:

let a solution with concentration $c(t)$ of a solution pass successive into "n" takes in which the initial concentration of the solutions are $c_1(0), c_2(0) \ldots\ldots c_n(0)$. The rate of inflow in each tank is the same of ratio out flow the tank

49 we have to find the concentration $c_1(t), c_2(t) \dots c_n(t)$. at the time "t".

we get the equations,

$$\left.\begin{aligned} V \cdot \frac{dc_1}{dt} &= RC - RC_1 \\ V \cdot \frac{dc_2}{dt} &= RC_1 - RC_2 \\ &\vdots \\ V \cdot \frac{dc_n}{dt} &= RC_{n-1} - RC_n \end{aligned}\right\} \rightarrow (1)$$

By solving the 1st of these eqn', we get

$$C(t)\, V \cdot \frac{dc_1}{dt} = RC - RC_1$$

$$\frac{dc_1}{R(C-C_1)} = \frac{dt}{V}$$

$$\frac{dc_1}{C-C_1} = \frac{R}{V} \cdot dt$$

$\int$ing on both sides. we get

$$-\log(C-C_1) = \frac{R}{V} \cdot t + a$$

$$\log(C-C_1) = -\frac{R}{V} \cdot t + (-a)$$

Taking exponential on both sides. we get

$$C - C_1 = e^{-\frac{R}{V} \cdot t} \cdot e^{-a}$$

Scanned with CamScanner

put $t=0$

$$C(0)-C_1(0)=e^{-a}$$

② $\Rightarrow$ $$C(t)-C_1(t)=\left[C(0)-C_1(0)\right]e^{-Rt/V}$$

$$C_1(t)=C(t)-\left[C(0)-C_1(0)\right]e^{-R/V \cdot t}$$

$$C_1(t)=C(t)-\left[C(0)-C_1(0)\right]e^{-Rt/V}$$

Sub the value of $C_1(t)$ & proceeding is the same way, we can find $C_2(t)$, $C_3(t) \ldots C_n(t)$.

$$\left[\because C_0=C(0)\right]$$

Simple Harmonic Motion :-

Here a particle moves in a straight line is such a manner that is acceleration is always proportional to its distance from the origin and is always directed towards the origin. So that,

$$v.\frac{dv}{dx} \propto x \qquad \text{[where } \mu \text{ is constant]}$$

$$v.\frac{dv}{dx}=-\mu x \longrightarrow ①$$

$$v.dv=-\mu x.dx$$

$\int$ing on both sides. we get

Scanned with CamScanner

[where consider a constant $\mu a^2/a$]

$$\frac{v^2}{2} = -\mu\cdot\frac{x^2}{2} + \mu\cdot\frac{a^2}{2}$$

$$\frac{v^2}{2} = -\mu\left[\frac{x^2}{2} - \frac{a^2}{2}\right]$$

$$v^2 = -\cancel{2}\mu\cdot\frac{1}{\cancel{2}}\left[x^2 - a^2\right]$$

$$v^2 = -\mu\left[x^2 - a^2\right]$$

$$v^2 = \mu(a^2 - x^2) \longrightarrow (2)$$

$$\left(\frac{dx}{dt}\right)^2 = \mu(a^2 - x^2) \qquad \left[\because v = \frac{dx}{dt}\right]$$

$$\frac{dx}{dt} = \sqrt{\mu}\cdot\sqrt{a^2 - x^2} \longrightarrow (3)$$

Since we take two negative sign, since velocity increase as x decrease.

Increase $\pi/2\mu$ $\sqrt{\mu}a$ decreas $\frac{\pi}{2}$ $\sqrt{\pi}$

$n = \frac{2\pi}{\ }$

Scanned with CamScanner

using the condition at $\boxed{t=0}$ $\boxed{x=a}$

$$\frac{a}{a} = \cos(0+C)$$

52

$$1 = \cos C$$

$$\cos C = 1$$

$$C = \cos^{-1}(1)$$

$$\boxed{C=0}$$

$[\because \cos(-\theta) = \cos\theta]$

④ $\Rightarrow$ $\frac{x}{a} = \cos(\sqrt{\mu}\,t)$

In diff
$\cos\theta = -\sin\theta$
$\sin\theta = \cos\theta$

In Inte
$\sin\theta = -\cos\theta$
$\cos\theta = \sin\theta$

$$x(t) = a.\cos\sqrt{\mu}\,t \rightarrow (5)$$

Diff we get

$$dx(t) = -a\sqrt{\mu}\,.\sin\sqrt{\mu}\,t.$$

(ie) $v(t) = -a\sqrt{\mu}\,.\sin\sqrt{\mu}\,t \rightarrow (6)$

Thus in simple harmonic motion both displacement and velocity are periodic function with period $\frac{2\pi}{\sqrt{\pi}}$.

Definition :-

The particle starts from A with zero velocity and move towards "O" with increasing velocity and reaches "O" at time $\frac{\pi}{2\sqrt{\mu}}$ with velocity $\sqrt{a}$.

It continuous to more in the same direction but now with ...

Scanned with CamScanner

$= (OA) = a$. where its

53 velocity is agains 0. It then begins movings towards "0" with increasing velocity and reaches "O" with velocity $\sqrt{\mu}$ and again comes to rest at "A" after a total time period $2\pi/\sqrt{\mu}$. The periodic motion the repeats itself.

Example of Simple Harmonic Motion :-

Consider a particle of mass "m" attached to one and of perfectly elastic the order other and of which is attaches to a fixed point "0" the particle moves under gravity in vacuum.

let "L_0" be the natural length of the solving and let "0" be its extension. when the particle is in equilibrium. so that by HOOKE'S LAW.

HOOKE'S Law :-

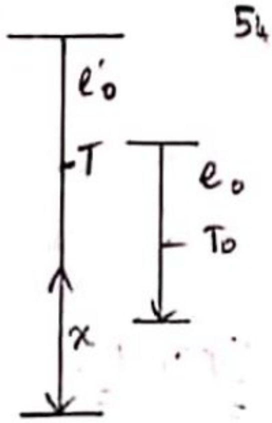
State that the surface is proportional to the strain the pull of spring is proportional to its expension and is given by, $T = \lambda \cdot \frac{L - L_0}{L_0}$.

where, L_0 is the original length of spring. λ is the constant, Then the modulus of velocity

$$mg = T_0 = \frac{\lambda a}{L_0} \longrightarrow ①$$

where λ is the co-efficient of elasticity. ~~Now, where~~ λ let the string by further streached a distances "C" and then the mass be left free, the equation of motion

= mass . λ acceleration in any direction D.



= force on the particle in that direction.

$$mv.\frac{dv}{dx} = mg. -\lambda \{F.ma\}$$

$$= mg - \frac{\lambda(a+x)}{l_0} \left\{T = \frac{\lambda(a+x)}{l}\right\}$$

$$① \Rightarrow mv.\frac{dv}{dx} = \frac{\lambda a}{l_0} - \frac{\lambda_0}{l_0} - \frac{\lambda x}{l_0} \quad [\text{using } ①]$$

$$mv.\frac{dv}{dx} = \frac{-\lambda x}{l_0} \quad \left[\because ① \Rightarrow \lambda = mg.\frac{l_0}{a}\right]$$

$$v.\frac{dv}{dx} = \frac{-\lambda x}{ml_0} = \frac{mgL_0}{a}.\frac{x}{mL_0}$$

$$v.\frac{du}{dx} = \frac{-g}{a}.x \quad \left[\because \frac{2\pi}{\mu} = \frac{2\pi}{\sqrt{g/a}}\right.$$

which gives a SHM with time period $= 2\pi a/g$.

Motion under gravity in a Resisting Medium :-

A particle falls under gravity in a medium in which the resistance is proportional to the velocity.

(ie) $\boxed{R \propto V}$

$$R = KV$$

The equation of the Motion, $\boxed{F = ma}$

$$mg - mKV = m.\frac{dv}{dt}$$

$$\cancel{m}(g - Kv) = \cancel{m}.\frac{dv}{dt}$$

$$g - KV = \frac{dv}{dt}$$

$$g = \frac{dv}{dt} + KV$$

$\div$ by K on both sides. we get

$$\frac{g}{k} = \frac{1}{k}.\frac{dv}{dt} + v$$

$$V - v = \frac{1}{k}.\frac{dv}{dt}$$

$$\frac{-dv}{V - v} = k.dt$$

$\int$ing on both sides. we get

$$-\log(V - v) = Kt + C$$

$$\therefore \boxed{-\log(V - v(t)) = Kt + C} \rightarrow (1)$$

sub $\boxed{t = 0}$, $\boxed{v(t) = 0}$ we get

$$-\log(V-v(t))+\log V = Kt$$

$$\log\left(\frac{V}{V-v(t)}\right) = Kt$$

Taking exponential on both sides.

$$\frac{V}{V-v(t)} = e^{Kt}$$

$$\frac{V}{e^{Kt}} = V-v(t)$$

$$v(t) = V - \frac{V}{e^{Kt}}$$

$$v(t) = V(1-e^{-Kt}) \rightarrow 3$$

So that the velocity goes on increasin and approaches the limiting velocity.

$\frac{g}{K}$ as $K \to \infty$

Replacing v by $\frac{dx}{dt}$ we get

$$\frac{dx}{dt} = V(1-e^{-Kt})$$

$$= V - Ve^{-Kt}$$

$$dx = (V - Ve^{-Kt}).dt$$

$\int$ing on both sides. we get

$$x = Vt - \frac{Ve^{-Kt}}{-K} + c_1 \rightarrow 4$$

using $\boxed{x=0}$ when $\boxed{t=0}$

$$0 = c_1 + \frac{v}{k}$$

$$\boxed{c_1 = \frac{-v}{k}}$$

sub c_1 value in (4) we get

$$x = v(t) + \frac{ve^{-kt}}{k} - \frac{v}{k}$$

(or)

$$x = v(t) + \frac{v}{k}\left[e^{-kt} - 1\right] \text{ //.}$$

Motion of a Rocket :-

we neglect both gravity and air resistance. A rocket moves forward, because of the large suspension velocity with which ~~goes~~ gases producted by the burning of the fuel inside the rocket come out of the converging, diverging nozzle of the Rocket.

let $m(t)$ be the mass of the rocket at times t and let it moves forward with the velocity $v(t)$.

So that the momentum at time,

$$V = m(t) . v(t)$$

In The interval at time $(t, t + \Delta t)$ the mass of the Rocket is $m(t + \Delta t)$

Scanned with CamScanner

$$m(t+\Delta t) = m(t) + \frac{dm}{dt}.\Delta t + O(\Delta t)$$

Since, the Rocket is losing mass $\frac{dm}{dt}$ is -ve and the mass of gasses;

$= -\frac{dm}{dt}.\Delta t$. 58

It moves with velocity "u" is relative to the rocket.

It moves with velocity "u" is relative to the earth $= v(t+\Delta t) - u$.

At the time, the total momentum of the rocket and the gasses is,

$$m(t+\Delta t).v(t+\Delta t) - \frac{dm}{dt}.\Delta t\left[v(t)+\Delta t - u\right]$$

Since, we are neglecting air resistance and gravity, there is no external force on the ~~fo~~ Rocket.

We know the momentum is conserved,

(ie) $m(t).v(t) = m(t+\Delta t).v(t+\Delta t) - \frac{dm}{dt}.\Delta t\left[v(t) - D(t) - u\right]$

$$= m(t) + \frac{dm}{dt}.\Delta t + O\Delta(t)\left[v(t) + \frac{dv}{dt}.\Delta t + O(\Delta t)\right] - \frac{dm}{dt}.\Delta t\left[v(t) + \frac{dv}{dt}.\Delta t + O(\Delta t) - u\right]$$

Scanned with CamScanner

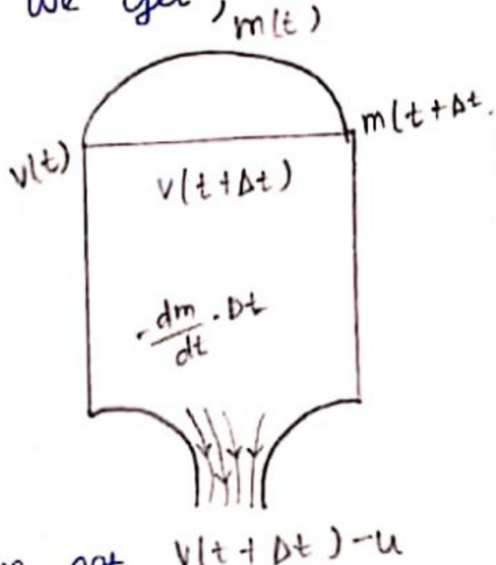
59

$$m(t)\cdot v(t) = m(t)\cdot v(t) + m(t)\,\frac{dv}{dt}\cdot\Delta t + v(t)\,\frac{dm}{dt}\cdot\Delta(t) - \frac{dm}{dt}\cdot\Delta t\, v(t) + u\cdot\frac{dm}{dt}\cdot\Delta(t) + O(\Delta t)^2$$

$$0 = m(t)\cdot\frac{dv}{dt}\cdot\Delta t + u\cdot\frac{dm}{dt}\cdot\Delta t + O(\Delta t)^2$$

$$\left[m(t)\cdot\frac{dv}{dt}\right]\Delta t = -u\cdot\frac{dm}{dt}\cdot\Delta t - O(\Delta t)^2$$

Dividing through out by Δt and taking limit as $\Delta t \to 0$ we get,



$$m(t)\cdot\frac{dv}{dt} = -u\cdot\frac{dm}{dt}$$

$$m(t)\,dv = -u\cdot dm$$

$$\frac{dv}{-u} = \frac{dm}{m}$$

$$\Rightarrow \frac{dm}{m} = \frac{-dv}{u}$$

∫ing on both sides. we get

$$\log m(t) = \frac{-V(t)}{u} + K \rightarrow ③$$

when, $\boxed{t=0}$, $\boxed{V(t)=0}$

$$\log m(0) = K$$

sub K value in ③ we get

$$\log m(t) = -\frac{V(t)}{u} + \log m(0)$$

$$\log m(t) - \log m(0) = \frac{-V(t)}{u}$$

$$\log\left[\frac{m(t)}{m(0)}\right] = \frac{-v(t)}{u}$$

Taking exponential on both sides. we get

$$\frac{m(t)}{m(0)} = e^{-v(t)/u}$$

$$m(t) = e^{-v(t)/u}.$$

Then, equ ④

$$\Rightarrow \log\left(\frac{m(t)}{m(0)}\right) = \frac{-v(t)}{u}$$

$$u.\log\left(\frac{m(t)}{m(0)}\right) = -v(t)$$

$$v(t) = -u.\log\left(\frac{m(t)}{m(0)}\right)^{-1}$$

(or)

$$v(t) = u.\log\left(\frac{m(0)}{m(t)}\right).$$

Assume that the Rocket starts with zero, velocity as the fuel burns the mass of the Rockets decreases initially the mass of the Rocket.

$$= m_p + m_F + m_s.$$

$m_p \rightarrow$ mass of the pay load.

$m_F \rightarrow$ mass of the fuel.

$m_s \rightarrow$ mass of the structure.

When, the fuel is complementary burnt

out MF is zero and if VB is the velocity of the Rocket at this stage. when the fuel is all burnt. Then the equ (3) gives. 61

$$VB = u \cdot \log \left(\frac{mp + mF + mS}{mp + mS} \right)$$

$$\therefore VB = u \log \left(\frac{1 + mF}{mS} \right)$$

This is the maximum velocity that the rocket can attains and it depends on the velocity u and ratio $\frac{mF}{mp+mS}$. For a best moderns fuels and structural materials, the maximum velocity, the gives 7km/sec.

In particle, it would be muss less. since we have neglected air resistance and gravity both of which tends to reduce the velocity. However if a Rocket is to place satellite in orbit. we require a velocity are more than 7km/sec.

Multistage Rocket :-

The fuel may be carried in a no. of. continuous and when fuel of a container is in thrown away so that rocket has not to carry and dead weight.

Thus, in a three stage rocket. let $m_{F_1}, m_{F_2}, m_{F_3}$ be the mass of the structure the velocity at end of the 1st stage in,

$$v_1 = u.\log \frac{m_p + m_{F_1} + m_{S_1} + m_{F_2} + m_{S_2} + m_{F_3} + m_{S_3}}{m_p + m_{F_2} + m_{S_2} + m_{F_3} + m_{S_3}}$$

At the end of 2nd stage. The velocity is,

$$V_2 = v_1 + u.\log \frac{m_p + m_{F_2} + m_{S_2} + m_{F_3} + m_{S_3}}{m_p + m_{F_3} + m_{S_3}}$$

At the end of the 3rd stage, the velocity is,

$$v_3 = v_2 + u.\log \frac{m_p + m_{F_3} + m_{S_3}}{m_p}$$

In this way, a much longer velocity is obtained that can be obtained by a single stage rocket.

Scanned with CamScanner

point (x, y) to the origin (0,0) that is
63 (y/x).

since, There lines orthogonal $m_1 \cdot m_2 = 1$

$$\frac{dy}{dx} \cdot \frac{y}{x} = 1$$

$$y \cdot dy = -x \cdot dx$$

$\int$ing on both sides. we get

$$\frac{y^2}{2} = \frac{-x^2}{2} + \frac{a^2}{2}$$

$$y^2 + x^2 = a^2$$

$$\therefore \boxed{a^2 = x^2 + y^2}$$

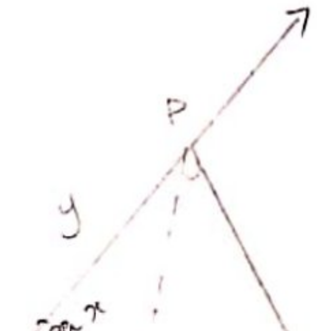
which is represent a family is

Geometric circle.

Q. Find curves for which the projection of the normal on the x-axis of constant length (or) orthogonal trajectories of the family of curve $y^2 = 4Cx$.

Soln:

W.K.T



$$\tan \phi = \frac{dy}{dx}$$

$$\tan \phi = \frac{LN}{PL}$$

$$\frac{dy}{dx} = \frac{k}{y}$$

$$y \cdot dy = k \cdot dx$$

$\int$ing on both sides. we get

Scanned with CamScanner

$$\frac{y^2}{2} = kx + c$$

$$y^2 = 2kx + c$$

where, c is constant.

$$y^2 = 2kx + c$$

which depends of family of parabola all with the some axis and same length of focus section.

3. Find curves for which tangent planes makes a constant angle with radius vector.

Soln: The angle made by the tangent at any point with radius vector.

(i.e) $r \cdot \frac{d\theta}{dr} = \tan\alpha$

$$\frac{d\theta}{\tan\alpha} = \frac{dr}{r}$$

$\int$ing on both sides. we get

$$\frac{\theta}{\tan\alpha} + \log A = \log r$$

$$\frac{\theta}{\tan\alpha} = \log r - \log A$$

$$\Rightarrow \theta \cdot \cot\alpha = \log r - \log A$$

$$\theta \cdot \cot\alpha = \log(r/A)$$

Taking exponential on both sides.

Scanned with CamScanner

$$\frac{r}{A} = e^{\theta \cdot \cot\alpha} \Rightarrow Ae^{\theta \cdot \cot\alpha} = r.$$

$$\therefore \boxed{r = Ae^{\theta \cdot \cot\alpha}}$$

which represent a family of equiangular spirals.

Orthogonal Trajectories:-

let $\boxed{F(x, y, a) = 0} \rightarrow (1)$

represent a family of curves one curve for each value of the parameter "a".

Diff' equ (1) we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{dy}{dx} = 0$$

Eliminating "a" from (1) & (2) we get the diff' equ'.

$$\phi\left(x, y, \frac{dy}{dx}\right) = 0$$

At a point of Intersection of the two curves x, y are the same. But the slope of the 2^{nd} curve is negative reciprocal of the slope of the 1^{st} curve.

we want to family of curves cutting every no. of (1) at right angles at all point of Intersection.

The diff' equ' of the family of orthogonal trajectories

$$\phi\left(x, y, -1/(dy/dx)\right) = 0$$

$\int$ing on both sides. we get

$$g(x, y, b) = 0$$

This is the orthogonal trajectories of family ①.

1) To find the orthogonal trajectories of family of a straight line through of the origin.

Soln: let $\boxed{y = mx}$ → ① is a family of straight line through origin. where, "m" is parameter.

$$dy = m.dx$$

$$\frac{dy}{dx} = m \rightarrow ②$$

Eliminating m from ① & ② we get

$$① \Rightarrow \boxed{m = \frac{y}{x}}$$

sub m value in ②, we get

$$② \Rightarrow \frac{dy}{dx} = \frac{y}{x}$$

$$① \Rightarrow y = x . \frac{dy}{dx}$$

This is the diff' equ' of the given family diff' equ' of orthogonal trajectories

$$y = x . \frac{-1}{dy/dx}$$

67

$$y \cdot \frac{dy}{dx} = -x$$

$$y \cdot dy = -x \cdot dx$$

$\int$ing on both sides. we get

$$\frac{y^2}{2} = -\frac{x^2}{2} + \frac{a^2}{2}$$

$$y^2 = -x^2 + a^2$$

$$a^2 = y^2 + x^2$$

$$\therefore \boxed{x^2 + y^2 = a^2} //$$

Which represent a family of circle.

2. Find the orthogonal trajectories of the family of the conformal coins.

Soln: The conformal coins equ is,

$$\boxed{\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1} \rightarrow (1)$$

where, λ is a parameter diff w.r to equ (1) we get.

$$\frac{2x}{a^2+\lambda} + \frac{2y}{b^2+\lambda} \cdot \frac{dy}{dx} = 0$$

Divided by (2) we get

$$\frac{x}{a^2+\lambda} + \frac{y}{b^2+\lambda} \cdot \frac{dy}{dx} = 0$$

where, $\boxed{p = dy/dx}$

Scanned with CamScanner

$$\frac{x}{a^2+\lambda} + \frac{y}{b^2+\lambda} \cdot p = 0 \longrightarrow (2)$$

Eliminating λ from (1) & (2) we get

$$(2) \Rightarrow \frac{x(b^2+\lambda) + yp(a^2+\lambda)}{(a^2+\lambda)(b^2+\lambda)} = 0$$

$$x(b^2+\lambda) + yp(a^2+\lambda) = 0$$

$$xb^2 + x\lambda + ypa^2 + yp\lambda = 0$$

$$x\lambda + yp\lambda = -(xb^2 + ypa^2)$$

$$(x+yp)\lambda = -(xb^2+ypa^2)$$

$$\therefore \quad \boxed{\lambda = \frac{-(xb^2+ypa^2)}{(x+yp)}}$$

From (1) $\Rightarrow$

$$x^2(b^2+\lambda) + y^2(a^2+\lambda) = (a^2+\lambda)(b^2+\lambda)$$

Take L.H.S of (3)

$$x^2(b^2+\lambda) + y^2(a^2+\lambda) = \frac{x^2\left[b^2-(xb^2-ypa^2)\right]}{x+yp}$$

R.H.S : $(a^2+\lambda)(b^2+\lambda) \quad + \quad \underline{y^2\left[a^2-(xb^2+ypa^2)\right]}$

Scanned with CamScanner

$\cancel{x+yp}$ $(x+yp)^2$

$$y-xp = \frac{-p(a^2-b^2)}{x+yp}$$

$$(x+yp)(y-xp) = -p(a^2-b^2)$$

$$-(x+yp)(y-xp) = p(a^2-b^2)$$

$$(x-y)(x+yp) = p(a^2-b^2) \rightarrow (4)$$

To get orthogonal trajectories. we replace by "p" by "$-1/p$".

$$\left[\frac{-x}{p}-y\right]\left[x-\frac{y}{p}\right] = \frac{-1}{p}(a^2-b^2)$$

$$\frac{-x-yp}{p}\cdot\frac{px-y}{p} = \frac{-1}{p}(a^2-b^2)$$

$$\frac{(x+py)(px-y)}{p} = -(a^2-b^2) \longrightarrow (5)$$

Hence, (4) & (5) are indertical as such that family of conformal coins orthogonal for every coins family there is another with same for which cuts it at right angles.

Scanned with CamScanner

3. Find the orthogonal trajectories of $r = 2a.\cos\theta$ where, a is the parameter.

Soln: Given the equ' of the curves,

$$r = 2a.\cos\theta \rightarrow (1)$$

Diff equ (1) w.r to "θ"

$$\frac{dr}{d\theta} = -2a\sin\theta \rightarrow (2)$$

Eliminating "a" from (1) & (2)

$$(1) \Rightarrow \quad a = \frac{r}{2}.\cos\theta$$

$$(2) \Rightarrow \quad \frac{dr}{d\theta} = -\cancel{2}\left(\frac{r}{\cancel{2}}.\cos\theta\right)\sin\theta$$

$\frac{\sin\theta}{\cos\theta} = \tan\theta$ $$\frac{dr}{d\theta} = -r\tan\theta$$

$\frac{1}{\tan\theta} = \cot\theta$ $$\frac{1}{\tan\theta} = -r.\frac{d\theta}{dr}$$

$$\cot\theta = -r.\frac{d\theta}{dr}$$

This is the diff' equ' if given family the orthogonal trajectories is given by,

$$\cot\theta = -\left[\frac{-1}{r.d\theta/dr}\right]$$

$$\cot\theta = \frac{1}{r}.\frac{dr}{d\theta} \quad \left[\because \frac{d\theta}{dr} = \frac{-1}{d\theta/dr}\right]$$

$$\cot\theta.d\theta = \frac{dr}{r}$$

$\int$ing on both sides. we get

Scanned with CamScanner

$$\log r = \log \sin\theta + \log b$$

$$\cancel{\log} r = \cancel{\log}(b\sin\theta)$$

$$\boxed{r = b\sin\theta} \rightarrow ③$$

This is the orthogonal family of ①,

let $r = 2a.\cos\theta$

W.K.T $\quad x = r\cos\theta \Rightarrow \boxed{\cos\theta = x/r}$.

$\quad y = r\sin\theta \Rightarrow \boxed{\sin\theta = y/r}$.

$\quad r = \sqrt{x^2+y^2}$,

Sub r in ① we get

$$① \Rightarrow r = 2a.\cos\theta \Rightarrow \sqrt{x^2+y^2} = 2a.\frac{x}{r}$$

$$\sqrt{x^2+y^2} = \frac{2ax}{\sqrt{x^2+y^2}}$$

$$x^2+y^2 = 2ax$$

$$x^2+y^2-2ax = 0$$

$$x^2+y^2-2ax+a^2-a^2 = 0$$

$$(x-a)^2+y^2 = a^2$$

$$\Rightarrow x^2+y^2-2ax+y^2 = a^2$$

This is the system of circle with centre on x-axis and touching x-axis.

$$③ \Rightarrow \sqrt{x^2+y^2} = b.\frac{y}{r}$$

$$\sqrt{x^2+y^2} = b.\frac{y}{\sqrt{x^2+y^2}}$$

Scanned with CamScanner

$$x^2+y^2=by$$

$$x^2+y^2-by=0$$

add and sub by $(b/2)^2$, we get

$$x^2+y^2-by+(b/2)^2-(b/2)^2=0$$

$$(y-b/2)^2+x^2=(b/2)^2$$

The orthogonal family of eqn ① is the system of circle with centre on y-axis and having x-axis a common tangents.

4/12/2020

The need for Mathematically Modelling through difference equation :-

1

Same simple Method :-

we need difference equ' models whether other the independent variable is discrete (or) It is mathematically conviniemt to total, it is a disciate variable.

Eg:- In genetics, the genetic characteristics change from generation to generation connect that variables representing a generation is a discrete variable.

In economics, The price change are from year (or) from month (or) from week to week (or) from day to day. In every cases the time is discretized.

In population dynamics, we consider the change in population from one age group to another and the variables representing the age group is a discrete variable.

Population growth Model :-

Scanned with CamScanner

proportional to the population at time 't'" we get

$$x(t+1) - x(t) = b \cdot x(t) - d x(t)$$

$$= (b-d) \cdot x(t)$$

2 $$x(t+1) = a x(t)$$

So that,

$$x(t) = a \cdot x(t-1) = a^2 x(t^2-1) = \cdots\cdots$$

This may be compared with the differential equation Model,

$$\frac{dx}{dt} = ax$$

$$x(t) = x(0) e^{at}.$$

Logistic growth Model :-

8m Apr-2019

This is given by,

$$x(t+1) - x(t) = ax(t) - bx^2(t).$$

This is not early to solve but given $x(0)$, we can find $x(1), x(2) \ldots$ is suggestion and we can get a fairly good idea of the behaviour of the Model with the help of pocket calculate.

Prey-predator Model :-

Scanned with CamScanner

Simple - Epidemics Model :-

This is given by,

$$x(t+1) - x(t) = -\beta x(t).y(t)$$

$$y(t+1) - y(t) = \beta x(t) \cdot y(t) \quad ; \ \beta > 0$$

Basic Theorey of Linear difference equ' with constant co-efficient :-

The Linear Difference Equation :-

An equ' of the form,

$$f[x_{t+n}, x_{t+n-1}, \dots x_t, t] = 0 \rightarrow (1)$$

is called "Difference equ of n^{th} order".

The equation,

$$f_0(t) \cdot x_{t+n} + f_1(t) \cdot x_{t+n-1} + \dots + f_n(t) \cdot x_t = \phi(t) \rightarrow (2)$$

is called a "Linear Difference equ'.

Since, it involves $x(t) \dots x_{t+n}$ only in 1^{st} degree.

The equation,

$$a_0 x_{t+n} + a_1 x_{t+n-1} + \dots + a_n x_t = \phi(t) \rightarrow (3)$$

is called a "Linear Difference equ" with constant co-efficient.

The equation,

$$a_0 x_{t+n} + a_1 x_{t+n-1} + \dots + a_n x_t = 0 \rightarrow (4)$$

is called a "homogeneous Linear Difference equ' with constant co-efficient".

let, $x_t = g_1(t) \cdot g_2(t) \cdots g_n(t)$ be "n" linearly independent solution of (4). Then,

$$x_t = A_1 g_1(t) + A_2 g_2(t) \cdots + A_n g_n(t) \rightarrow (5)$$

is also a linear solution of (4) where, $A_1, A_2 \ldots A_n$ are n arbitrary constant. This is most general solution of (4).

Again "t" can be show that if $G_1(t)$ is the solution of (4) containing "n" arbitary constants and $G_2(t)$ is any particular solution of (3) containing arbitary constant. Then, $G_1(t) + G_2(t)$ is the general solution of (3) $G_1(t)$ is called complementary function.

$G_2(t)$ is called particular solution.

i) The complementary function :-

we try the soln'.

$x_t = a\lambda^t$. If this satisfies the equ',

$$a_0 x_{t+n} + a_1 x_{t+n-1} + \cdots + a_n x_t = 0 \longrightarrow (1)$$

$$a_0 \lambda^{t+n} + a_1 \lambda^{t+n-1} + \cdots + a_n \lambda^t = 0$$

$$\lambda^t \left[a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n \right] = 0$$

$$g(\lambda) = a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n = 0 \longrightarrow (2)$$

The algebraic equ' of nth degree has "n" roots $\lambda_1, \lambda_2 \ldots \lambda_n$ real (or) complex, the complementary function is given by.

Scanned with CamScanner

② $\Rightarrow a_0 C(t+n)\lambda_1^{t+n} + a_1 C(t+n-1)\lambda_1^{t+n-1} + \cdots +$

$(a_{n-1})\lambda + C(t+1)\lambda_1^{t+1} + a_n C\,t\,\lambda_1^{t} = 0$

$a_0 C t \lambda_1^{t+n} + a_0 C n \lambda_1^{t+n} + a_1 C\, t\, \lambda_1^{t+n-1} +$ 5

$a_1 C(n-1)\lambda_1^{t+n-1} + \cdots + a_{n-1} C\, t\, \lambda_1^{t+1} + a_{n-1} C \lambda_1^{t+1}$

$+ a_n C t \lambda^t = 0$

$Ct\lambda_1^t \left[a_0 \lambda_1^n + a_1 \lambda_1^{n-1} + \cdots + a_{n-1}\lambda\right] +$

$C\lambda_r^t \left[a_0 n \lambda_1^n + a_1 (n-1)\lambda_1^{n-1} + \cdots + a_n \lambda\right] = 0$

$Ct\lambda_1^t \left[g(\lambda)\right] + C\lambda_1 t \left[g'(\lambda_1)\right] = 0$

$\boxed{t \cdot g(\lambda) + g'(\lambda_1) = 0}$. //

ii) The particular Solution :-

let $a_0 x_{t+n} + a_1 x_{t+n-1} + \cdots + a_n x_t = \phi(t)$ → ①

Here the solution of ① not containing any arbitary constant.

case i) :-

let $\boxed{\phi(t) = AB^t}$; B is not root of $g(\lambda = 0)$ → ② we try the solution CB^t → ⊛

① $\Rightarrow a_0 x_{t+n} + a_1 x_{t+n-1} + \cdots + a_n x_t = \phi(t)$

put,

$x_t = CB^t \cdot a_0 CB^{t+n} + a_1 CB^{t+n-1} + \cdots + a_n CB^t = AB^t$

$CB^t \left[a_0 B^n + a_1 B^{n-1} + \cdots + a_n\right] = A\cancel{B^t}$ → ③

$C \left[a_0 B^n + a_1 B^{n-1} + \cdots + a_n\right] = A$

Scanned with CamScanner

$$C = \frac{A}{a_0 B^n + a_1 B^{n-1} + \dots + a_n}$$

6 and the particular solution is,

$$= \frac{AB^t}{a_0 B^n + a_1 B^{n-1} + \dots + a_n} \rightarrow (5)$$

(using *)

case ii) :-

let $\boxed{\phi(t) = AB^t}$ B is an non-repeated root of $g(\lambda) = 0 \rightarrow (6)$

we try the solution,

$$x_t = C^t \cdot B^t \rightarrow (A)$$

sub in (1),

$$C^t \cdot B^t g(B) + CB^t g'(B) = AB^t$$

$$B^t \left[Ctg(B) + Cg'(B) \right] = AB^t \rightarrow (7)$$

Since, $g(B) = 0$, $g'(B) \neq 0$

$$Cg'(B) = A$$

$$C = \frac{A}{g'(B)} \rightarrow (8)$$

So, that the particular solution is,

$$A^t \cdot B^t \rightarrow (9)$$

$① \Rightarrow a_0 x_{t+n} + a_1 x_{t+n-1} + \dots + a_{n-1} a_{t+1} + a_n x_t = \phi(t)$

$a_0 C^t B^{t+n} + a_0 C n B^{t+n} + a_1 C^t B^{t+n-1} + a_1 C(n-1) B^{t+n-1}$

$+ \dots + a_{n-1} C B^{t+1} + a_n C^t B^t = AB^t$

$[a_0 C^t B^{t+n} + a_1 C^t B^{t+n-1} + \dots + a_n C^t B^t] +$

$[a_0 C n B^{t+n} + a_1 C(n-1) B^{t+n-1} + \dots + a_{n-1} C B^{t+1}] = AB^t$

$C^t B^t [a_0 B^n + a_1 B^n + \dots + a_n] + CB^t [a_0 n B^n +$

$a_1 (n-1) B^{n-1} + \dots + a_{n-1} B] = AB^t$

$C^t B^t . g(B) + CB^t g'(B) = AB^t$.

case iii) :-

let $\phi(t) = AB^t$, $g(B) = 0$, $g'(B) = 0$

$g^{k-1}(B) = 0$, $g^k(B) \neq 0 \rightarrow ⑩$ then the particular soln' is $\dfrac{At^{k-1} B^t}{g^k B} \rightarrow ⑪$

case iv) :-

let $\phi(t) = At^k$

we try the solu' is,

$d_0 t^k + d_1 t^{k-1} + d_2 t^{k-2} + \dots + d_k = 0$

sub in ① we get

$a_0 [d_0 (t+n)^k + d_1 (t+n)^{k-1} + d_2 (t+n)^{k-2} + \dots + d_k] +$

$a_1 [(t+n-1) k d_0 + d_1 (t+n-1)^{k-1} + \dots + d_k] + \dots +$

$\dots + a_n [d_0 t^k + d_1 t^{k-1} + \dots + d_k] = 0$

∴ Equating the co-efficient of $t^k, t^{k-1} \dots + t^n$ on both sides. we get $(K+1)$ equ'

which is general will enable us to determine $d_0, d_1 \dots d_n$ thus the particular

8 soln' will be determined.

Obtaining complementary function by use of Matrices :-

10m imp'

let, $x(t) = x_1(t)$

$$\left.\begin{array}{l} x(t+1) = x_2(t) - x_1(t+1) \\ x(t+2) = x_3(t) - x_2(t+1) \\ \vdots \\ x(t+n) = x_{n+1}(t) - x_n(t+1) \end{array}\right\} \rightarrow ①$$

so that,

$$a_0 x_{t+n} + a_1 x_{t+n-1} + \dots + a_n x_t = 0$$

$$a_0 x_{n+1}(t) + a_1 x_n(t) + \dots + a_n x_1(t) = 0 \quad (\text{using } ①$$

$$a_0 x_{n+1}(t) = -a_1 x_n(t) - a_2 x_{n-1}(t) \dots - a_n x_1(t)$$

$$a_0 x_n(t+1) = -a_1 x_n(t) - a_2 x_{n-1}(t) \dots - a_n x_1(t) \rightarrow ②$$

Equ ① gives,

$$x_1(t+1) = x_2(t)$$

$$x_2(t+1) = x_3(t)$$

$$\vdots \qquad \vdots$$

$$x_{n-1}(t+1) = x_n(t)$$

$$② \Rightarrow x_n(t+1) = \frac{-a_1}{a_0} \cdot x_n(t) - \frac{a_2}{a_0} x_{n-1}(t) \dots - \frac{a_n}{a_0} x_1(t).$$

which can be written in matrix form,

Scanned with CamScanner

Scanned with CamScanner

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \\ \vdots \\ x_n(t+1) \end{bmatrix} = \begin{bmatrix} x_1(t) & x_2(t) & \cdots & x_n(t) \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ -\frac{a_n}{a_0} & -\frac{a_{n-1}}{a_0} & \cdots & -\frac{a_1}{a_0} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \rightarrow (3)$$

$$x(t+1) = A \cdot x(t) \rightarrow (4)$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix} \; ; \; A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ -\frac{a_n}{a_0} & -\frac{a_{n-1}}{a_0} & & \cdots & -\frac{a_1}{a_0} \end{bmatrix} \rightarrow (5)$$

Applying (4) respectively,

$$x(K) = A^K \cdot x(0)$$

where,

$$x(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \\ \vdots \\ x_n(0) \end{pmatrix} = \begin{pmatrix} x_1(0) \\ x_2(1) \\ \vdots \\ x_1(n-1) \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$$

Thus knowing the values of x_1 at times $0, 1, 2 \ldots n_{-1}$.

we can find its value at all Subsequence times,

Solution of a system of Linear Homogenous difference equ' with constant co-efficient:-

Let the system be, given by,

$$x_1(t+1) = a_{11}x_1(t) + a_{12}x_2(t) + \cdots + a_{1n}x_n(t) \Big\}$$

$$\left.\begin{aligned} x_2(t+1) &= a_{21}x_1(t)+a_{22}x_2(t)+\dots+a_{2n}x_n(t) \\ &\vdots \\ x_n(t+1) &= a_{n1}x(t)+a_{n2}x_2(t)+\dots+a_{nn}x_n(t) \end{aligned}\right\} \textcircled{1}$$

This is can be written in an matrix form,

$$x(t+1) = Ax(t) \rightarrow \textcircled{2}$$

where,

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \quad ; \quad n = \begin{pmatrix} a_{11} & a_{12} \dots\dots & a_{1n} \\ a_{21} & a_{22} \dots\dots & a_{2n} \\ \vdots & & \\ a_{n1} & a_{n2} \dots\dots & a_{nn} \end{pmatrix} \rightarrow \textcircled{3}$$

Applying ② we get,

$$x(k) = A^k . x(0)$$

Solution of Linear difference equation by using Laplace Transform :-

Let the linear difference equ be,

$$a_0 f(t) + a_1 f(t-1) + \dots + a_n f(t-n) = \phi(t) \rightarrow \textcircled{1}$$

$$f(t) = 0 \quad \text{when} \quad t < 0$$

Let $\bar{f}(\lambda)$ be the laplace transform of $f(t)$. So that,

$$\bar{f}(\lambda) = L[f(t)] = \int_0^\infty e^{-\lambda t} . f(t-1) dt$$

$$= e \int_0^\infty e^{-\lambda t} f(t) dt$$

$$= e^{-\lambda t} . f(\lambda)$$

Scanned with CamScanner

$$\bar{f}(\lambda) = L[f(t)] = \int_0^\infty e^{-\lambda t} f(t)\,dt \longrightarrow (2)$$

Then,

$$L[f(t-1)] = \int_0^\infty e^{-\lambda t} f(t-1)\,dt$$

$$= e^{-\lambda t} \int_0^1 e^{-\lambda t} \cdot f(t)\,dt$$

$$= e^{-\lambda t} \cdot f(\lambda)$$

$$L[f(t-2)] = \int_2^\infty e^{-\lambda t} f(t-2)\,dt$$

$$= e^{-2} \int_2^\infty e^{-\lambda t} f(t)\,dt$$

$$= e^{-2\lambda} \cdot f(\lambda) \longrightarrow (3)$$

and so on. So that taking replace transform on both sides of (1) we get,

$$(1) \Rightarrow a_0 f(t) + a_1 f(t-1) + \dots + a_n f(t-n) = \phi(t)$$

$$a_0 L\{f(t)\} + a_1 L\{f(t-1)\} + \dots + a_n L\{f(t-n)\} = L\{\phi(t)\}$$

$$[a_0 + a_1 e^{-\lambda} + a_2 e^{-2\lambda} + \dots + a_n e^{-n\lambda}]\, \bar{f}(\lambda) = L\{\phi(t)\} = \phi(\lambda) \rightarrow (4)$$

So that, $\bar{f}(\lambda)$ is know inverting the laplace transform we get $f(t)$. In this case "t" is regarded as "a" invrete variate $\ni f(t) = 0$. when, $t < 0$. If f is discreate variable it is better to use transform.

Solution of linear difference equation by using Z-transforms :-

let $\{u_n\}$ be an infinite sequence. Thus

12 its Z-transforms is defined by,

$$Z(u_n) = \sum_{n=0}^{\infty} u_n Z^{-n}$$

whenever, this infinite series converges. If $\{u_n\}$ is a probability distance and $z = \frac{1}{3}$ it will be same as the probability generating function.

The following results can be easily establised,

i) If $k>0$, $Z(u_{n-k}) = \sum_{n=0}^{\infty} u_{n-k} \cdot Z^{-n}$

$$= \sum_{n=0}^{\infty} u_{n-k} Z^{-n+k-k} \Rightarrow \sum_{n=0}^{\infty} u_{n-k} Z^{-(n-k)} \cdot r$$

$$= Z^{-k} \sum_{n=0}^{\infty} u_{n-k} \cdot Z^{-(n-k)} \Rightarrow Z^{-k} Z(u_n).$$

ii) If $k>0$, $Z(u_{n+k}) = \sum_{n=0}^{\infty} u_{n+k} \cdot Z^{-n}$

$$= \sum_{n=0}^{\infty} u_{n+k} \cdot Z^{-n+k-k} \Rightarrow \sum_{n=0}^{\infty} u_{n+k} \cdot Z^{-(n+k)} \cdot Z^{k}$$

$$= Z^{k} \sum_{n=0}^{\infty} u_{n+k} \cdot Z^{-(n+k)}$$

$$= Z^{k} \left[u_k Z^{-k} + u_{k+1} \cdot x^{-(k+1)} \right]$$

$$= Z^{k} \left[\sum_{n=0}^{\infty} u_n \cdot Z^{-n} - \sum_{n=0} u_m Z^{-m} \right]$$

Scanned with CamScanner

iii) u_n : 1 a^n e^{an}

13

14

$z(u_n)$: $\frac{z}{z-1}$ $\frac{z}{z-a}$ $\frac{z}{z-e^{an}}$

If $u_n = 1$; $z(u_n) = \sum_{n=0}^{\infty} u_n z^{-n}$.

$$= \sum_{n=0}^{\infty} 1 \cdot \frac{1}{z^n} = 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots$$

$$= \left(1 - \frac{1}{z}\right)^{-1}$$

$$= \frac{1}{1-\frac{1}{z}} \Rightarrow \frac{z}{z-1}.$$

If $u_n = a^n$; $z(u_n) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} a^n \cdot \frac{1}{z^n}$.

$$= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \cdots$$

$$= \left(1 - \frac{n}{z}\right)^{-1} = \frac{1}{\left(1-\frac{n}{z}\right)} = \frac{z}{z-n}$$

If $u = e^{an}$; $z(u_n) = \sum_{n=0} e^{an} \cdot z^{-n}$.

$$= \sum_{n=0}^{\infty} e^{an} \cdot \frac{1}{z^n}.$$

$$= 1 + \frac{e^a}{z} + \frac{e^{a^2}}{z^2} + \cdots = \left[1 - \frac{e^a}{z}\right]^{-1}$$

$$= \frac{1}{1-\frac{e^a}{z}} = \frac{z}{z-e^a}$$

Taking Z-transforms on both sides of linear difference equ' we can find $z(u_n)$ and expanding it in powers of $\frac{1}{z}$ & finding the co-efficients of z^{-n}, we get u_n.

Solutions of non-linear difference equ's reducible to linear equ's :-

14

The equations,

$$y_{n+1} = \sqrt{y_n} \rightarrow ①$$

$$y_n \cdot y_{n+2} = y_{n+1}^2 \rightarrow ②$$

become linear an sub $u_n = \log y_n \rightarrow ③$

Also, $u_{n+1} = \log y_{n+1}$

$$\log y_{n+1} = \log (y_n)^{1/2} = \frac{1}{2} \log y_n$$

$$u_{n+1} = \frac{1}{2} u_n \quad (\text{using } ②)$$

$$2u_{n+1} = u_n$$

② $\Rightarrow$ $\log (y_n \cdot y_{n+2}) = \log y_{n+1}^2$

$$\log y_{n+1} + \log y_{n+2} = 2 \cdot \log y_{n+1}$$

$$u_n + u_{n+2} = 2u_{n+1} \quad (\text{using } ③)$$

W.K.T

$$y_{n+2} = \frac{y_n \cdot y_{n+1}}{y_n + y_{n+1}}$$ becomes linear on substitution.

$$u_n = \frac{1}{y_n} \Rightarrow y_n = \frac{1}{u_n}$$

∵ consider, $y_{n+2} = y_n - y_{n+1}$

$$y_{n+1} \cdot y_{n+2} = y_{n+1} [y_n - y_{n+1}] = y_{n+1} \cdot y_n - y_{n+1}^2$$

$$y_{n+1} \cdot y_{n+2} = y_n \cdot y_{n+1} - y_n \cdot y_{n+2}$$

$$y_{n+1} \cdot y_{n+2} + y_n \cdot y_{n+2} = y_n \cdot y_{n+1}$$

$$y_{n+2} [y_{n+1} + y_n] = y_n \cdot y_{n+1}$$

$$y_{n+2} = \frac{y_n \cdot y_{n+1}}{y_{n+1} + y_n}$$

Scanned with CamScanner

$$\Rightarrow \frac{1}{u_{n+2}} = \left(\frac{\frac{1}{u_n} \cdot \frac{1}{u_{n+1}}}{\frac{1}{u_{n+1}} + \frac{1}{u_n}} \right) \quad \text{(using ③)}$$

15

$$\Rightarrow \frac{1}{u_{n+2}} = \frac{(1/u_n \cdot 1/u_{n+1})}{\left(\frac{u_n + u_{n+1}}{u_{n+1} \cdot u_n}\right)} = \frac{(1/u_n \cdot u_{n+1})}{\left(\frac{u_n + u_{n+1}}{u_{n+1} \cdot u_n}\right)}$$

$$\frac{1}{u_{n+2}} = \frac{1}{u_n + u_{n+1}}$$ (Taking reciprocal) we get

$$\therefore \boxed{u_{n+2} = u_n + u_{n+1}} \quad //$$

Stability : Theory of difference equation:-

Condition of stability Theory,

If $x_t = K$ satisfies.

$$f(x_t, x_{t+n}, \dots x_{t+n}) = 0 \rightarrow ①$$

Then gives an equilibrium position. To find its stability. we sub $x_t = K + u_t$ in ① and simplity neglecting sequence and product and higher power of "u_t" to get a linear equations.

$$a_0 u_{t+n} + a_1 u_{t+n} + \dots + a_K u_t = 0 \rightarrow ②$$

We try the soln' $u_t = A\lambda^t$ and get the characteristic equ'

$$a_0 A \lambda^{t+n} + a_1 A \lambda^{t+n-1} + \dots + a_n A \lambda^t = 0$$

$$A\lambda^t [a_0 \lambda^n + a_1 \lambda^{n-1} + \dots + a_n] = 0$$

$a_0 \lambda^n + a_1 \lambda^{n-1} + \dots + a_n = 0$

If the absolute value of each of the "n" roots of this eqn less than unity. Then "u_t" would tends to zero as $t \to \infty$ for all small initial disturbance & the equilibrium position would be locally asymptotically stable.

The condition for all the roots having magnitude less than unity are given by schur's criterion is that all the solution determinated should be positive.

$$A_1 = \begin{vmatrix} a_0 & a_n \\ a_n & a_0 \end{vmatrix}, \quad A_2 = \begin{vmatrix} a_0 & 0 \dots a_n & a_{n-1} \\ a_1 & a_0 \dots 0 & a_n \\ \vdots & \vdots \quad \vdots & \vdots \\ a_n & 0 \dots a_0 & a_1 \\ a_{n-1} & a_n \dots 0 & a_0 \end{vmatrix}$$

$$= \begin{vmatrix} a_0 & 0 & 0 \dots a_n & a_{n-1} \dots a_1 \\ a_1 & 0 & 0 \dots 0 & a_n \dots a_2 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ a_{n-1} & a_n & a_0 \dots a & 0 \dots a_n \\ a_n & 0 & 0 \dots a_0 & a_1 \dots a_{n-1} \\ a_n & a_{n-1} & 0 \dots 0 & a_0 \dots a_{n-2} \\ a_1 & a_2 & a_n \dots 0 & 0 \dots a_0 \end{vmatrix}$$

//

Scanned with CamScanner

Mathematical Modelling through difference equation in Economics and Finance:

The Harrod Model :- 17

Assumption model in Harrod Model.

let $S(t)$, $y(t)$ and $I(t)$ denotes the savings, National income, Inverstment respectively. we how make the following assumption.

i) saving Model by the people in a country depend on the national income.

$$\boxed{S(t) = \alpha \cdot y(t)} \quad \therefore \alpha > 0 \quad \rightarrow (1)$$

ii) The inverstment demands on the difference b/w the income of the current year and the last year.

Scanned with CamScanner

$$\beta \cdot y(t-1) = y(t)[\beta-\alpha]$$

$$y(t) = \frac{\beta \cdot y(t-1)}{\beta-\alpha}$$

$$y(t) = \frac{\beta}{\beta-\alpha} \cdot y(t-1) \rightarrow (4)$$

This is a linear difference equ.

Its complementary fun' is of the form,

$$y(t) = A\lambda^t \rightarrow (5)$$

$$A\lambda^t = \frac{\beta}{\beta-\alpha} \cdot A\lambda^{t-1}$$

$$\cancel{A\lambda^t} = \frac{\beta}{\beta-\alpha} \cdot \cancel{A\lambda^t} \cdot \lambda^{-1}$$

$$1 = \frac{\beta}{\beta-\alpha} \cdot \lambda^{-1}$$

$$\frac{\beta}{\beta-\alpha} \cdot \frac{1}{\lambda} = 1$$

$$\boxed{\lambda = \frac{\beta}{\beta-\alpha}} \rightarrow (6)$$

put, $t=0$, $y(0) = A$ sub in (5) we get

$$(5) \Rightarrow \quad y(t) = A\lambda^t$$

$$y(t) = y(0) \cdot \left(\frac{\beta}{\beta-\alpha}\right)^t \rightarrow (7)$$

Assuming that $y(t)$ is always +ve.

$$\beta > \alpha, \quad \frac{\beta}{\beta-\alpha} > 1 \longrightarrow (8)$$

So that the national income increases

Scanned with CamScanner

with "t". The national income at diff times 0, 1, 2... from a geometrical progression. 19

Thus, if all the savings are investment savings are proportional to National Income and the investment is proportional to the excess of the current year incomes over the proceeding years income. Then the National income increases geometrically.

The cobwele Model :-

Apr 2m 2018

let p_t = Price of commodity in the year "t". q_t = Amount of commodity available in the market in year "t". Then, we make the following assumptions.

i) Amount of the commodity produced this year and available for sale is a linear function, the price of the commodity in the last year.

(ie) $\boxed{q_t = \alpha + \beta \cdot p_{t-1}} \longrightarrow (1)$

where; $\beta > 0$. Since, if the last year price has high the amount available this year, would also be high.

ii) The price of commodity this year is at linear function of the amount available this year, (ie) $p_t = \gamma + \delta q_t$

Scanned with CamScanner

$\delta q_t = p_t - \gamma$

where, $\delta' > 0$ since, if γ_t is Large. The price would be have,

$$② \Rightarrow p_t = \gamma + \delta q_t$$

$$\delta q_t = p_t - \gamma$$

$$q_t = \frac{p_t - \gamma}{\delta} \rightarrow ③$$

sub in ① we get.

$$\frac{p_t - \gamma}{\delta} = \alpha + \beta \cdot p_{t-1}$$

$$p_t - \gamma = \delta[(\alpha + \beta) \cdot p_{t-1}]$$

$$p_t - \gamma = \delta\alpha p_{t-1} + \delta\beta p_{t-1}$$

$$p_t - \delta\beta p_{t-1} = \gamma + \delta\alpha p_{t-1} \rightarrow ④$$

This is the linear difference eqn'.

the complementary fun' is,

$$\text{let } \boxed{p_t = a\lambda^t} \longrightarrow ⑤$$

$$④ \Rightarrow a\lambda^t - \beta\delta a\lambda^{t-1} = 0$$

$+(\delta\delta$

$$a\lambda^{t-1}[\lambda - \beta\delta] = 0$$

$$\lambda - \beta\delta = 0$$

$$\boxed{\lambda = \beta\delta}$$

Sub in ⑤ we get the complementary function is, $p_t = a(\beta\delta)^t \longrightarrow ⑥$

Hence, the particular solution is constant

$$\phi(t) = (\gamma + \alpha\delta)t^0.$$

$= \gamma + \alpha\delta$ (using n)

putting in ④ & let $P_t = P_{t-1} = A$

21

22

④ $\Rightarrow$ $A - \beta.\delta A = \gamma + \alpha\delta$

$$A[-\beta\delta + 1] = \gamma + \alpha\delta$$

$$A = \frac{\gamma + \alpha\delta}{1-\beta\delta}$$

The complete solution is, $P_t = C.F + P.I$

$$P_t = a(\beta\delta)^t + \frac{\gamma + \alpha\delta}{1-\beta\delta} \rightarrow ⑦$$

put $\boxed{t=0}$

$$P_0 = a + \frac{\gamma + \alpha\delta}{1-\beta\delta}$$

$$a = P_0 - \frac{\gamma + \alpha\delta}{1-\beta\delta}$$

⑦ $\Rightarrow$ $P_t = \left[P_0 - \frac{\gamma+\alpha\delta}{1-\beta\delta}\right](\beta\delta)^t + \left(\frac{\gamma+\alpha\delta}{1-\beta\delta}\right)$

$$P_{t-1} - \frac{\gamma+\alpha\delta}{1-\beta\delta} = P_0 - \left[\frac{\gamma+\alpha\delta}{1-\beta\delta}\right](\beta\delta)^{t-1} \rightarrow ⑧$$

Multiply by $\beta\delta$, we get

corre.. $\left[P_{t-1} - \frac{\gamma+\alpha\delta}{1-\beta\delta}\right](\beta\delta) = \left[P_0 - \frac{\gamma+\alpha\delta}{1-\beta\delta}\right](\beta\delta)^t \rightarrow ⑨$

Equating ⑧ & ⑨ we get

$$\left[P_t - \frac{\gamma+\alpha\delta}{1-\beta\delta}\right] = \left[P_{t-1} - \frac{\gamma+\alpha\delta}{1-\beta\delta}\right](\beta\delta) \rightarrow ⑩$$

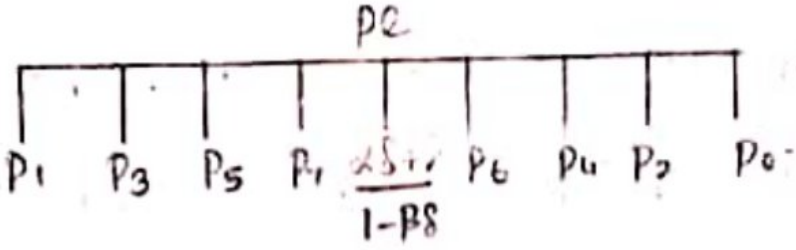
Scanned with CamScanner

since, $\beta\delta$ is (-ve) P_1, P_2 are alternating greater & less than $\left(\frac{\alpha\delta+\gamma}{1-\beta\delta}\right)$.

From (10) If $|\beta\delta|>1$. the derivative of P_t from $\frac{\alpha\delta+\gamma}{1-\beta\delta}$ goes on force increasing on the otherhand if $|\beta\delta|<1$. This derivation goes decreasing and ultimately.

$$P_L \rightarrow \frac{\alpha\delta+\gamma}{1-\beta\delta} \quad \text{as} \quad t \rightarrow \infty$$

Here, $P = \dfrac{\alpha\delta+\gamma}{1-\beta\delta}$



$(\therefore P_0 > pe)$

pe $(\therefore P_0 < pe)$

$$q(t) = \alpha + \beta(\gamma + \delta q_{t-1})$$

$$q(t) = \alpha + \beta\gamma + \beta\delta q_{t-1}$$

$$q_t - \beta\delta q_{t-1} = \alpha + \beta\gamma \rightarrow (11)$$ 23

This is linear difference equ'. Its complementary fun' of the form.

let, $q_t = b\lambda^t \rightarrow (12)$

$(10) \Rightarrow \quad b\lambda^t - \beta\delta\lambda^{t-1} = 0$

$$b\lambda^{t-1}[\lambda - \beta\delta] = 0$$

$$\lambda - \beta\delta = 0$$

$$\lambda = \beta\delta$$

sub in (12) we get complementary function is $b(\beta\delta)^t$. Here then the particular integral is constant.

(ie) $\phi(t) = (\alpha + \beta\gamma)^t = \alpha + \beta\gamma$

putting in (11) & let $q_t = q_{t-1} = \beta$

$(4) \Rightarrow \quad \beta - \delta\beta = \alpha + \beta\gamma$

$$\beta(1 - \beta\delta) = \alpha + \beta\gamma$$

$$\beta = \frac{\alpha + \beta\gamma}{1 - \beta\delta}$$

The complete solution is, $q_t = C.F + P.I$

$$q_t = b(\beta\delta)^{t=0} + \left(\frac{\alpha + \beta\gamma}{1 - \beta\delta}\right) \rightarrow (13)$$

Put $t = 0$

$$q_0 = b + \frac{\alpha + \beta\gamma}{1 - \beta\delta}.$$

Scanned with CamScanner

$$b = q_0 - \frac{\alpha + \beta\gamma}{1-\beta\delta}$$

Sub in (13) we get

24

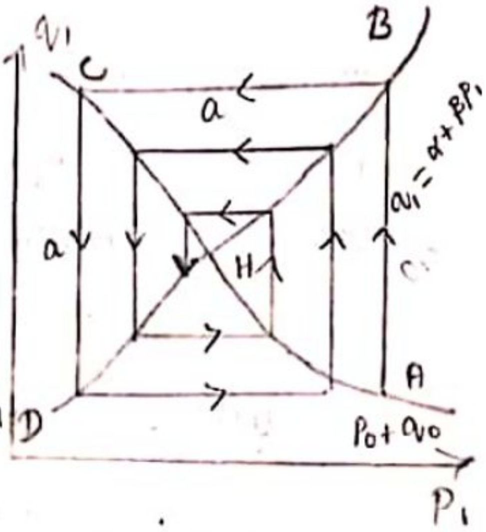
$$q_t = \left[q_0 - \frac{\alpha+\beta\gamma}{1-\beta\delta}\right](\beta\delta)^t + \frac{\alpha+\beta\gamma}{1-\beta\delta}$$

$$q_t - \frac{\alpha+\beta\gamma}{1-\beta\delta} = \left[q_0 - \frac{\alpha+\beta\gamma}{1-\beta\delta}\right](\beta\delta)^t.$$

So that q_t also oscillate about the equilibrium quantity level.

$$q_t = \frac{\alpha+\beta\gamma}{1-\beta\delta} \quad \text{if} \quad |\beta\delta|<1.$$

The fig show that the variation of both price and quantity is shown simultaneously.



Suppose we start in year zero with price p_0 and quantity q_1 is given by, $\alpha + \beta p_0$ and the price $p_1 = \gamma + \delta q_1$. This begins as to the pt. C in 2 steps via B.

The path of price and quantities in thus given by the cobweb path ABCDEF and the equilibrium price and quantity are given by the intersection of the 2 straight line.

Scanned with CamScanner

Samulson's Intersection Models :-

5 mark

The basic equ' for the first intersecting model is, 25

$$y(t) = C[t) + I(t) \rightarrow (1)$$

$$C(t) = \alpha \cdot y(t-1) \rightarrow (2)$$

$$I(t) = \beta\,[C(t) - C(t-1)] \rightarrow (3)$$

Here the +ve constant α is marginal propensity to consume w.r to income and the previous year to the +ve constant β is the reaction given by the acceleration, principle of (ie) β is the increases in investment per unit of excess of this year consumption over the last years.

Sub (2) & (3) in (1) we get

$$y(t) = \alpha \cdot y(t-1) + \beta\,[C(t) - C(t-1)]$$

$$= \alpha \cdot y(t-1) + \beta\,[\alpha \cdot y(t-1) - \alpha y(t-2)]$$

$$= \alpha \cdot y(t-1) + \alpha\beta \cdot y(t-1) - \alpha\beta\, y(t-2)$$

$\alpha y(t-1)[1+\beta]$ $= \alpha[$

$$= \underline{\alpha}\,[(1+\beta)]\,\underline{y(t-1)} - \alpha\beta y(t-2)$$

common

$$y(t) - \alpha(1+\beta)\,y(t-1) + \alpha\beta y(t-2) = 0 \rightarrow (4)$$

In the 2nd intraction Model there is an additional investment by the government and this investment is assume to be constant γ.

In this case (4) is Modified to

Scanned with CamScanner

$y(t) - \alpha(1+\beta) y(t-1) + \alpha\beta y(t-2) - \gamma = 0 \rightarrow (5)$

The Solution (4) & (5) can show either increasing trend is $y(t)$ (or) a decreasing trend in $y(t)$ (or) oscillating trend. Then "α" in it.

Application of Acturial Science :-

One important aspect of Acturial science is called Mathematics of Financial (or) Mathematic of investments.

If a sum "S_0" is invested at component interest of i per unit amount per unit time & S_t is the amount at the end of the time "t" then we get the difference equ.

$$S_{t+1} = S_t + iS_t$$

$$S_{t+1} = S_t(1+i) \rightarrow (1)$$

This is a linear difference equ'. Its Complementary fun' is of the form,

$$S_t = a\lambda^t \rightarrow (2)$$

$$(1) \Rightarrow \quad a\lambda^{t+1} = (1+i)a\lambda^t$$

$$a\lambda^{t+1} - (1+i)a\lambda^t = 0$$

$$a\lambda^t [\lambda - (1+i)] = 0$$

$$\lambda - (1+i) = 0 \Rightarrow \boxed{\lambda = 1+i}$$

Scanned with CamScanner

sub ② we get the complementary fun' is, $a(1+i)^t = S_t$

27

put $t=0$,

$$\boxed{S_0 = a}$$

$\therefore$ The soln' is $S_t = S_0(1+i)^t \rightarrow$ ③

which is the well known formula for compound interest.

Suppose a person barrows a sum "S_0" at compound interest i and want to his dept i.

(ie) He wants to pay the amount & interest back by payment of "n" equal in statements say R the first payment to be made at the end the first year.

let S_t be the amount due but the end of "t" years. Then the difference eqn'.

$$S_{t+1} = S_t + iS_t - R$$

$$\Rightarrow S_{t+1} - S_t(1+i) = -R$$

$$S_{t-1} = S_t(1+i) - R \rightarrow (4)$$

$$S_{t-1} - S_t(1+i) = -R \rightarrow (*)$$

This is a linear difference eqn' its corresponding fun' is of the form,

let $S_t = b\lambda^t \rightarrow$ ⑤

$(*) \Rightarrow b\lambda^{t+1} - (1+i)b\lambda^t = 0$

$b\lambda^t [\lambda - (1+i)] = 0$

$\lambda - (1+i) = 0$

$\boxed{\lambda = 1+i}$

Sub ⑤ we get the complementary fun',

(ie) $b(1+i)^t \cancel{= b\lambda^t}$

The particular soln' is a constant

(ie) $\phi(t) = -R$

Putting in ④, $A = S_t$

④ $\Rightarrow$ $S_{t-1} = S_t(1+i) - R$

$A = A(1+i) - R$

$A - A(1+i) = -R$

$A[1 - (1+i)] = -R$

$$A = \frac{-R}{1-(1+i)} = \frac{-R}{1-1-i} = \frac{\cancel{-}R}{\cancel{-}i} = \frac{R}{i}$$

$$\boxed{A = \frac{R}{i}}$$

The complete soln' is

$S_t = C.F + P.I$

$S_t = b(1+i)^t + \frac{R}{i} \rightarrow ⑥$

Put $\boxed{t=0}$

$S_0 = b + \frac{R}{i}$

$b = S_0 - \frac{R}{i}$

Scanned with CamScanner

$$⑥ \Rightarrow S_t = \left(S_0 - \frac{R}{i}\right)(1+i)^t + \frac{R}{i}$$

$$= S_0(1+i)^t - \frac{R}{i}(1+i)^t + \frac{R}{i}$$

$$= S_0(1+i)^t - R\left[\frac{(1+i)^t - 1}{i}\right] \rightarrow ⑦$$

put $\boxed{t = n}$

$$S_n = S_0(1+i)^n - R\left[\frac{(1+i)^n}{i} - 1\right] \rightarrow ⑧$$

In the amount is paid back in n years. let, $S_n = 0$ (using 8)

$$0 = S_0(1+i)^n - R\left[\frac{(1+i)^n - 1}{i}\right] \rightarrow ⑨$$

$$\therefore S_0(1+i)^n = R\left[\frac{(1+i)^n - 1}{i}\right]$$

$$\frac{R}{i} = \frac{S_0(1+i)^n}{(1+i)^n - 1} \Rightarrow R = \frac{i.S_0(1+i)^n}{(1+i)^n - 1}$$

$$R = \frac{i.S_0(1+i)^n}{(1+i)^n\left(1 - \frac{1}{(1+i)^n}\right)}$$

$$= \frac{i.S_0}{1-(1+i)^n}$$

$$R = \frac{S_0}{(a_{n}/i)} \rightarrow ⊛⊛$$

where, $\frac{\overline{a_n}}{i} = \frac{1-(1+i)^n}{i}$

$\frac{\overline{a_n}}{i}$ is called the amorlization factors

is the i present value of an ⬭ of one per unit time for "n" periods after interest i.

The functions $\frac{\overline{a_n}}{i}$ and $\left(\frac{\overline{a_n}}{i}\right)^{-1}$ are tabulated for common values of n & i ~~R is deposited at the end~~

Suppose an amount R is deposited at the end of every period in a bank and let S_t be the amount of at the end,

$$S_{t+1} = S_t(1+i) + R \rightarrow (10)$$

$S_0 = 0$ (8) becomes,

$$S_n = R\left[\frac{(1+i)^n - 1}{i}\right] = R\frac{\overline{S_n}}{i} \rightarrow (11)$$

From (3) $S_t = S_0(1+i)^t$

put $t=n$, $S_n = S_0(1+i)^n \rightarrow (12)$

equating (11) & (12) we get

$$S_0(1+i)^n = \frac{R\overline{S_n}}{i}$$

$$(1+i)^n \cdot \frac{\overline{a_n}}{i} = \frac{\overline{S_n}}{i}$$

$$S_0(1+i)^n = \frac{S_0}{\overline{a_n}\, i} \cdot \frac{\overline{S_n}}{i}$$

$$(1+i)^n \frac{\overline{a_n}}{i} = \frac{\overline{S_n}}{i}$$

Scanned with CamScanner

$$\frac{\overline{s_n}}{i} = (1+i)^n \frac{\overline{a_n}}{i} \rightarrow (13)$$

Taking reciprocal on both sides.

$$\frac{1}{\left(\frac{\overline{s_n}}{i}\right)} = \frac{1}{(1+i)^n \frac{\overline{a_n}}{i}} = \frac{(1+i)^{-n}}{\frac{\overline{a_n}}{i}} \rightarrow (14)$$

If a person, to pay an amount 'S' at the end of "n" years be can do it by paying into a sinking find an amount R per period.

where, $R = S \cdot \frac{1}{\overline{s_n}} \rightarrow (15)$

where, $\frac{1}{\left(\frac{\overline{s_n}}{i}\right)}$ is the sinking find factor & can be tabulated by using (*).

1. Solve $x_{t+2} - 2x_{t+1} + 2x_t = 0$

Soln: let the solution be,

$x_t = a\lambda^t$, $x_{t+1} = a\lambda^{t+1}$,

$x_{t+2} = a\lambda^{t+2}$

$a\lambda^{t+2} - 2a\lambda^{t+1} + 2a\lambda^t = 0$

$a\lambda^t [\lambda^2 - 2\lambda + 2] = 0$

$\lambda^2 - 2\lambda + 2 = 0$

Quadratic formula, $\lambda = -b \pm \sqrt{b^2 - 4ac}$

Scanned with CamScanner

$$= \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm i\sqrt{4}}{2} = \frac{2 \pm i2}{2}$$

$$\lambda = \frac{\cancel{2}(1 \pm i)}{\cancel{2}} \Rightarrow \lambda = 1 \pm i$$

$\lambda = 1$

$$\boxed{\lambda = 1+i \ , \ 1-i}$$

Formula :- The most general solution is

$$\boxed{G(t) = (\alpha^2+\beta^2)^{t/2} \left[d_1 . \cos^t\theta + d_2 \sin^t \theta \right]}$$

$$= (\alpha^2+\beta^2)^{t/2} \left[d_1 \cos\theta + d_2 \sin\theta \right]^t$$

Here, $\boxed{\alpha = 1}$ $\boxed{\beta = 1}$

$$G(t) = (1+1)^{t/2} \left[d_1 \cos\theta + d_2 \sin\theta \right]^t$$

$$= 2^{t/2} \left[d_1 \cos\theta + d_2 \sin\theta \right]^t$$

$$\tan\theta = \beta/\alpha = 1/1 = 1$$

$$\theta = \tan^{-1}(1) = \frac{\pi}{4}$$

$\theta = \pi/4$

$\tan\theta$

$0° = 0$

$30° = \frac{1}{\sqrt{3}}$

$45° = 1$

$60° = \sqrt{3}$

$90° =$ not defined.

$\pi = 180$

$\frac{\pi}{4} = \frac{180}{4} = 45$

$$= (\sqrt{2})^t \left[d_1 \cos \pi/4 + d_2 \sin \pi/4 \right]^t$$

$2^{t/2} = 2^{1/2} = (\sqrt{2})^t$

$$= (\sqrt{2})^t \left[\frac{d_1}{\sqrt{2}} + \frac{d_2}{\sqrt{2}} \right]^t$$

$\cos 45° = \frac{1}{\sqrt{2}}$

$\sin 45° = \frac{1}{\sqrt{2}}$

$$= \frac{(\sqrt{2})^t}{(\sqrt{2})^t} (d_1 + d_2)^t$$

$\cos 0° = 1$

$\cos 90° = 0$

$\sin 0° = 0$

$\sin 90° = 1$

$$\boxed{G(t) = (d_1 + d_2)^t} \quad //$$

Q. Solve $x_{t+2} - x_{t+1} + x_t = 0$

Soln: let the solution be,

$$x_t = a\lambda^t$$

$$a\lambda^{t+2} - a\lambda^{t+1} + a\lambda^t = 0$$

$$a\lambda^t [\lambda^2 - \lambda + 1] = 0$$

$$\lambda^2 - \lambda + 1 = 0$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad \begin{matrix} a = 1 \\ b = -1 \\ c = 1 \end{matrix}$$

$$\lambda = \frac{1 \pm \sqrt{1-4}}{2} \Rightarrow \frac{1 \pm \sqrt{i3}}{2}$$

$$\lambda = \frac{1}{2} \pm i\sqrt{3}/2 .$$

Here, $\alpha = 1/2$, $\beta = \sqrt{3}/2$

The most general solution is,

$$G(t) = (\alpha^2 + \beta^2)^{t/2} [d_1 \cos\theta^t + d_2 \sin\theta t]$$

$$= \left(\frac{1}{4} + \frac{3}{4}\right)^{t/2} [d_1 \cos\theta + d_2 \sin\theta]^t$$

$$= [d_1 \cos\theta + d_2 \sin\theta]^t \qquad \frac{\sqrt{3}}{2} \times \frac{2}{1}$$

$$\tan\theta = \beta/\alpha = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}/1 = \sqrt{3}$$

$$\theta = \tan^{-1}(\sqrt{3})$$

$$\boxed{\theta = \pi/3}$$

$$G(t) = (d_1 \cos\pi/3 + d_2 \sin\pi/3)^t$$

$$= (d_1 \cdot 1/2 + d_2 \cdot \sqrt{3}/2)^t$$

34

3 S

S

4

$$= \left(\frac{d_1 + d_2\sqrt{3}}{2}\right)$$

$$G(t) = \left(\frac{d_1 + d_2\sqrt{3}}{2}\right)^t \quad //$$

where, d_1 & d_2 are constants.

Solve $x_{t+2} - 7x_{t+1} + 12x_t = 0$.

Soln: Defn z-transform also find the soln of linear eqn by using z-transform method & this one also

let the soln be

$$x_t = a\lambda^t$$

$$a\lambda^{t+2} - 7a\lambda^{t+1} + 12a\lambda^t = 0$$

$$a\lambda^t [\lambda^2 - 7\lambda + 12] = 0$$

$$\lambda^2 - 7\lambda + 12 = 0$$

$$(\lambda - 3)(\lambda - 4) = 0$$

$$\boxed{\lambda = 3, 4}$$

12 → −3, −4

The must general soln is

Formula:- $\boxed{G(t) = d_1 \cdot \lambda_1^t + d_2 \lambda_2^t}$

Scanned with CamScanner

$$a\lambda^t[\lambda^3-5\lambda^2+7\lambda-3]=0$$

$$\lambda^3-5\lambda^2+7\lambda-3=0$$

$$\lambda^2(-5\lambda^2+7\lambda-3)=0$$

$$(\lambda-1)(\lambda^2-4\lambda-3)=0$$

$$(\lambda-1)(\lambda-1)(\lambda-3)=0$$

$$\boxed{\lambda=1,1,3}$$

$$\begin{array}{r|rrrr} 1 & 1 & -5 & 7 & -3 \\ & 0 & 1 & -4 & 0 \\ \hline & 1 & -4 & 3 & \boxed{0} \end{array}$$

The most general soln' is

$$G(t)=\boxed{(At+B)\lambda^t}+C\lambda_1^t$$

$$=(At+B)(1)^t+C(3)^t$$

$$G(t)=(At+B)+C(3)^t$$

where A, B & C are constant.

$$8x_{t+3}-12x_{t+2}+6x_{t+1}-2x_t=0$$

Soln: let the solution is,

$$x_t=a\lambda^t$$

$$8a\lambda^{t+3}-12a\lambda^{t+2}+6a\lambda^{t+1}-2a\lambda^t=0$$

$$a\lambda^t[8\lambda^3-12\lambda^2+6\lambda-2]=0$$

$$8\lambda^3-12\lambda^2+6\lambda-2=0$$

$$8\lambda^3-8\lambda^2-4\lambda^2+4\lambda+2\lambda-2=0$$

$$8\lambda^2(\lambda-1)-4\lambda(\lambda-1)+2(\lambda-1)=0$$

$$(\lambda-1)(8\lambda^2-4\lambda+2)=0$$

$$8\lambda^2-4\lambda+2=0$$

Scanned with CamScanner

36

$$\lambda = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\lambda = \frac{4 \pm \sqrt{16-64}}{16} \Rightarrow \frac{4 \pm \sqrt{-48}}{16}$$

$$= \frac{4 \pm 4\sqrt{-3}}{16} \Rightarrow \frac{1 \pm i\sqrt{3}}{4}$$ //

$\boxed{\alpha = \frac{1}{4}}$ & $\boxed{\beta = \sqrt{3}/4}$.

$$\tan\theta = \beta/\alpha$$

$$\theta = \tan^{-1}(\beta/\alpha)$$

$$= \tan^{-1}\left(\frac{\sqrt{3}/4}{1/4}\right)$$

$$= \tan^{-1}(\sqrt{3})$$

$$\boxed{\theta = \pi/3}$$

The most general soln' is

$$G(t) = d_1 \lambda_1^t + (\alpha^2+\beta^2)^{t/2}[d_2 \cos\theta + d_3 \sin\theta]^t$$

$$= d_1(1)^t + \left(\frac{1}{16}+\frac{3}{16}\right)^{t/2}[d_2 \cos(\pi/3) + d_3 \sin(\pi/3)]^t$$

$$= d_1 + (1/4)^{t/2}[d_2(1/2) + d_3(\sqrt{3}/2)]^t$$

$$= d_1 + \frac{1}{(\sqrt{4})^t} \cdot \frac{(d_2+\sqrt{3}\cdot d_3)^4}{2^t}$$

$$= d_1 + \frac{1}{2^t} \cdot \frac{(d_2+\sqrt{3}\cdot d_3)^t}{2^t}$$

$$G(t) = d_1 + \frac{1}{4^t}(d_2+\sqrt{3}\,d_3)^t$$ //

Scanned with CamScanner

Mathematical Modelling through difference equ' in populations of Dynamics and Genetics :-

37

Non-Linear difference equ' Model for population growth :- (or)

Non-Linear difference equations :-

let x_t be the population at time "t" and the let births and deaths in time interval $(t, t+1)$ be proportional to x_t, Then the population x_{t+1} at time $(t+1)$ is given by,

$$x_{t+1} = x_t + bx_t - dx_t = x_t[1+(b-d)]$$

where,
$$= x_t(1+a) \rightarrow ①$$

$a = b-d$. This has the soln' $x_t = x_0(1+a)^t$ So that the population increase (or) decrease exponentially according as $a>0$ (or) $a<0$.

We now consider the generalization when birth and death "b" & "d" per unit population depend linearly on x_t.

So that,

$$x_{t+1} = x_t + (b_0 - b_1 x_t)x_t - (d_0 + d_1 x_t)x_t$$

$$x_{t+1} = x_t + b_0 x_t - b_1 x_t^2 - d_0 x_t - d_1 x_t^2$$

$$= [1 + b_0 - d_0]x_t + [-b_1 - d_1]x_t^2$$

$$= [1 + b_0 - d_0]x_t - [b_1 + d_1]x_t^2$$

Scanned with CamScanner

$$= mx_t - rx_t^2$$

where, $m = 1 + b_0 - d_0$; $r = b_1 + d_1$

$$x_{t+1} = mx_t\left(1 - \frac{r}{m} \cdot x_t\right) \rightarrow (2)$$

This is the simplest non linear generalization of (1) and gives the discrete version of the logistic law of population growth, However this model shows many features not present in the centre version of the logistic Model.

let, $\frac{rx_t}{m} = y_t$

when (2) becomes

$$x_t = \frac{my_t}{r}$$

$$x_{t+1} = \frac{my_{t+1}}{r}$$

$$(2) \Rightarrow \frac{my_{t+1}}{r} = \frac{m(my_t)}{r}[1 - y_t]$$

$$y_{t+1} = my_t[1 - y_t] \rightarrow (3)$$

1) one period fixed points and this Stability :-

A one period fixed point of This eqn is that value y_t for which $y_{t+1} = y_t$.

(ie) $y_{t+1} = my_t(1 - y_t)$ [using (3)]

$$y_t = my_t(1 - y_t) = my_t - my_t^2$$

$$my_t^2 + y_t - my_t = 0$$

$$y_t[my_t - m + 1] = 0$$

$y_t = 0$, $my_t - m + 1 = 0$ $\Rightarrow \cancel{m(y_t - 1)} = 0$

$my_t = m - 1 \Rightarrow y_t = \frac{m-1}{m}$.

So that, there are ~~2~~ one period fixed point "0" and $\frac{m-1}{m}$. If $y = 0$ then $y_1, y_2 \ldots$ are all zero and the population remains fixed at zero value.

If $y_0 = \frac{m-1}{m}$, then, $y_1, y_2 \ldots y_3$ are all equal to $\frac{m-1}{m}$. The 2^{nd} fixed points only of $m > 1$. we now ~~consider~~ discuss the stability of equilibrium of each of 3 equilibrium positions.

putting $y_t = u_t$ in (3) & neglect squares & higher power of u_t. we get

$$u_{t+1} = mu_t .$$

$\therefore$ put $y_t = u_t$ in (3) we get

$$y_{t+1} = my_t(1 - y_t)$$

$$u_{t+1} = mu_t(1 - u_t)$$

$$= mu_t - mu_t^2$$

$$= mu_t \quad \text{(neglecting square)}$$

Since, $m > 0$ the 1^{st} equilibrium position is one of unstable equilibrium.

Again putting $y_t = \frac{m-1}{m} + u_t$ in (3)

Scanned with CamScanner

and neglecting square & higher powers of $u_t < m$. we get

$$u_{t+1} = (2-m)u_t$$

put $y_t = \frac{m-1}{m} + u_t$ in ③

$$y_{t+1} = m y_t (1-y_t)$$

$$\frac{m-1}{m} + u_{t+1} = m\left[\frac{m-1}{m} + u_t\right]\left[1 - \left\{\frac{m-1}{m} + u_t\right\}\right]$$

$$= [m-1+mu_t]\left[\frac{1 - m-1+mu_t}{m}\right]$$

$$= [m-1+mu_t]\left[\frac{m-m-1+mu_t}{m}\right]$$

$$= [m-1+mu_t]\left[1 - \frac{mu_t}{m}\right]$$

$$= \frac{1}{m}\left[m - m^2 u_t - 1 + mu_t + mu_t - m^2 u_t^2\right]$$

$$u_{t+1} = \frac{1}{m}\left[m - m^2 u_t - 1 + mu_t + mu_t - m^2 u_t^2\right]$$

$$= \frac{1}{m}\left[m - m^2 u_t - 1 + 2mu_t - m^2 u_t^2\right]$$

$$= \frac{1}{m}\left[-m^2 u_t + 2mu_t - m^2 u_t^2\right]$$

$$= -mu_t + 2u_t - mu_t^2$$

$$= -mu_t + 2u_t \quad (\text{neglect square})$$

$$= (2-m)u_t$$

So that the 2^{nd} position equilibrium is stable only if $-1 < 2, -m < 1$ (or) if

Scanned with CamScanner

Scanned with CamScanner

$1 > m-2 > -1$ (or) If $1 < m < 3$. Thus if $0 < m < 1$

There is only one period fixed pt's & it is unstable.

41

If $1 < m < 3$. There are 2 one period fixed points, the 1st is unstable & 2nd is stable.

If $m > 3$. There are 2 one period fixed points both of which are unstable.

ii) two period fixed point and their stability:-

A point is called a 2 period fixed pt's of it repeat itself after 2 periods.

(ie) $y_{t+2} = y_t$ $\quad y_{t+1} - my_{t+1})^2$

③ $\Rightarrow y_{t+1} = my_t(1-y_t) = my_{t+1}(1-y_{t+1})$

$$= m[my_t(1-y_t)][1-\{my_t(1-y_t)\}]$$

$$= m^2 y_t(1-y_t)[1-my_t+my_t^2]$$

If $y_{t+2} = y_t$

$$y_{t+2} = m^2 y_t(1-y_t)[1-my_t+my_t^2] - y_t$$

$$y_{t+2} = m^2 y_t(1-y_t)[1-my_t+my_t^2] - y_t =$$

This is a 4th degree eqn and two of these are the same as one period fixed points. This is obvious from the concentration that every one period fixed point is also a two period fixed pt.

The genuine two period fixed pts are

42 $\frac{m-1}{m}$ obtained by solving the equⁿ,

$$m^2 y_t^2 - m(1+m)y_t + (1+m) = 0$$

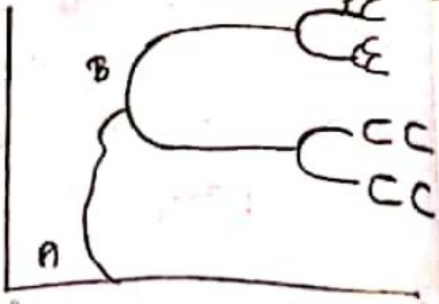
If roots are real if $m > 3$. Thus if $m > 3$ the 2 one period fixed pts becomes unstable but two new period can be shown that if $m_2 < m_1 < m_4$. where $m_2 = 3$ & m_4 is a number slightly > than 3. Then, the 2 period fixed pts are stable but if $m > m_4$ all the four one and two periods become unstable but four new four period fixed pts exists. which are stable if the $m_4 < m < m_8$ and become unstable in $m > m_8$.

iii) Third period fixed point and this stability :-

It can be shown that there exists an increasing infinite sequence in real numbers $m_2, m_4 \dots m_{2n} + m_{2n+1} \dots$ such that when ~~m~~ $m_{2n} < m < m_{2n+1}$. There are 2^n period fixed points out of which 2^n fixed points are also fixed points of lower order time periods and all these are unstable & the remaining 2^n periods are genuine 2^{n+1} period.

Scanned with CamScanner

where m lies b/w m_1 & m_2 there is one stable one period fixed pt. when m lies b/w m_2 & m_4 there are 2 stable 2 period fixed pts. when m lies b/w m_4 & m_2 there are four stable 4 period fixed pts and so on.



43

iv) Fixed points of other periods :-

The seq. $m_2, m_4 \ldots m_6$ is bounded above by a fixed number m^*.

If $m > m^*$ there can be a there period fixed pt and if there is a 3 period fixed pt will also be fixed pt of periods.

$3, 5, 7, 9 \ldots$

$2.3, 2.5, 2.7, 2.9 \ldots$

$2^2.3, 2^2.5, 2^2.7 \ldots$

This is expressed by saying that the period 3 means choose.

v) characteristic behaviour of the Non-Linear Model :-

If m lies b/w m_3 and m_{in} there will be eight, 16 period. stable fixed pts

Scanned with CamScanner

If a population size starts from any one of these value, it will oscillate through fifteen other value of return of the original value this pattern will go on repeating itself.

If we draw graph, it will show rapid oscillations and will look like the graph representing a random phenomenon our model is perfectly deterministic through its behaviour may appear to be random and stochastic.

+4

vi) Special features of Non-Linear difference equ' Models :-

The simple method illustrates the diff' in behaviour b/w diff and equ' models. The problem of existence and uniqueness of solutions of the stability of equilibrium position are all diff' due to the basic fact that inspite of similarities the discrete and the continuous are really different.

Age structured population Model :-

let $x_1(t), x_2(t) \dots x_p(t)$ population sizes of "q" reproduction age group at time "t". let $x_{p+q+1}, x_{p+q+2t} \dots x_{p+q+n}(t)$ be the population sizes of "r" post reproductive age groups at time 't'.

let $b_{p+1}, b_{p+2} \dots b_{p+q}$ be the birth rates.

Scanned with CamScanner

(ie) The no. of births per unit time out individual in the reproductive age group.

let $m_1, m_2 \ldots m_{p+q+r}$ be the rates of emigration to the $x(t)$ age groups. Then we get the system of difference eqn'. 45

$$x_1(t+1) = b_{p+1} x_{p+1}(t) + \cdots + b_{p+q} x_{p+q}(t) - (d_1+m_1) x_1(t)$$

$$x_2(t+1) = m_1 x_1(t) - (d_2+m_2) x_2(t)$$

$$x_{p+q+r-1}(t-1) = m_{p+q-2}\, x_{p+q+r-2} - [d_{p+q+r-1} + m_{p+q+r-1}][x_{p+q+r-1}]^2$$

$$x_{p+q+r}(t+1) = m_{p+q+r-1} \cdot x_{p+q+r-1}(t) - (d_{p+q+r}) x_{p+q+r}$$

which can be written in the matrix form,

$$x_{t+1} = Lx(t) \longrightarrow (2)$$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_{p+q+r}(t) \end{bmatrix}$$

$$L = \begin{bmatrix} -(d_1+m_1) & 0 & 0 & 0 & b_{p+1} \ldots b_{p+q} & 0 & 0 \\ m_1 & -(d_2+m_2) & 0 & 0 & 0 & 0 & 0\ 0 \\ 0 & m_2 & (-d_2+m_2) & 0 & 0 & 0 & 0\ 0 \\ \vdots & \vdots & \vdots & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0\ 0 \end{bmatrix}$$

where, $p+q+r=n$.

L is called lastie matrix all the elements of its main diagonal are -ve and all the elements of its main subdiagonal are +ve.

In addition "q" elements in 1st row are +ve and the rest of the elements are all zero the soln of ⑧ can be written as,

$$X(t) = L(t) \cdot X(0)$$

Now, the [lesdie matrix has the property] that it has a determinant eigen value in which is real and +ve which is the greater to absolute value then any other eigen vector value and for which the corresponding 2 +ve.

In this dominant eigen value is > than unity. Then the populations of all age groups will increase exponentially and if it is less than unity. The population of all age group will die out.

In this dominant eigen value is unity the population can have a stable age structure.

The lestie Model interms of a system of linear diff' eqn'. If we take the effects of the eqn' are non-linear.

Mathematical Modelling through Difference

Scanned with CamScanner

of an individual like height or colour of the hair is determined by a pair of genetics one obtained from the father and mother obtained from the mother.

47

Every genetics occurs in two form a dominant [denote "d" by a capital letter say G] and a recessive [denote by the corresponding small letter say "g"].

Thus, with respect to a characteristic an individual may be a dominant [GG] a hybrid (Gg (or) g(G)) (or) a recessive (gg)

In the n^{th} generation let the property of "dominants" "hybrids" and "recessive" be p_n q_n r_n so that,

$\boxed{p_n + q_n + r_n = 1}$; $p_n \geq 0, q_n \geq 0, r_n \geq 0$

we assume that individuals in this generation made at random.

p_{n+1} = The probability than an individuals in the $(n+1)^{th}$ generation is a dominant (GG).

= Probability that this individual get G'. G' from the father (probability that the individual get a G from the mother)

$$p_{n+1} = (p_n + \tfrac{1}{2}.q_n)(p_n + \tfrac{1}{2}.q_n)$$
$$= (p_n + \tfrac{1}{2}q_n)^2 \rightarrow (1)$$

$||^{ly}$, $q_{n+1} = 2(p_n + \tfrac{1}{2}.q_n)(r_n + \tfrac{1}{2}q_n)$

$$r_{n+1} = (r_n + \tfrac{1}{2}q_n)^2$$

So that,

$$p_{n+1} + q_{n+1} + r_{n+1} = (p_n + \tfrac{1}{2}q_n)^2 + (2(p_n + \tfrac{1}{2}.q_n)).(r_n + \tfrac{1}{2}.q_n) + (r_n + \tfrac{1}{2}.q_n)^2.$$
$$= p_n + \tfrac{1}{2}.q_n + r_n + \tfrac{1}{2}q_{n-1} \rightarrow (2)$$
$$[\because (p_n + q_n + r_n)^2 = (1)^2 = 1]$$

as excepted similarly,

$$p_{n+2} = (p_{n+1} + \tfrac{1}{2}.q_{n+1})^2$$
$$= [(p_n + \tfrac{1}{2}q_n)^2 + (p_n + \tfrac{1}{2}.q_n)(r_n + \tfrac{1}{2}.q_n)]$$
$$= (p_n + \tfrac{1}{2}.q_n)^2 (p_n + \tfrac{1}{2}.q_n + \tfrac{1}{2}.q_n + r_n)^2$$
$$= [p_n + \tfrac{1}{2}.q_n]^2 = p_{n+1} \rightarrow (3)$$

$\therefore \boxed{p_{n+1} = p_{n+2}}$ $\boxed{q_{n+2} = q_{n+1}}$, $\boxed{r_{n+2} = r_{n+1}}$

So that the properties of dominants, hybrids ane recessive in the $(n+1)^{th}$ generation are same as in the $(n+1)^{th}$ generation.

Thus, in any populations in which random take place w.r to a characteristic of dominant, hybrids & recessive do not change after the 1st generation. This is known as Hardy-weinburry Law) after the mathematician Hardy and genestic weinbery jointly discovered it the eqn (1) & (2)

is a set of diff equ of the 1st order.

b) Improvement of planets through elimination of Recessive :-

suppose the recessive and undesirable & as such we do not allow the recessives in any generation to spread.

let p_n, q_n be the proportions of dominates hybrids and recessive elimination of recessive elimination. Then,

$$\frac{p_n'}{p_n} = \frac{q_n'}{q_n} = \frac{p_n'+q_n'}{p_n+q_n} = \frac{1}{1-r_n}$$

Now, we allow random meeting and let $p_{n+1}, q_{n+1}, r_{n+1}$ be the proportions in the next generations before eliminations of recessive. Then using ①–②.

$$p_{n+1} = (p_n' + \tfrac{1}{2}.q_n')^2$$

$$q_{n+1} = 2(p_n' + \tfrac{1}{2}.q_n')(\tfrac{1}{2}q_n')$$

$$= q_n'(p_n' + \tfrac{1}{2}.q_n')$$

$$r_{n+1} = (\tfrac{1}{2}.q_n')^2$$

$$r_{n+1} = \tfrac{1}{4}(q_n')^2$$

After elimination of recessive. let the new proportions be p_{n+1}, q'_{n+1} so that.

$$\frac{p_{n+1}'}{p_{n+1}} = \frac{q'_{n+1}}{q_{n+1}} = \frac{1}{p_{n+1}+q_{n+1}} = \frac{1}{1-\frac{1}{4}q_n'^2}$$

Scanned with CamScanner

$$1 - \frac{1}{4} \cdot a_n'^2 \qquad (1 - \frac{1}{2} \cdot a_n')(1 + \frac{1}{2} a_n'$$

$$\boxed{a'_{n+1} = \frac{a_n'}{1 + \frac{1}{2} \cdot a_n'}}$$ //

This is a non-linear diff' eqn of the 1st order to solve it we sub.

$$a_n' = \frac{1}{u_n}.$$

$$u_{n+1} = u_n + \frac{1}{2}$$

which has the solution $u_n = A + \frac{1}{2} A$ (or)

$$a_n' = \frac{1}{A + \frac{1}{2} \cdot n} \rightarrow (4)$$

So that, $a_n' \rightarrow 0$ & $p_n' \rightarrow 0$ at $n \rightarrow 0$.

Thus, ultimately we should be left will all dominates eqn (4) determine the rate of which hybrids disappear.

Mathematical Modelling through diff' eqn' in

Scanned with CamScanner

the System is in state j at time t
(j=1,2...n) then at time t+1 it can be is any one of the state 1,2 ... n. 51

It can in the ith state at time t+1 in exclusive ways. since it would have been in any one of the n states 1,2....n at time t and it could have transisted from the state to ith state i time interval (t, t+1).

By using The theorem, of total & compound Propability.

$$\boxed{P_i(t+1) = \sum_{j=1}^{n} P_{ij} \cdot P_j(t)} \quad i=1,2...n$$

$$P_1(t+1) = P_{11} P_1(t) + P_{21} P_2(t) \cdots P_{n1} P_n(t)$$

$$P_2(t+1) = P_{12} P_2(t) + P_{22} P_2(t) \cdots P_{n2} P_n(t)$$

$$\vdots$$

$$P_n(t+1) = P_{1n} P_n(t) + P_{2n} P_n(t) \cdots P_{nn} P_n(t)$$

(or)

$$\boxed{P(t+1) = A P(t)}$$

Where, P(t) is a "probability vector" and A is a matrix all the elements lie b/w zero and unity (since there are all probability).

Further are the sum of elements of every columns is unity. since the sum of elements of ith column is $\sum_{i=1}^{n} P_{ij}$ as this denote the sum of the probability of the system going from the ith state to any other state and this sum must be unity.

The solution of the matrix diff' equ ①
is $\boxed{P(t) = A^t . P(0)}$ If all the eigen values $\lambda_1, \lambda_2 \dots \lambda_n$ of A are distinct. we can write, $A = S\Lambda S^{-1}$.
where,

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \dots 0 \\ 0 & \lambda_2 & 0 \dots 0 \\ \vdots & & \\ 0 & 0 & 0 \dots 0 \end{bmatrix}$$

so that, $A^t = (S\Lambda S^{-1})(S\Lambda S^{-1}) \dots (S\Lambda S^{-1})$

$$= S\Lambda^t S^{-1}.$$

$$= S \begin{bmatrix} \lambda_1 t & 0 & 0 \dots 0 \\ 0 & \lambda_2 t & 0 \dots 0 \\ \vdots & & \\ 0 & 0 & 0 \dots 0 \end{bmatrix} S^{-1}$$

The probability vector will not change, if $\boxed{P(t+1) = P(t)}$ so that from ①.

$$\boxed{(I - A).P(t) = 0}$$

Thus if P is the eigen vectors of the matrix is corresponding to unit eigen value then P does not change.

(ie) If the system state with probability vector P at time 0. It will always remains in this state.

Even if the system starts from

another probability vector it will ultimately be described by the probability vector p as $1 \Rightarrow C_0$. 53

suppose we have a machine which can be in two states working (or) non-working.

let the probability of this transition from working to networking be α of its transition from ~~non making~~ non-working to β. Then the transition probability matrix A is obtained from

$$\begin{array}{c c} & \begin{array}{cc} \text{working} & \text{Non-working} \end{array} \\ \begin{array}{c} \text{working} \\ \text{Non-working} \end{array} & \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix} \end{array}$$

The system of diff' equ' is

$$p_1(t+1) = p_1(t)(1-\alpha) + p_2(t)\beta$$

$$p_2(t+1) = p_1(t)\alpha + p_2(t)(1-\beta) \qquad \text{(or)}$$

$$\begin{bmatrix} p_1(t+1) \\ p_2(t+2) \end{bmatrix} = \begin{bmatrix} 1-\alpha & \beta \\ \alpha & 1-\beta \end{bmatrix} \begin{bmatrix} p_1(t) \\ p_2(t) \end{bmatrix}$$

The eigen values of the matrix A is given by,

$$\begin{vmatrix} 1-\alpha-\lambda & \beta \\ \alpha & 1-\beta-\lambda \end{vmatrix} = 0 \quad \text{(or)}$$

$$(\lambda-1)(\lambda-1-\alpha-\beta) = 0$$

The eigen vector corresponding to the unit eigen value is $\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}$ and as such

Scanned with CamScanner

ultimately the probability of the machines being found is working order is $\frac{\beta}{\alpha+\beta}$ and
54 the probability of its being found in a non-working state is $\frac{\alpha}{\alpha+\beta}$.

Gambler's Rain problem :-

let a gambler's of his winning (or) Lossing a unit dollar in any same be p & q res/. where $p+q=1$ and let p_n be the probability of his being ultimately rain etc.

At the next game the pr of his winning is a "p" and if the wins capital would become $n+1$ and the pr of his ultimate rains would be p_{n+1}.

If he loses at the next game the probability for which is q his captial would become $n-1$ and the probability of his ultimates rain would be p_{n-1}. So that we get the linear diff equ of 2nd order.

$$p_n = p \cdot p_{n+1} + q \cdot p_{n-1} \longrightarrow (1)$$

$\therefore$ The auxillary equ of this is

$$p\lambda^2 - \lambda + (1-p) = 0 \quad (or)$$

$$p(\lambda - 1)(\lambda - 1 - p/p) = 0$$

As such the solution (1) is

$$p_n = A + B\,(q/p)^n \longrightarrow (2)$$

let the ——— decide to stop this game then this capital becomes a dollars. So that the pr of his being rained. when his starting capital is a dollars is zero.

(ie) $\boxed{p_a = 0}$ $\rightarrow$ (3)

55

In the same way when the starting capital is zero. He is already rained. so we put $p_0 = 1$, $P_a = 0$ $\rightarrow$ (4)

Equ (2) gives,

$$P_n = \frac{(q/p)^a - (q/p)^n}{(q/p)^a - 1} \rightarrow (5)$$

let D_n denote the expected no. of games before the gambler is rained. If he wins at the next game his capital becomes $n+1$ and the expected no. of games would than be D_{n+1} and if we loss his capital become $n-1$ and the expected no of game would be only D_{n-1}.

$$D_n = D, \quad D_{n+1} \mp q_{n-1} + 1 \rightarrow (6)$$

Co boundary conditions,

$$D_0 = 0, \quad D_a = 0 \rightarrow (7)$$

This gives the Soln's,

$$D_n = \frac{n}{q-p} - \frac{a}{q-p} - \frac{(1-q/p)^n}{(1-q/p)^a}$$

16/2/20

Scanned with CamScanner

Scanned with CamScanner

Mathematical Modelling through ODE of 2^{nd} order :-

central force (or) centre of force :-

A force always directed along a line to a fixed point is called central force, the fixed point is called the centre of force.

central orbit :-

A path is discribed by a particle under a center force is called central orbit.

Mathematical Modelling of Plantary Motions needs for the study of [Motion under the central forces] :-

Every planet moves mainly under the gravitational attraction force excerted by the sun.

If S and P are the masses of the sun and planet and G is the universal constant of gratitational. Then the growitational attraction on the sun and the planets are both $\frac{GSP}{r^2}$

where, r is the distance b/w the sun and planet.

Scanned with CamScanner

According the acceleration of the sun towards the planet is $\frac{GP}{r^2}$ and the acceleration of the planet towards the sun is $\frac{GS}{r^2}$.

The acceleration of the planet relative to the sun is $\frac{G(S+P)}{r^2} = \frac{\mu}{r^2}$.

Now, we take the sun as fixed then the planet can be said to move under a central force $\frac{\mu}{r^2}$ per unit mass.

(ie) under a force which always directed towards a fixed center S.

We shall for the present also regard P as a particle so that to study the motion of the planet we have to study the motion of the particle moving under a central force.

We can take "S" as origin. So that, the central form is always along the radius vector.

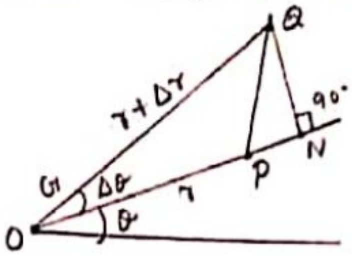
To study this motion it is convelent to use polar co-ordinates and to find the components of the velocity and acceleration along ⊥r to the rediction vector.

5m **Components the velocity and Acceleration vector along Radial and Transverse Direction :-**

As the particle moves P and ...

displacement along the radius vector = ON − OP

$$\cos \Delta\theta = \frac{ON}{r+\Delta r}$$



$$ON = \cos\Delta\theta\,(r+\Delta r)$$

$$OP = r$$

$$ON - OP = \cos\Delta\theta\,(r+\Delta r) - r \rightarrow (1)$$ and the radial component u of velocity is,

$$u = \lim_{\Delta t \to 0} \frac{ON-OP}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{(r+\Delta r)\cos\Delta\theta - r}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{r\cos\Delta\theta + \Delta r\cos\Delta\theta - r}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{r + \Delta r - r}{\Delta t} \qquad [\because \cos\Delta\theta = 1]$$

$$= \lim_{\Delta t \to 0} \frac{\Delta r}{\Delta t}$$

$$u = \frac{dr}{dt} = r'$$

$$u = r' \longrightarrow (2)$$

Similarly, the displacement ⊥r to the radius vector.

$$\sin\Delta\theta = \frac{\Delta S}{r+\Delta r}$$

$$\Delta S = (r+\Delta r)\sin\Delta\theta \longrightarrow (3)$$

and the transverse components of the velocity is given by,

$$V = \lim_{\Delta t \to 0} \frac{\Delta S}{\Delta t}$$

$$V = \lim_{\Delta t \to 0} \frac{(r+\Delta r)\sin\Delta\theta}{\Delta t}$$

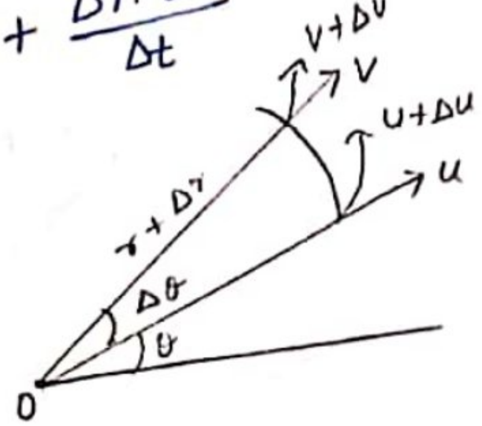
Scanned with CamScanner

$$= \lim_{\Delta t \to 0} \frac{(r+\Delta r)\sin\Delta\theta}{\Delta t \cdot \Delta\theta}$$

$$= \lim_{\Delta t \to 0} \frac{r+\Delta r \cdot \Delta\theta}{\Delta t} \cdot \frac{\sin\Delta\theta}{\Delta\theta}$$

$$= \lim_{\Delta t \to 0} \frac{(r+\Delta r)\cdot\Delta\theta}{\Delta t} \cdot \frac{\sin\Delta\theta}{\Delta\theta}$$

$$= \lim_{\Delta t \to 0} \frac{r\cdot\Delta\theta}{\Delta t} + \frac{\Delta r \cdot \Delta\theta}{\Delta t}$$



$$v = \lim_{\Delta t \to 0} r \cdot \frac{\Delta\theta}{\Delta t}$$

$$v = r \cdot \frac{d\theta}{dt}$$

$$\boxed{v = r\cdot\theta'} \rightarrow (4)$$

As such that [velocity components polar co-ordinates].

$$u = \frac{dr}{dt} = r' \quad , \quad v = \frac{d\theta}{dt} = r\cdot\theta' \rightarrow (5)$$

Now, the change in the velocity along the radius vector = velocity (of the radial direction) - velocity of P.

$$= (u+\Delta u)\cos\Delta\theta + (v+\Delta v)\cos(90^\circ+\Delta\theta) - u$$

$$= (u+\Delta u)\cos\Delta\theta - (v+\Delta v)\sin\Delta\theta - u$$

and the radial component of acceleration.

$$= \lim_{\Delta t \to 0} \frac{(u+\Delta u)\cos\Delta\theta - (v+\Delta v)\sin\Delta\theta - u}{\Delta t}$$

$$= \lim \quad u\cos\Delta\theta + \Delta u\cdot\cos\Delta\theta - v\sin\Delta\theta - \Delta v\sin\Delta\theta - u$$

Scanned with CamScanner

$$= \frac{du}{dt} - v.\frac{d\theta}{dt}$$ (neglecting $\Delta v . \Delta\theta$) ($\cos\Delta\theta = 1$, $\sin\Delta\theta = \Delta\theta$)

$$= \frac{d}{dt}(r') - (r\theta')\theta' \quad \text{(By ⑤)}$$

$$= r'' - r\theta'^2 \longrightarrow ⑥$$

Similarly, change in the velocity along the radius vector.

$\Rightarrow = (u + \Delta u)\cos(90° - \Delta\theta) + (v + \Delta v)\cos\Delta\theta - v$

$\Rightarrow = (u + \Delta u)\sin\Delta\theta + (v + \Delta v)\cos\Delta\theta - v$

and the transverse components acceleration.

$$= \lim_{\Delta t \to 0} \frac{(u+\Delta u)\sin\Delta\theta + (v+\Delta v)\cos\Delta\theta - v}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{u\sin\Delta\theta + \Delta u \sin\Delta\theta + v\cos\Delta\theta + \Delta v\cos\Delta\theta - v}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{u\sin\Delta\theta + \Delta u\sin\Delta\theta + v\cos\Delta\theta + \Delta v . \cos\Delta\theta - v}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{u.\Delta\theta + \Delta u.\Delta\theta + \cancel{v} + \Delta v - \cancel{v}}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{u.\Delta\theta + \Delta v}{\Delta t} = u.\frac{d\theta}{dt} + \frac{dv}{dt}$$

$$= r'\theta' + d/dt\,(r\theta') \qquad uv = uv' + vu'$$

$$= r'\theta' + r\theta'' + r'\theta'$$

$$= 2r'\theta' + r\theta'' \longrightarrow ⑦$$

Then the radial transverse component of acceleration are,

$$r'' - r\theta'^2 \quad (or) \quad \frac{1}{r}.\frac{d}{dt}(r^2\theta') \longrightarrow ⑧$$

Motion under a central force :-

let the force acting on a particle of unit mass m be $mF(x)$ and let it can be directed towards the origin. then the equ' of Motion are,

$$m(r'' - r(\theta')^2) = -mF(r) \rightarrow (1)$$

$$m\left[\frac{1}{r} \cdot \frac{d}{dt}(r^2\theta')\right] = 0 \rightarrow (2)$$

From (2) $\Rightarrow \frac{d}{dt}(r^2\theta') = 0$

$\int$ing we get

$$r^2\theta' = h \text{ (constant)} \rightarrow (3)$$

Then the equ' (1) gives

$$r'' - r\theta'^2 = -F(r) \rightarrow (4)$$

we can eliminate "t" b/w (3) & (4). to get a diff' equ' r & θ. we can it convenient to use $u = 1/r$ instead of r. So that making use of (3) we get.

$$\boxed{r = \frac{1}{u}} \rightarrow (5)$$

$$\frac{dr}{dt} = \frac{-1}{u^2} \cdot \frac{du}{dt}$$

$$= \frac{-1}{u^2} \cdot \frac{du}{d\theta} \cdot \frac{d\theta}{dt} \qquad \because \left(\frac{d\theta}{dt} = \frac{h}{r^2}\right)$$

$$= \frac{-1}{u^2} \cdot \frac{du}{d\theta} \cdot \theta'$$

$$= \frac{-1}{u^2} \cdot \frac{du}{d\theta} \cdot \frac{h}{r^2} \rightarrow \text{(using (3))}$$

$$= \frac{-1}{u^2} \cdot \frac{du}{d\theta} \cdot hu^2$$

$$\boxed{r' = -h \cdot \frac{du}{d\theta}} \rightarrow (6)$$

Scanned with CamScanner

$\frac{d}{dt}(\dot r)$

$$= \frac{d}{dt}\left(-h \cdot \frac{du}{d\theta}\right)$$

$$= \frac{d}{d\theta}\left(-h \cdot \frac{du}{d\theta}\right) \cdot \frac{d\theta}{dt} \quad (\text{using } 6)$$

$$= -h \cdot \frac{d^2u}{d\theta^2} \cdot \frac{h}{r^2} \qquad \frac{d\theta}{dt} = \theta' = \frac{h}{r^2}$$

$$\boxed{r'' = -h^2u^2 \cdot \frac{d^2u}{d\theta^2}} \rightarrow 7$$

$r = \frac{1}{u}$

(4) $\Rightarrow$ $$-Fr = -h^2u^2 \cdot \frac{d^2u}{d\theta^2} - \frac{1}{u} \cdot h^2 \cdot u^4 \qquad \frac{h}{r^2} = \frac{h}{(1/u^2)}$$

$$= -h^2u^2\left[\frac{d^2u}{d\theta^2} + \frac{1}{\cancel{u}} \cdot u^{\cancel{2}}\right] \qquad = (hu^2)^2 = h^2u^4$$

$$\cancel{-}Fr = \cancel{-}h^2u^2\left[\frac{d^2u}{d\theta^2} + u\right] \quad (\text{or})$$

$$\frac{d^2u}{d\theta^2} + u = \frac{F(r)}{h^2u^2}.$$

where, F can be easily expressed as a function of u. This is the diff' equ' of 2nd order. whose integration will give the relation b/w u & θ (or) relation b/w r and θ.

The equ' of the both described by a particle to moving under a central force F per unit mass.

Motion under the inverse square Law :-

If the central force per unit mass $\frac{\mu}{r^2}$ (or) μu^2 equation gives,

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2} \rightarrow 1$$

equ 1 $\Rightarrow$ $$\frac{d^2u}{d\theta^2} + u = \frac{\mu \cancel{u^2}}{h^2\cancel{u^2}}$$

Scanned with CamScanner

$$\frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2} \longrightarrow (2)$$

$\int$ ing this linear equ' with constant co-efficient. we get

$$(m^2+1)u = 0$$
$$m^2+1 = 0$$
$$m^2 = -1$$
$$m = \pm 1$$

$\alpha = 0$ assume that,

$$A = A\cos\alpha + B\sin\alpha$$
$$C.F = (A\cos\alpha.\cos\theta + A\sin\alpha.\sin\theta)$$
$$= A(\cos\alpha\cos\theta + \sin\alpha\sin\theta)$$
$$= A[\cos(\alpha-\theta)]$$
$$C.F = A[\cos(\theta-\alpha)]$$
$$P.I = \frac{1}{D^2+1}\cdot\frac{\mu}{h^2}e^0 = \frac{\mu}{h^2}$$
$$u = C.F + P.I$$
$$u = A\cos(\theta-\alpha) + \mu/h^2$$
$$\frac{1}{r} = A\cos(\theta-\alpha) + \frac{\mu}{h^2}$$

$\frac{\mu}{h^2}$ Multiple and divided.

$$\frac{1}{r} = \frac{\mu}{h^2}\left[\frac{Ah^2}{\mu}\cdot\cos(\theta-\alpha)+1\right]$$
$$\frac{h^2/\mu}{r} = \frac{Ah^2}{\mu}.\cos(\theta-\alpha)+1 \longrightarrow (*)$$
$$\left[L = \frac{h^2}{\mu} \;\&\; e = \frac{Ah^2}{\mu},\; h^2 = L\mu\right] \rightarrow (4)$$

sub (4) in (*) we get

$$\frac{L}{r} = e\cos(\theta-\alpha)+1$$

$$\frac{L}{r} = 1+e\cos(\theta-\alpha)$$

$$Lu = 1+e\cos(\theta-\alpha)$$

$$u = \frac{1+e\cos(\theta-\alpha)}{L} \rightarrow (5)$$

$$\frac{du}{d\theta} = \frac{-e\sin(\theta-\alpha)}{L} \rightarrow (6)$$

which represents a conic with a focus of the centre of force. If a particle moves under a force μ/r^2 per unit mass. The path is conic section with a focus at the centre.

The centre can be ellipse, parabola (or) hyperbola according as $e \leq 1$ (or) $e \geq 1$.

Now, the velocity "v" at a particle is given by, $v^2 = r'^2 + r^2\theta'^2$.

$$v^2 = \left(\frac{dr}{dt}\right)^2 + \frac{1}{u^2}\cdot\theta'^2$$

$$= \left(\frac{dr}{dt}\right)^2 + \frac{1}{u^2}(hu^2)^2$$

$$= \left(\frac{dr}{du}\cdot\frac{du}{d\theta}\cdot\frac{d\theta}{dt}\right)^2 + \frac{1}{u^2}(h^2u^4)$$

$\therefore h^2 = \mu L$

$$= \left(\frac{du}{d\theta}\right)^2\left[\frac{-1}{u^2}\cdot hu^2\right] + h^2u^2$$

$$= h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right]$$

$\because \left[r = \frac{1}{u}\cdot\frac{dr}{du} = \frac{-1}{u^2}\right]$

$$= h^2\left[\left(\frac{e^2\sin^2(\theta-\alpha)}{\ldots}\right) + \left(\frac{1+e\cos(\theta-\alpha)}{\ldots}\right)^2\right]$$

$$v^2 = h^2 \left[\frac{e^2 \sin^2(\theta-\alpha) + 1 + e^2\cos^2(\theta-\alpha) + 2e\cos(\theta-\alpha)}{L^2} \right]$$

$$= \frac{h^2}{L^2}\left[e^2(\sin^2(\theta-\alpha) + \cos^2(\theta-\alpha)) + 1 + 2e\cos(\theta-\alpha)\right]$$

$\therefore (h^2 = \mu L)$

$$= \frac{\cancel{L}\mu}{L^{\cancel{2}}}\left[e^2(1) + 1 + 2e\cos(\theta-\alpha)\right]$$ 10

Add and subtract "1"

$$= \frac{\mu}{L}\left[e^2 + 1 - 1 + 1 + 2e\cos(\theta-\alpha)\right]$$

$$= \frac{\mu}{L}\left[(e^2-1) + 2 + 2e\cos(\theta-\alpha)\right]$$

$$= \frac{\mu}{L}\left[(e^2-1) + 2(1+e\cos(\theta-\alpha))\right]$$

$$= \frac{\mu}{L}\left[(e^2-1) + 2Lu\right]$$

$$= \frac{\mu}{L}\left[(e^2-1) + \frac{\mu}{\cancel{L}}(2\cancel{L}u)\right]$$

$$= \frac{\mu}{L}(e^2-1) + 2\mu u \longrightarrow (7)$$

If the path is an ellipse,

$L = a(1-e^2)$ [from (7)]

$$v^2 = \frac{\mu(e^2-1)}{a(1-e^2)} + 2\frac{\mu}{r}$$

$$= \frac{-\mu(1-e^2)}{a(1-e^2)} + \frac{2\mu}{r}$$

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a}$$

$$v^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right] \longrightarrow (8)$$

If the path is hyperbola.

$L = a(e^2-1)$

Scanned with CamScanner

$\cos(\theta-\gamma)$

$1+2e\cos(\theta-\alpha)$

10

11

From ⑧ $\rightarrow v^2 = \frac{\mu(e^2-1)}{a(e^2-1)} + \frac{2\mu}{r}$

$$v^2 = \frac{\mu}{a} + \frac{2\mu}{r}$$

$$v^2 = \mu\left[\frac{1}{a} + \frac{2}{r}\right] \rightarrow (9)$$

If the path is parabola.

$$e = 1 \quad (\text{From } (7))$$

$$v^2 = \frac{\mu}{L}(0) + \frac{2\mu}{r}$$

$$= 0 + \frac{2\mu}{r}$$

$$v^2 = \mu(2/r) \rightarrow (10)$$

Thus, if the particle is projected with velocity v from a point at a distance are from the centre of force. The path will be hyperbola, parabola (or) ellipse according as,

$$v^2 = \frac{2\mu}{r}$$

$$v^2 = \frac{-2\mu}{r}.0 \longrightarrow (11)$$

we have proved that if the central force is μ/r^2 per unit mass, the path is conic section with the centre of force at 1 focus.

conversely, If we know that the path is a conic section.

$$\frac{1}{r} = Lu = 1 + e\cos(\theta-\alpha) \rightarrow (12)$$

with a focus at the centre of force, then the force per unit mass is given by,

$$\frac{d^2u}{d\theta^2} + u = \frac{F}{h^2u^2}$$

$$F = h^2u^2\left(\frac{d^2u}{d\theta^2} + u\right)$$

$$= h^2u^2\left[\frac{-e\cos(\theta-\alpha)}{L} + \frac{1+e\cos(\theta-\alpha)}{L}\right]$$

$$= \frac{h^2u^2}{L}\left[\cancel{-e\cos(\theta-\alpha)} + 1 + \cancel{e\cos(\theta-\alpha)}\right]$$

$$F = \frac{h^2u^2}{L}(1) \Rightarrow F = \frac{h^2u^2}{L}.$$

$$F = \frac{\mu L}{L}\cdot\frac{1}{r^2}$$

$$\boxed{F = \frac{\mu}{r^2}} \rightarrow (13)$$

So, that the centre follows the inverse square Law. since all planets are observed to move is elliptic orbits with the sun at one focus it follows that the law of attraction b/w different planets and must be the inverse square Law.

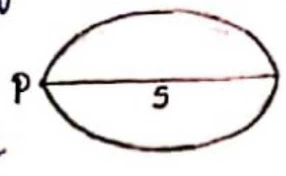
Kepler's Law of planetary Motion :-

Kepler deduced three laws of motion as following,

i) Every planet describes an ellipse with a sun at one focus.

ii) The radius vector from the sun to a planet describes equal areas in equal intervals

iii) The squares of periodic time of planets are proportional to the cubes of the semi major areas the orbits of the planets.



We can deduced all these three laws from the Mathematical of planetary motion discussed above. when the law of attraction is the inverse square Law.

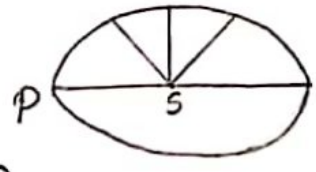
First Law :-

We have already seen that the inverse square law, the path has to be a conic section and this includes elliptic orbit.

Second Law :-

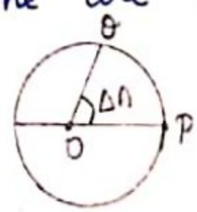
prit-2011 Since, $r^2\theta' = h$ (say)

Motion of centra



$$\lim_{\Delta t \to 0} \frac{1}{2} \cdot r^2 \frac{\Delta\theta}{\Delta t} = \frac{1}{2} h \rightarrow (1)$$

From the figure the area ΔA bounded by radius vectors OP and $O\theta$ an the arc $P\theta$ is $\frac{1}{2} r^2 \cdot \sin \Delta\theta$.



So that eqn (1) gives,

$$\Delta A = P\theta = \frac{1}{2} \cdot r^2 \sin \Delta\theta \quad [\because \Delta A = \Delta\theta]$$

$$\lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{2} \cdot r^2 \cdot \frac{\sin \Delta\theta}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \lim_{\Delta t \to 0} \frac{1}{2} \cdot r^2 \cdot \frac{\Delta\theta}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{1}{2} \cdot h \Rightarrow \boxed{\frac{d\theta}{dt} = \frac{1}{2} \cdot h}$$

and the ratio of description of sectional area is constant and equal area are describes in equal intervals of time. This is Kepler's second law. 14

Third Law :-

2m The total area of the ellipse is πab and since the area velocity is $\frac{1}{2}h$. The periodic time "T" is given by,

$$T = \frac{\pi ab}{\frac{1}{2}\cdot h} = \frac{2\pi ab}{h} = \frac{2\pi ab}{\sqrt{\mu l^2}}$$

$$= \frac{2\pi ab}{\sqrt{\mu}\cdot\sqrt{b^2}/a} = \frac{2\pi ab}{\sqrt{\mu}\cdot\frac{\sqrt{b^2}}{\sqrt{a}}}$$

$$T = \frac{2\pi a\cancel{b}\sqrt{a}}{\sqrt{\mu}\cdot\cancel{b}} = \frac{2\pi\cdot a^{3/2}}{\sqrt{\mu}}$$

$= a\cdot\sqrt{a}$
$= a^1\cdot a^{1/2}$
$= a^{3/2}$

$$\boxed{T = \frac{2\pi\cdot a^{3/2}}{\sqrt{\mu}}} \rightarrow (3)$$

For the different planets of masses P_1, P_2 and semi area of orbit S. a_1, a_2 this given,

$$\frac{T_1}{T_2} = \frac{2\pi a_1^{3/2}/\sqrt{\mu_1}}{2\pi a_2^{3/2}/\sqrt{\mu_2}} \rightarrow (4)$$

$$\frac{T_1}{T_2} = \frac{2\pi a_1^{3/2}}{\sqrt{\mu_1}}\cdot\frac{\sqrt{\mu_2}}{2\pi a_2^{3/2}}$$

$$\frac{T_1}{T_2} = \frac{\sqrt{\mu_2}}{\sqrt{\mu_1}}\cdot\frac{a_1^{3/2}}{a_2^{3/2}}$$

Scanned with CamScanner

15

$$\frac{T_1}{T_2} = \frac{\sqrt{G(S+P_2)}}{\sqrt{G(S+P_1)}} \cdot \frac{a_1^{3/2}}{a_2^{3/2}} \Rightarrow \frac{\sqrt{GS(1+P_2)}}{\sqrt{\cancel{GS}(1+P_1)}} \cdot \frac{a_1^{3/2}}{a_2^{3/2}}$$

Multiply and divided on S.

$$\frac{T_1}{T_2} = \frac{\sqrt{S(1+P_2/S)}}{\sqrt{S(1+P_1/S)}} \cdot \frac{a_1^{3/2}}{a_2^{3/2}}$$

squaring on both sides,

$$\frac{T_1^2}{T_2^2} = \frac{\cancel{S}(1+P_2/S)}{\cancel{S}(1+P_1/S)} \cdot \frac{a_1^3}{a_2^3} \rightarrow (5)$$

$$\frac{T_1^2}{T_2^2} = \frac{(1+\cancel{P_2/S})}{(\cancel{1+P_1/S})} \cdot \frac{a_1^3}{a_2^3}$$

$$\therefore \boxed{\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}} \rightarrow (6)$$

Since, P_1 & P_2 are very small compared with S. This gives as a very good approximation

$$\boxed{\frac{T_1^2}{T_2^2} = \frac{a_1^3}{a_2^3}} \rightarrow (7)$$

which is the Kepler's law of planetary motion. Hence the Kepler's law is proved.

Mathematical Modelling of circular motion and Motion of satellites :.

Circular Motion :-

When a particle moves in a circle of a radius "a". So that $r = a$, the radial component of velocity $r' = 0$.

The transverse component of acceleration,

$$\boxed{r\theta' = a\theta'} \rightarrow (1)$$

The radial component of acceleration,

$$\frac{1}{r}\cdot\frac{d}{dt}(r^2\theta') = \frac{1}{a}\cdot\frac{d}{dt}(a^2\theta')$$

$$= \frac{a^2}{a}\cdot\frac{d}{dt}(\theta')$$

$$= a\theta'' \longrightarrow (2)$$

Thus the velocity is $a\theta''$ along the tangen & the acceleration of two component $a\theta''$ along the tangent and $a\theta''$ along the normal. If a particles moves in a circle of radius "a".

Its equation of Motion are,

$ma\theta''$ = External force in the direction of the tangent.

$ma\theta'^2$ = External force in the direction of the inward normal. $\rightarrow (3)$

Thus, it "a" particle is attached to one, end of a string.

The other end of which is fixed and the particle moves in a vertical circle.

The equ' of motion are,

$$ma\theta'' = -mg\sin\theta \rightarrow (4)$$

$$ma\theta'^2 = T - mg\cos\theta \rightarrow (5)$$

If θ is small equ (4) we get

$$ma\theta'' = -mg\sin\theta$$

$$a\theta'' = -g\sin\theta$$

$$a\theta'' = -g\theta \qquad (\therefore \sin\theta = \theta)$$

$$\boxed{\theta'' = \frac{-g\theta}{a}} \qquad (\theta \text{ is small})$$

which is the equ' for a SHM. Thus for small oscillation of Simple pendulum.

Scanned with CamScanner

The time period is,

$$\boxed{T = 2\pi\sqrt{a/g}} \rightarrow (7)$$

17 If θ is not necessary small.

Using eqn (4), we get

$$\cancel{m}a\theta'' = -\cancel{m}g \sin\theta$$

$$\frac{a d^2\theta}{dt^2} = -g \sin\theta$$

Both sides are Multiple by $2.\frac{d\theta}{dt}$.

$$2.\frac{d\theta}{dt} . a\frac{d^2\theta}{dt^2} = 2.\frac{d\theta}{dt}(-g\sin\theta)$$

$$\frac{d}{dt}(a\theta'^2) = -2g\sin\theta . \frac{d\theta}{dt}$$

$$\int \frac{d}{dt}(a\theta'^2) = -2g\int \sin\theta . \frac{d\theta}{dt}$$

$$a\theta'^2 = -2g(-\cos\theta) + c$$

$$\boxed{a\theta'^2 = 2g\cos\theta + c} \rightarrow 8 \quad (c = \text{constant}$$

If a particle is projected from the lowest point with velocity "u". Then,

$$a\theta' = u.$$

when $\theta = 0$. So that

$$a\theta' = u$$

$$(a\theta')^2 = u^2$$

$$a^2\theta'^2 = u^2$$

$$a\theta'^2 = \frac{u^2}{a}$$

$$2g\cos\theta + c = \frac{u^2}{a}$$

$$c = \frac{u^2}{a} - 2g\cos\theta \quad (\text{when } \theta = 0)$$

Scanned with CamScanner

sub ⑨ in ⑧ we get

$$a\theta'^2 = 2g\cos\theta + \frac{u^2}{a} - 2g$$

$$a\theta'^2 = \frac{u^2}{a} - 2g + 2g\cos\theta$$

$$a\theta'^2 = \frac{u^2}{a} - 2g(-\cos\theta + 1)$$

$$a\theta'^2 = \frac{u^2}{a} - 2g(1-\cos\theta) \rightarrow (10)$$

when, v is the velocity of the particle.

so that, $\boxed{a\theta'^2 = \frac{v^2}{a}}$

$$(10) \Rightarrow \frac{v^2}{a} = \frac{u^2}{a} - 2g(1-\cos\theta)$$

$$\frac{v^2}{a} = \frac{u^2 - 2ga(1-\cos\theta)}{a}$$

$$v^2 = u^2 - 2ga(1-\cos\theta)$$

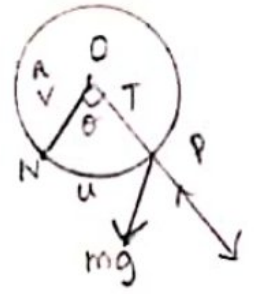
(or)

Multiple on both sides by $\frac{1}{2}m$.

$$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 - \frac{2ga(1-\cos\theta)}{2}.m$$

From Figure,

$$\cos\theta = \frac{ON}{a}$$

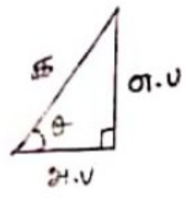


$$ON = a.\cos\theta$$

$$h = a - ON$$

$$= a - a\cos\theta$$

$$h = a(1-\cos\theta)$$



$$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 - \frac{\cancel{2}mgh}{\cancel{2}}$$

$$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 - mgh \rightarrow (12)$$

travelled.

Scanned with CamScanner

directly from the principle of conservation of energy equ ⑤ gives.

19

$$T = ma\theta'^2 + mg\cos\theta \qquad a\theta'^2 = \frac{v^2}{a}$$

$$= \frac{mv^2}{a} + mg\cos\theta$$

$$= \left[m\left(\frac{u^2}{a} - 2g(1-\cos\theta)\right) + mg\cos\theta\right] \text{ (by ⑪)}$$

$$= m\left[\frac{u^2}{a} - 2g + 2g\cos\theta\right] + mg\cos\theta$$

$$= \frac{mu^2}{a} - 2mg + 2mg\cos\theta + mg\cos\theta$$

$$= \frac{mu^2}{a} - 2mg + 3mg\cos\theta$$

At the highest point $\theta = \pi$ and

$$T = \frac{mu^2}{a} - 2mg + 3mg\cos\pi$$

$$(\because \cos\pi = -1)$$

$$T = \frac{mu^2}{a} - 2mg - 3mg$$

$$T = \frac{mu^2}{a} - 5mg.$$

If $u^2 \geq 5mg$ the particle will have move in the complete verticle circle again and again.

However, if $u^2 \leq 5mg$ tension will vanish before the particle reaches the highest when the tension the particle begins to move freely under the gravity & describes a parabolic path, all the string again becomes tight and the circular motion is started again.

Motion of a particle on a smooth (or) Rough verticle wire :-

a) If the particle moves on the inside of a smooth wire, the equ of motion are,

$$ma\theta'' = -mg\sin\theta \longrightarrow (1)$$ 20

$$ma\theta'^2 = R - mg\cos\theta \longrightarrow (2)$$

If θ is small, equ (1) gives,

$$\cancel{m}a\theta'' = -\cancel{m}g\sin\theta$$

$$a\theta'' = -g\sin\theta \qquad (\theta \text{ is small, } \sin\theta = \theta)$$

$$\boxed{\theta'' = \frac{-g\theta}{a}} \longrightarrow (3)$$

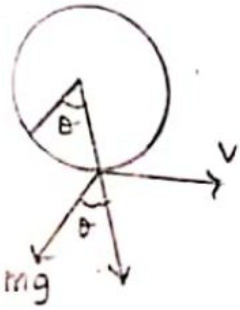
which is the equ of the SHM. Thus for small oscillation for single pendulum the normal Motion.

$$\boxed{R = 2\pi\sqrt{a/g}} \longrightarrow (4)$$

If θ is not necessarly small.

sing in equ (1), we get

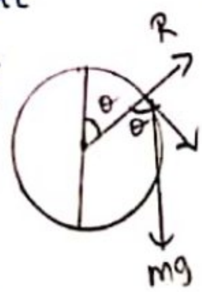
$$\cancel{m}a\theta'' = -\cancel{m}g\sin\theta$$



$$a\frac{d^2\theta}{dt^2} = -g\sin\theta$$

Both sides are Multiply by $2\cdot\frac{d\theta}{dt}$

$$2a\cdot\frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2} = -2g\sin\theta\cdot\frac{d\theta}{dt}$$



$$\frac{d}{dt}(a\theta'^2) = -2g\sin\theta\cdot\frac{d\theta}{dt}$$

$$a\int d(\theta'^2) = -2g\int\sin\theta\cdot d\theta$$

$$a\theta'^2 = 2g\cos\theta + c \to (5)$$

If the particle is projected from the lowest point with velocity "u" Then,

$a\theta' = u$ (when $\theta = 0$)

$(a\theta')^2 = u^2$

21 $a^2\theta'^2 = u^2$

$\boxed{a\theta'^2 = u^2/a}$

(5) $\Rightarrow$ $c = \frac{u^2}{a} - 2g\cos\theta$ (when $\theta = 0$)

$c = \frac{u^2}{a} - 2g \rightarrow$ (6) $\quad \therefore \cos 0 = 1$

Sub eqn (6) in eqn (5) we get

$a\theta'^2 = 2g\cos\theta + \frac{u^2}{a} - 2g$

$a\theta'^2 = \frac{u^2}{a} + 2g(-1+\cos\theta)$

$a\theta'^2 = \frac{u^2}{a} - 2g(1-\cos\theta) \rightarrow$ (7)

when, v is the velocity of the particle,

$a\theta'^2 = \frac{v^2}{a}$

(7) $\Rightarrow$ $\frac{v^2}{a} = \frac{u^2}{a} - 2g(1-\cos\theta)$

$\frac{v^2}{\cancel{a}} = \frac{u^2 - 2ga(1-\cos\theta)}{\cancel{a}}$

$v^2 = u^2 - 2ga(1-\cos\theta) \rightarrow$ (8)

Multiply both sides by $\frac{1}{2}M$.

$\frac{1}{2}Mv^2 = \frac{1}{2}Mu^2 - \underline{2ga(1-\cos\theta)}$

Scanned with CamScanner

$$h = a - ON$$
$$= a - a\cos\theta$$

$$h = a(1-\cos\theta)$$

$$\frac{1}{2}Mv^2 = \frac{1}{2}mu^2 - \frac{\cancel{2}ghm}{\cancel{2}}$$

$$\frac{1}{2}mv^2 = \frac{1}{2}mu^2 - mgh \longrightarrow (9)$$

where 'h" is the verticle distance travelled by the particle equ (9) can be obtained directly from the principle of conservation of energy equ (8). Then gives,

$$ma\theta'^2 = R - mg\cos\theta$$
$$R = ma\theta'^2 + mg\cos\theta$$
$$= \frac{v^2}{a}.m + mg\cos\theta$$
$$= \frac{mv^2}{a} + mg\cos\theta$$
$$= m\left[\frac{u^2}{a} 2g(1-\cos\theta)\right] + mg\cos\theta$$
$$= \frac{mu^2}{a} - 2mg(1-\cos\theta) + mg\cos\theta$$
$$= \frac{mu^2}{a} - 2mg + 2mg\cos\theta + mg\cos\theta$$
$$R = \frac{mu^2}{a} - 2mg + 3mg\cos\theta \rightarrow (10)$$

At the highest point $\theta = \pi$

$$R = \frac{mu^2}{a} - 2mg + 3mg\underbrace{\cos\pi}_{(1)}$$
$$= \frac{mu^2}{a} - 2mg - 3mg$$
$$R = \frac{mu^2}{a} - 5mg$$

As such if $u^2 \geq 5mg$ the particle makes a identity number of complete rounds of the circular wire.

23 If $u^2 \leq 5mg$ the reaction vanish before the particle reaches the highest point, the particle leaves circular wire again and it again describes a circular path motion is repeated again and again.

b) If the particle moves on the outside of the verticle wire. The equ of motion are,

$$ma\theta''^2 = mg\sin\theta \rightarrow (1)$$

$$ma\theta'^2 = -R + mg\cos\theta \rightarrow (2)$$

$$(1) \Rightarrow \cancel{m}a\theta'' = \cancel{m}g\sin\theta$$

$$a\theta'' = g\sin\theta$$

$$a\frac{d^2\theta}{dt^2} = g\sin\theta$$

Multiple both sides by $2 \cdot \frac{d\theta}{dt}$.

$$2a\frac{d\theta}{dt} \cdot \frac{d^2\theta}{dt^2} = 2g\sin\theta \cdot \frac{d\theta}{dt}$$

$$\frac{d}{dt}(a\theta'^2) = 2g\sin\theta \cdot \frac{d\theta}{dt}$$

$$\int d(a\theta'^2) = 2g\int \sin\theta \cdot d\theta$$

$$a\theta'^2 = 2g(-\cos\theta) + c \qquad (c = \text{constant})$$

$$a\theta'^2 = -2g\cos\theta + c \rightarrow (3)$$

If the path is projected from the lower point with velocity u.

$$a\theta' = u \qquad (\because \theta = 0)$$

$$(a\theta')^2 = u^2$$

Scanned with CamScanner

$$a^2 \cdot \theta'^2 = u^2$$

$$a\theta'^2 = \frac{u^2}{a}$$

③ $\Rightarrow$ $\frac{u^2}{a} = -2g\cos\theta + c$

$$c = \frac{u^2}{a} + 2g\cos\theta \quad (\because \theta = 0)$$

$$\boxed{c = \frac{u^2}{a} + 2g} \rightarrow ④ \quad (\because \cos 0 = 1)$$

sub equ ④ in equ ③ we get

$$a\theta'^2 = -2g\cos\theta + \frac{u^2}{a} + 2g$$

$$a\theta'^2 = \frac{u^2}{a} + 2g(1-\cos\theta) \rightarrow ⑤$$

$$a\theta'^2 = \frac{u^2}{a} + \frac{2ga(1-\cos\theta)}{a}$$

$$a\theta'^2 = \frac{u^2 + 2ga(1-\cos\theta)}{a} \rightarrow ⑥$$

Equ ② then gives,

$$-R = ma\theta'^2 - mg\cos\theta$$

$$R = mg\cos\theta - m\left[\frac{u^2}{a} + 2g(1-\cos\theta)\right]$$

(by ⑤)

$$= mg\cos\theta - \frac{mu^2}{a} - 2mg + 2mg\cos\theta$$

$$= 3mg\cos\theta - \frac{mu^2}{a} - 2mg \rightarrow ⑦$$

At the highest point $\theta = 0$.

⑦ $\Rightarrow$ $R = 3mg\cos\theta - \frac{mu^2}{a} - 2mg$ $(\cos 0 = 1)$

$$= 3mg - \frac{mu^2}{a} - 2mg$$

$$= mg - \frac{mu^2}{a} \rightarrow ⑧$$

25

At the point A, $\theta = \pi/2$

(7) $\Rightarrow R = 3mg\cos\pi/2 - mu^2 - 2mg$ $(\cos\pi/2 = 0)$.

(7) $= R = -mu^2 - 2mg \rightarrow$ (9)

If $u^2 > mg$, the particle leaves constant with the wire immediately and describes a parabolic path.

If $u^2 < mg$, the particle remains in constan for some distance but leaves constant, when, R vanishes.

(ie) Before it reaches A and then it describes a parabolic path.

c) If the particle moves on the inside of rough vertical circular wire, then there is additional frictional for μR along the tangent oppositing the motion.

The equation of Motion are,

$ma\theta'' = -mg\sin\theta - \mu R \rightarrow$ (1)

$ma\theta'^2 = -mg\cos\theta + R \rightarrow$ (2)

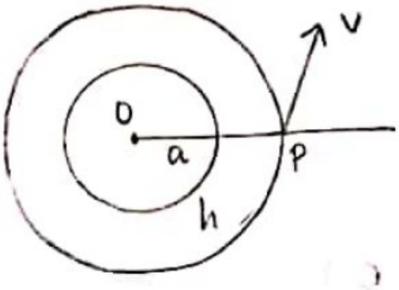
We can again eliminate R, R solve for, "θ'" and "$\ddot{\theta}$" and find the value of "θ" when R vanishes.

Circular Motion of satellites:-

Just as planets moves in elliptic orbit with the sun in one focus. The non-made artifical satellites move elliptic orbits with the earth at one focus.

If the earth is of mass m & radius "a" and a satellites of mass M (< M) is projected from a point "p" at a height "h" above the earth with velocity v at right angles to "op" it will move under a central force $\frac{GmM}{r^2}$.

Since the central force of a circular orbit is $\frac{mv^2}{r}$. we get the path is to be circular.



$$\frac{mv^2}{r} = \frac{GmM}{r^2}$$

Take $r = a+h$

$$\frac{\cancel{m}v^2}{\cancel{a+h}} = \frac{G\cancel{m}M}{(a+h)^{\cancel{2}}}$$

$$\boxed{v^2 = \frac{GM}{a+h}} \rightarrow (1)$$

If g is the acceleration due to gravity then the gravitational force on a particle of mass m on the surface of the earth is mg.

Alternating from Newton's inverse square law, it is $\frac{GmM}{a^2}$.

So that, $$\frac{G\cancel{m}M}{a^2} = g\cancel{m}$$

$$\boxed{GM = ga^2} \rightarrow (2)$$

Sub (2) in (1) we get

$$\boxed{v^2 = ga^2/a+h} \longrightarrow (3)$$

$$v = \frac{\sqrt{a^2 g}}{\sqrt{a+h}} \text{ //}$$

$$\boxed{v = \frac{a\sqrt{g}}{\sqrt{a+h}}} \rightarrow (4)$$

This gives a velocity of satellite describing a circular orbit to a height "h" above the surface of the earth. Its time period is,

$$T = \frac{2\pi(a+h)}{v} = \frac{2\pi(a+h)}{a\sqrt{g}/\sqrt{a+h}} \quad (\because \text{by } (4))$$

$$= \frac{2\pi(a+h)(a+h)^{1/2}}{a\sqrt{g}}$$

$$T = \frac{2\pi(a+h)^{3/2}}{a\sqrt{g}} \longrightarrow (5)$$

The earth complete one revolution about its axis is twenty four hours as such is T is 24 hrs

The satellite would have the same period as the earth and would appear stationary to an observe on the earth.

Now taking,

$$g = 32\ ft/sec^2, \quad a = 4000 \text{ miles}$$

$$T = 24 \text{ hrs}$$

$$[(4000+h)\times 5280]^{3/2} = \frac{24\times 60\times 60\times\sqrt{32}\times 4000\times 1760\times 3\times 7}{2\times 22}$$

If h is measured in miles

$$(5) \Rightarrow (a+h)^{3/2} = \frac{Ta\sqrt{g}}{2\pi}$$

[1 yd = 3ft
1 mile = 1760 yd]

$$(4000+h)\times(1760\times 3)^{3/2} = \frac{(24\times 60\times 60)\sqrt{32}\,(4000\times 1760\times 3)}{2\times\frac{22}{7}}$$

$$= 24\times 60\times 60\times\sqrt{32}\times 4000\times 1760\times 3\times 7$$

Scanned with CamScanner

$= 1642207416 \times 10^{0}$ $= 1642207416 \times 10^{6}$

$(4000 \times h) \times 5280 = 13919 \cdot 3409 \times 10^{4}$ $= 1642207.416 \times 10$

$4000 \times h = 26.36 \times 38788 \times 10^{3}$ $= (1642207.416)$

$= \sqrt{2.696845197 \times 10}$

$= 26362.38788$ $= \frac{139193409.4}{5}$

26 $h = 22362.38788$ miles

This gives the height of the synchronous (or) synchron satellite which is very usefull for communication purpose.

$4000 + h = 26362.3882$

$h = 26362.3882 - 4000$

5m ## Elliptic Motion of Satellites :- $h = 22362.3882$

PRTI - 2018

If a satellite is projected at a height $\boxed{a+h}$ above centre of the earth with a velocity different $\boxed{\frac{a\sqrt{g}}{\sqrt{a+h}}}$ (or) if it is not projected at right angles to the radius vector, the orbit will not be circular but can be elliptic parabolic (or) hyperbolic depending on $\underset{\sim}{v}$ and the angle of projection.

If the angle of projection is $\underset{\sim}{90^\circ}$ and the orbit is an elliptic with $\boxed{\text{semi major axis}}$

Scanned with CamScanner

29

$r = a'(1-e)$ and $a = a'$

$\therefore v^2 = \mu\left(\frac{2}{a'(1+e)} - \frac{1}{a'}\right)$; $a'(1+e) = a+h$ → ①

(cross multiply)

$v^2 = \mu\left(\frac{2-(1+e)}{a'(1+e)}\right)$

$v^2 = \mu\left(\frac{2}{a'(1-e)} - \frac{1}{a'}\right)$; $a'(1-e) = a+h$ → ②

$v^2 = \mu\left(\frac{1-e}{a'(1+e)}\right)$

$= \frac{a^2g(1-e)}{a+h}$

$v^2 = \frac{\mu}{a'(1+e)} \cdot (1-e)$ $\therefore a'(1+e) = a+h$

$\therefore (\mu = a^2 g)$

$v^2 = \frac{\mu(1+e)}{a'(1-e)}$

$v^2 = \frac{a^2 g}{a+h} \cdot (1-e)$ $\therefore \left(v_0^2 = \frac{a^2 g}{a+h}\right)$

$= \frac{a^2g(1+e)}{a+h}$

$\boxed{v^2 = v_0^2 (1-e)}$ → ③

② $\Rightarrow$ $v^2 = \frac{\mu(2-(1-e))}{a'(1-e)}$

$\mu\left(\frac{2a' - a'(1+e)}{a'(1+e)\,a'}\right)$

$= \frac{\mu(2-1+e)}{a'(1-e)}$

$\mu\left(\frac{a'(2-1-e)}{(a')^2(1+e)}\right)$

$= \frac{\mu(1+e)}{a'(1-e)}$ $(\because \mu = a^2 g)$

$\mu \frac{1-e}{a'(1+e)}$

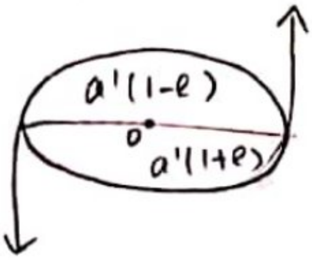
$\mu \frac{1-e}{a+h}$

$= \frac{a^2 g}{a+h} \cdot (1+e)$

Scanned with CamScanner

$$T = \frac{2\pi a^{3/2}}{\sqrt{\mu}} = \frac{2\pi a^{3/2}}{\sqrt{a^2 g}} \quad (\because \mu = a^2 g)$$

$$= \frac{2\pi a^{3/2}}{a\sqrt{g}}$$



30

where, $v < v_0$

③ $\Rightarrow 1 - e = \frac{v^2}{v_0^2}$

$\left[\because b^2 = a^2(1-e^2) \Rightarrow \frac{b^2}{a^2} = (1-e^2)\right]$

$$e^2 = 1 - \frac{b^2}{a^2}$$

$$e = \sqrt{1 - \frac{b^2}{a^2}} \quad \Rightarrow 1 - e = \frac{v^2}{v_0^2}$$

$$e = \sqrt{1 - \frac{v^2}{v_0^2}}$$

$$e = \sqrt{\frac{v_0^2 - v^2}{v_0^2}} \quad //$$

① $\Rightarrow a'(1+e) = a + h$

$$a' = \frac{a+h}{1+e}$$

$$a' = \frac{a+h}{1 + \sqrt{\frac{v_0^2 - v^2}{v_0^2}}} \quad // \rightarrow$$

and if $v > v_0$.

④ $\Rightarrow 1 + e = \frac{v^2}{v_0^2}$ $\quad e^2 = \frac{v^2}{v_0^2} - 1$

$$e = \sqrt{\frac{v^2}{v_0^2} - 1}$$

$$e = \sqrt{\frac{v^2 - v_0^2}{v_0^2}}$$

Scanned with CamScanner

(2) $\Rightarrow a'(1-e) = a+h$

$$a' = \frac{a+h}{1-e}$$

31

$$= \frac{a+h}{1-\sqrt{\frac{V^2-V_0^2}{V_0^2}}} \longrightarrow (5)$$

If 5 max and h min are the maximum and minimum height of a satellite above the earth's surface and "a" is the radius of the earth.

$a'(1+e) = a+h$ max [Further points of the orbit it to the centre of the earth].

$a'(1-e) = a+h$ min [Nearest point of the orbit it to the centre of the earth]

we get,

$$\frac{\cancel{a'}(1+e)}{\cancel{a'}(1-e)} = \frac{a+h\,max}{a+h\,min}$$

$$\frac{1+e}{1-e} = \frac{a+h\,max}{a+h\,min}$$

$$\left.\begin{aligned}\frac{1+e}{a+h\,max} &= \frac{1-e}{a+h\,min}\\ &= \frac{2}{2a+h\,max+h\,min}\end{aligned}\right\} \longrightarrow (A)$$

$1+e \quad -(1-e), \quad 2e \quad (B)$

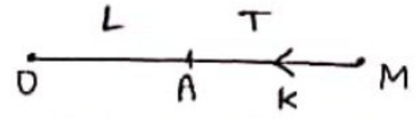
Equating (A) & (B) $\qquad \frac{1+e-1+e}{a+h_{max}-a+h_{min}}$

$$\frac{2e}{h_{max}-h_{min}} = \frac{2}{2a+h_{max}+h_{min}}$$

$$e = \frac{h_{max}-h_{min}}{2a+h_{max}+h_{min}}.$$

Mathematical Modelling through Linear differential equation of 2^{nd} order :-

Rectilinear Motion :-



let one end "O" of an elastic string of natural length $L = (OA)$ be fixed and let the other end to which a particle of mass "m" is attached to be stoched a distance "a" and then released.

At time "t" let $x(t)$ be the extension then the equ' of motion of the particle, (i.e)

$$m\frac{d^2x}{dt^2} = \frac{-\lambda x}{L} = -kx \rightarrow (1)$$

where k is the elastic constant, if the particle makes in a resisting medium with resistance proportional to the velocity equ (1) becomes,

$$mx'' + cx' + kx = 0 \rightarrow (2)$$

$$x'' + \frac{c}{m}\cdot x' + \frac{k}{m}\cdot x = 0$$

Which is the linear diff' equ' of 2^{nd} order,

Scanned with CamScanner

Its Solution is

$$x(t) = A_1 e^{\lambda_1(t)} + A_2 e^{\lambda_2(t)} \longrightarrow (3)$$

where, λ_1, λ_2 are the roots of

$$m\lambda^2 + CA + K = 0 \longrightarrow (4)$$

Here, $\lambda_1 + \lambda_2 = -\frac{c}{m}$, $\lambda_1, \lambda_2 = \frac{k}{m}$

The sum of the root is -ve & the products of the root is +ve.

case i) :-

$\boxed{c^2 > 4km}$ the roots are real & distinct and are -ve. As such $x(t) \Rightarrow 0$ as $t \Rightarrow \infty$, the motion is said to be over damped.

case ii) :-

April 2018

$\boxed{c^2 = 4km}$ the roots are real & equal and $x(t) = (A_1 + A_2 t) e(-c/2m, t) \rightarrow (5)$ & again $x(t) \rightarrow 0$ as $t \rightarrow \infty$.

2m

In this case, the motion is said to be "critically damped".

case iii) :-

$\boxed{c^2 < 4km}$ the roots are complete conjugate with the real parts of the roots -ve $x(t)$ always oscillate but as oscillation are damped out and tends to zero. In this case of motion is said to be under damped.

Next, we consider the case when there is an external force $m \cdot F(t)$ acting on the particle (2) becomes,

Scanned with CamScanner

$$mx'' + cx' + kx = mF(t) \rightarrow (6)$$ 36

A particular case of interest is given by the model $x'' + w_0^2 x = F\cos wt \rightarrow (7)$

(ie) when in the absence of the external forces of motion is SHM with period $2\pi/w_0$ and the external force is periodic with period $2\pi/w$.

The section of (1) is given by,

$$x(t) = A\cos(wt - \alpha) + \frac{F\cos wt}{(w_0^2 - w)}$$

$$= A\cos(w_0 t - \alpha) + \frac{F\cos wt}{w_0^2 - w}$$

$$x(t) = A\cos(w_0 t - \alpha) + \frac{F}{2w_0} + \sin w_0 t \rightarrow (8)$$

$$\left[\because x'' + w_0^2 x = 0 \Rightarrow m^2 + u_0^2 = 0 \Rightarrow m = I_i w_0.\right.$$

$$\left. x = A\cos w_0 t + B\sin w_0 t \right]$$

where, $w = w_0$ the 1st term is periodic and its amplitude never exceeds $|A|$.

However as $t \to \infty$ along a sequence for which $\sin w_0 t = I \cdot 1$. The magnitude of the 2nd form of term approaches infinity.

The phenomenon we have discussed have here is known as to pure (or) undamped Resonance. It occur when $c = 0$ & the input and natural frequencies are equal.

we shall get a similar pheonomenon

35 when θ is small. The forcing function $F \cos \omega t$ is then said to be in resonance will the system.

Bridges, cars, planes, ships are vibrating systems and an external periodic force with the same frequency as their natural frequency can damage them.

This is the reason why soldiers crossing a bridge are not allowed to much in step. However resonance phenomenon can also be used to advantages.

Example :-

In upr ading frees (or) in getting a corout of a ditch.

when, ω and ω_0 differ only slightly the solution represents super position of two sinusoidal makes whose period differ only slightly and this leads to occurance of beats.

Electrical circuits :-

The current $i(t)$ amphere's represents the time rate of change "q" following in the circuit. So that,

$$\frac{dq}{dt} = i(t) \rightarrow (1)$$

1) There is a resistance of R ohms in the circuit. This may be provided by a light bulb. An electric heater (or) any other

electrical device oppositing the motion of the charge and causing a potential drop of magnitude. $E_R = Ri$ volts.

There is a capacitance "F" which is an induction of inductance "t" henrys which produces a potential drop. 36

$$E_L = L . \frac{di}{dt}$$

There is a capacitance "C" which produce a potential drop.

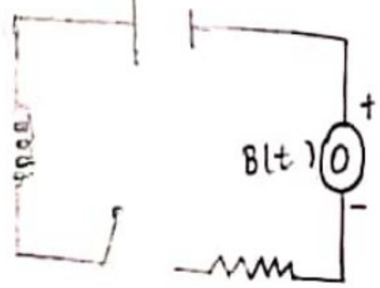
$$E_C = \frac{1}{C} . q$$

All these potential drops are balanced by the battery which produces a voltage "E" volts.

Now, according to Kirchofts 2nd law, the algebraic sum of the voltage drops round a closed circuit is zero.

So that,

$$L . \frac{di}{dt} + Ri + \frac{1}{C} . q = E(t) \rightarrow (2)$$



Differentiate and using (1),

$$L . \frac{d^2 i}{dt^2} + R . \frac{dq}{dt} + \frac{1}{C} . \frac{dq}{dt} = \frac{dE}{dt}$$

$$L . \frac{d^2 i}{dt^2} + R . \frac{dq}{dt} + \frac{1}{C} . i = \frac{dE}{dt} \rightarrow (3) \quad (1) \Rightarrow i = \frac{dq}{dt}$$

(using (1))

Also sub (1) in (2) we get

$$L . \frac{d^2 q}{dt^2} + R . \frac{dq}{dt} + \frac{1}{C} . q = E(t) \longrightarrow (11)$$

Scanned with CamScanner

By 3 & 4 represent linear diff' eqn' with constant co-efficient & their solution will determine i(t) and q(t).

comparing the eqn' are,

$$mx'' + Lx' + kx = mF(t) \longrightarrow (5)$$

37 $$L \cdot \frac{d^2q}{dt^2} + R \cdot \frac{dq}{dt} + \frac{1}{C} \cdot q = E(t)$$

we get the correspondance,

mass "m" $\longleftrightarrow$ inductance "L" Friction co-efficient "C" $\Longleftrightarrow$ resistance "R".

spring constant "K" $\Leftrightarrow$ inverse capacitance $\frac{1}{C}$ impressed force "F" $\Leftrightarrow$ impressed voltage "E" displacement "x" $\Leftrightarrow$ charge q velocity "v" $= \frac{dx}{dt} \leftrightarrow$ current $i = \frac{dq}{dt}$.

This shows the correspondance b/w mechanical and electrical systems. This forms the basis of analog computers.

A linear differential eqn' of the 2nd order can be solved by formed.

A electrical circuit and measuring the electric current in it similar, analogues exist b/w hydrodynamical & electrical systems

Mathematical Modelling brings out the ... structure for

Scanned with CamScanner

we can have analogues of ④ & ⑤ in economic system when $x(t)$ represent the excess of the capital inverted over the equilibrium external inversements. 3e

Phillip's stabilization Model for a closed Economy :-

5M The assumption adjust the national production "Y" of the Models are,

i) The producers adjust the national production "Y" of a product. According to the aggregate demand "D".

If $D > Y$ they increase production and if $D < Y$ they decrease production.

So that,

$$\frac{dy}{dt} = \alpha (D - Y) \quad ; \alpha > 0$$

where, "α" is a reaction co-efficient representing the velocity of adjustment.

ii) Aggregate demand "D" is the sum of private demand, government demand "G" & an exogeneous disturbance "u" the private demand is proportional to the national income (or) output. So that,

$$D = (1-L)Y + G - u$$

where $(1-L)$ is the marginal propensity to spend. (ie) It is marginal propensity to consume plus the propensity to invest.

iii) The government adjust its demand to bring the national output to a described

39 level, which without loss of generality may be taken as zero.

The government decides its demand according to are of the following policies,

a) Proportionals stabilization policy according in which $G^* = +f_p y$

where, $f_p > 0$ is the co-efficient of proportionality and we use the -ve sign on the right hand side. If the output is less than the described government will come out with a +ve demand.

b) Derivative stabilization policy according to which $G^* = -f_d y'$.

where, $f_d > 0$ & the government demand is proportional to y'.

c) Mixed proportional derivative policy according to which,

$$G^* = f_p y - f_d y' \rightarrow (5)$$

d) Integral stabilization policy according to which,

$$G^* = -f_I \int_0^t y \, dt \quad , \quad f_I > 0 \rightarrow (6)$$

iv) G^* is the potential demand which the government may like to make but the actual demand "G" will be gradually adjusted.

so that, $G' = \beta(G^* - G) \rightarrow$ (7)

where, β is the reaction co-efficient

$\beta > 0$ increase the demand to reach G^*. 40

Now from (1) & (2)

$$\frac{dy}{dt} = \alpha\left[(1-L)y + G - u - y\right] \rightarrow (8)$$

$$\frac{dy}{dt} = \alpha\left[y - Ly + G - u - y\right]$$

$$\frac{dy}{dt} = \alpha\left[G - (Ly + u)\right] \rightarrow (A)$$

Diff' equ (8) we get

$$\frac{d^2y}{dt^2} = -\alpha L \frac{dy}{dt} + \alpha \frac{dG}{dt} \rightarrow (9)$$

$$\frac{d^2y}{dt^2} = \alpha\left[-L\frac{dy}{dt} + G'\right] \rightarrow (B)$$

eliminating G b/w (A) & (B) using (7).

$$(7) \Rightarrow \frac{d^2y/dt^2}{\alpha} + L\cdot\frac{dy}{dt} = \beta\left[G^* - \frac{dy/dt}{\alpha} + (Ly + G)\right] \rightarrow (10)$$

$$\frac{d^2y}{dt^2} + \alpha L\frac{dy}{dt} = \alpha\beta\left[G^* - \frac{dy/dt}{\alpha} - (Ly + u)\right]$$

(Multiply by α)

$$= \alpha\beta G^* - \beta\frac{dy}{dt} - \alpha\beta Ly - \alpha\beta u$$

$$\frac{d^2u}{dt^2} + \frac{dy}{dt}(\alpha L + \beta) + \alpha\beta Ly + \alpha\beta u = \alpha\beta u^* \rightarrow (11)$$

If we sub G^* from (3) & (4), (5) we get a linear diff' equ' of 2nd order with constant co-efficient.

Scanned with CamScanner

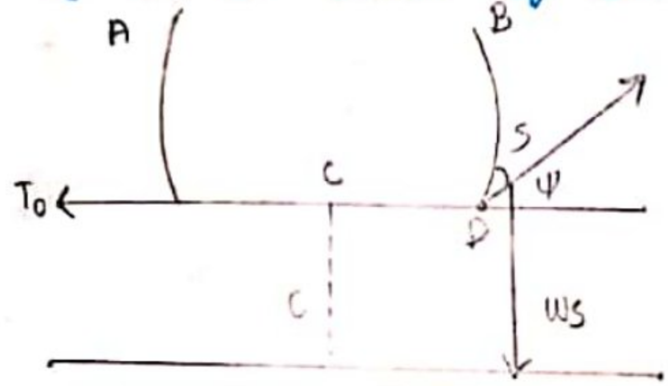
Scanned with CamScanner

If however the government these integral stabilization policy we use (6) to we get the 3^{rd} order diff' equ'.

41 $$\frac{d^3y}{dt^3} + (\alpha I + \beta)\frac{d^2y}{dt^2} + \alpha\beta\frac{dy}{dt} + \alpha\beta F_I y = 0 \quad \to (12)$$

The equ (11) & (12) can be easily solved even without solving these & stability of the solutions and their behaviour as $t \to \infty$ can be easily obtained.

Miscollaneous Mathematical Models through O.D.E of the 2^{nd} order of catenary :-



A perfectly inflexible string is suspended under gravity from 2 fixed point A and B.

consider the equilibrium of the part "CD" of the string of length "S" where "C" is the lowest point of the string at which the tangent is horizontal.

The force acting on this part of the string are,

i) Tension T_0 at C.

Scanned with CamScanner

ii) Tension T at point D along tangent at D.
iii) weight "ws" of the string equating the horizontal and vertical components of force. we get

42 43

$$T\cos\psi = T_0\cos 0^\circ + ws\cos 90^\circ$$

$$T\cos\psi = T_0 \rightarrow ①$$ $\cos 0^\circ = 1$, $\cos 90^\circ = 0$

$$T\sin\psi = ws \rightarrow ②$$

let T_0 be the equal to weight of length "c" of the string then ① gives,

$$\Rightarrow \frac{\cancel{T}\sin\psi}{\cancel{T}\cos\psi} = \frac{ws}{T_0} \quad [\because T_0 = wc]$$

$$\tan\psi = \frac{\cancel{w}s}{\cancel{w}c} = \frac{s}{c} \rightarrow ③$$

$$\frac{ds}{d\psi} = \rho = c\sec^2\psi \rightarrow ④$$

where "ρ" is radius of curvature of the string at D. So that

$$\left[\frac{1+(dy/dx)^2}{d^2y/dx^2}\right]^{3/2} = c\left[1+\left(\frac{dy}{dx}\right)^2\right]$$

(or)

$$c\left(\frac{d^2y}{dx^2}\right) = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{1/2} \rightarrow ⑤$$

which is a non-linear differential equ' of 2nd order.

If $\frac{dy}{dx} = p$. Then, ⑤ gives.

Scanned with CamScanner

Scanned with CamScanner

$$\left[\because \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(p)\right]$$

$$c \cdot \frac{dp}{dx} = (1+p^2)^{1/2}$$

43 $$c \cdot \frac{dp}{\sqrt{1+p^2}} = dx$$

$\int$ing on both sides. we get

$$\int \frac{dp}{\sqrt{1+p^2}} = \int dx \cdot \frac{1}{c} \quad \left[\because \int \frac{dx}{\sqrt{1+x^2}} = \sinh^{-1}(x)\right]$$

$$\sinh^{-1}(P) = \frac{x}{c} + A \rightarrow (7)$$

when, $x=0$, $P=0$ so that $A=0$ &

$$\sinh^{-1}(P) = \frac{x}{c}$$

$$P = \sinh(x/c)$$

$$\frac{dy}{dx} = \sinh(x/c) \longrightarrow (8)$$

$\int$ing on both sides.

$$\int dy = \int \sinh(x/c)\, dx$$

$$y = \cosh(x/c) / \tfrac{1}{c}$$

$$y = c \cdot \cosh(x/c) \longrightarrow (9)$$

where, we choose x-axis in such that $y=0$ when $x=0$, this is the cartesion equ' of the common catenary.

It may be noted that here we get a diff' equ' of 2nd order from a problem statics rather than from a problem of dynamics.

Scanned with CamScanner

A curve of pursuit :-

A ship at the point $(a,0)$ slights a ship at $(0,0)$ moving along y-axis with a uniform velocity, "ku" $(0 < k < 1)$ persure ships "B" with a velocity "u" always moving in the direction of ship "B". So that at any time "AB" is along the tangent to the path "A". 44 45

From figure, $\tan(\pi, \psi) = \frac{BM}{AM} = \frac{OB - OM}{AM}$

$$= \frac{kut - y}{x}$$

$$\Rightarrow \tan\psi = \frac{kut - y}{x}$$

$$\frac{-dy}{dx} = \frac{kut - y}{x}$$

$$-x \cdot \frac{dy}{dx} = kut - y$$

Diff' w.r to "x" we get

$$-x \cdot \frac{d^2y}{dx^2} - 1 \cdot \frac{dy}{dx} = ku \cdot \frac{dt}{dx} - \frac{dy}{dx}$$

$$-x \cdot \frac{d^2y}{dx^2} - \cancel{\frac{dy}{dx}} + \cancel{\frac{dy}{dx}} = ku \cdot \frac{dt}{dx}$$

Multiply by $(-)$

$$\Rightarrow x \frac{d^2y}{dx^2} = -ku \cdot \frac{dt}{dx} \longrightarrow (1)$$

Now, $\frac{dx}{dt}$ = velocity along horizontal direction.

$$\frac{dx}{dt} = u \cdot \cos(\pi - \psi)$$

Scanned with CamScanner

ights a
with a
sure 44
ys moving
at
nt to

$\frac{-OM}{AM}$

$\frac{-y}{x}$

45

$$\frac{dx}{dt} = -u\cos\psi$$

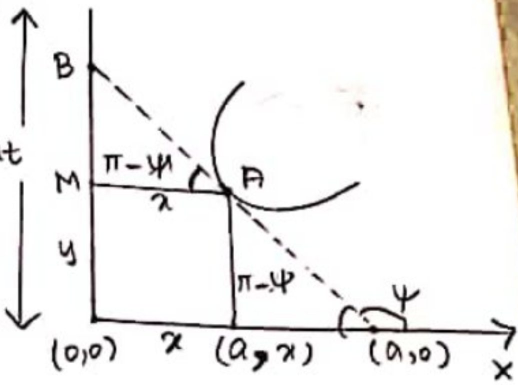
$$\frac{dt}{dx} = -\frac{1}{u}\cdot\sec\psi$$

$$= \frac{-1}{u}\sqrt{1+\tan^2\psi}$$

$$= u\cdot\frac{dt}{du} = \sqrt{1+\tan^2\psi} \rightarrow (5)$$

$$(1) \Rightarrow x\cdot\frac{d^2y}{dx^2} = k\sqrt{1+\tan^2\psi} \quad \left(\because \tan\psi = \frac{dy}{dx}\right)$$

$$x\cdot\frac{d^2y}{dx^2} = k\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$



put $P = dy/dx$

$$x\cdot\frac{dP}{dx} = k\sqrt{1+P^2}$$

$$\frac{dP}{\sqrt{1+P^2}} = k\cdot\frac{dx}{x}$$

$\int$ing on both sides.

$$\int\frac{dP}{1+P^2} = k\int\frac{dx}{x}$$

$$\sinh^{-1}(P) = k\log x + k_1 \rightarrow (3)$$

At $\boxed{x=a,}$ $\boxed{y=0=P}$

$$\sinh^{-1}(0) = k\log a + k_1 \rightarrow (3)$$

$$k_1 = -k\log a$$

$$\sinh^{-1}(P) = k\log x - k\log a$$

$$\sinh^{-1}(P) = k\log(x/a)$$

$$P = \sinh\left[k\log(x/a)\right]$$

Scanned with CamScanner

Scanned with CamScanner

$$\frac{dy}{dx} = \sin h\left[k \log(x/a)\right]$$

$$dy = \sin h\left[k \log(x/a)\right]dx \rightarrow (4)$$

$\int$ing once again. we get a function of "x".

put $\boxed{\log(x/a) = t}$

$$\frac{x}{a} = e^t$$

$$x = ae^t$$

$$dx = ae^t.dt \qquad \left[\sin hx = \frac{e^x - e^{-x}}{2}\right]$$

$$(4) \Rightarrow dy = \sin hk.ae^t.dt$$

$$dy = ae^t\left[\frac{e^{kt} - e^{-kt}}{2}\right]dt$$

$$dy = \frac{a}{2}\left[e^{(k+1)t} - e^{(1-k)t}\right]dt$$

$\int$ing on both sides. we get

$$y = \frac{a}{2}\left[\frac{e^{(k+1)t}}{k+1} - \frac{e^{(1-k)t}}{1-k}\right] + k_2 \rightarrow (5)$$

put $\boxed{y=0}$, $\boxed{t=0}$

$$0 = \frac{a}{2}\left[\frac{e^0}{k+1} - \frac{e^0}{1-k}\right] + k_2$$

$$0 = \frac{a}{2}\left[\frac{1}{k+1} - \frac{1}{1-k}\right] + k_2$$

$$0 = \frac{a}{2}\left[\frac{\cancel{1} - k - k - \cancel{1}}{1-k^2}\right] + k_2$$

47

$$0 = \frac{a}{2}\left[\frac{-2k}{1-k^2}\right] + k_2$$

$$0 = a\left[\frac{-k}{1-k^2}\right] + k_2$$

$$= \frac{-ak}{(1-k^2)} + k_2$$

$$\boxed{k_2 = \frac{ak}{1-k^2}}$$

⑤ $\Rightarrow$ $$y = \frac{a}{2}\left[\frac{e^{(k+1)t}}{k+1} - \frac{e^{(1-k)t}}{1-k}\right] + \frac{ak}{1-k^2}$$

$\therefore$ $$y = \frac{a}{2}\left[\left(\frac{e^{k+1}\cdot\log(x/a)}{k-1} - \frac{e^{(1-k)}\cdot\log(x/a)}{1-k}\right) + \frac{ak}{1-k^2}\right]$$

4/12/2020

Population Dynamics:

i) Prey Predator models:

Let $x(t)$ be the population of the prey special and $y(t)$ be the population of the predator species at time "t" we assume that.

i) If the are no predators, the prey speciour will grow at a rate proportional to the population of the prey- species.

ii) If there are no prey to predator species will decline at a rate proportional to the population of the predator species.

iii) The presents of both prodlors and prey's is benefical to growth of predator species and is harmful of growth of prey species. more specifically the predator species increase and prey. species decrease at a rates proportional to the product of the two populations.

$$\frac{dx}{dt} = ax(t) - bx(t) - y(t)$$

$$= x(t)\left[a - b \cdot y(t)\right]$$

$$= b \cdot x(t)\left[a/b - y(t)\right] \longrightarrow ①, \text{where } a, b > 0$$

$$\frac{dy}{dt} = -p \cdot y(t) + q \cdot x(t) \cdot y(t)$$

$$= -y(t)\left[p - q \cdot x(t)\right]$$

Scanned with CamScanner

$= -q\,y(t)\left[P/q - x(t)\right] \rightarrow (2)$ ②

Now, dx/dt, dy/dt both vanish, when $P, q \neq 0$.

i) If $dx/dt = 0$

$dx/dt = b\,x(t)\left[a/b - y(t)\right] = 0$

$x(t) \neq 0$

$\Rightarrow \left[a/b - y(t)\right] = 0$

$\Rightarrow y(t) = a/b$

ii) If. $dy/dt = 0$

$-y(t).q\left[P/q - x(t)\right] = 0$

$y(t) \neq 0$

$\left[P/q - x(t)\right] = 0$

$x(t) = P/q$

If the initial populations of Prey and Predator species are P/q and a/b respectively the populations will not change with time.

(i) The equilibrium size of the populations to the two species, (0.0) is the other equilibrium size of the population.

From ① and ②

$$2/1 \Rightarrow \frac{dy/dt}{dx/dt} = \frac{-\cancel{q}.y(t)\left[P/\cancel{q} - x(t)\right]}{\cancel{b}.x(t)\left[a/\cancel{b}.y(t)\right]}$$

Scanned with CamScanner

(3)

$$\frac{dy}{dx} = \frac{-y(t)[p \cdot q\, x(t)]}{x(t)[a - b\, y(t)]}$$

$$\left[\frac{a - b\,y(t)}{y(t)}\right] dy = -\left[\frac{p \cdot q\, x(t)}{x(t)}\right] dx$$

$$\Rightarrow \frac{a - by}{y} \cdot dy = \frac{-p + qx}{x} dx$$

$\int$ing on both sides, we get

$$\int \frac{a - by}{y}\, dy = -\int \frac{p + qx}{x} \cdot dx$$

$$\int a/y \cdot dy - \int b \cdot \cancel{y}/\cancel{y} \cdot dy = -\int p/x\, dx + \int q \cdot \cancel{x}/\cancel{x}\, dx$$

$$a \cdot \log y - by = -p \log x + qx + c$$

$$\Rightarrow a \cdot \log y(t) - b \cdot y(t) = -p \log x(t) + q \cdot x(t) + c \longrightarrow (3)$$

put. $t = 0$. we get

$$a \cdot \log y(0) - b \cdot y(0) = -p \log -x(0) + q(x) \cdot \cos x + c$$

$$a \cdot \log y(0) - b \cdot y(0) + p \log x(0) - q \cdot (x)\, 0 = c$$

sub "c" value in (3) we get.

$$a \cdot \log y(t) - b \cdot y(t) = -p \log x(t) + q \cdot x(t) + a \cdot \log y(0)$$
$$- b \cdot y(0) + p \log x(0) - q(x) 0.$$

$$a \cdot \log y(t) - a \cdot \log y(0) + p \cdot \log x(t) - b \log x(0) =$$
$$b \cdot y(t) - b \cdot y(0) + q(x)(t) - q(x) 0.$$

$$a[\log \cdot y(t) - \log y(0)] + p[\log x(t) - \log x(0)] =$$
$$[b \cdot y(t) - y(0)] + q[x(t) - x(0)]$$

Scanned with CamScanner

$$a\left[\log\frac{y(t)}{y(0)}\right] + p\left[\log\frac{x(t)}{x(0)}\right] = b[y(t)-y(0)] \quad (1)$$
$$q[x(t)-x(0)]$$

Thus through every point of 1st quadrant of the xy plane. there is a unique trajectory.

now. the trajectories can intersect. since intersection will imply two different steps at pass the same point.

The equilibrium points (0.0) or (P/q, a/b) present points trajectories.

If x increase while y remains zero. It can be represented as a line trajectory (+ve) side of x-axis.

IIIy if y decrease while x-remains zero it can be represented as a line trajectory.

(ie) downward direction of y-axis. since no. two trajectories intersect no trajectory starting from a point situated with in the 1st quadrant will intersect the x-axis and y-axis trajectories

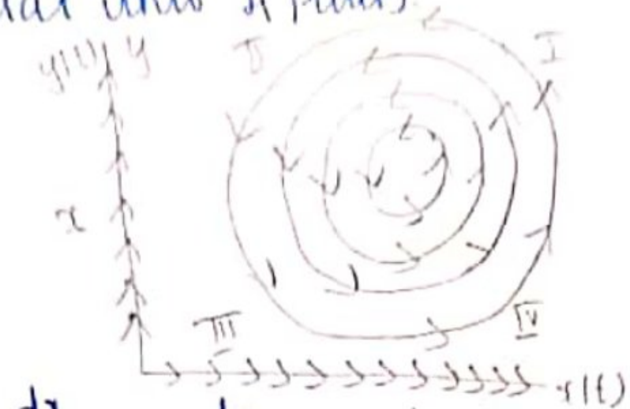
Thus will be all trajecties cover p corresponding to a +ve initial populations will lies strictly with in the 1st quardrant. Thus if the initial populations are +ve. the populations will be always +ve.

If the population of one (or both) species

Scanned with CamScanner

is initially zero it will always remain zero.

consider the line through the point $\{p/q, a/b\}$ parallel to the axis of co-ordinates divide of the 1st quadrant into 4 parts) (5)



i) In I $\frac{dx}{dt} < 0$, $dy/dt > 0$, $dy/dx < 0$

$\Rightarrow x \downarrow$ & $y \uparrow$

ii) In II $dx/dt < 0$, $dy/dt < 0$, $dy/dx > 0 \Rightarrow x \downarrow$, $y \downarrow$

iii) In III $dx/dt > 0$, $dy/dt < 0$, $dy/dx < 0 \Rightarrow x \uparrow$ & $y \downarrow$

iv) In IV $dx/dt > 0$, $dy/dt > 0$, $dy/dx > 0 \Rightarrow x \uparrow$ & $y \downarrow$

Each trajectory is a closed convex curve these trajectory appear relatively cramped near the axis.

Competition models of population dynamics:-

Let $x(t)$ at $y(t)$ be the population of the two species competing for the some resources. then each species grows in the absence. the other species, and the rate of growth of each species decrease due to the presence of the other species.

This gives the system of differential equation,

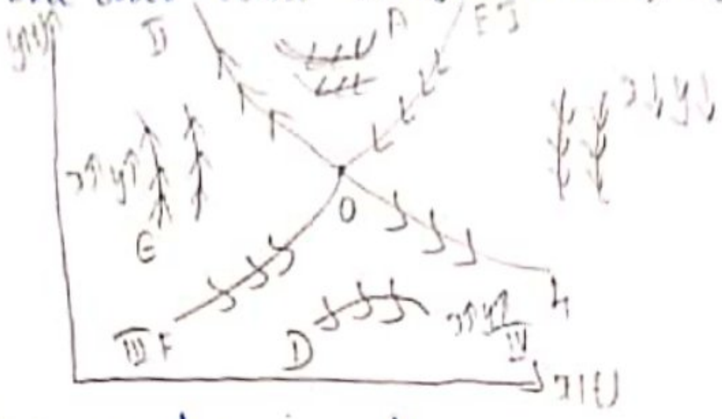
$$dx/dt = ax - by \Rightarrow x(a - bxy)$$

$$= bx(a/b - y) \longrightarrow (1) \quad a, b > 0.$$

$dy/dt = py - qxy = y(p - qx)$

$= qy(p/q - x) \longrightarrow (2) \quad p, q > 0$ (5)

These are two equilibrium positions $(0,0)$ and $(p/q, a/b)$. There are 2 trajections $(0,0)$ and $(p/q, a/b)$ and these are two line trajectories $x = 0$ and $y = 0$



In I $dx/dt < 0$, $dy/dt < 0$, $dy/dx > 0$

In II $dx/dt < 0$, $dy/dt > 0$. $dy/dx < 0$

⋮

In III $dx/dt > 0$, $dy/dt > 0$. $dy/dx > 0$.

In IV $dx/dt > 0$, $dy/dt < 0$, $dy/dx < 0$.

From (1) and (2)

$$dy/dx = \frac{y(p - qx)}{x(a - by)} \quad (\text{or}) \quad \frac{a - by}{y}\, dy = \frac{p - qx}{x}\, dx$$

$\int$ing on both sides.

$$a - by/y \, dy = p - qx/x \, dx$$

$$\int a/y \, dy - \int b/y\,(y)\, dy = \int p/x \, dx - \int q \cdot x/x \, dx$$

$$a \log y - by = p \log x - qx + c \longrightarrow (3)$$

put $t = 0$.

$$a \log y(0) - by(0) = p \log x(0) - qx(0) + c.$$

Scanned with CamScanner

$a \log y(0) - by(0) - p \log x(0) + q\, x(0) = c$ (7)

sub "c" value in (3) we get.

$a \log y(t) - by(t) = p \log x(t) - q(x)(t) + a \log y(0)$
$- by(0) - p \log x(0) + q\, x(0)$

$a \log y(t) - a \log y(0) - by(t) + by(0) = p \log x(t) - p \log x(0)$
$- q\, x(t) + q\, x(0)$

$a[\log y(t) - \log y(0)] - b[y(t) - y(0)] = p[\log x(t) - \log x(0)]$
$- q[x(t) - x(0)]$

$$a\left[\log \frac{y(t)}{y(0)}\right] - b[y(t) - y(0)] = p\left[\log \frac{x(t)}{x(0)}\right] - q[x(t) - x(0)]$$

The trajectory which passes through $(p/q, a/b)$ is

$$a\left[\log \frac{by(t)}{a}\right] - b[y(t) - a/b] = p\left[\log \frac{qx(t)}{p}\right] - q[x(t) - p/q]$$

$$a\left[\log \frac{by(t)}{a}\right] - b[y(t)] + a = p\left[\log \frac{qx(t)}{p}\right] - qx(t) + p.$$

If the initial populations correspondes to the point A. ultimately the 1st species die out and 2nd species increases in size to infinity.

If the initial populations corresponds to the point C. the 1st species dies out 2nd species goes to infinitly and if the initial populations corresponds to point D. The 2nd species dies out and the 1st species goes to infinitly.

multi - species models :-

consider the model represented by the (8)

system of differential equation.

$$\frac{dx_1}{dt} = a_1x_1 + b_{11}x_1^2 + b_{12}x_1x_2 + \cdots\cdots + b_{1n}x_1x_n.$$

$$\frac{dx_2}{dt} = a_2x_2 + b_{21}x_2x_1 + b_{22}x_2^2 + \cdots\cdots + b_{2n}x_2x_n$$

$$\frac{dx_3}{dt} = a_3x_3 + b_{31}x_3 + b_{32}x_3x_2 + b_{33}x_3^2 + \cdots + b_{3n}x_3x_n$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_nx_n + b_{n1}x_nx_1 + b_{n2}x_nx_2 + \cdots\cdots + b_{nn}x_n^2.$$

Where $x_1(t), x_2(t) \ldots x_n(t)$ represent the populations of the n- species.

i) $a_i > 0$, if i^{th} species grows with respresent to the absense of the other.

ii) $a_i < 0$, if i^{th} species decays.

iii) $b_{ij} > 0$, if i^{th} species is benefit by the presence of the j^{th} species.

iv) $b_{ij} < 0$, if i^{th} species is harmed by the presence of the i^{th} species.

v) $b_{ij} < 0$. since each number of the i^{th} species also complete among themselve for limited resource

let, $\frac{dx_i}{dt} = 0$, for $i = 1, 2, \ldots n \longrightarrow (1)$

R.H.S of equation (1).

Scanned with CamScanner

Solving these n-algebraic equations for $x_1, x_2 \ldots x_n$ which gives the equilibrium position some of these species are disappear & finally we have the x_is are zero implies some of these equilibrium position in which all the species can disappear.

Let $x_{10}, x_{20} \ldots\ldots x_{n0}$ we be the equilibrium position. Then its local stability is.

$$\left.\begin{aligned} x_1 &= x_1^0 + u_1 \\ x_2 &= x_2^0 + u_2 \\ &\vdots \\ x_n &= x_n^0 + u_n \end{aligned}\right\} \longrightarrow (2)$$

Sub (2) in (1) we get

$$\frac{d}{dt}(x_{10}+u_1) = a_1(x_{10}+v_1) + b_{11}(x_{10}+v_1)^2 + \ldots b_{1n}(x_{10}+v_1)x_n.$$

After, neglecting squares, products and power v_is we get

$$\left.\begin{aligned} \frac{dv_1}{dt} &= c_{11}v_1 + c_{12}v_2 + \ldots\ldots + c_{1n}v_n \\ \frac{dv_2}{dt} &= c_{21}v_1 + c_{22}v_2 + \ldots\ldots + c_{2n}v_n \\ &\vdots \\ \frac{dv_n}{dt} &= c_{n1}v_1 + c_{n2}v_2 + \ldots\ldots + c_{nn}v_n \end{aligned}\right\} \longrightarrow (3)$$

Assume that,

$v_1 = A_1 e^{\lambda t}$, $u_2 = A_2 e^{\lambda t} \ldots\ldots v_n = A_n e^{\lambda t}$

Scanned with CamScanner

$$\left.\begin{aligned} \frac{dv_1}{dt} &= A_1 e^{\lambda t}, \quad \lambda = \lambda v_1 \\ \frac{dv_2}{dt} &= A_2 e^{\lambda t}, \quad \lambda = \lambda v_2 \\ &\vdots \\ \frac{dv_n}{dt} &= A e^{\lambda t}, \quad \lambda = \lambda v_n \end{aligned}\right\} \longrightarrow (4)$$

sub (4) in (3)

$$\left.\begin{aligned} c_{11}v_1 + c_{12}v_2 + \dots\dots + c_{1n}v_n - \lambda v_1 &= 0 \\ c_{21}v_1 + c_{22}v_2 + \dots\dots + c_{2n}v_n - \lambda v_2 &= 0 \\ c_{31}v_1 + c_{32}v_2 + \dots\dots + c_{3n}v_n - \lambda v_3 &= 0 \\ &\vdots \\ c_{n1}v_1 + c_{n2}v_2 + \dots\dots + c_{nn}v_n - \lambda v_n &= 0 \end{aligned}\right\} \longrightarrow (5)$$

This can be written as,

$$\begin{vmatrix} c_{11}-\lambda & c_{12} & c_{13} & \dots\dots & c_{1n} \\ c_{21} & c_{22}-\lambda & 23 & \dots\dots & c_{2n} \\ c_{31} & c_{32} & c_{33}-\lambda & \dots\dots & c_{3n} \\ \vdots & & & & \\ c_{n1} & c_{n2} & c_{n3} & \dots\dots & c_{nn}-\lambda \end{vmatrix} = 0 \longrightarrow (6)$$

Thus the equilibrium position would be stable if the real part of all the eigen values of the matrix. $[c_{ij}]$ are -ve

since, $|v| = |Ae^{\lambda t}|$

$= \lambda |v| \rightarrow 0$ as only when $\lambda \neq 0$.

Scanned with CamScanner

Routh Hurwitz criterion considers all the roots.

$a_0 x^n + a_1 x^{n-1} + \dots + a_n = 0, \; a_0 > 0 \longrightarrow (7)$

(7) ⇒ will have -ve real point iff, $T_0, T_1, \dots$ are +ve.

where $T_0 = a_0$, $T_1 = a_1$

$$T_2 = \begin{vmatrix} a_1 & a_0 \\ a_3 & a_2 \end{vmatrix} \text{ and } T_3 = \begin{vmatrix} a_1 & a_0 & 0 \\ a_3 & a_2 & a_1 \\ a_5 & a_4 & a_3 \end{vmatrix}$$

$$T_4 = \begin{vmatrix} a_1 & a_0 & 0 & 0 \\ a_3 & a_2 & a_1 & 0 \\ a_5 & a_4 & a_3 & a_2 \\ a_7 & a_6 & a_5 & a_4 \end{vmatrix}$$

This is true iff $a_1 > 0$ and either all even num T_k (or) all odd number T_k all the two consider,

$a_1 x^{n-1} + a_2 x^{n-2} + \dots - \frac{a_0}{a_1} a_3 x^{n-2} - \frac{a_0}{a_1} a_3 x^{n-1} = 0 \longrightarrow (8)$

eqn (7) will have all roots with -ve real parts iff the true for the $(n-1)^{th}$ degree equation.

Age-structured population models:-

Let $x_1(t), x_2(t), \dots x_p(t)$ be the population of the p. pre-reproductive age groups. Let $x_{p+1}(t), x_{p+2}(t) \dots x_{p+q}(t)$ be the population of "q" productive age group and let.

$x_{p+q+1}(t), x_{p+q+2}(t) \dots x_{p+q+r}(t)$. be population of r post reproductive age group.

Scanned with CamScanner

Let d_i be the death rates in the ith age group $(i=1,2 \dots p+q+r)$ and let m_i be the age group migration from the ith age group to the $(i+1)^{th}$ age group $(i=1,2, \dots p+q+r-1)$ Then we let the system of differential equation.

$$\frac{dx}{dt} = (b-d)x$$

$$\left.\begin{aligned}
\frac{dx_1}{dt} &= b_{p+1} x_{p+1} + \dots + b_{p+q} x_{p+q} - (d_1+m_1) \\
\frac{dx_2}{dt} &= m_1 x_1 - (d_2-m_2)x_2 \\
&\vdots \\
\frac{dx_n}{dt} &= m_{n-1} d_n x_n, \quad n = p+q+r
\end{aligned}\right\} \longrightarrow (1)$$

The above equation can be written in the matrix as follows,

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} = \begin{bmatrix} -(d_1+m_1) & 0 & b_{p+1} & b_{p+q} & 0 & p \\ m_1 & -(d_2-m_2) & 0 & 0 & 0 & 0 \\ 0 & m_2 & 0 & 0 & 0 & 0 \\ \vdots & - & - & - & - & - \\ \vdots & - & - & - & - & m_{n-1} \end{bmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} \longrightarrow (2)$$

$$\Rightarrow \frac{dx(t)}{dt} = Ax(t)$$

$$\frac{dx(t)}{dt} = A\,dt$$

(12)

Scanned with CamScanner

ting we get

$$\log x(t) = At + c \longrightarrow (*)$$

put $t=0$. we get

$$\log x(0) = 0 + c$$

sub "c" value in (*) we get

$$\log x(t) = At + \log x(0) \Rightarrow \log x(t) - \log x(0) = At.$$

$$\log\left(\frac{x(t)}{x(0)}\right) = At$$

Taking exponential on both sides we get

$$\frac{x(t)}{x(0)} = e^{At}$$

$$\therefore x(t) = x(0) \cdot e^{At}$$

where A is the matrix all of whose diagonal elements are -ve all of whose main sub diagonal elements are +ve & other elements of the 1st row are +ve and all the other elements are zero.

equation (1) has the solutions.

$$x(t) = e^{At} \cdot x(0).$$

mathematical modelling of epidemics through system of O.D.E of 1st order:.

1) A simple epidemics models:.

let $S(t)$ and $I(t)$ be the nbr of the epidemics.

(i) Those who can get decrease and interact

ii) these who can have definitely get the decease initially. Let these be n susceptible person in the system so that 11

$S(t)+I(t)=n+1,\ S(0)=n,\ I(0)=1 \longrightarrow (1)$

The nbr of infected persons grows at a rate proportional to the product of susceptible infected persons grows and the nbr of subception persons decreases of the same rate so that we get.

The system of differential equations.

$$\frac{dS}{dt}=-\beta SI,\quad \frac{dI}{dt}=\beta SI \longrightarrow (2)$$

so that.

$$\frac{dS}{dt}+\frac{dI}{dt}=0$$

$S(t)+I(t)=\text{constant}=n+1 \longrightarrow (3)$

$$\frac{dS}{dt}=-\beta S(n+1-S) \longrightarrow (4)$$

$$\frac{dI}{dt}=\beta I(n+1-I) \longrightarrow (5)$$

eqn (4) $\Rightarrow \dfrac{dS}{dt}=-\beta S(n+1-S)$

$$\frac{dS}{S(n+1-S)}=-\beta\, dt$$

consider.

$$\frac{1}{S(n+1-S)}=\frac{A}{S}+\frac{B}{n+1-S}$$

$$1=A(n+1-S)+B(S)$$

put $S=0$.

$$1 = A(n+1-0) + B(0)$$

$$A = \frac{1}{n+1}$$

put, $S = 1$

$$1 = A(n+1-1) + B$$

$$1 = A(n) + B$$

$$1 = \frac{1}{n+1}(n) + B$$

$$B = 1 - \frac{n}{n+1} \Rightarrow \frac{\cancel{n}+1-\cancel{n}}{n+1} = \frac{1}{n+1}$$

$$\frac{1}{S(n+1-S)} = \frac{1}{(n+1)S} + \frac{1}{(n+1)(n+1-S)}$$

$$\frac{1}{n+1}\left[\int ds/s + \int \frac{ds}{n+1-s}\right] = -\beta \int dt$$

$$\log s - \log(n+1-S) = -\beta(n+1)t + A$$

$$\log\left(\frac{S(t)}{n+1-S(t)}\right) = -\beta(n+1)t + A \longrightarrow (1)$$

put, $t = 0$, $S(0) = n$ we get

$$\log\left(\frac{S(0)}{n+1-S(0)}\right) = A$$

Sub A" value in (1) we get

$$\log\left(\frac{S(t)}{n+1-S(t)}\right) = -\beta(n+1)t + \log n$$

$$\log\left[\left(\frac{S(t)}{n+1-S(t)}\right) - \log n\right] = -\beta(n+1)t.$$

$$\log\left[\frac{\frac{S(t)}{n+1} - S(t)}{n}\right] = -\beta(n+1)t.$$

Scanned with CamScanner

Taking exponential we get.

$$\frac{S(t)}{n(n+1)-S(t)} = e^{-\beta(n+1)t}$$ (15)

$$S(t) = e^{-\beta(n+1)t} \cdot n\left[(n+1)-S(t)\right]$$

$$S(t) = n(n+1)e^{-\beta(n+1)t} - nS(t)e^{-\beta(n+1)t}$$

$$S(t) + S(t)ne^{-\beta(n+1)t} = n(n+1)e^{-\beta(n+1)t}$$

$$S(t)\left[1+ne^{-\beta(n+1)t}\right] = n(n+1)\cdot e^{-\beta(n+1)t}$$

$$S(t) = \frac{n(n+1)e^{-\beta(n+1)t}}{1+ne^{-\beta(n+1)t}}$$

$$= \frac{n(n+1)e^{-\beta(n+1)t}}{e^{-\beta(n+1)t}\left[\frac{1}{e^{-\beta(n+1)t}}+n\right]}$$

$$S(t) = \frac{n(n+1)}{e^{\beta(n+1)t}+n}$$

eqn (5) $\Rightarrow$ $\frac{dI}{dt} = \beta I(n+1-I)$

$$\frac{dI}{I(n+1-I)} = \beta . dt$$

Consider.

$$\frac{1}{I(n+1-I)} = \frac{A}{I} + \frac{B}{n+1-I}$$

$$1 = A(n+1-I)+BI$$

put $I=0$.

$$1 = A(n+1) = 0$$

$$A = \frac{1}{n+1}$$

Scanned with CamScanner

put $I=1$,

$$1 = A(n+1-1) + B(1)$$

$$1 = \frac{1}{n+1}(n) + B$$

$$1 - \frac{n}{n+1} = B \Rightarrow \frac{n+1-n}{n+1} = B, \quad B = \frac{1}{n+1}$$

$$\frac{1}{I(n+1-I)} = \frac{1}{(n+1)\cdot I} + \frac{1}{(n+1)(n+1-I)}$$

$$\frac{dI}{I(n+1-I)} = \frac{1}{n+1}\left[\frac{dI}{I} + \frac{dI}{(n+1-I)}\right] = \beta \cdot dt$$

$$\frac{dI}{I} : \frac{I}{(n+1-I)} = \beta(n+1)dt$$

$\int$ing on both sides we get,

$$\log I - \log(n+1-I) = \beta t(n+1) + G$$

$$\log\left(\frac{I}{n+1-I}\right) = \beta t(n+1) + G \longrightarrow \text{II}$$

put $t=0$, $I(0)=1$

$$\log\left(\frac{1}{n+1-1}\right) = G$$

$$G = \log(1/n)$$

sub 'G' value in II

$$\log\left(\frac{I}{n+1-I}\right) = \beta \cdot t(n+1) + \log(1/n)$$

$$\log\left(\frac{I}{n+1-I}\right) - \log(1/n) = \beta(n+1)$$

$$\log\left[\frac{I/(n+1-I)}{1/n}\right] = \beta t(n+1)$$

(12)

$$\log\left[\frac{In}{n+1-I}\right] = \beta t(n+1).$$

Taking exponential on both sides we get

$$\frac{nI}{n+1-I} = e^{\beta t(n+1)}$$

$$\therefore I = \frac{n+1-I}{n} e^{\beta t(n+1)}$$

$$I = \frac{n+1}{n} e^{\beta t(n+1)} - I/n\, e^{\beta t(n+1)}$$

$$I + I/n\, e^{\beta t(n+1)} = \frac{n+1}{n} e^{\beta t(n+1)}$$

$$\frac{In + I e^{\beta t(n+1)}}{n} = \frac{n+1}{n} e^{\beta t(n+1)}$$

$$I(t)\left(n + e^{\beta t(n+1)}\right) = (n+1)e^{\beta t(n+1)}$$

$$I(t) = \frac{(n+1)e^{\beta t(n+1)}}{n + e^{\beta t(n+1)}}$$

$$\left.\begin{array}{l} t \to n+1 \text{ as } t \to \infty \\ t^{lt} \to \infty \text{ as } S(t) = 0 \text{ and} \\ t^{lt} \to \infty\ I(t) = n+1 \end{array}\right\} \to (6)$$

2) A susceptible infected susceptible (SIS) model:.

(or)

A simple epidemices models equations:.

Here a subsciptible person can be becomes infected at a rate proportional to S·I and an infected person can removes and becomes susceptible again at a rate γI. (19)

$$\frac{ds}{dt} = -\beta SI + \gamma I \longrightarrow (1)$$

$$\frac{dI}{dt} = \beta SI - \gamma I \longrightarrow (2)$$

Adding (1) and (2) we get,

$$\frac{ds}{dt} + \frac{dI}{dt} = -\beta SI + \gamma I + \beta SI - \gamma I$$

$$\frac{ds}{dt} + \frac{dI}{dt} = 0$$

$$d/dt\,(S+I) = 0$$

$$S+I = n+1$$

$$I = (n+1) - S$$

$$(2) \Rightarrow \frac{dI}{dt} = \beta\left[(n+1) - I\right]I - \gamma I$$

$$= \beta\left[(n+1)I - I^2 - \gamma I\right]$$

$$\frac{dI}{dt} = \left[\beta(n+1) - \gamma\right]I - \beta I^2$$

$$\frac{dI}{dt} = I\alpha - \beta I^2$$

when $\alpha = \beta(n+1) - \gamma$

$$\frac{dI}{dt} = (\alpha - \beta I)I.$$

$$\frac{dI}{(\alpha - \beta I)I} = dt \longrightarrow (3)$$

consider,

$$\frac{1}{(\alpha - \beta I)I} = \frac{A}{\alpha - \beta I} + B/I$$

Scanned with CamScanner

$$1 = A(I) + B(\alpha - \beta)I$$

Put $I = 0$,

$$1 = A(0) + B(\alpha)$$

$$1 = B(\alpha)$$

$$B = 1/\alpha$$

(25)

Put $I = \alpha/\beta$.

$$1 = A(\alpha/\beta) \cdot 1/\alpha\,(\alpha - \beta(\alpha/\beta))$$

$$1 = A(\alpha/\beta)$$

$$A = \beta/\alpha.$$

$$\frac{1}{(\alpha - \beta I)I} = \frac{\beta}{\alpha(\alpha - \beta I)} + \frac{1}{\alpha I}$$

sub in (5) we get

$$\frac{dI}{(\alpha - \beta I)I} = \frac{\beta dt}{\alpha(\alpha - \beta I)} + \frac{dt}{dI}$$

$$\left[\frac{\beta . dt}{(\alpha - \beta t)} + \frac{dI}{t}\right] = \alpha . dt$$

$\int$ing on both sides we get

$$-\log(\alpha - \beta I) + \log I = \alpha t + c$$

$$-\log\left(\frac{I}{\alpha - \beta I}\right) = \alpha t + c \longrightarrow (6)$$

Put $t = 0$, $I(0) = 1$ we get

$$\log\left(\frac{I(0)}{\alpha - \beta I(0)}\right) = \alpha(0) + c.$$

Scanned with CamScanner

$$\log\left(\frac{1}{\alpha-\beta}\right) = c$$

sub "c" value we get.

$$\log\left(\frac{I}{\alpha-\beta I}\right) = \alpha t + \log\left(\frac{1}{\alpha-\beta}\right)$$

$$\log\left(\frac{I(\alpha-\beta)}{\alpha-\beta I}\right) = \alpha t$$

taking exponetial on both sides we get

$$\frac{I(\alpha-\beta)}{\alpha-\beta I} = e^{\alpha t}$$

$$I(\alpha-\beta) = e^{\alpha t}(\alpha-\beta I)$$

$$I(\alpha-\beta) = \alpha e^{\alpha t} - \beta I e^{\alpha t}$$

$$I(\alpha-\beta) + \beta I e^{\alpha t} = \alpha e^{\alpha t}$$

$$I(\alpha-\beta) + \beta I e^{\alpha t} = \alpha e^{\alpha t}$$

$$I\left[(\alpha-\beta) + \beta e^{\alpha t}\right] = \alpha e^{\alpha t}$$

$$I = \frac{\alpha e^{\alpha t}}{\left[\alpha-\beta+\beta e^{\alpha t}\right]}$$

$$I(t) = \frac{\left[\beta(n+1)-\gamma\ e^{(\beta(n+1)-\gamma)t}\right]}{\left[\beta(n+1)-\gamma-\beta+\beta e^{(\beta(n+1)-\gamma)t}\right]}$$

$$I(t) = \frac{\left[\beta(n+1)-\gamma\right] e^{(\beta(n+1)-\gamma)t}}{\beta n+\beta-\gamma-\beta-\beta e^{(\beta(n+1)-\gamma)t}}.$$

$$I(t) = \frac{(\beta(n+1)-\gamma)\, e^{(\beta(n+1)-\gamma)t}}{\beta n+\cancel{\beta}-\gamma-\cancel{\beta}-\beta e^{(\beta(n+1)-\gamma)t}}$$

Scanned with CamScanner

$$I(t) = \frac{(\beta(n+1)-\gamma)\, e^{(\beta(n+1)-\gamma)t}}{\beta n-\gamma+\beta\, e^{(\beta(n+1)-\gamma)t}}$$

22

3) SIS model with constant nbr of carries:

Here infection is spread both by infective and a constant nbr c of carries we know that.

$$\frac{dI}{dt} = \beta SI - \gamma t$$

$$\frac{ds}{dt} = -\beta SI + \gamma t$$

It becomes

$$\frac{dI}{dt} = \beta(I+c)S - \gamma I \quad \text{——} \rightarrow (1)$$

$$\frac{ds}{dt} = -\beta(I+c)S + \gamma I \quad \text{——} \rightarrow (2)$$

Add (1) and (2) we get

$$\frac{ds}{dt} + \frac{dI}{dt} = 0$$

$$d/dt\,(S+I) = 0$$

S+I is a constat.

$$S+I = n+1$$

$$S = n+1-I$$

$$I = n+1-S$$

$$(1) \Rightarrow \frac{dI}{dt} = \beta(I+c)(n+1-I) - \gamma I$$

$$= \beta I(\beta c(n+1)-I) - \gamma I$$

$$= \beta I(n+1) - \beta I^2 + \beta c(n+1) - \beta c I - \gamma t$$

$$= \beta c(n+1) + (\beta(n+1) - \beta c - \gamma) I - \beta I^2.$$

$$dI/dt = \beta c(n+1) + \beta\,[(n+1) - c - \gamma/\beta]\, I - \beta I^2.$$

Scanned with CamScanner

b) Simple epidemic model with carriers:

In this model only carriers spread the disease and their nbr decreases exponentially with time as these are identified and eliminated so that we get,

$$\frac{dI}{dt} = \beta S(t) \cdot C(t) - \gamma I(t) \longrightarrow (1)$$

$$\frac{dS}{dt} = -\beta S(t) \cdot C(t) + \gamma I(t) \longrightarrow (2)$$

$$\frac{dC}{dt} = \alpha C \longrightarrow (3)$$

$$(3) \Rightarrow \frac{dC}{C} = -\alpha dt$$

$\int$ing on both sides we get

$$\log C = -\alpha t + a.$$

Taking exponential on both sides.

$$C = e^{-\alpha t + a}$$

$$C = e^{-\alpha t} \cdot e^{a}$$

put $t=0$. we get

$$C(0) = e^{a}$$

Sub e^{a} value we get $C(0) = C_0$

$$C(t) = e^{-\alpha(t)} \cdot C(0)$$

we know that.

$$S(t) + I(t) = S_0 + I_0 = N (say).$$

$$S(t) + I(t) = N$$

$$I(t) = N - S(t)$$

$$S(t) = N - I(t)$$

Sub these value in (1) we get

$$dI/dt = \beta(N-I) C_0 e^{\alpha t} - \gamma I$$

$$= \beta N C_0 e^{-\alpha t} - \beta I C_0 e^{-\alpha t} - \gamma I$$

$$\frac{dI}{dt} = \beta N C_0 e^{-\alpha t} - [\beta C_0 e^{-\alpha t} + \gamma] I \quad (6)$$

Model with removal:

Here infected person are removed by death (or) Hospitalisation at a rate proportional to the No. of infectives so that the model is.

$$\frac{dI}{dt} = \beta S I - \gamma I$$

$$\frac{dI}{dt} = I(\beta S - \gamma)$$

$$= I(\beta(S - \gamma/\beta))$$

$$\frac{dI}{dt} = \beta I (S - \gamma/\beta)$$

with initial condition

$S(0) = S(0) > 0$

$I(0) = I(0) > 0$

$R(0) = R(0) = 0$

$S(0) + I(0) = N.$

Compartment model through system of O.D.E :.

Pharmokinetics also called drug kinetics (or) tracer kinetics (or) multiplication compartment analysis deals with the distribution of drugs chemical tracer (or) radio active substance among various compartments of the body. where compartment are real (or) fictitious space for drug.

Scanned with CamScanner

Let $x_i(t)$ be the amount of drug in the i^{th} compartment at time 't' we shall assume that can be transferred from i^{th} compartment to the j^{th} compartment $(i \neq j)$ in the time interval $(t, t+\Delta t)$ is $k_{ij}.(x_i(t)+o'u(t))$.

25

where k_{ij} is called the transformation co-eff from the i^{th} compartment to the j^{th} compartment

The total change Δx_i is the time Δt is given by the amound entering the i^{th} compartment from other compactments which is reduced by the amount entering the i^{th} compartment for other compartment inducing zero i^{th} compert the denotes the outside system. Thus we get.

$$\Delta x_i = \sum_{j=0}^{n} k_{ij}.x_i \Delta t + \sum_{j=1}^{n} k_{ij}.x_j \Delta t + o(\Delta t) \longrightarrow (1)$$

Dividing by Δt and taking unit as $\Delta t \to 0$ we get.

$$\frac{dx_i}{dt} = -x_i \sum_{j=1}^{n} k_{ij} + \sum_{j=1}^{n} k_{ij} x_j \longrightarrow (2)$$

$$= \sum_{\substack{j=1 \\ j\neq 1}}^{n} k_{ij} x_i , \quad (i=1,2,\ldots\ldots n)$$

where,
$$k_{ij} = \sum_{\substack{j=1 \\ j\neq 1}}^{n} k_{ij} x_i , \left(i=1,2,\ldots\ldots n\right)$$

consider.
$$k_{ij} = \sum_{j=1}^{n} k_{ij} \quad [i=1,2,\ldots\ldots n]$$

Scanned with CamScanner

where,

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \quad K = \begin{bmatrix} k_{11}, k_{21} \cdots\cdots k_{n1} \\ k_{12}, k_{22} \cdots\cdots k_{n2} \\ \vdots \\ k_{1n}, k_{2n} \cdots\cdots k_{nn} \end{bmatrix}$$

If $x = Be^{\lambda t}$, where B is a column matrix

$$\Rightarrow \frac{d}{dt} Be^{\lambda t} = KBe^{\lambda t}$$

$$B\lambda e^{\lambda t} = KBe^{\lambda t}$$

This gives the consistant system of equation to determine B if $|K - \lambda I| = 0$.

where I is $a \times n$ unit matrix of the matrix K.

we know that all the diagonal elements of K are non-negative and the sum of zero. for such a matrix it can be show that the real part of the eigen value always ≤ 0. The imaginary part is the non-zero only when the real part is strictly less than 0.

Thus if $\lambda_1, \lambda_2 \ldots\ldots \lambda_n$ are eigen value, Then $Re(\lambda_i) \leq 0$. $Im(\lambda_i) \neq 0$ only if $Re(\lambda_i) < 0$. If the drug is ~~of~~ infected at t constant rate given by the column vector (1) with components (3) gives.

$$\frac{dx}{dt} = Kx + D \quad \text{—(4)}$$

equation (3), (4) consists the basis equation for the analysis of the drug distribution in the compartment system.

Scanned with CamScanner

1) Domar Macro model: (37)

Let $S(t), I(t), Y(t)$ be the savings, investment and national income at the time t. Then it is assumed that,

i) Savings are proportional to national income so that,

$$S(t) \propto y(t)$$

$$S(t) = \alpha y(t), \ \alpha < 0 \longrightarrow (1)$$

ii) Investment is proportional to the rate of increase of national income so that,

$$I(t) \propto y'(t)$$

$$I(t) = \beta y'(t) \longrightarrow (2)$$

iii) All savings are invested so that,

$$S(t) = I(t) \longrightarrow (3)$$

we get a system of three O.D.E of 1^{st} order for determining $S(t), y(t), I(t)$ solving we get,

$$S(t) = I(t)$$

$$S(t) = \alpha y(t) \longrightarrow (4)$$

sub this value in (2) we get.

$$\alpha y(t) = \beta y'(t).$$

$$\frac{dy(t)}{y(t)} = \frac{\alpha}{\beta \cdot dt}$$

$\int$ing on both sides

$$\log y(t) = \frac{\alpha}{\beta t} + c$$

Put. $t=0$ we get

$\log y(0) = c.$

sub "c" value.

$$\log y(t) = \frac{\alpha}{\beta t} + \log y(0)$$

$$\log y(t) - \log y(0) = \frac{\alpha}{\beta t}$$

$$\log\left(\frac{y(t)}{y(0)}\right) = \frac{\alpha}{\beta t}$$

Taking exponential on both sides we get.

$$\frac{y(t)}{y(0)} = e^{\alpha/\beta - t}$$

$$y(t) = y(0) \cdot e^{\beta/\beta - t} \longrightarrow (2)$$

sub (5) in (4) we get.

$$I(t) = \alpha\left(y(0) \cdot e^{\alpha/\beta - t}\right)$$

$$S(t) = I(t) = \alpha\left(y(0) - e^{\alpha/\beta t}\right)$$

Hence, the $S(t)$, $I(t)$ and savings all increase exponentially.

2) Domar 1st department company model:-

let $D(t)$, $y(t)$ denote the total national department and total national income respectively then we assume that.

i) Rate of which national department change is proportional to national income.

So that,

$$D'(t) \propto y(t)$$

$$D'(t) = \alpha y(t) \longrightarrow (1)$$

Scanned with CamScanner

ii) National income increase at a constant rate so that $y'(t) = \beta$ (29)

$$dy/dt = \beta$$

$$dy = \beta . dt$$

$\int$ing on both sides, we get

$$y = \beta t + c$$

put $t = 0$. $y(0) = c$

sub "c" value

$$y(t) = \beta t + y(0) \longrightarrow (2)$$

sub (1) in (2) we get

$$D'(t) = \alpha [\beta t + y(0)]$$

$$\frac{dD(t)}{dt} = \alpha \beta t + \alpha y(0)$$

$$\Rightarrow \int dD(t) = [\alpha \beta^t + \alpha y(0)] dt$$

$\int$ing on both sides we get

$$\int d(D(t)) = \int [\alpha \beta t + \alpha y(0)] dt$$

$$D(t) = \alpha \beta t^2/2 + \alpha y(0) + c$$

put $t = 0$. $D(0) = c$

sub "c" value.

$$D(t) = \alpha \beta t^2/2 + \alpha y(0) + D(0) \longrightarrow (3)$$

eqn (2) divided by eqn (3) we get

$$\frac{D(t)}{y(t)} = \frac{\alpha \beta t^2/2 + \alpha y(0) + D(0)}{\beta(t) + y(0)}.$$

In this model the ratio of national debt to

ease

ly

Scanned with CamScanner

Let $D(t)$, $y(t)$ denote the total national dept and national income respectively then assume that

i) Rate at which national dept change is proportional to the national incomes. so that.

$$D'(t) \propto y(t)$$

$$D'(t) = \alpha y(t) \longrightarrow (1)$$

ii) The rate of increases of national income is proportional to the nation income.

$$y'(t) \propto y(t)$$

$$y'(t) = \beta y(t) \longrightarrow (2)$$

solving, $\dfrac{y'(t)}{y(t)} = \beta.$

$\int$ing on both sides we get

$$\int \frac{y'(t)}{y(t)} \cdot dt = \int \beta \cdot dt$$

$$\log y(t) = \beta t + C$$

put $t = 0$, we get

$$\log y(0) = C$$

sub "C" value

$$\log y(t) = \beta t + \log y(0).$$

$$\log y(t) - \log y(0) = \beta t.$$

$$\log\left(\frac{y(t)}{y(0)}\right) = \beta t.$$

Taking exponential on both sides,

$$\frac{y(t)}{y(0)} = e^{\beta t}$$

$$y(t) = y(0)\, e^{\beta t} \longrightarrow (3)$$

$(2) \Rightarrow D'(t) = \alpha y(t)$

$$D'(t) = \alpha y(0)\, e^{\beta t} \quad [\text{by } (3)].$$

jing w.r.to "t" we get

$$\int D'(t)dt = \int \alpha \cdot y(0) \cdot \frac{e^{\beta t}}{\beta} + C_1$$

put $t=0$, we get

$$D(0) = \alpha/\beta\, y(0) + C_1$$

$$C_1 = D(0) - \alpha/\beta\, y(0).$$

Sub "C" value we get

$$D(t) = \alpha y(0) \frac{e^{\beta t}}{\beta} + D(0) - \alpha/\beta\, y(0)$$

$$D(t) = \alpha/\beta \cdot y(0)\left[e^{\beta t} - 1\right] + D(0) \longrightarrow (4)$$

eqn (4) divided by eqn (3) we get

$$\frac{D(t)}{y(t)} = \frac{\alpha/\beta\, y(0)\left[e^{\beta t} - 1\right] + D(0)}{y(0) \cdot e^{\beta t}}$$

$$= \frac{y(0) \cdot \alpha/\beta \left[e^{\beta t} - 1\right]}{y(0) \cdot e^{\beta t}} + \frac{D(0)}{y(0) \cdot e^{\beta t}}$$

$$= \frac{\alpha/\beta \left[e^{\beta t} - 1\right]}{e^{\beta t}} + \frac{D(0)}{y(0) \cdot e^{\beta t}}$$

$$= \left[1/\alpha \left\{1 - e^{\beta t}\right\}\right] + \frac{D(0)}{y(0)} \cdot e^{-\beta t}$$

In this case $\frac{D(t)}{Y(t)} \to \alpha/\beta$ as $t \to \infty$. Thus when dept increase at a rate propose to income then if the rates of dept to income is not increase infinitelly income must increase exponentially

4) Allen's speculative model:-

Let $d(t), s(t), p(t)$ denote the demand supply price of a commodity then this model is given by

$$d(t) = \alpha_0 + \alpha_1 p(t) + \alpha_2 p'(t) \longrightarrow (1)$$

$\alpha_0 > 0, \alpha_1 > 0, \alpha_2 > 0.$

$$s(t) = \beta_0 + \beta_1 p(t) + \beta_2 p'(t) \longrightarrow (2)$$

$\beta_0 > 0, \beta_1 > 0, \beta_2 > 0.$

If $\alpha_2 = 0, \beta_2 = 0$. this given even price adjustment model in which $\alpha_1 < 0$ since. when price increase supply increase.

In Allen's model co-efficient $\alpha_2 \beta_2$ account for the co-efficient of speculation.

If the price is increasing demand increase in the expectation of the further increase in price and supply decreases for the same reason.

For dynamics equilibrium.

$$d(t) = s(t)$$

$$d(t) - s(t) = 0 \longrightarrow (3)$$

(2) - (1)

$$s(t) - d(t) = (\beta_0 - \alpha_0) + (\beta_1 - \alpha_1) p(t) + (\beta_2 - \alpha_2) p'(t).$$

$$0 = (\beta_0 - \alpha_0) + (\beta_1 - \alpha_1) p(t) + (\beta_2 - \alpha_2) p'(t).$$

by (30).

$$-(\beta_0 - \alpha_0) = (\beta_1 - \alpha_1)\, p(t) + (\beta_2 - \alpha_2)\, p'(t)$$

$\Rightarrow (\beta_2 - \alpha_2)$ we get.

$$p'(t) + \frac{(\beta_1 - \alpha_1)}{(\beta_2 - \alpha_2)} \cdot p(t) = \frac{\alpha_0 - \beta_0}{\beta_2 - \alpha_2}$$

multeply and divided by $(\beta_1 - \alpha_1)$ on R.H.S

$$p'(t) = \frac{(\alpha_1 - \beta_1)}{(\beta_2 - \alpha_2)}\, p(t) = \left[\frac{(\alpha_0 - \beta_0)}{(\beta_2 - \alpha_2)}\right] \cdot \left[\frac{\beta_1 - \alpha_1}{\beta_1 - \alpha_1}\right]$$

$$= \frac{\alpha_0 - \beta_0}{\beta_1 - \alpha_1} \left[\frac{\beta_1 - \alpha_1}{\beta_2 - \alpha_2}\right]$$

$$p'(t) = \left(\frac{\alpha_1 - \beta_1}{\beta_2 - \alpha_2}\right) \cdot p(t) = \left(\frac{\alpha_0 - \beta_0}{\beta_1 - \alpha_1} \cdot \frac{\alpha_1 - \beta_1}{\beta_2 - \alpha_2}\right)$$

$$p'(t) - \lambda \cdot D(t) = -pe\lambda:$$

when, $\lambda = \dfrac{\alpha_1 - \beta_1}{\beta_2 - \alpha_2}$

$$pe = \frac{\alpha_0 - \beta_0}{\beta_1 - \alpha_1} \Rightarrow \text{equilibrium position}$$

$$\Rightarrow \frac{dp(t)}{dt} - \lambda\, p(t) = -pe\lambda.$$

$\left\{ \because dy/dx + py = Q.\ y = p(t).\ P = -\lambda.\ Q = -pe\lambda \right\}$

jing factor,

$$ye^{\int p\,dx} = \int Q e^{\int p\,dx}\, dx + c$$

$$pe^{-\lambda t} = \int -pe\,\lambda e^{-\lambda t} \cdot dt + c$$

Scanned with CamScanner

$$Pe^{-\lambda t} = -Pe\frac{\lambda e^{-\lambda t}}{-\lambda} + C \rightarrow$$

$$P(t)\, e^{-\lambda t} = Pe\, e^{-\lambda t} + C \rightarrow (*)$$

put $t = 0$.

$$P(0) = Pe + C$$

$$C = P(0) - Pe.$$

$$(*) \Rightarrow P(t)\, e^{-\lambda t} = Pe \cdot e^{-\lambda t} + P(0) - Pe$$

÷ by $e^{-\lambda t}$ both sides we get

$$P(t) = Pe + \left(\frac{P(0) - P(e)}{e^{-\lambda t}}\right)$$

$$P(t) = Pe + [P(0) - P(e)]\, e^{\lambda t}$$

The behaviour of $P(t)$ depends on whenever $P(\alpha)$ (or) Pe is large and whether $h<0$ (or) $h>0$ speculative model is highly unstable.

5) Samilson's investment model:

5m (7) Let $K(t)$ represented the capital and $I(t)$ represent the investment at time "t" then, we assume that.

i) The investment gives the rate of increase of capital so that.

$$I(t) = \frac{dk}{dt} \rightarrow (1)$$

$I(t) = \frac{dk}{dt}$.

The deficiency of capital below a certain equilibrium level leads to acceleration of the rate of invesment Proportional to this equilibrium level of investment duration of the rate of investment again Proportional to the surplus. so that,

$$\frac{dI}{dm} = -m\,(K(t) - Ke)$$

(33)

where Ke is the capital equilibrium level if

$K(t) = K(t) - Ke$ we get

$$\frac{dI}{dt} = -m(K, t)$$

$$I(t) = \frac{dK}{dt} \longrightarrow (2)$$

$$\frac{dI}{dK} \cdot \frac{dK}{dt} = -m\,K(t)$$

$$\frac{dI}{dK} \cdot I(t) = -mK(t) \quad \text{using } (1)$$

$$dI \cdot I(t) = -mK(t)\,dK$$

$$I \cdot dI = -mK\,dK$$

Jing on both sides, we get

$$I^2/2 = \frac{-mK^2}{2} + c^2/2$$

$$I^2 = -mK^2 + c^2 \longrightarrow (3)$$

$$I^2(t) = -mK^2(t) + c^2$$

put $t = 0$

$$I^2(0) = -mK^2(0) + c^2$$

$$0 = -mK^2 0 + c^2$$

$$c^2 = mK_0^2$$

$$(3) \Rightarrow I^2 = -mK^2 + mK_0^2 = m$$

$$I^2 = -m\left[K^2 - K_0^2\right]$$

$$I(t) = I = -\sqrt{m(K^2 - K_0^2)}$$

Scanned with CamScanner

$(1) \Rightarrow -\sqrt{m}\sqrt{x^2-x_0^2} = \frac{dx}{dt}$

$$-\sqrt{m}\,dt = -\frac{dx}{\sqrt{x^2-x_0^2}}$$

$$-\sqrt{m}\,dt = \frac{-dx}{\sqrt{x^2-x_0^2}}$$

$$\sqrt{m}\,dt = \frac{dx}{\sqrt{x^2-x_0^2}}$$

jing on both sides we get

$$\sqrt{m}.\ t = \cos^{-1}(x/x_0) + c_1 \longrightarrow (4)$$

put $t=0$,

$$0 = \cos^{-1}\left(\frac{x(0)}{x_0}\right) + c_1$$

$$0 = \cos^{-1}\left(\frac{x(0)}{x_0}\right) + c_1$$

$$0 = \cos^{-1}(1) + c_1$$

$$0 = 0 + c_1$$

$$\boxed{c_1 = 0}$$

$(4) \Rightarrow \left[m t = \cos^{-1}(x/x_0)\right]$

$$x/x_0 = \cos\sqrt{m}t$$

$$x = x_0\cos\sqrt{m}t$$

diff w.r to "t" we get

$$\frac{dx}{dt} = -x_0 \sin\sqrt{m}.t.\sqrt{m}$$

$$I(t) = -x_0\sqrt{m}\sin\sqrt{m}t \quad (\because \text{using } (4))$$

So that both $x(t)$ and $I(t)$ osillate with a time period $2\pi/\sqrt{m}$.

It will be noted that it can put

$K(t) = X(t)$

$I(t) = V(t)$

equation (1) are the equation for simple harmonic motion thus the mathematical models for the oscillation of a particle in a simple harmonic motion and for the oscillation of capital about its equilibrium value are the same.

Samulesens modified investment model:

In this case the rate of investment is slowed not only by excess capital as before but it is also slowed by a high investment level and so that.

$$\frac{dk}{dt} = I(t)$$

$$\frac{dI}{dt} = -mk(t) - nI(t) \longrightarrow (1)$$

$$\frac{dI}{dt} \cdot \frac{dk}{dt} = -mk(t) - nI(t)$$

$$I\frac{dI}{dk} + nI(t) + mk(t) = 0 \longrightarrow (2)$$

$$(1) \Rightarrow d/dt\left(dk/ds\right) = -mk(t) - nI(t)$$

$$\frac{d^2k}{dt^2} = -mk(t) - nI(t)$$

$$\frac{d^2k}{dt^2} = -mk(t) - nI(t)$$

$$\frac{d^2k}{dt^2} + mk(t) + nI(t) = 0 \longrightarrow (3)$$

which are the eqn for damped harmonic eqn

Scanned with CamScanner

corresponding to the case when a particle performs show in as by a resistance force proportional velocity

eigen

Stability of market equilibrium:

Let $P_r(t)$, $S_r(t)$ and $d_r(t)$ be the price supply & demand of a commodity in the r^{th} market. So that evan's price adjustment model mechanism suggests

$$\frac{dp_r}{dt} = -\mu_r (S_r - d_r), \quad r = 1, 2, \ldots n \longrightarrow (1)$$

Now we assume that supply and demand of the commodity in the r^{th} market depend upon its price in all the market.

$$S_r - d_r = C_r + \sum_{s=1}^{n} d_{rs} P_s \longrightarrow (2), \quad r = 1, 2, \ldots n \longrightarrow (2)$$

where r's and d_{rs}'s are constant from (1) & (2)

$$\frac{dp_r}{dt} = -\mu_r \left[C_r + \sum_{s=1}^{n} d_{rs} P_s \right] \longrightarrow (3)$$

If $P_{1}e, P_{2}e, \ldots P_{n}e$ are the equilibrium price in the n^{th} market and $P_r = P_r - P_e$, we get.

$$\frac{dp_r}{dt} = \mu_r \sum_{s=1}^{n} d_{rs} P_s$$

$\cancel{\mu_r}\{-\mu_r (r=0)\}$

$$= \sum_{s=1}^{n} P_{rs} P_s$$

where $e = \mu_r d_{rs}$.

Sup $P_r = A_r e^{\lambda t}$ and eliminating $A_1, A_2 \ldots \ldots A_n$ we get.

$$|\lambda_1 - E| = 0$$

$$E = [P_{rs}]$$

Thus the equilibrium will be stable if all the eigen values of the matrix have +ve real parts.

If $d_{rs}=0$. when $r \neq s$ the market's are independent. so that non-zero values of some (or) all these d_r's introduce depends among markets. (39)

Leontief's open and closed Dynamical system for inter-industry relations:

using ODE 1st order:

we consider n industries

let x_r = contribution from the r^{th} industry + consumers per unit time.

x_{rs} = contribution from the r^{th} industry + the s^{th} industry per unit time.

x_r = Total output of the r^{th} industry per unit time

ε_r = Input of topocille in the r^{th} industry.

p_r = price per unit of the product of the r^{th} industry.

w = wage per unit of labour per unit time.

y = total dapour input into the system.

S_{rs} = stock by the product of the r^{th} industry hold by the r^{th} industry.

S_r = stock of the r^{th} industry.

Thus we get the following equation,

i) From the principles of continuity, the rate of change at stock of the r^{th} industry equal to the excess of total output of the r^{th} industry equal to the excess of total output of the r^{th} industry per unit time over the contribution of the r^{th} industry to consumers and other industries per unit time.

$$\frac{d}{dt}(S_r) = x_r - x_r - \sum_{s=1} x_{rs}$$

and assume, $S_r = \sum_{s=1}^{n} S_{rs}$ → (2)

$$\frac{d}{dt}\sum_{r=1}^{n} S_{rs} = x_r \cdot x_r - \sum_{s=1}^{n} x_{rs},\ r = 1,2,\dots n \rightarrow (3)$$

ii) since the total labour input into system equal to the sum of labour into all industries

$$Y = \sum_{r=1}^{n} \leq p_r \rightarrow (4)$$

iii) Assuming the condition of perfect competition and no profit in each industry we should have for each industry the value of input equal to the value of output so that.

$$P_r x_r = \sum_{s=1}^{n} x_{rs} + w\, x_r,\quad r = 1,2,\dots\dots n \rightarrow (5)$$

iv) we further assume that the input co-eff

$$a_{rs} = \frac{x_{rs}}{x_s},\ b_{rs} = \frac{S_{rs}}{x_s},\ b_r = \frac{E_r}{x_r} \rightarrow (6)$$

$r = 1,2,\dots n$

are constants.

substitute this value in (3) and (5)

$$\frac{d}{dt}\sum_{s=1}^{n} b_{rs} \cdot x_s = x_r - \cancel{x_r} \cdot \sum_{s=1}^{n} a_{rs} \cdot x_s \rightarrow (7)$$

$$Y = \sum_{s=1}^{n} b_r x_r \rightarrow (8)$$

$\div$ x_r by x_r.

$$(5) \Rightarrow P_r = \sum_{s=1}^{n} \frac{S_r}{x_r} + w\frac{E_r}{x_r}$$

$$P_r = \sum_{s=1}^{n} a_{sr} + w b_r \rightarrow (9)$$

we assume that the constant a_{rs}, b_{rs}, b_r are known we also assume that $x_1, x_2 \dots\dots x_n$ and w_r given to

is and $P_1 P_2$

are each ...

is the function of time then eqn determine $x_1, x_2 \ldots x_n$ and the eqn (3) determine y and finally eqn (1) determine $p_1, p_2 \ldots p_n$.

Thus if the consumer demand from all industries are known as fn of time. we find the out find with each industry must give. the totally labour force required at any time knowing the buoge rate or any time. we can find the projects at difference industries.

chapter - 5.

mathematical modelling in medicine Arms. races battles and international trade interms of system of O.D.E :-

i) A model for Dibates melitus:-

let $x(t), y(t)$ be the blood sugar and insulin level in the blood stream at time "t" the rate of changes of dy/dt of insulin level is proportional to.

i) The express $x(t) - x(0)$ of the sugar in blood over its fasting level. since, this excess makes the pancreas secrete insulin into the blood. streams and

$$\frac{dy}{dt} \propto (x(t)) - x(0)$$

ii) The amount $y(t)$ of insulin. since insulin, itself to itself tends to decay at rate proportional to it amount and $dy/dt \propto y(t)$.

iii) The insulin dose $d(t)$ injected per unit time this gives $dy/dt \propto d(t)$.

$$\frac{dy}{dt} = a_1(x-x_0) + (x-x_0) - a_2 y + a_3 d(t) \longrightarrow (1)$$

$$= a_1(x-x_0) - x_2 y + a_3 d(t),$$

where,

$a_1, a_2, a_3 \ldots\ldots$ are constant

Scanned with CamScanner

H(x) - step fn

H(x) taken the value unity, when $x \geq 0$ and takes the value zero otherwise.

(i.e) $\begin{cases} H(x)=1 & ; x \geq 0 \\ H(x)=0 & ; x < 0 \end{cases}$

This across in (1) becomes if blood sugar level is less than x_0 there is no secreaction of insulin from the pancreas.

Again the rate of change dx/dt of sugar level is proportional to.

i) This product xy: since the higher the levels of sugar and insulin. The higher is metabolism of sugar and $dx/dt \propto xy$.

ii) $x_0 - x$, since, if sugar level falls below fasting level - sugar is related from the level sloves to raise the sugar level to normal and $dx/dt \propto x_0 - x$.

iii) $x - x_0$. Since if $x \geq x_0$ there is a natural decay in sugar level proportional to its excess over fasting level and $dx/dt \propto x - x_0$.

iv) fns of $t - t_0$, where t_0 is the time at which food is taken $dx/dt \propto t - t_0$.

$$dx/dt = -b_1 xy + b_2(x_0 - x_1) . H(x_0 - x_1) - b_3(x - x_0) + (x_3 - x_0)$$

Where "a" suitable from for $z(t - t_0)$ can be

$z(t - t_0) = 0, \; t < t_0.$

$$= a . e^{-\alpha(t - t_0)}, \; t \geq t_0 \longrightarrow (3)$$

eqn (1) & (2) give two simultaneous differential eqn be determine $x(t) . y(t)$.

Richardson's model for arm weapon's race:-

Let $x(t), y(t)$ be an expenditure on arms by two continuous A and B. Then the rate of change dx/dt of the expenditure by the country A has a term (+3) proportional to y

since the large the expenditure in arms by B. The larger will be the rate of expenditure of arm by A

Similarly it has a term proportional to $(-x)$ since its own arms expenditure has an inhibiting fact on the rate of expenditure on arms by A. It may be also contain a terms independent of the expenditure dependent on mutual suspicions (or) natural good will with these considerations Richardson give the mode dx.

$$\left.\begin{aligned} \frac{dx}{dt} &= ay - mx + r \\ \frac{dy}{dt} &= bx - ny + s \end{aligned}\right\} \longrightarrow (1)$$

Here a, b, m, n are all >0 r and s will be +ve in case of mutual suspicious and -ve in the case of mutual good will be.

A position of equilibrium $x_0 y_0$ if at exists will be given by,

$$\left.\begin{aligned} mx_0 - ay_0 - r &= 0 \\ bx_0 - ny_0 + s &= 0 \end{aligned}\right\} \longrightarrow (2)$$

Solving we get.

Scanned with CamScanner

$$\frac{x_0}{-as-nr} = \frac{y_0}{-br-ms} = \frac{1}{-mn+ab}$$

$$\frac{x_0}{-as-nr} = \frac{1}{-mn+ab}$$

$$x_0 = \frac{-as-nr}{-(mn-ab)}$$

$$x_0 = \frac{(as+nr)}{mn-ab} \longrightarrow (3)$$

IIIly,

$$y_0 = \frac{-(br-ms)}{-mn+ab} = \frac{-(br+ms)}{-(mn-ab)}$$

$$y_0 = \frac{br+ms}{mn-ab} \longrightarrow (4)$$

If r, s are +ve a position of eqn exists $ab < mn$.

If. $X = x - x_0$

$Y = y - y_0$ we get.

$$\left\{ \begin{aligned} \because\ y &= y + y_0 \\ x &= x + x_0 \end{aligned} \right\}$$

$$dx/dt = a(y+y_0) - m(x+x_0) + r$$

$$= ay + ay_0 - mx - mx_0 + r$$

$$= ay - mx - (mx_0 - ay_0 - r)$$

$$= ay - mx$$

$$\left. \begin{aligned} dx/dt &= ay - mx \\ dy/dt &= bx - ny \end{aligned} \right\} \longrightarrow (5)$$

$X = Ae^{\lambda t}$, $Y = Be^{\lambda t}$ will satisfy these equations.

$$f = \begin{vmatrix} \lambda+m & -a \\ -b & \lambda+n \end{vmatrix} = 0 \quad (5)$$

$\Rightarrow (\lambda+m)(\lambda+n)-ab=0$

$\Rightarrow \lambda^2+\lambda n+\lambda m+mn-ab=0$

$\lambda^2+\lambda(m+n)+mn-ab=0 \longrightarrow (6)$

Now the following cases arises:

i) $mn-ab=0$, $r>0$, $s>0$. In this case $x_0>0$, $y_0>0$ and from (5) $\lambda_1<0$, $\lambda_2<0$. As such there is a position of equilibrium and it is stable.

ii) $mn-ab>0$, $r<0$, $s<0$. there is no position of equilibrium. Since $x_0<0$, $y_0<0$. However since $\lambda_1, \lambda_2 < 0$.

$x(t) \to 0$, $y(t) \to 0$ as $t \to \infty$ so that.

$x(t) \to x_0$, $y(t) \to y_0$.

However x_0 and y_0 are -ve and population cannot becomes -ve.

In any case to becomes -ve they have to pass through zero values.

As such as $x(t)$ becomes zero(1) is modified to $dy/dt = -ny+s$ and.

since. $s<0$, $y(t)$ decreases till the reaches zero(0) is modified to.

$$dx/dt = -mx+r \text{ and}$$

since. $r<0$, $x(t)$ decreases till at reaches. zero. Thus at $mn-ab>0$, $n>0$, $s<0$. there will ultimate be complete disarmament.

Scanned with CamScanner

iii) $mn-ab<0$, $\gamma>0$, $\delta<0$ then gives $x_0<0$, $y_0<0$ One of λ_1, λ_2 is the other is -ve In this case there will be a run always arms rate

iv) $mn-ab<0$, $\gamma<0$, $\delta<0$ These gives $x_0>0$, $y_0>0$ on of the λ_1, λ_2 is +ve and the other is -ve.

In this case λ_1, λ_2 +ve and the other is -ve

In this case there will be a run away arms ray (or) disarmament on the initial expenditure (or) arm

Lanchester's Compact model:

Let $x(t)$ and $y(t)$ be the strenghts of two forces engaged in combact and let M and N be the fighting power of individuals depending on physical. ~~fit~~ findness types of arms and traling then lenchester postulated that the reduction in strenghth of each force is proportional to the effective fighting strength of the opposite force we get.

$$\left.\begin{aligned} dx/dt \propto yN &\Rightarrow dx/dt = -a\,yN \\ dy/dt \propto xm &\Rightarrow dy/dt = -a\,xm \end{aligned}\right\} \rightarrow (1)$$

from (1) $\Rightarrow a = dx/dt \cdot -1/yN$

$$a = dy/dt \cdot -1/xm$$

$$dx/dt \cdot -1/ny = dy/dt \cdot 1/xm$$

$$\frac{dx}{yN} = \frac{dy}{xm}$$

$\Rightarrow xm\ dx = yN\ dy$

... on both sides.

$$\frac{Mx^2}{2} = \frac{Ny^2}{2} + c/2$$

$$Mx^2 = Ny^2 + c$$

$$Mx^2 - Ny^2 = c$$

If the proportional reduction of strength in the two forces are the sum.

$$\frac{1}{x} \cdot dx/dt = \frac{1}{y} \cdot dy/dt$$

$$\frac{1}{x}(-ayN) = \frac{1}{y}(-axm)$$

$$\frac{yN}{x} = \frac{xM}{y}$$

$$\Rightarrow y^2 N = x^2 m.$$

This is the square law of fighting strength of an arms depends on the square of its numerical strength and directly on the fighting quality of individual.

International Trade model:

Since, international trade is benefical to all parties we can consider the model:

$$\frac{dx_1}{dt} = a_{21} x_1 x_2 + a_{13} x_1 x_3 + \cdots\cdots + a_{1n} x_1 x_n.$$

$$\frac{dx_2}{dt} = a_{21} x_2 x_1 + a_{23} x_2 x_3 + \cdots\cdots + a_{2n} x_2 x_n$$

$$\vdots$$

$$\frac{dx_n}{dt} = a_{n1} x_n x_1 + a_{n2} x_n x_2 + \cdots\cdots + a_{n-n,} x_n x_{n-1}.$$

Scanned with CamScanner

where all a_{ij} are ive on equilibrium position is (0,0, ... 0) and this is stable

mathematical modelling in Dynamics through system of O.D.E of 1st order:

modelling in Dynamics:

If a particle moves in two dimensional space we want to determine $x(t)$ and $y(t)$ and $u(t), v(t)$ be the velocity components at the same time

Since the equation of motion are based on the principle mass x acceleration = force in the direction.

(ie) ma = force in that direction.

2) motion of a projective:

A particle of mass m. is projective from the region in vacum with velocity V inclined at an angle α to the horizontal and velocity components are $u(t)$ and $v(t)$ its position $x(t), y(t)$ at time "t" respectively.

Two eqn of motion are $m.\,du/dt = 0$.

$$m.\,dv/dt = -mg$$

$$u = V\cos\alpha \; ; \; v = V\sin\alpha - g(t)$$

$$dx/dt = V\cos\alpha \; ; \; dy/dt = V\sin\alpha - gt.$$

∫ing we get,

$$dx = V\cos\alpha\, dt \; ; \; dy = (V\sin\alpha - gt)\,dt$$

$$x = V\cos\alpha\, t \; ; \; y = V\sin\alpha t - \frac{gt^2}{2} \rightarrow (*)$$

Eliminating "t" b/w two equations

$$y = \frac{Vx}{x}\sin\alpha t - \frac{gt^2}{2} - x^2/x^2.$$

using

from

Scanned with CamScanner

$$Y = \frac{Vx}{V\cos\alpha t} \cdot \sin\alpha t - \frac{gt^2}{2} \frac{x^2}{v^2\cos^2\alpha t^2}$$

$$Y = x \cdot \tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha} \rightarrow (1)$$

which is parabola since the term of 2nd degree from a perfect square the parabola is $Y=0$.

when, $x\tan\alpha - \frac{gx^2}{2v^2\cos^2\alpha} = 0$

$$x\left(\tan\alpha - \frac{gx}{2v^2\cos^2\alpha}\right) = 0$$

$x = 0$,

$$\tan\alpha - \frac{gx}{2v^2\cos^2\alpha} = 0$$

$$\tan\alpha = \frac{gx}{2v^2\cos^2\alpha}$$

$$\frac{\sin\alpha}{\cancel{\cos\alpha}} = \frac{gx}{2v^2\cos^2\alpha}$$

$$x = \frac{2v^2\sin\alpha \cdot \cos\alpha}{g}$$

$$x = \frac{v^2\sin 2\alpha}{g}$$

corresponding to position 'O' and 'A'. so that the range of the particle is given by,

$$R = v^2\sin 2\alpha / g$$

putting $y=0$ in eqn ⊛

$$V \cdot \sin\alpha t - (1/2) g t^2 = 0.$$

$t = 0$,

$$V\sin\alpha - (1/2) g t = 0.$$

Scanned with CamScanner

Scanned with CamScanner

$v \sin\alpha = \frac{1}{2} g t$

$t = 2 \sin\alpha \, v/g$.

This gives the time 'T' of the flight since the horizontal velocity is constant and equal to $v \cos\alpha$. The total horizontal distance is travelled is

$u = S/t$.

$S = ut$.

$$S = v\cos\alpha \cdot \frac{2 \sin\alpha v}{g}$$

$$S = \frac{v^2 \sin^2\alpha}{g}.$$

which gives in the same range.

External Ballistics of gun shells:

To study the motion of the Gun shells the following additional factors to be takes into amount

i) Air resitance. which may be proportional to v. but the power n can be different for different range so.

ii) wind velocity humidity and pressure.

iii) Ratation of the earth.

iv) The fact that shell is a right body and as s both motion about the center of gravity have to be studied when the shell comes out the gun it is rotating with a large angular velocity.

It is obvious that the problem's will be quite complex but all these problem have been solved and powerful compute have been developed to solve these proble because of their importance to define.

In the case of interconditional ballistics r

aises heating and air dynamic effects have also

to be consider.

51

UNIT-III

Scanned with CamScanner

Unit-V

(5)

mathematical modelling through graphs:

situations that can be modelled through graphs:

qualitative relative in Applied mathematics:

It has been stated that "Applied mathematics" is nothing but solution of differential equations. This statement is wrong on many counts.

Applied mathematics also deals with solution of difference, differential difference, integral integre differential functional and algebraic equations.

Applied mathematics is also concerned with mathematical modelling in fact mathematical also deals with situations which cannot be modelled interms of equations (or) inequations. one such set of situations is concerned with qualitative relations.

mathematics deals with both quantitative and qualitative relationship. Typical qualitative relations are y likes x, y hates x, y is superior to x, y is subordinate to x, y belongs to some pointical partyas x, set y has a non-null intersection with set x. point y is joined to point x by a road. state y can be transformed into state x. team y has defeated team x, y is. Further of course y is a pro-requeste for course x operation y has to be done before operation x. species y eats species x, y and x

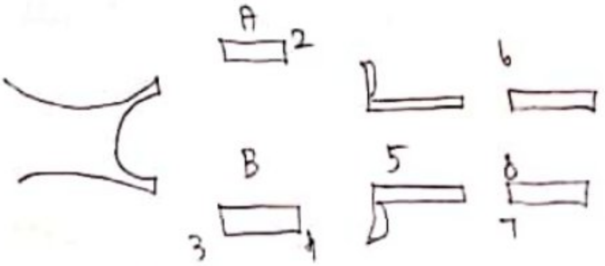
Scanned with CamScanner

are connected by an airline y has a healthy Influence on x. any increase of y. leads to a decrease in x. y belongs to same caste as x. y and x have different nationalities and so on.

Such relationship are very conveniently represented by graphs where a graph consists of a set of vertices and edges joining some or all pair of these vertices. To motivate the typical problem situations which can be modelled through graphs. we consider the 1st problem. so historically modelled viz. The Problem of seven bridges of Konigberge

The (5) 5m Seven bridge problem:

There are four land masses A, B, C, D which are connected by 7 bridges numbered 1 to 7 across a river. The prolem is to start from any point in one of the land masses cover each of the 7 bridges one and once only and return to the starting point.



Here A and B are connected by 2 bridges 1 and 2. B and C are connected by 2 bridges 3 and 4 A and D are connected bridges 6. C and D are connected by bridge 7.

Scanned with CamScanner

Scanned with CamScanner

All these facts are represented by the graphs with 4 vertices and 7 edges (5)

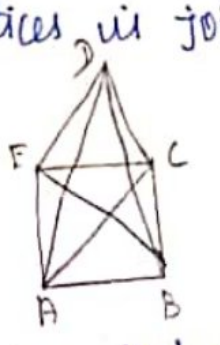
If we can trace this graph in such a way that we start with any vertex and return to the same vertex and trace every edge once and onece only without lifting the pencil from the paper. the problem can be solved again trival and error method cannot be satisfactorily use to show that no solution is possible.

The number of edges meating of a vertex is called the degree of the vertex here the degrees of A.B.C.D are 3.5.3.3 respectively and each of these is an odd number. If we have to start from a vertex and return to it we need an even no.of that Konigberg bridges problem cannot be solved.

Some types of graphs:-

Definition:-

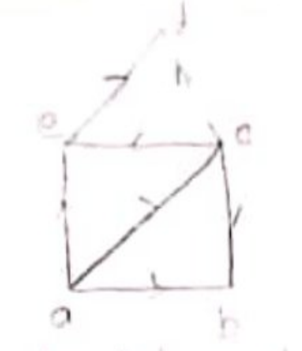
A graph h is called complete of every pair of its vertices is joined by an edge.



N-2013
2m

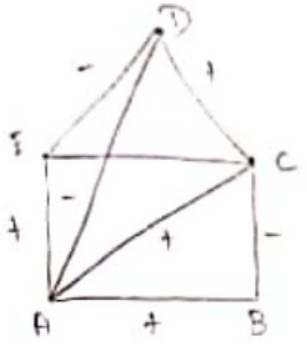
A graph is called a directed graph or a digraph it every edge is directed with an arrow

The an edges is left undirected in a graph. It will be assumed to be directed path ways.



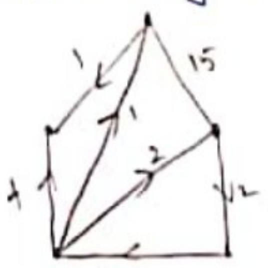
directed graph

A graph is called a signed graph is every edge has a either a plus or minus sign associated with it.



signed graph.

A diagraph is called a "weighted diagraph" of e directed edges has a weight we may also have diagr with +ve and -ve numbers associated with edges there will be called weighted signed diagraphs.



weighted graphs

N-2013
3m

Mathematical models interms of Directed graphs respresenting Results of Tournaments:-

The graph shows that

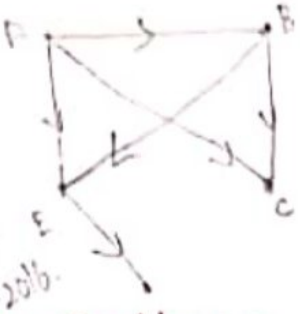
Scanned with CamScanner

i) Team A has defeated team B, C, E

ii) Team B has defeated team C, E

iii) Team E has defeated D

iv) matches b/w A and D, B and C, C and D and C and E have yet to be played.



10m (or) 5m A-15 N-2016.

one-way traffic Problems :-

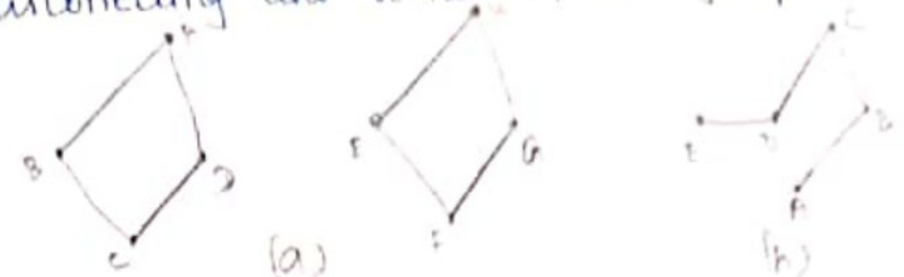
The road map of a city can be represented by a directed graph. If only one way traffic is allowed from point a to point b we draw an edge directed from c to b.

If traffic is allowed both ways, we can either draw two edges, one directed from a to b and the other directed from b to a or simply draw an undirected edge b/w a and b.

The problem is to find whether we can introduce one way traffic on some or all the roads without preventing persons from any point of the city to anyother point. In otherwords we have to find when the edges of a graph can be given direction in such a way that there is a directed path from any vertex to every other.

Scanned with CamScanner

It is easily seen that one way traffic on the road D.E cannot be introduced without disconecting the vertices of the graph (b)



(a) (b)

Fig(a):-

DE can regarded as a bridge connecting two region of the town.

Fig(b):-

DE can be regarded as a blind street on which a two way traffic in necessary edges like DE are called separating edges two way traffic should be permitted. It can also be shown that this is sufficient In other words the following therem can be established

If G is an undirected connected graph then one can always direct the circuit edges of G and leave the separating edges undirected (or both way directed so that there is a directed path from any given vector to any other vertex.

N-2016 5M

Genetic graphs:-

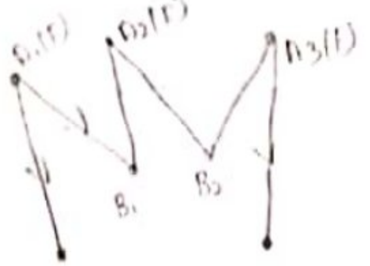
A/M-2019 2M

In a genetic graph we draw a directed edges from A to B indicated that B is the child of in general each vertex will have two incoming edges one from the vertex representing the mother.

Scanned with CamScanner

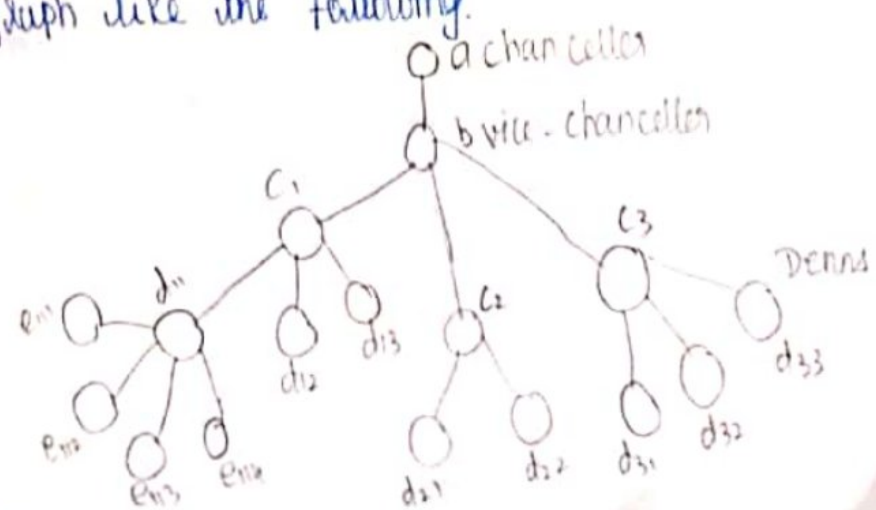
If the father or mother is unknown there may be less than two in coming edges thus in genetic graph the local degree of incoming edges at each vertex must be less than or equal to two this is necessary condition for directed graph to be a genetic but it is not a sufficient condition.

The figure does not give a genetic graph inspite of the fact that the no of incoming edges at each vertex does not exceed two suppose A_1 is male then AS now AS must be male. since A_1, A_3 have a child B_2. A_1, A_3 being males cannot have a child B_3.



Senior - Subordinate Relationship:-

If a is senior to b we write $a \leq b$ and draw a directed edge from a to b. Thus be organisational structure of a graph may be represented by a graph like the following.



The relationship S Satisfies the following properties. (8)

i) $\sim(aSa)$ i.e. no one is his own senior be implies ii) $aSb \Rightarrow \sim(bSa)$ i.e. a is senior to be implies that b is not senior.

iii) $aSb, bSc \Rightarrow aSc$ i.e. If a is senior to b and b is senior to c. then a senior to c.

The following theorem can easily be proved the necessary and sufficient condition that the above three requirements hold is that the graph of an organisation should be free of cycles.

We want now to develop a measure for the status of each person. The status $m(x)$ of the individual should satisfy the following reasonable requirement.

$m(x)$ is always a whole number. If x has no subordinate $m(x)=0$.

If without otherwise changing the structure we add a new individual structure we add a new individual subordinates to x then $m(x)$ increases.

A measure satisfying all these criteria was proposed by harary we define the level of seniority of x over y as the length of the shortest path from x to y. To find the measure of status of x we find n the number of individuals who are the no. of

Scanned with CamScanner

individual who are x level below x. Then the theory measure h(x) is defined by

(9)

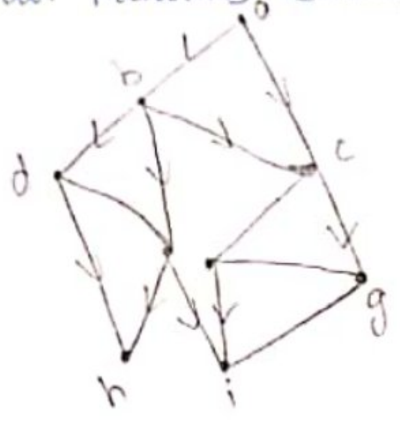
$$h(x) = \sum_k k n_k \longrightarrow (1)$$

It can be show that among all the measure which satisfy the four requirements given above many measure is the least

It however we defines the level of seniority of x over y as the length of the longest path from x to y and the find.

$$H(x) = \sum_K k h_x$$

we get another measure which will be the largest any all measures satisfy the four requirements.



$h(a) = 1.2 + 4.2 + 2.3 = 16$

$h(b) = 1.3 + 2.4 = 11$

$h(c) = 1.2 + 1.2 = 4$

$h(d) = 1.1$ $h(k) = 0$

$h(e) = 1.3$ $h(i) = 0$

$h(f) = 1.1$

$h(g) = 1.2$

$H(a) = 1.1 + 3.2 + 2.3 + 2.4 = 21$

$H(b) = 2.1 + 2.2 + 2.4 + 1.4 = 16$

$H(c) = 1.1 + 1.2 + 1.3 = 6$

$H(d) = 1.1 = 1$

$H(e) = 1.2 + 1.2 = 4$

$H(F) = 1.1 = 1$ $H(K) = 0$

$H(g) = 1.2 = 1$ $H(i) = 0.$

Food webs

Here ash if a eats b and we draw a directed edge from a to b. Here also N(a 3 b) and a3b ash ⇒ N(b 3 a). However the transitive law need not hold. Thus consider the food web in fig. Here fox eats birds, bird grass, but fox does not eats grass.

we can however calculate measure of the status of each species in this food web by using
h (insect) = 1, h (grass) = 0, h(bird) = 2, h(fox) = 4, h(deas) = 1.

Communication networks:

A directed graph can serve as a model for a communication network. Thus consider the network given in fig. In an edge is directed from a to b. It means that a can communicate directly with b, but b can communicate however with e only indirectly through c and d.

However every individual can communicate with every other individual.

Our problem is to determine the importance of each individual in this network. The importance can be measured by faction of the measure on an average that pass through him.

In the absence of any other knowledge we can assume that if an individual can send message to any one of them with probability $1/n$.

In the present example the communication probability matrix is.

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array}\begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} 0 & 1/2 & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \end{array} \longrightarrow (1)$$

No individual is to send a message to himself and so all diagonal elements are zero.

Since all elements of the matrix are non-negative and the sum of elements of every row is unity, the matrix is a stochastic matrix and one of its eigen values is unity.

$\Rightarrow 1$ is an eigen value of A.

$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ represent the left eigen value of the matrix A w.r.t (1)

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \quad A = \begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

(2)

$$A^T \cdot (x_1, x_2\, x_3\, x_4\, x_5) = (x_1, x_2, x_3, x_4, x_5)$$

$$\begin{pmatrix} 0 & 1/2 & 1/3 & 0 & 0 \\ 1/2 & 0 & 1/3 & 0 & 0 \\ 1/2 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/3 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 \end{pmatrix}$$

$$(x_1, x_2, x_3, x_4, x_5) = (x_1, x_2, x_3, x_4, x_5)$$

$$\begin{pmatrix} 1/2\, x_2 + 1/2\, x_3 + 0 + 0 \\ 1/2\, x_1 + 1/2\, x_3 \\ x^1/3 + x^2/3 + x^4/3 \\ x^3/2 + x^3/2 \\ x^2 \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$$

The corresponding normalized eigen values is $(4/45, 13/45, 3/10, 1/10, 1/18)$. In this long run these fraction of message will pass through a, b, c, d, e, respectively. Thus we can calculate that in this network is the most importance person.

If in a network an individual cannot communication with every other individual either directly or indirectly the mental chain is not ergodic and the process of finding the importance of each individual breaks down (13)

matrices Associated with a directed graph:

For a directed graph with n vertices we define the $n \times n$ matrix $A_i = (a_{ij})$ by $a_{ij} = 1$ if there is an edge directed from i to j $a_{ij} = 0$ if there is no edge directed from i to j. Thus the matrix associated with the graph of fig is given by.

$$A = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \longrightarrow (1)$$

we not that.

i) The diagonal elements of the matrix are all zero.

ii) The numbers of non-zero elements is equal to the number edge.

iii) The number of non-zero elements in any row is equal to the local outward degree in the vertex corresponding to the row.

iv) The number of non-zero elements in a column is equal to the local inwards degree of the vertex corresponding to the column. now,

$$A^2 = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 2 & 1 & 1 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 1 & 2 & 0 \\ 1 & 2 & 1 & 0 \end{bmatrix} \end{array} = (a_{ij})^2 \longrightarrow (2)$$

Scanned with CamScanner

The element $(a_{ij})^2$ gives the no of 2 chains from i to j. Thus from vertex 2 to vertex 1. there are 2-chains viz vertex 2 and 3.

We can generalize this result in the form of a theorem viz. The element a_{ij}^2 of A^2 gives the no. of 2 chains (ie) the no. of paths with the edges from vertex i to vertex 1. (14)

It is also easily seen that the i^{th} diagonal element of A^2 gives the no. of vertices with which i has symmetrical relationship.

From the matrix A of a graph a symmetric matrix can be generated by taking the elements wise product of A with its transpose so that in our case.

$$S = A \times A^T \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(S obviously is the matrix of the graph) from which all unreciprocated connections have been eliminated

from which all unreciprocated connections have been eliminated. In the matrix {as well as in S.S.....}

the elements in the row and column corresponding to a vertex which has no symmetric relation with any other vertex all zero. (15)

Application of directed graph to direction and cliques:

A subset of persons in a socio-physiological group will be said to form a clique if

i) Every number of this subset has a symmetrical relation with every other member of this subset.

ii) No other group member has a symmetric relation with all the member of the subset (otherwise it will be included in the clique)

iii) The subset has atleast three members (If the words or clique) can be defined as a maximal completely connected subset of the ~~of~~ orthogonal graph containing atleast three persons).

Thus subset should not be property contain in any larger completely connected subset.

If the graph consists of n persons we reason. The graph by n vertices of a graph the structure is provided by persons known (or) being connected to other persons.

If a person i known j. we can draw a directed edge from i to j if i known j and j know i then. we have a symmetrical relation b/w i and j with this undeperation the graph of fig [Previous fig] shows that persons 1, 2, 3 from clique with every small groups. me an find clique by currentully observing the corresponding graphs. For larger groups analytical.

method based on the following results are useful.

i) i is a member of a clique results are useful element of S^3 is different from zero. (15)

ii) If there is only one clique of k member in the groups the corresponding k elements of S^3 will be $(k-1)(k-2)$ and the result of the diagonal element will be zero.

iii) If there are only two clique with k and m members respectively and there is no element common to these cliques then k elements of S^2 will be $\frac{(k-1)(k-2)m}{2}$ element will be zero.

iv) If there are m disjoint cliques with $k_1 k_2 \ldots k_m$ member then the trace of S^3 is $\frac{1}{2}\sum_{i=1}^{m} k_i(k_i-1)(k_i-2)$, $i=1,2,\ldots$

v) A member is non-cliqual iff the corresponding row and column of $S^3 \times S$ consists entirely of zero's.

mathematical model interms of singed graphs:-

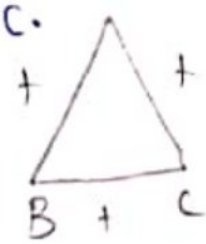
Balance of signed graphs:-

If every edge of a graphs has a +ve (or) -ve sign assigociated with it then the graph is said to be a signed graph.

Example:1

let +ve sign the represent friendship and -ve sign represented enemies. there A is a field of B. A is a friend of C. B is a fiised of C.

It is an normal behaviour and thes elation is nalanced.



Scanned with CamScanner

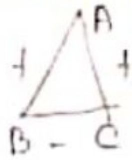
Scanned with CamScanner

Example: 2 (17)

Here A is a friend of B. A is an enemy of C. B is an enemy of C. Therefore A is a friend of B and A and B are enemies at where an normal behaviour of cities an normal behaviour and these relation is balanced.

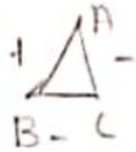
Example: 3

Here A is a friend of B and C. B is an enemy of C it is not normal behaviour of e it is unbalanced it creates the tension in the system.



Example: 4

Here is an enemies of and c is an enemy of C. It is unbalanced it caused the tension.



Definition :-

The sign of a cycles as the Product of the sign of components edges.

Note :-

i) The sign of a cycles in of 1 and 2 are positive both and balanced.

ii) The sign of a cycle in 3 and 4 are -ve both are unbalanced.

iii) The cycle of length 3 or a ~~*~~ triangle is balanced iff it is 3 sign is positive.

Definition :-

A complete algebric graph is a completed graph such that b/w any edges of it there is a +ve (or) -ve sign.

A complete algebraic graph is balanced if its triangles are balanced (or) A complete algebraic graph is balanced if all its cycles are positive. (18)

Theorem:

All triangles are balanced iff all cycles are +ve.

Proof:

All cycles are positive

All edge of cycles are +ve

<=> all edge of triangles are +ve

signs of the triangles are +ve

Triangles are balanced.

Definition:-

i) A graph is said to be totally balanced affected vertex a if all the cycles passing through are balanced

ii) If all the point of a graph is balanced then a graph is locally bans balanced.

iii) A graph is said to be m balanced its cycle of length m are +ve.

iv) If all the cycles of an incomplete graph is +ve then the graph is balanced.

NOV 13, 12, 16.

structure theorem and its application:-

The following four conditions are equivalent.

i) The graph is balanced ie) every date in point +ve

ii) All closed is line - sequence (cycle) in the graph are (+ve) ie) Any sequence of edges starting from a given vertex and ending on it and possible passing through the same vertex more then once is +ve.

Scanned with CamScanner

iii) Any 2 line seq/. b/w 2 verticies have the same sign

iv) The set of all points of the graph can be partitiones into 2 disjoint set such that every the sign connectes 2 pts in the same set and every (-ve) sign connectes 2 pts in the different sets.

Fodex :-

It states the it is an group of persons and are only 2 possible relationship (ie) liking and daliking assume that their relationship is balanced. then the group will be break up into 2 separable parties such that person with in one party like one another and person of one party disliked then other party.

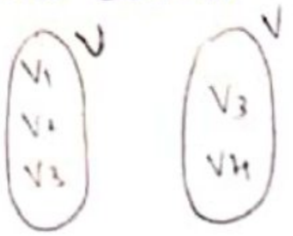
If a balanced situation is stable ⇒ two party political system is stable.

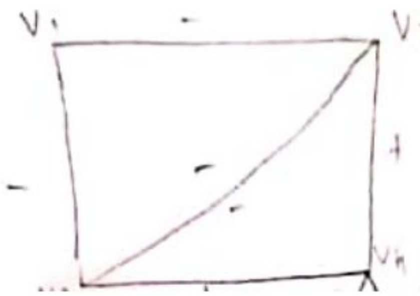
Antibalanced and punbalance of a graph:-

Definition :-

If every cycles in an algebraic graph has an even no of the edges then the graph is said to be antibalance.

Analgebraic graph is antibalanced id its vertices can be separted into 2 disjoint classes. Such that each (-ve) edges joints 2 verticies of the same class and each 2 edges join person from different classes.

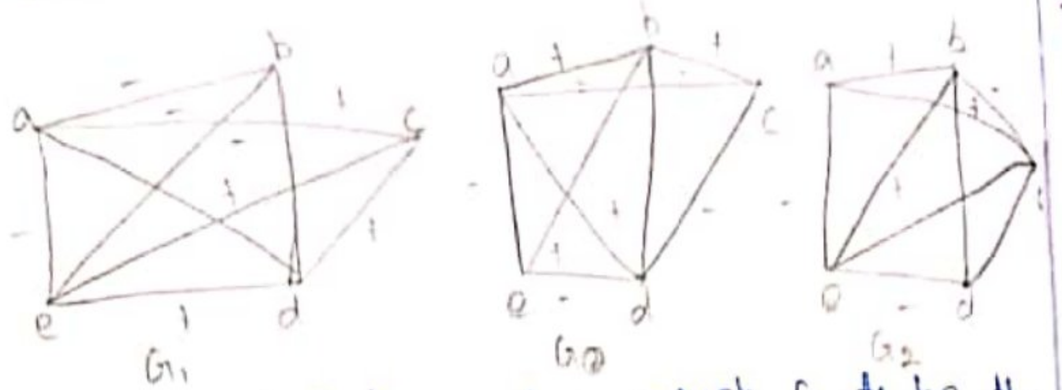




Definition:

A signed graph is said to duobalanced if it is both balanced and antibalanced. The degree of unbalanced of a graph. (20)

Definition:



The degree of balance of a graph G to be the ratio of the (+ve) cycles to the total numbers of cycles in G. It always lies b/w 0 and 1.

Note:

i) The degree of balance of G is less than the degree of balance of G2.

$$G_1 = \frac{\text{no. of the in } G_1}{\text{Total no. of graph in } G_1}$$

ii) In order to get balanced graph from G we have to change the sign of only 2 edges namely bc and d.

iii) To get the balanced graph G2 we have to change the signs of 2 edges.

iv) Both G1 and G2 are equally balanced.

Definition:

The degree of unbalanced of an algebraic graph as the no. of smallest set of edges of G.

whose changes of sign produces balanced graph.

Example :.

21

i) degree of unbalanced graph $G_1 = 2$.

ii) degree of unbalanced graph $G_2 = 2$.

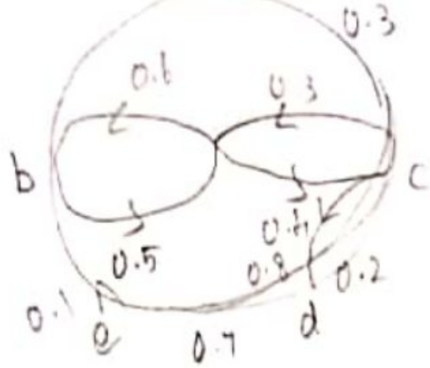
Definition :.

The degree of an anti balanced complete algebraic graph is given by $[n(n-2)+k]/4$. where k - if n is odd and $k=0$ if n is even where n is a no. of vertices.

10th April - 18

mathematical modelling interms of weighted diagraph communication newtons with known probability of communication :.

In the communication graph w.r.t a can communication b and c. only and the absence of any other knowledge. we assign equation probability to us communicating b and c.



How every we may have period knowledge that a channel of commulcative with b and c are in the ratio 3:2 then we assign probability to also community with b and 0.4 also communicate with a similarly we can associated a probability with every directed edge we get the weight diagraph with the associated matrix.

Scanned with CamScanner

$$\begin{array}{c} a \\ b \\ c \\ d \\ e \end{array}\begin{bmatrix} 0 & 0.6 & 0.4 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 & 0 \\ 0.4 & 0.3 & 0 & 0.3 & 0.7 \\ 0 & 0 & 0.3 & 0 & 0.7 \\ 0 & 1.0 & 0 & 0 & 0 \end{bmatrix}$$

we note that the elements are all non-negative and the sum of the elements of matrix and unity is one of its eigen values the eigen vector corresponding its eigen vector and. so that relative importance of the individual depends both on the directed edges well as on the weights associated with the edges.

weighted diagraphs and markov chains :-

A markov system is characterized by a transition probability matrix. Thus if the states of a system are represented by $1, 2 \ldots n$ and P_{ij} gives the probability of transition from the i^{th} state j^{th} states the system is characterized by the transition probability matrix:

$$T = \begin{bmatrix} P_{11} & P_{12} \cdots & P_{ij} \cdots & P_{1n} \\ P_{21} & P_{22} \cdots & P_{2j} \cdots & P_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ P_{n1} & P_{n2} & P_{nj} & P_{nn} \end{bmatrix}$$

since $\sum_{j=1}^{n} P_{ij}$ represents the problem of the system going from 1 state to any other states or remaining i^{th} the same. state this sum must be equal unit. Thus the sum of elements of every row of a transition probability matrix is unity.

Scanned with CamScanner

consider a set of N ∃ markov system where N is large and suppose of any instant $Np_1, Np_2 \ldots Np_n$ of these $(P_1 + P_2 + \ldots + P_{n-1})$ we P_n states 1.2. . n respectively. After one step let the proportion in these states denoted by $P_1', P_2' \ldots P_n'$ then. (2)

$$\left.\begin{aligned} P_1' &= P_1 P_{11} + P_2 P_{21} + \ldots + P_n P_{n1} \\ P_2' &= P_1 P_{12} + P_2 P_{22} + \ldots + P_n P_{n2} \\ &\vdots \\ P_n' &= P_1 P_{1n} + P_2 P_{2n} + \ldots + P_n P_{nn} \end{aligned}\right\} \longrightarrow (2)$$

$$P' = PT \longrightarrow (3)$$

where P and P' are row matrices representity the propositions of system in various states before and after the step and 'T' is the transition prob. matrix we assume that the system has been in operation for a long time and the propetions $P_1 P_2 \ldots P_n$ have reached equilibrium values. In this case.

$$P = PT \text{ (or) } P[I - T] = 0 \longrightarrow (4)$$

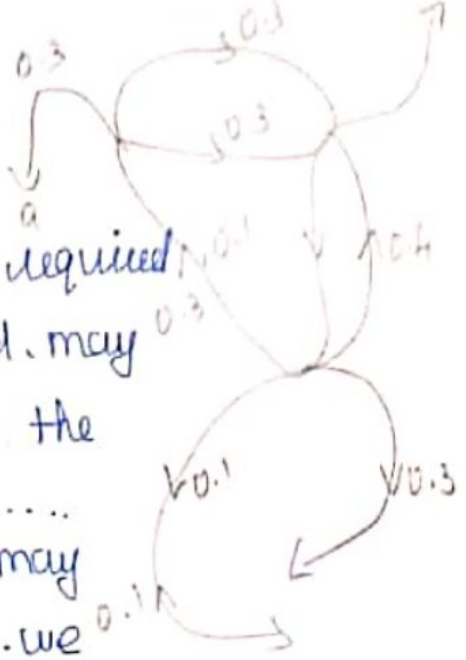
where "I" is the unit matrix. The represents system of equations for determining the equilibrium values $P_1, P_{21} \ldots P_n$. If the eqn %. we consistent the determined of the co-effi must vanish.

i) $ST = T = 0$. This requires unit must o be an eigen values of T. A markov system can be represента by a weighted directed graph. The considered the markov an system with the stochastic matrix.

$$
\begin{array}{c}
a \\ b \\ c \\ d
\end{array}
\begin{bmatrix}
0.75 & 0.25 & 0 & 0 \\
0.3 & 0.6 & 0.1 & 0 \\
0.25 & 0.25 & 0.375 & 0.25 \\
0 & 0 & 0 & 1
\end{bmatrix}
$$

22

In this example d is an absorbing state or a states of a/. once it system reaches the state d. it state there for ever.



It is clear from fig (ii) which ever state the system the no. of steps that may be required to reach depends reach and d. may be 1.2. . . . . starting from to the steps to reach d may 2,3,4 . . . . . starting for the no. of steps may be 3,4,5, ... In each case. we can find the Probability that the number of steps required is n and then we can find the expected of steps to reach if. Thus for the matrix.

$$
\begin{array}{cc} a & b \end{array}
$$
$$
\begin{bmatrix} 1 & 0 \\ 1/3 & 2/3 \end{bmatrix} \longrightarrow \text{(B)}
$$

a is an absorbing state starting from. we can reach a in 1, ... n steps with Problems!

$(1/3)\,(1/3)(2/3)(1/3)\,(2/3)^2 \cdots\cdots (1/3)(2/3)^{n-1} \cdots\cdots$ so that the expected number of steps is

$$
\sum_{n=1}^{\infty} n(1/3)(2/3)^{n-1} = 1/3\left[\sum_{n=1}^{\infty} 1.n\,(2/3)^{n-1}\right]
$$

$$
= 1/3\left[1.(2/3)^0 + 2(2/3)^1 + 3(2/3)^2\right]
$$

$= \frac{1}{3}\left[1+2(2/3)^1+3(2/3)^1 \cdots \right]$

$= \frac{1}{3}\left[1+2(2/3)+3(2/3)^1 \cdots \right] \to (5)$

$= \frac{1}{3}(1-2/3)^{2}$

$= \frac{1}{3}(1/3)^{-2}$

9 $= (1/3)\, 3^2$

$= 3 \longrightarrow (6)$

General communication networks:-

So far we have considered communication networks in which the weighted associated with directed edge represent the probability of communication along that edge we can however have more general network.

Example:-

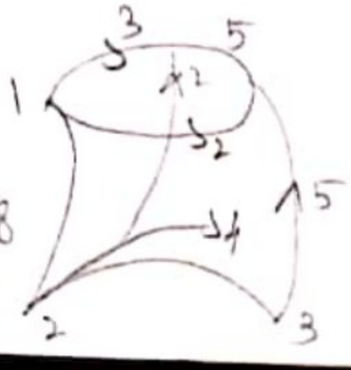
a) for communication of message where directed edges represents the channel and the weight represents the capacity of the channel say in bits per second.

b) for communication of gas in pipe lines where the weights are the capacities say in gallons per hour.

An interesting problem is to find the maximum flow rate of when ever is being communicated for many vertex of the communication network to another useful graph orthogonal algorithms for this have been developed by Elias-Peinstain and Shannon.

2m NOV-17

More general weighted diagraph:



In the must general case the weight associated with a directed edges can be(+ve) and (-ve). Thus fig means that a unit change at vertex at time t at vertex 4 and of 2 unit at vertex 4 and of 3 units vertex 5 at time t'

26

Illy change at 1 unit at vertex 2 causes a change of 3 units at 3 vertex 4 unit at vertex 4 and of 9 units vertex 5 and so on gives the values at all vertices at time t' we can find the values at time. $t+1, t+2, \ldots\ldots$

The process of doing this asymmetrically is known as the plus Rule. These general weighted diagraphs are usefull representing energy flow monetary liquid flows and changes in evironmental conditions.

Nov-16. AP 14 3m

Singal flow group:-

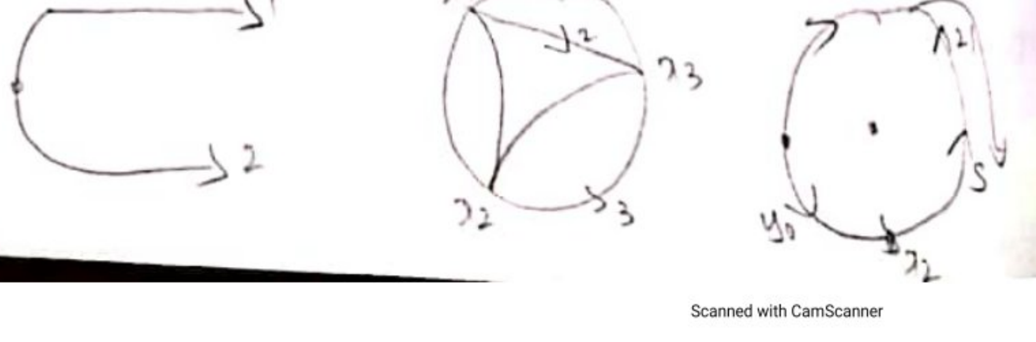
The system of algebric equations,

$$\left.\begin{aligned} x_1 &= 4y_0 - 6x_2 - 2x_3 \\ x_2 &= xy_0 - 2x_1 + 2x_3 \\ x_3 &= 2x_1 - 2x_2 \end{aligned}\right\} \rightarrow (1)$$

Can be represented by the weighted diagraph for solving for x. we successively eliminate x_3 and x_0 to get the graphs and finally we get.

$$x_1 = 4y_0$$

we can Illy represent the solution of any number of limit equation.



weighted bipartitic diagraph and difference equations
consider the system of difference equation.

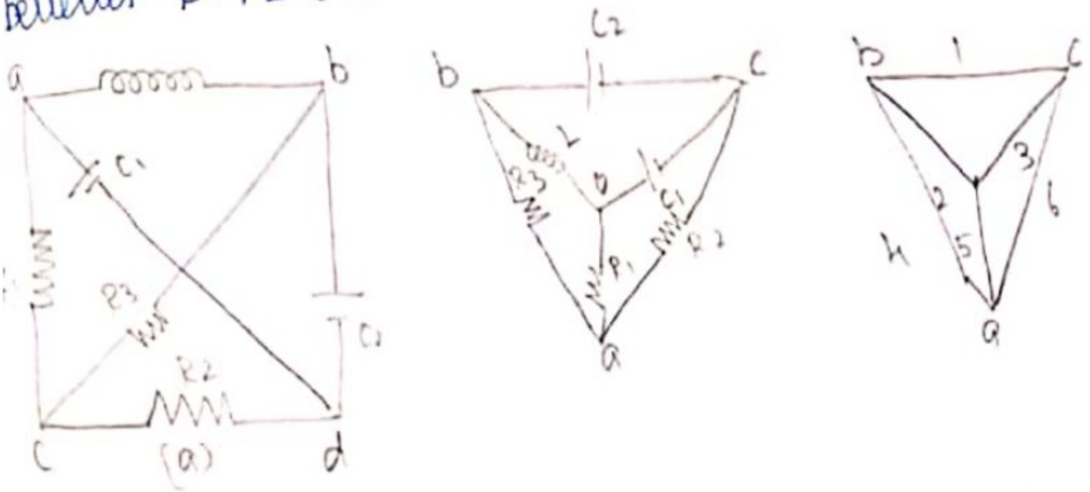
$$\left.\begin{aligned} x_{t+1} &= a_{11}x_t + a_{12}y_t + y_{13}z_t \\ y_{t+1} &= a_{21}x_t + a_{22}y_t + a_{23}z_t \\ z_{t+1} &= a_{31}x_t + a_{32}y_t + a_{33}z_t \end{aligned}\right\} \rightarrow (1)$$

This is can be represended by a weighted bipartitic diagraph other weights can be (+ve) (0,1) (-ve)

mathematical modelling and kirchoff Laws:

Electrical networks and kichoffs laws:.

A electrical circuit consist of register $R_1 R_2 \ldots$ inductances $L_1 L_2 \ldots$ Capacitor C_1 and C_2 and betteries $B_1 B_2$ etc.



The network diagram represent two independent aspects of an electrical network. The 1st gives the inter connection b/w components and the 2nd gives the voltage - current relationship of each component

Scanned with CamScanner

The 1st aspect is called network topology a not can be modeled graphically (25)

This aspect is independent of voltages and current and I modeled through different eqns.

For topological purpose, length and shapes of connections are not important and graphs of figures are isomorphic.

For starting kirchoffs law. we need 2 insidence matrices associated with the graph. If v and e denote the no. of vertices and edges res/. we define the vertex (or) indicidence matrix A = [aij] as follows aij = 1, if the edge it is incident at vertex i. aij = 0, if the edge j is not incident at vertex i. This consists of v rows e columns.

$$A = \begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

We note that every column has 2 non-zero. elements

Illy we define the circuit B = [bkj] as follows.

bkj = 1, if elements j is in limit k

bkj = 0, if element j is not limit k

The matrix B contains as many row as there are circuits and it has a common.

$$B = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Now kirchof is law can be written in the matrix from is follows.

$AI = 0$ [kirchoff current law] → (3)

$BV = 0$ [kirchofs ~~cu~~ voltage law] → (4)

where I is an ex1. column matrix giving the e current and v is on ex1. column matrix giving e voltage matrices A and B depend on the graph only now we can find the rank of A is v-1 and the rank of B is e-v+1 Thus (v-1) and e-v+1 are the no of linearly independent kirchofs current and voltage equation

The graph theoratic methods can be now used to

i) Establish the velocity of the circuit and vertex equation and find their generaliozation

ii) condition under which unique solutions of these equation exit.

iii) Justify the duality procedures used in network theory

iv) Develop short-cut methods for writting equal

v) Develop techinque for network system

Scanned with CamScanner

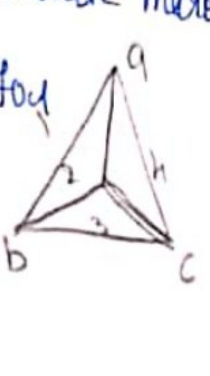
Lamped mechanical system:

If the linear graph represents a mechanical system with the vertices representing rigid bodies. matrices A and B are for networks force and displacement equations respectively and v-1 and e-v+1 represent the numbering linearly independent force and displacement equations.

map-Colouring Problems:

The four-colour problems that every plane map however complex, can be coloured with four colour is such a may two neighbourhood regions get different colours challenged and fasciated mathematicians for over one hundred years till it was finally solved by appak and taken in 1976 by using our 1000 hours of compare of the essentically theoretic since the size and shaples of regions are not important that four colour are necessary is easily seen by considering simple graph in fig. It was the proof of the sufficiently that took more that hundred years. However the effects to solve these problem let to that developted of may other graph theoretic models.

Illy map colouring problem arise for colouring of maps on surface of a sphere force or there surface.



planar graphs :-

In printing of T.V and radius circuit. we want that the wires at lying in a plane should not intersent. In the graphs of fig(1)a will appears to interset level. we can find e, an not intersect. A graph isomorphic to it in which edges do not intersects is called a "planar graphs".

A complete graph with five vertices is not planar fig(3) we can draw nine it the edges. so that these does not interset fig(4). but however we may draw me cannot draw of all then the edges without atleast two of then intersetring. The proof of this depends on of Jordons theorem that every simple based curves divides the plane into two regions one inside the curve and one outside the curve ABCDE in a fig(4) it closed Jordan curve and we cannot draw three edges either inside outside it without intersetting.

Scanned with CamScanner

Eulers formula for polygon graphs :

A polygon graph with n vertices and n straight at curved edges has n vertices n edges and two faces (one inside and one outside)

so that for this graph.

$V\ E+F \longrightarrow (1)$

If we add on one edge another polynomial region of v vertices. we increases the no. vertices by r-2 the no.of edges by r-1 colours and no.of face by 1. so that the not increase V-E+F is zero and the formula i remains valid it can be show by using the principle of induction that (1) is valid for any polynomial graph with any no.of regions.

The draw the dual graph G^* of G. we take a inside each region and draw on edge through it intersecting once of the edges of the region. It is obvious that for this dual graph the no.of vertices edges and takes of the α.

$$V^* = F, \ E = E^*, \ F^* = V \longrightarrow (2)$$

So that,

$$V^* - E^* + F^* = F - E + V = 2 \longrightarrow (3)$$

Regular solids :

A polygon graph G is said to be completely Regular (ii) If both G and its dual G^* are regular (i.e) it the degree of reach vertex of G. vertex of G^* is the same [say e] form this definition points follow

$2E = 1V = p^* F \longrightarrow (1)$

(or)

$E = \frac{1}{2} PV = F = P/p^* V \longrightarrow (2)$

Sub (2) in $V - E + F = 2$

$V - \frac{1}{2} PV + P/p^* V = 2 \longrightarrow (3)$

$V[2P + 2p^* - p^*] = 4p^*$

Since V, P, p^* are the integers

$2P + 2p^* = Pp^* > 0$ (or) $(P-2)(p^*-2) < 4 \longrightarrow (5)$

If $P > 2$, $p^* > 2$ the only one solution in the inequality 5 are,

$P = 3,\ p^* = 3,\ P = 3,\ p^* = 4,\ P = 3,\ p^* = 4$

$P = 4,\ p^* = 3,\ P = 5,\ p^* = 4.$

Sub (2) and (4) we get the table and graphs

	P	V	E	F	p*	v*	E	F
i)	3	4	6	4	3	4	6	4
ii)	3	8	12	6	4	6	12	8
iii)	3	20	30	12	5	12	30	20
iv)	4	6	12	8	3	8	12	6
v)	5	12	30	30	3	20	30	12

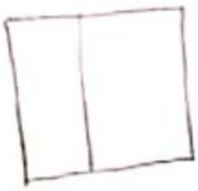
The corresponding graph are gives in fig(1). It is obvious that tetrahedian graph dual to itself Cube is dual of octahedron and dodacedhedran and icosahedron are dual to each other.

Scanned with CamScanner

Tetrahedron cube Octahedron

34

The five graphs corresponding to five Platonic regular solides.

i) 

ii) 

iii) 

iv) 

v) 

vi) 

vii) 

There is a another solution of 5 viz $P = 2P^* = 2,3$

The corresponding graphs G and G* are shown in fig 3.