UNIT - I

Random Variables :

Consider a function whose domain is the set of possible outcomes and whose range is the subset of the real such a function is called a random variable.

Ex: 1

Consider a experiment 'H' like tossing two coins the random variable which is the no. of outcomes.

outcomes : HH HT TH TT

Variable : 2 1 1 0

Random variables : $= X(w) = (0, 1, 2)$

To each outcome 'w' these corresponding the real number $X(w)$.

Ex : 2

If the coin is tosses then

$$S = \{HT\} = \{w_1, w_2\}$$

$$X(w) = \begin{cases} 1 & \text{if } w = H \\ 0 & \text{if } w = T \end{cases}$$

$X(w)$ is a bernouli random variable.

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Stochastic process :

Families of random variable which are function of time 't' are known as stochastic process (or) random process (or) random functions.

Ex : 1

Consider a random event occur in time such as number of telephone calls received at the switch board.

Suppose that $x(t)$ is random variables which represent the number of incoming calls is an interval $(0, T)$ of duration 't' units.

The no. of calls with in a fixed interval of specified duration the unit of time is an random variable $x(t)$ and the family $\{x(t), t \in T\}$ constitutes the stochastic process $[T = (0, \infty)]$

Ex : 2

Consider a simple experiment like throwing a die suppose that x_n is outcomes of the n^{th} throw $n \geq 1$ then $\{x_n, n \geq 1\}$ is the family of random variables.

Such that for distinct value of $n = 1, 2 \ldots$ one get the distinct random variables $x_n \{x_n, n \geq 1\}$ constitutes stochastic process is known as bernouli process.

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Ex: 3

Suppose that x_n is the number of sixes in the first n^{th} throws for a distinct value of $n = 1, 2, 3 \ldots$

we get a distinct variable is known as stochastic process.

State space :

The set of possible value of a single random variable X_n of a stochastic process $\{X_n, n \geq 1\}$ is known as the state space.

Discrete state space :

The state space is discrete if it contains a finite (or) denumerable infinity of point.

otherwise it is continuous state space

Ex:

The X_n is the total no. of six appearing in the first n throwing of a die the set of possible value of X_n is the finite set of non - negative integer $(0, 1, 2 \ldots n)$. Here the state space of X_n is discrete.

We can write $X_n = Y_1 + Y_2 + \ldots + Y_n$.

where Y_i is a discrete random variable denoting the outcomes of the i^{th} throw & $Y_i = 1$ (or) 0 According as the i^{th} throw status six (or). not.

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Consider $X_n = Z_1 + Z_2 + \cdots Z_n$ where Z_i as a continuous random variable assuming value in $[0, \alpha]$ we have assumed that the value assumed by the random variable.

X_n (or) $X(t)$ are one dimensional but the process $[x_n]$ may be multi-dimensional.

One dimensional process can be classified by a 4-types of process.

Discrete time : Discrete state space

Discrete time : Continuous state space

Continuous time : Discrete state space

Continuous time : Continuous state space

All the 4-types may be represented by $[X(t), t\in T]$.

In case of discrete time the parameter generally used in 'n'

i.e. The family is represented by

$$\{X_n, n = 0, 1, 2 \ldots\}$$

Incase of continuous time both the symbol

$\{X_t, t \in T\}$ & $\{X(t), t\in T\}$ are used.

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Process with independent Increment:

If for all $t_1, t_2 \ldots t_n$

$t_1 < t_2 < t_3 \ldots < t_n$ the random variables

$x(t_2) - x(t_1)$; $x(t_3) - x(t_2) \ldots x(t_n) - x(t_{n-1})$

are independent then $\{x(t), t \in T\}$ is said to be a process with independent increment.

Discrete parameter Case:

Consider a process in discrete time which independent Increment $T = \{0, 1 \ldots\}$

$t_i = i - 1$

$x(t_i) = x_{t_i} = x_{i-1}$

$z_i = x_i - x_{i-1}$

$z_0 = x_0$

We have a sequence of independent random variables $\{z_n, n \geq 0\}$.

Markov process:

If $\{x(t), t \in T\}$ is a stochastic process such that given value $x(s)$ the value of $x(t)$ $t > s$ do not independent upon the values of $x(u)$ $u < s$ then the process is said to be markov process

If for $t_1 < t_2 \ldots < t_n < t$

$$P_r\{a \leq x(t) \leq b / x(t_1) = x_1 \ldots x(t_n) = x_n)$$

$$= P_r\{a \leq x(t) \leq b / x(t_n) = x_n\}$$

Then the process $\{x(t), t \in T\}$ is a markov process.

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Note :

Discrete parameter markov process is known as markov chain.

Stationary process :

Second order stationary process

A stochastic process $\{X(t), t \in T\}$ is called the second order stationary process if $E\{X(t)^2\} < \infty$ it is a collection of second order random variable (ie, Random variable with finite second order moments)

Co-variance stationary :

The second order stationary process is called co-varient stationary process (or) weekly stationary process (or) wide sense stationary process. if

i) The mean function is defined by

$$M(t) = E[X(t)]$$

ii) The co-variance function is defined by

$$Cov(X(t), X(s)) = E\{X(t) . X(s)\} - E\{X(t)\}$$

$$Cov(X(t), X(s)) = E\{X(t) . X(s)\} - E\{X(t)\} . E\{X(s)\}$$

The co-variance function is denoted by $C(s,t)$ (or) $C_{s,t}$.

The co-variance function is also called auto co-variance function.

[Mean function $M(t)$ is independent of t and its co-variance function $C(s,t)$ is a function only of the time difference $|t-s|$ $\forall$ t, s.

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ie, $C(s,t) = F(t-s)$

for any t_0,

$$\text{cov}(s+t_0, t+t_0) = E\{X(s+t_0)\, X(t+t_0)\} - E\{X(s+t_0)\}. E\{X(t+t_0)\}.$$

$$= E\{X(s).X(t)\} - E\{X(s)\} E\{X(t)\}$$

$$= \text{cov}(s,t)$$

$$= C(s,t)$$

properties of co-variance function :

i) It is symmetric in t & s

ie, $C(s,t) = C(t,s) \;\forall\; s,t \in T$

ii) Application of schwartz inequality yields.

ie, $C(s,t) \le \sqrt{C(s,s)\; C(t,t)}$

iii) It is a non-negative definite

ie. $\sum_{j=1}^{n} \sum_{k=1}^{n} a_j a_k \, C(t_j, t_k) = E\{\sum_{j=1}^{n} a_j X t_j\} \ge 0$

where $a_1, a_2 \ldots a_n$ is a set of all real number and $t_j \in T$.

closure property:

The sum and the product of two co-variance function is a co-variance function.

stationary of order n :

If for arbitary $t_1, t_2 \ldots t_n$ the join distribution of the vector random variable $\{X(t_1), X(t_2) \ldots X(t_n)\}$ and $[X(t_1+h), X(t_2+h) \ldots X(t_n+h)]$ are the same for all $h>0$. The stochastic process

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$\{X(t), t \in T\}$ is said to be stationary of order n

It is also known as strictly stationary of order n for any integer n.

Note :

i) If the mean of the process $X(t)$ exists then $E(X(t))$ must be equal to $E(X(t+h))$ for all h. So that $E[X(t)]$ must be constant m. independent of t.

ii) Assume $m=0$ then the co-variance function $C(s,t)$ (or) $C_{s,t}$ exists.

$$Cov\{X(t), X(s)\} = E\{X(t).X(s)\} - E\{X(t)\}.E\{X(s)\}$$

$$= E\{X(t) X(s)\}$$

$$= E\{X(t+h) X(s+h)\} \text{ for any } h.$$

$$= E\{X(t-s) X(0)\}$$

This shows that $C(s,t)$ is a function of time difference $|t-s|$.

iii) A strictly stationary process will not necessarily be a weekly stationary process and be necessarily strictly stationary process.

Evaluationary process :

A process which is not stationary is said to be Evaluationary process (Eh) poisson process.

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Gaussian process :

If the distribution of $x(t_1), x(t_2) \ldots x(t_n)$ for all $t_1, t_2 \ldots t_n$ is an multivariate normal Then the process $\{x(t), t \in T\}$ is said to be a Gaussian process.

If a Gaussian process $x(t)$ is co-variance stationary then it is strictly stationary process. multivariate normal distribution of $x(t_1), x(t_2) \ldots x(t_n)$ is completely determinant by its mean vector. $\mu_1, \mu_2 \ldots \mu_n$ where $\mu_i = E\{x(t_i)\}$ and the co-variance matrix $c(i, j)$ whose elements are

$$c(i,j) = \text{cov}\,[x(t_i), x(t_j)] \quad , i, j = 1, 2 \ldots n.$$

If the Gaussian process is co-variate & stationary then $E[x^2(t_i)]$. $E[x^2(t_j)]$ are finite and the co-variance function $c(t_i, j)$ is a function only of the time difference i, j.

problem :

Consider the process $\{x(t), t \in T\}$ whose probability distribution under a certain condition is given by

$$P_r\{x(t) = n\} = \begin{cases} \dfrac{(at)^{n-1}}{(1+at)^{n+1}} & n = 1, 2 \ldots \\ \dfrac{at}{1+at} & n = 0. \end{cases}$$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3$

Sol:

Mean $= M(t) = E\{x(t)\}$ $\quad (1-x)^{-n} = 1 + nx + \dfrac{n(n+1)}{2!}x^2.$

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$$= \sum_{n=0}^{\alpha} x(t)\ P_r \{x(t) = n\}$$

$$= \sum_{n=0}^{\alpha} n \left\{ \frac{(at)^{n-1}}{(1+at)^{n+1}} \right\}$$

$$= 0.\ \frac{(at)^{-1}}{(1+at)} + \sum_{n=1}^{\alpha} \frac{n(at)^{n-1}}{(1+at)^{n+1}}$$

$$= \frac{1.\ (at)^0}{(1+at)^2} + 2\frac{(at)}{(1+at)^3} + 3\frac{(at)^2}{(1+at)^4} + \cdots$$

$$= \frac{1}{(1+at)^2} + \frac{2(at)}{(1+at)^3} + \frac{3(at)^2}{(1+at)^4} + \cdots$$

$\frac{1}{(1+at)^2}$ is the common outside

$(1-x)^{-2} = 1+2x+3x^2+\ldots)$

$$= \frac{1}{(1+at)^2}\left[1 + \frac{2(at)}{(1+at)} + \frac{3(at)^2}{(1+at)^2} + \cdots\right]$$

$$= \frac{1}{(1+at)^2}\left[1 - \frac{at}{1+at}\right]^{-2}$$

$$= \frac{1}{(1+at)^2}\left[1 - \frac{at}{1+at}\right]^{-2}$$

$$= \frac{1}{(1+at)^2}\left[\frac{1+at-at}{1+at}\right]^{-2}$$

$$= \frac{1}{(1+at)^2}\left[\frac{1}{1+at}\right]^{-2}$$

$$= \frac{1}{(1+at)^2}\ \frac{(-1)^{-2}}{(1+at)^{-2}} = \frac{1}{(1+at)^2} \times \frac{1}{(1+at)^{-2}}$$

$$= \frac{(1+at)^2}{(1+at)^2} = 1$$

$M(t) = 1$

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$$V(x) = E(x^2) - [E(x)]^2$$

$$\text{var}\{x(t)\} = E(x^2(t)) - [E(x(t))]^2$$

$$[E(n^2(t))] = \sum_{n=0}^{\infty} x^2(t)\, Pr\{x(t)=n\}$$

$$= \sum_{n=0}^{\infty} x^2(t) \frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$= \sum_{n=0}^{\infty} n^2 \frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$= \sum_{n=0}^{\infty} [n(n-1)+n] \frac{(at)^{n-1}}{(1+at)^{n+1}}$$

$$= \sum_{n=0}^{\infty} n(n-1) \frac{(at)^{n-1}}{(1+at)^{n+1}} + \sum_{n=0}^{\infty} \frac{n(at)^{n-1}}{(1+at)^{n+1}}$$

$$= \sum_{n=1}^{\infty} n(n-1) \frac{(at)^{n-1}}{(1+at)^{n+1}} + E\{x(t)\}$$

$$= \sum_{n=2}^{\infty} n(n-1) \frac{(at)^{n-1-2+2}}{(1+at)^{n+1-2+2}} + 1$$

$$= \sum_{n=2}^{\infty} n(n-1) \frac{(at)^{n-2} \cdot (at)}{(1+at)^{n-2}(1+at)^3} + 1$$

$$= \frac{(at)}{(1+at)^3} \sum_{n=2}^{\infty} n(n-1) \frac{(at)^{n-2}}{(1+at)^{n-2}} + 1$$

$$= \frac{at}{(1+at)^3} \left[2 + 3 \cdot 2 \frac{at}{1+at} + 4 \cdot 3 \frac{(at)^2}{(1+at)^2} + \cdots \right] + 1$$

$$= \frac{2\,at}{(1+at)^3} \left[1 + 3 \frac{at}{(1+at)} + 2 \cdot 3 \frac{at}{(1+at)^2} + \cdots \right] + 1$$

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$$= \frac{2at}{(1+at)^3}\left[1-\frac{at}{1+at}\right]^{-3}+1$$

$$= \frac{2at}{(1+at)^3}\left[\frac{1+at-at}{1+at}\right]^{-3}+1$$

$$= \frac{2at}{(1+at)^3}\left[\frac{1}{1+at}\right]^{-3}+1$$

$$= \frac{2at}{(1+at)^3}\left[\frac{1}{1+at}\right]^{-3}+1$$

$$= \frac{2at}{(1+at)^3}(1+at)^{3}+1$$

$$= 2at+1$$

$$E\left[x^2(t)\right] = 2at+1$$

$$Var\,(x(t)) = 2at \cancel{+1-1}$$

$$Var\,(x(t)) = 2at$$

This process is non-stationary. This process is Evolutionary.

Stationary process definition:

Ex: 1

Let $x_n,\ n \geq 1$ be uncorrected random variables with mean 0 and variable 1.

Then $E(x_n) = 0$

$V(x_n) = 1$

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$$V\{x_n\} = E\{x_n^2\} - \{E(x_n)\}^2$$

$$1 = E\{x_n^2\} - 0$$

$$1 = E\{x_n^2\}$$

$$E\{x_n^2\} = 1$$

$$C(n,m) = cov(x_n, x_m)$$

$$= E\{x_n, x_m\} - E\{x_n\} . E\{x_m\}$$

$$= E\{x_n, x_m\}$$

$$C(n,m) = E\{x_n, x_m\} = 1 \quad \text{if} \quad n = m$$

$$= 0 \quad \text{if} \quad n \neq m$$

Thus it is co-variance stationary

since, i) $E\{x_n\} = 0$

mean of $\{x_n\}$ is independent of n.

ii) $cov(n,m) = f(n, m)$

which is a function difference between

the parameter m & n.

In particular if x_n's are Identically

distributed then $\{x_n, n \geq 1\}$ is strictly stationary

process.

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poission process :

Consider the process $\{x(t), t \in T\}$ and $pr\{x(t) = n\}$

$$= \frac{e^{-at}(at)^n}{n!} \quad \text{where } a>0,\ n=0,1,2\ldots$$

sol :

Given $Pr\{x(t)=n\} = \dfrac{e^{-at}(at)^n}{n!}$

poisson process $p(x=r) = \dfrac{e^{-\lambda}\lambda^r}{r!}$

We know that

$$M(t) = E\{x(t)\}$$

$$= \sum_{n=0}^{\infty} x(t)\ Pr\{x(t)=n\}$$

$$= \sum_{n=0}^{\infty} n \frac{e^{-at}(at)^{n}}{n!}$$

$$= \sum_{n=0}^{\infty} n. \frac{e^{-at}(at)^{n-1}(at)}{n(n-1)!}$$

$$= at \sum_{n=1}^{\infty} \cancel{n} \frac{e^{-at}(at)^{n-1}}{\cancel{n}(n-1)!}$$

$$= at\ e^{-at} \sum_{n=1}^{\infty} \frac{(at)^{n-1}}{(n-1)!}$$

$$= at\ e^{-at} \left[1 + at + \frac{(at)^2}{2!} + \ldots\right]$$

$$= at\ \cancel{e^{-at}}.\cancel{e^{at}}$$

$E(x(t)) = M(t) = at.$

$Var[x(t)] = E\{x^2(t)\} - [E(x(t))]^2$

$$[E(x^2(t))] = \sum_{n=0}^{\infty} x^2(t)\ Pr\{x(t)=n\}$$

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$$= \sum_{n=0}^{\infty} n^2 \frac{e^{-at}(at)^n}{n!}$$

$$= \sum_{n=0}^{\infty} (n(n-1)+n) \frac{e^{-at}(at)^n}{n!}$$

$$= \sum_{n=0}^{\infty} n(n-1) \frac{e^{-at}(at)^n}{n!} + \sum_{n=0}^{\infty} \frac{n e^{-at}(at)^n}{n!}$$

$$= \sum_{n=0}^{\infty} n(n-1) \frac{e^{-at}(at)^n}{n!} + E(x(t))$$

$$= \sum_{n=0}^{\infty} n(n-1) \frac{e^{-at}(at)^{n-2+2}}{n(n-1)(n-2)!} + at$$

$$= (at)^2 e^{-at} \sum_{n=2}^{\infty} n\cancel{(n-1)} \frac{(at)^{n-2}}{n\cancel{(n-1)}(n-2)!} + at$$

$$= e^{-at}(at)^2 \sum_{n=2}^{\infty} \frac{(at)^{n-2}}{(n-2)!} + at$$

$$= e^{-at}(at)^2 \left[1 + at + \frac{(at)^2}{2!} + \dots\right] + at$$

$$= e^{-at}(at)^2[e^{at}] + at$$

$$= e^{-at+at}(at)^2 + at$$

$$= e^{0}(at)^2 + at$$

$$= (at)^2 + at$$

$$E(x(t)) = (at)^2 + at$$

$$Var(x(t)) = \cancel{(at)^2} + at - \cancel{(at)^2}$$

$$Var(x(t)) = at$$

$\therefore$ This process is non-stationary so this process is evolutionary

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Example: Find mean and co-variance

1. Consider the process $X(t) = A_1 + A_2(t)$. where A_1 & A_2 are independent random variables with $E(A_i) = a_i$, $V(A_i) = \sigma_i^2$, $i = 1, 2 \ldots$

Sol:

$$\text{Given } X(t) = A_1 + A_2(t)$$

$$\text{Mean} = E(X(t))$$

$$= E[A_1 + A_2 t]$$

$$\text{Mean} = E(A_1) + E(A_2)t$$

$$\text{put } E(A_1) = a_1 ; \quad E(A_2) = a_2$$

$$\text{Mean} = a_1 + a_2 t$$

$$M(t) = a_1 + a_2 t$$

$$Cov(s,t) = Cov\{X(t), X(s)\}$$

$$= E\{X(t) . X(s)\} - E\{X(t)\} . E\{X(s)\}$$

$$= E[\{A_1 + A_2 t\}\{A_1 + A_2 s\}]$$

$$= E[A_1^2 + A_1 A_2 s + A_1 A_2 t + A_2^2 ts]$$

$$= E(A_1^2) + E(A_1 A_2) s + E(A_1 A_2) t + E(A_2)^2 ts$$

$$= E(A_1^2) + (s+t) E(A_1 A_2) + E(A_2)^2 ts$$

$$E\{X(t), X(s)\} = E(A_1^2) + (s+t) E(A_1) E(A_2) + E(A_2)^2 ts$$

$$V(A_j) = \sigma_i^2$$

$$\text{Var } X = E(X^2) - [E(x)]^2$$

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$$\sigma_i^2 = E(A_i^2) - [E(A_i)]^2$$

$$\sigma_i^2 + [E(A_i)]^2 = E(A_i^2)$$

$$E(A_i^2) = \sigma_i^2 + a_i^2$$

$$E\{X(t) \cdot X(s)\} = E(A_i^2) + (s+t)\, E(A_1) E(A_2) + E(A_2^2)\, ts$$

$$= (\sigma_1^2 + a_1^2) + (s+t)(a_1)(a_2) + (\sigma_2^2 + a_2^2)\, ts$$

$$Cov(X(t), X(s)) = E(X(s)\, X(t)) - E(X(s))\, E(X(t))$$

$$Cov(s,t) = [(\sigma_1^2 + a_1^2) + (s+t)(a_1)(a_2) + (\sigma_2^2 + a_2^2)\, ts]$$

$$- (a_1 + a_2 t)(a_1 + a_2 s)$$

$$= \sigma_1^2 + a_1^2 + t a_1 a_2 + s a_1 a_2 + \sigma_2^2 ts + a_2^2 ts - a_1^2 - a_1 a_2 s - a_1 a_2 t - a_2^2 st$$

$$Cov(s,t) = \sigma_1^2 + \sigma_2^2 st$$

Thus we see that $M(t) = a_1 + a_2 t$ which depends on 't'. $cov(s,t) = \sigma_1^2 + \sigma_2^2 st$ which is not a function of time difference $(t-s)$ $\{X(t)\ t \in T\}$ is not weekly stationary.

$\therefore$ It is not stationary

$\therefore$ The process $\{X(t)\ t \in T\}$ is evaluationary.

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2 Consider the process $X(t) = A \cos \omega t + B \sin \omega t$ where A and B are uncorrected random variable with mean 0 and variance 1 and ω is a +ve constant.

Sol

Given that

$$X(t) = A \cos \omega t + B \sin \omega t$$

$$\left.\begin{array}{ll} E(A) = 0, & E(B) = 0 \\ Var(A) = 1 & var(B) = 1 \end{array}\right\} \rightarrow (1)$$

$$1 = E(A^2) - 0$$

$$E(A^2) = 1$$

$$Var(B) = E(B^2) - [E(B)]^2$$

$$1 = E(B^2) - 0$$

$$E(B^2) = 1$$

$$\text{Mean} = M(t) = E(X(t))$$

$$= E[A\cos\omega t + B\sin\omega t]$$

$$= E[A\cos\omega t] + E[B\sin\omega t]$$

$$= \cos\omega t\, E(A) + \sin\omega t\, E(B)$$

$$M(t) = 0$$

$$C(s,t) = \operatorname{cov}\{X(t), X(s)\}$$

$$= E\{X(t).X(s)\} - E[x(t)].E[x(s)]$$

$$= E[(A\cos\omega t + B\sin\omega t).(A\cos\omega s + B\sin\omega s)]$$

$$\operatorname{cov}[x(t), X(s)] = \cos\omega t\cos\omega s\, E(A^2) + [(\cos\omega t\sin\omega s)(\sin\omega t\cos\omega s)]E(AB)$$
$$+ \sin\omega t\sin\omega s\, E(B^2) \rightarrow (2)$$

$$Var(A) = E(A^2) - [E(A)]^2$$

$$1 = E(A^2) - 0$$

$$E(A^2) = 1 \quad \text{and} \quad E(B^2) = 1$$

Given that A and B are uncorrected random variable

$$V(A,B) = 0$$

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$$\frac{cov(A,B)}{\sigma_A \cdot \sigma_B} = 0$$

Since $\sigma_A \cdot \sigma_B = 1$

$$E(A) = E(B) = 0$$

$$cov(A,B) = 0$$

$$cov(A,B) = E(A,B) - E(A).E(B) = 0$$

equation (i) becomes.

$$cov(x(t), x(s)) = \cos\cos\cos\omega t + \sin\cos\sin\omega t$$

$$= \cos(\omega s - \omega t)$$

$$cov(x(t), x(s)) = \cos\omega(s-t)$$

which is a function of the time difference.

The process is co-variance stationary.

3. Let $x(t) = A_0 + A_1 t + A_2 t^2$ where A_i, $i = 0,1,2 \ldots$ are uncorrected random variable with mean 0 and variance 1 find the mean value function and co-variance function $[x(t), t \in T]$.

Sol: Given that

$$x(t) = A_0 + A_1 t + A_2 t^2$$

$$E(A_0) = 0, \quad E(A_1) = 0, \quad E(A_2) = 0$$

$$Var(A_0) = 1, \quad var(A_1) = 1, \quad var(A_2) = 1$$

$$var(A_0) = E(A_0^2) - [E(A_0)]^2$$

$$1 = E[A_0^2] - 0$$

$$\therefore E(A_0^2) = 1$$

$$var(A_1) = E(A_1^2) - [E(A_1)]^2$$

$$1 = E(A_1^2) - 0$$

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$\therefore\ E(A_1^2) = 1$

$$Var(A_2) = E(A_2^2) - [E(A_2)]^2$$

$$1 = E(A_2^2) - 0$$

$$\therefore\ E(A_2^2) = 1$$

$$Mean = E[x(t)]$$

$$= E[A_0 + A_1 t + A_2 t^2]$$

$$= 0.$$

$$Cov[x(t), x(s)] = E[x(t).x(s)] - E(x(s)).E(x(t))$$

$$= E[(A_0 + A_1 t + A_2 t^2)(A_0 + A_1 s + A_2 s^2)] - (A_0 + A_1 t + A_2 t^2) \cdot (A_0 + A_1 s + A_2 s^2).$$

$$= E(A_0^2) + A_0 A_1 s + A_0 A_2 s^2 + A_1 A_0 t + A_1^2 st + A_1 A_2 st^2 + A_2 A_0 t^2 + A_1 A_2 t^2 s + A_2^2 t^2 s^2].$$

$$= E(A_0^2) + s E(A_0 A_1) + s^2 E(A_0 A_2) + E(A_1 A_0) t + E(A_1^2) st + ts^2 E(A_1 A_2) + t^2 E(A_2 A_0) + t^2 s E(A_1 A_2) + E(A_2^2) t^2 s^2$$

$$= 1 + st + s^2 t^2.$$

$$Cov[x(t), x(s)] = 1 + st + s^2 t^2 - st + st$$

$$= 1 + 2st + s^2 t^2 - st$$

$$= (1 + st)^2 - st$$

$$\therefore\ Cov[x(t), x(s)] = (1 + st)^2 - st.$$

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4. Let $x(t) = \sum_{r=1}^{K} [A_r \cos\theta_r t + B_r \sin\theta_r t]$ where A_r, B_r are uncorrected random variable with mean 0 and variance σ^2 and θ_r are constant show that $[x(t), t \geq 0]$ is co-variance stationary.

Sol:

$$\text{Given } x(t) = \sum_{r=1}^{k} [A_r \cos\theta_r t + B_r \sin\theta_r t]$$

$$E(A_r) = 0 \quad , \quad E(B_r) = 0$$

$$var(A_r) = \sigma^2 \quad , \quad var(B_r) = \sigma^2$$

$$var(A_r) = E(A_r^2) - [E(A_r)]^2$$

$$\sigma^2 = E(A_r^2) - 0$$

$$E(A_r^2) = \sigma^2$$

$$var(B_r) = E(B_r^2) - [E(B_r)]^2$$

$$\sigma^2 = E(B_r^2) - 0$$

$$\therefore E(B_r^2) = \sigma^2$$

$$\text{Mean } M(t) = E[x(t)]$$

$$= E\left[\sum_{r=1}^{K} A_r \cos\theta_r t + B_r \sin\theta_r t\right]$$

$$= \sum_{r=1}^{k} [\cos\theta_r t \, E(A_r) + \sin\theta_r t \, E(B_r)]$$

$$M(t) = 0$$

$$cov[x(t), x(s)] = E[x(t), x(s)] - E(x(t)) . E(x(s))$$

$$= E\left[\sum_{r=1}^{K} (A_r \cos\theta_r t + B_r \sin\theta_r t) . (A_r \cos\theta_r s + B_r \sin\theta_r s)\right]$$

$$= E\left[\sum_{r=1}^{k} (A_r \cos\theta_r s + B_r \sin\theta_r s) . (A_r \cos\theta_r t + B_r \sin\theta_r t)\right]$$

$$= E\left[\sum_{r=1}^{k} (A_r^2 \cos\theta_r s \cos\theta_r t + A_r B_r \cos\theta_r s \sin\theta_r t + B_r A_r \sin\theta_r s \cos\theta_r t + B_r^2 \sin\theta_r . s \sin\theta_r t\right].$$

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$$= E\left[\sum_{r=1}^{K} A_r^2 \cos\theta_r t \cos\theta_r s + A_r B_r (\cos\theta_r s \sin\theta_r t + B_r A_r (\sin\theta_r s \cos\theta_r t) + B_r^2 (\sin\theta_r s \sin\theta_r t)\right]$$

$$= \sum_{r=1}^{K} [\cos\theta_r s \cos\theta_r t \, E(A_r^2) + E(A_r B_r) \cos\theta_r s \sin\theta_r t + E(B_r A_r) \sin\theta_r s \cos\theta_r t + E(B_r^2) \sin\theta_r s \sin\theta_r t]$$

$$= \sum_{r=1}^{K} [\cos\theta_r s \cos\theta_r t \, \sigma^2 + \sigma^2 \sin\theta_r s \sin\theta_r t]$$

$$= \sigma^2 \sum_{r=1}^{K} (\cos\theta_r s \cos\theta_r t + \sin\theta_r s \sin\theta_r t)$$

$$= \sigma^2 \sum_{r=1}^{K} (\cos(\theta_r s - \theta_r t))$$

$$\text{Cov}(s,t) = \sigma^2 \sum_{r=1}^{K} \cos\theta_r (s-t)$$

which is independent of t

The Given process $[X(t), t \geq 0]$ is co-variance stationary.

Markov chain :

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The stochastic process $\{x_n, n = 0, 1, \dots n\}$ is called a markov chain. If for $j, K, j_1, j_2 \dots j_{n-1} \in N$

$$P_r\{X_n = K / X_{n-1} = j, x_{n-2} = j_1, x_{n-3} = j_2 \dots x_{n-n} = j_{n-1}\}$$

$$= P_r\{X_n = K / x_{n-1} = j\}$$

$$= P_{jK}^{(1)}$$

$x_n, n = \dots$ $\{x(t), t \in T\}$ j, K j_1, j_2 $j_n \in N$

Whenever the 1st transition matrix is define

The outcomes are called the state of the Markov chain.

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Order of the Markov chain:

The Markov chain x_n is said to be a order of 's'. where $s = 0, 1, 2 \ldots$ if for all 'n'.

order : 1

The markov chain x_n is said to be a order 1 probability $(P_r)\{x_n = k / x_{n-1} = j, x_{n-2} = j_1 \ldots x_{n-s} = j_{s-1}\}$

problem:

1. consider the markov chain $\{x_n\}$ $n \geq 1$ with states 0, 1, 2 with transition

5M 10M

$$P = \begin{array}{c c} & \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \end{array} & \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix} \end{array}$$ with intial distribution

$$P\{x_0 = i\} = 1/3 \quad i = 0, 1, 2 \ldots$$

Find i) $P_r\{x_1 = 1 / x_0 = 2\}$

ii) $P_r\{x_2 = 2 / x_1 = 1\}$

iii) $P_r\{x_2 = 2, x_1 = 1 / x_0 = 2\}$

iv) $P_r\{x_2 = 2, x_1 = 1, x_0 = 2\}$

v) $P_r\{x_3 = 1, x_2 = 2, x_1 = 1, x_0 = 2\}$

sol:

Given $P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$ with initial distribution. $P(x_0 = i) = 1/3 \quad i = 0, 1 \ldots$

i) $P_r\{x_1 = 1 / x_0 = 2\} = P_{21}^{(1)} = 3/4$

ii) $P_r\{x_2 = 2 / x_1 = 1\} = P_{12}^{(1)} = 1/4$

iii) $P_r\{x_2 = 2, x_1 = 1 / x_0 = 2\} = P_r\{x_2 = 2 / x_1 = 1\} P_r\{x_1 = 1 / x_0 = 2\}$

$= P_{12}^{(1)} P_{21}^{(1)} = 1/4 \cdot 3/4 = 3/16$

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iv) $Pr\{X_2=2, X_1=1, X_0=2\}$

$= Pr\{X_2=2/X_1=1\}\ Pr\{X_1=1/X_0=2\} . Pr\{X_0=2\}$

$= P_{12}^{(1)} . P_{21}^{(1)} . 1/3 \quad = 1/4 . 3/4 . 1/3$

$= 1/16.$

v) $Pr\{X_3=1, X_2=2, X_1=1, X_0=2\}$

$= Pr\{X_3=1/X_2=2\}\ Pr\{X_2=2/X_1=1\}\ Pr\{X_1=1/X_0=2\}\ Pr\{X_0=2\}$

$= P_{21}^{(1)}\ P_{12}^{(1)}\ P_{21}^{(1)} . 1/3$

$= 3/4 . 1/4 . 3/4 . 1/3$

$= 3/64.$

2 Consider a markov chain $\{X_n\}$ $n = 0, 1, 2 \dots$ 3 states 0, 1, 2 with transition probability matrix.

$$P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$$

with initial distribution $P\{X_0 = i\} = 1/3$ $i = 0, 1, 2..$

Find i) $Pr\{X_{n+2} = 1 / X_n = 0\}$

ii) $Pr\{X_2 = 1 / X_0 = 0\}$

Sol:

Given $P = \begin{pmatrix} 3/4 & 1/4 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 3/4 & 1/4 \end{pmatrix}$

with initial distribution $P(X_0 = i) = 1/3$

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Radha

UNIT – II Stochastic Process

classification of states and chain:

The state j, $j = 0,1,2,\ldots$ of a Markov chain $\{x_n, n \geq 0\}$ can be classified in a distinctive manner according to some fundamental property of system. 1

communication Relation: communication relation b/w two states?

If $P_{ij}^{(n)} > 0$ for some $n \geq 1$ then we say that state j can be reached (or) state j is accessible from state i. the relation is denoted by $i \rightarrow j$ $P_{ij}^{(n)} > 0$

If for all n, $P_{ij}^{(n)} = 0$ then j is not accessible from i, In notation $i \not\rightarrow j$. If two state i and j are such that each is accessible from the other. Then we say that the two state communicate. It is denoted by $i \leftrightarrow j$ then there exists an integer m and n $\ni P_{ij}^{(n)} > 0$ and $P_{ij}^{(m)} > 0$.

The relation $\rightarrow$ is transitive ie) If i transitive to j ($i \rightarrow j$) and $j \rightarrow k$ then $i \rightarrow k$

From chapmon kolomogorov equation

$$P_{ik}^{(m+n)} = \sum_{r} P_{ir}^{(m)} \cdot P_{rk}^{(n)}$$

when a communication relation is said to be transitivity

we get $P_{ik}^{(m+n)} \geq P_{ij}^{(m)} \cdot P_{jk}^{(n)}$ — and symmetric?

The relation $\longleftrightarrow$ is

i) Transitive ie) $i \rightarrow j$, $j \rightarrow k \Rightarrow i \rightarrow k$

ii) symmetric ie) $i \leftrightarrow j$ and $j \leftrightarrow i$

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class property :

A class of states is a subset of the state space such that every state of the class communicates with every other and there is no other state outside the class which communicates with all other states in the class. A property defined for all states of a Markov chain is a class property if its possession by one state in a class ⇒ its possession by all states of some class. One such property is periodicity of a state.

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periodicity :

state i is a return state if $P_{ii}^{(n)} > 0$ for some $n \geq 1$. The period d_i of a return state i is defined as the G.C.D of all m such that $P_{ii}^{(m)} > 0$ Thus $d_i = G.C.D \{m, P_{ii}^{(m)} > 0\}$

The state i is said to be aperiodic if $d_i = 1$ and periodic if $d_i > 1$. clearly state i is aperiodic if $P_{ii} \neq 0$.

classification of chain :

The P_r $P_{jk}^{(n)}$ is non-zero for some $n \geq 1$ then we say that the state k can be reached from the state j. If every state can be reached from every other state (in any no. of transition). The chain is said to be ~~irre~~ $\overset{uc}{}$ ~~irredible~~ irreducible. The TPM is the irreducible.

Definition :

closed set :

If c is a set of states such that no state outside c can be reached from any state in c, then c is said to be closed. if c is closed and $j \in c$, which $k \notin c$. Then $P_{jk}^{(n)} = 0 \ \forall n$

ie) c is closed

iff $\sum_{j \in c} P_{ij} = 0 \ \forall i \in c$

The submatrix $P_1 = (P_{ij})$, $i, j \in c$ is also stochastic and can be expressed in the canonical form as

$$P = \begin{pmatrix} P_1 & 0 \\ R_1 & Q \end{pmatrix}$$

properties of the closed set :

i) A closed set may contain one or more states. If a closed set contains only one state j then it is called absorbing state.

ie) state j is absorbing iff $P_{jj} = 1$ and $P_{jk} = 0$, $k \neq j$

ii) Every finite Markov chain contain atleast one closed set.

iii) If a markov chain does not contain any other closed set (with expection of set of all state) then the chain is called irreducible. The chain which are not irreducible is said be reducible or non-reducible.

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Remark :

i) The irreducible matrices may be subdivided into two classes.

i) primitive and

ii) Imprimitive

A Markov chain is primitive iff the corresponding matrix is primitive.

ii) In an irreducible chain all the states belong to the same class.

AUBARASI

Classification of states :

Suppose that a system starts with the state j. Let $f_{jk}^{(n)}$ be the Pr that it reaches the state k for the 1st time at the n^{th} step (or after n-transitions) and let $P_{jk}^{(n)}$ be the probability that it reaches state k (not necessarily for the first time after n transitions.

Let j_k be the 1st passage time state k

ie) $j_k = \min \{n \geq 1, x_n = k\}$ and $\{f_{jk}^{(n)}\}$ be the distribution of j_k given that the chain starts at state j. A relation can be established b/w $f_{jk}^{(n)}$ and $P_{jk}^{(n)}$ by means of first entrance.

Theorem :

First Entrance theorem

Whatever be the state j and k

$$P_{jk}^{(n)} = \sum_{r=0}^{n} f_{jk}^{(r)} P_{kk}^{(n-r)}, \quad n \geq 1$$

with $P_{kk}^{(0)} = 1$, $f_{jk}^{(0)} = 0$, $f_{jk}^{(1)} = P_{jk}$.

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Proof :

The probability that starting with j state k is reached for the first time at the r^{th} step and again after that $(n-r)^{th}$ steps is given by $f_{jk}^{(r)} \; P_{kk}^{(n-r)} \quad \forall \, r \leq n$

These cases are mutually exclusive.

hence we have

$$P_{jk}^{(n)} = \sum_{r=0}^{n} f_{jk}^{(r)} \, P_{kk}^{(n-r)}$$

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Remark :

The above result can also be written as

$$P_{jk}^{(n)} = \sum_{r=1}^{n-1} f_{jk}^{(r)} \, P_{kk}^{(n-r)} + f_{jk}^{(n)} \quad , \quad n \geq 1$$

$$\Big[P_{jk}^{(n)} = f_{jk}^{(0)} \, P_{kk}^{(n)} + \sum_{r=1}^{n-1} f_{jk}^{(r)} \, P_{kk}^{(n-r)} + f_{jk}^{(n)} \, P_{kk}^{(0)}$$

$$= 0 \cdot P_{kk}^{(n)} + \sum_{r=1}^{n-1} f_{jk}^{(r)} \, P_{kk}^{(n-r)} + f_{jk}^{(n)}$$

$$= \sum_{r=1} f_{jk}^{(r)} \, P_{kk}^{(n-r)} + f_{jk}^{(n)} \Big]$$

First passage time distribution :

Let F_{jk} denote the Pr that starting with state j the system will ever reach state k.

clearly,

$$F_{jk} = \sum_{n=1}^{\infty} f_{jk}(n) .$$

we have,

$$\sup_{n \geq 1} P_{jk}^{(n)} \leq F_{jk} \leq \sum_{m \geq 1} P_{jk}^{(m)} \quad \forall \, n \geq 1$$

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we have to consider two cases 6

$F_{jk} = 1$ and $F_{jk} < 1$

when $F_{jk} = 1$ it is certain that the system starting with state j will reach state k.

In this case $\{f_{jk}^{(n)}, n = 1, 2, \ldots\}$ is a proper probability distribution and this gives the 1st passage time distribution for k given that the system starts with j.

The mean (1st passage) time from state j to state k is given by

$$\mu_{jk} = \sum_{n=1}^{\infty} n \cdot f_{jk}^{(n)}$$

In particular when $k = j$.

$\{f_{jj}^{(n)}, n = 1, 2, \ldots\}$ will represent the distribution of the recurrence time of j and $F_{jj} = 1$ will imply that the return to the state j is certain. In this case

$$\mu_{jj} = \sum_{n=1}^{\infty} n \cdot f_{jj}^{(n)}$$

is know as the mean recurrence time for the state j.

Definition:

A state j is said to be persistent (the word recurrent is also used by some authors)

if $F_{jj} = 1$

ie) (return to state j is certain) and transient if $F_{jj} < 1$ ie) return to state j is uncertain.

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A persistent state j is said to be null persistent iff $\mu_{jj} = \infty$.

i.e) If the mean recurrence time is infinite and is said to be non-null presistent if $\mu_{jj} < \infty$

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A persistent non-null and a periodic state of a Markov chain is said to be ergodic.

Theorem:

state j is persistent iff $\sum_{n=0}^{\infty} P_{jj}^{(n)} = \infty$.

Proof:

consider the generating function of the sequences $\{f_{jj}^{(n)}\}$ and $\{P_{jj}^{(n)}\}$

$\{f_{ji}^{(n)}\}$ and $P_{ji}^{(0)}$

They are given by

$$F_{jj}(s) = \sum_{n=0}^{\infty} f_{jj}^{(n)} s^n \qquad |s| < 1$$

$$= \sum_{n=1}^{\infty} f_{jj}^{(n)} s^n \qquad [\because f_{jj}^{(0)} = 0]$$

$$P_{jj}(s) = \sum_{n=0}^{\infty} P_{jj}^{(n)} s^n \qquad |s| < 1$$

$$= 1 + \sum_{n=1}^{\infty} P_{jj}^{(n)} s^n \qquad [\because P_{jj}^{(0)} = 1]$$

$$\Rightarrow P_{jj}(s) - 1 = \sum_{n=1}^{\infty} P_{jj}^{(n)} s^n \longrightarrow (1)$$

From First entrance theorem we have

$$P_{jj}^{(n)} = \sum_{r=0}^{n} f_{jj}^{(r)} P_{jj}^{(n-r)}$$

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Multiply both sides by $\sum_{n=1} s^n$ and $\forall n \geq 1$ we get

$$\sum_{n=1}^{\infty} P_{jj}^{(n)} s^n = \sum_{n=1}^{\infty} \left[\sum_{r=0}^{n} f_{jj}^{(r)} P_{jj}^{(n-r)} \right] s^n \longrightarrow (2)$$

$$= \sum_{n=1}^{\infty} \left[\sum_{r=0}^{n} f_{jj}^{(r)} P_{jj}^{(n-r)} s^r s^{-r} \right] s^n$$

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$$= \sum_{n=1}^{\infty} \left[\sum_{r=0}^{n} f_{jj}^{(r)} s^r P_{jj}^{(n-r)} s^{n-r} \right]$$

$$\Rightarrow P_{jj}^{(s)} - 1 = f_{jj}^{(s)} P_{jj}^{(s)} \quad [\text{From } (1)]$$

$\therefore$ R.H.S. of (2) is convolution of the $\{F_{jj}\}$ and $\{P_{jj}\}$ and that the generation function of the convolution is the product of two generating function.

$$\therefore P_{jj}^{(s)} - F_{jj}^{(s)} \cdot P_{jj}^{(s)} = 1$$

$$\Rightarrow P_{jj}^{(s)} \left[1 - F_{jj}^{(s)} \right] = 1$$

$$\Rightarrow P_{jj}^{(s)} = \frac{1}{1 - F_{jj}^{(s)}} \longrightarrow (s)$$

Assuming j is persistent we have

$$F_{jj}^{(s)} = 1$$

[(a) If $\sum_{k=0}^{\infty} a_k$

then $\lim_{s \to t} \sum_{k=0}^{\infty} a_k s^k = \sum_{k=0}^{\infty} a_k = a$

b) If $a_k > 0$ and if $\lim_{s \to t} \sum_{k=0}^{\infty} a_k s^k = \infty$

then $\sum_{k=0}^{\infty} a_k = \lim_{N \to \infty} \sum_{k=0}^{N} a_k = a$]

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using Abel's theorem we have (Lemma)

$$\lim_{s \to 1} F_{jj}^{(s)} = 1$$

(9)

$$\Rightarrow \lim_{s \to 1} P_{jj}^{(s)} = \infty$$

But $P_{jj}^{(s)} = \sum_{n=0}^{\infty} P_{jj}^{(n)} s^n$

So, $\lim_{s \to 1} P_{jj}^{(s)} = \lim_{s \to 1} \sum_{n=0}^{\infty} P_{jj}^{(n)} s^n$

since the co-efficient of $P_{jj}^{(s)}$ are non-negative Abel's lemma applies (part b) and we get

$$\sum P_{jj}^{(n)} = \infty$$

conversely,

If $\sum_{n=0}^{\infty} P_{jj}^{(n)} s^n = \infty$

Then we have to prove that $F_{jj} = 1$

From equation (3) $P_{jj}(s) = \dfrac{1}{1 - F_{jj}(s)}$

$$\lim_{s \to 1} P_{jj}(s) = \lim_{s \to 1} \sum_{n=0}^{\infty} P_{jj}^{(n)} s^n = \infty$$

$$\Rightarrow \frac{1}{1 - F_{jj}(s)} = \infty \text{ as } s \longrightarrow 1$$

$$F_{jj}(s) = 1 \text{ as } s \longrightarrow 1$$

$$\Rightarrow F_{jj} = 1$$

$\Rightarrow$ The state j is persistent.

Hence the proof.

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ARULMOZHI

Theorem:

The state j is transient when $\sum_{n=0}^{\infty} P_{jj}^{(n)} < \infty$

proof:

consider the generating function of $\{f_{jj}^{(n)}\}$ and $\{P_{jj}^{(n)}\}$

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They are given by

$$F_{jj}(s) = \sum_{n=0}^{\infty} f_{jj}^{(n)} s^n \quad , \quad |s| < 1.$$

$$P_{jj}(s) = \sum_{n=0}^{\infty} P_{jj}^{(n)} s^n \quad , \quad |s| < 1$$

Now,

$$P_{jj}(s) = 1 + \sum_{n=1}^{\infty} P_{jj}^{(n)} s^n$$

$$P_{jj}(s) - 1 = \sum_{n=1}^{\infty} P_{jj}^{(n)} s^n \longrightarrow (1)$$

we have,

$$P_{jj}^{(n)} = \sum_{r=0}^{n} f_{jj}^{(r)} P_{jj}^{(n-r)} \quad [\text{C-K equation}]$$

Multiplying by s^n and adding $\forall\, n \geq 1$ we get

$$\sum_{n=1}^{\infty} P_{jj}^{(n)} s^n = \sum_{n=1}^{\infty} \left[\sum_{r=0}^{n} f_{jj}^{(r)} P_{jj}^{(n-r)} \right] s^n$$

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$$\Rightarrow P_{jj}(s) - 1 = F_{jj}(s)\, P_{jj}(s)$$

$$P_{jj}(s) - F_{jj}(s)\, P_{jj}(s) = 1$$

$$P_{jj}(s) = \frac{1}{1 - F_{jj}(s)} \longrightarrow (2)$$

Suppose that the state j is transistent then by Abel's lemma we get,

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$$\lim_{s \to 1} F_{jj}(s) < 1$$

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$$\Rightarrow 1 - F_{jj} > 0$$

From equation (2) $P_{jj}(s)$ is finite

ie) $P_{jj}(s) < \infty$

But $P_{jj}(s) = \sum_{n=0}^{\infty} P_{jj}^{(n)} \cdot s^n < \infty$

$\because P_{jj}(s) < \infty$

$= \sum_{n=0}^{\infty} P_{jj}^{(n)} s^n$ is an convergent series.

$\Rightarrow \sum_{n=0}^{\infty} P_{jj}^{(n)} < \infty$ (as $s \to 1$)

Hence the proof.

Note :

By combining the above two theorems we can say ~~that the~~

i) state j is persistent if $\sum P_{jj}^{(n)} = \infty$

ii) state j is transient if $\sum P_{jj}^{(n)} < \infty$

SANUBATHI

Example :

Consider a Markov chain with TPM

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

The chain is irreducible as the chain is irreducible.

[$\because$ state 1 can be reached from state 0 with probability 1 and state 0 can be

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reached from state 1 with Pr 1/2 state 2 can be reached from state 1 with probability 1/2 and state 1 can be reached from state 2 with probability 1].

Solution :

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$$P = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0+1/2+0 & 0+0+0 & 0+1/2+0 \\ 0+0+0 & 1/2+0+1/2 & 0+0+0 \\ 0+1/2+0 & 0+0+0 & 0+1/2+0 \end{pmatrix}$$

$$= \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix}$$

$$P^3 = \begin{pmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1/2+0+1/2 & 0 \\ 0+1/2+0 & 0 & 0+1/2+0 \\ 0 & 1/2+0+1/2 & 0 \end{pmatrix}$$

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$$= \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{pmatrix} = P$$

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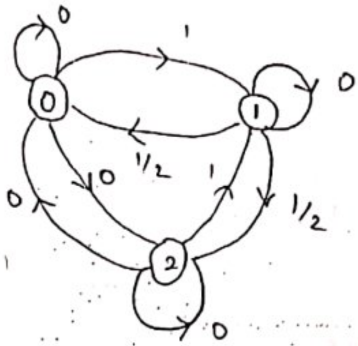
$$P^4 = P^3 \cdot P = P \cdot P = P^2$$



In general

$$P^{2n} = P^2$$

$$P^{2n+1} = P$$

$P^2 = P^{2n}$

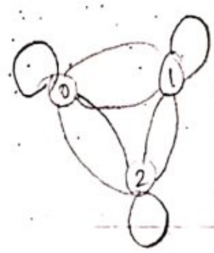
$P^{2n+1} = P$

$P_{11}^{(3)} \ldots 1$

$$P_{11}^{(2)} = 1 \quad (\text{from } P^2)$$

$$P_{11}^{(3)} = 0 \quad (\text{from } P^3)$$

$$P_{11}^{(4)} = 0 \quad (\text{from } P^4)$$



In general

$$P_{11}^{2n} = 1 \text{ and } P_{11}^{2n+1} = 0$$

so that

$$P_{ii}^{2n} > 0 \text{ and } P_{ii}^{(2n+1)} = 0 \text{ for each } i$$

we see that

$$f_{11}^{(1)} = 0$$

$$f_{11}^{(2)} = 1$$

$$F_{11} = \sum_{n=1}^{\infty} f_{11}^{(n)} = f_{11}^{(1)} + f_{11}^{(2)} = 0 + 1 = 1$$

$$\Rightarrow F_{11} = 1$$

$\Rightarrow$ state 1 is persistent and hence other states 0 and 2 are also presistent

$$\mu_{11} = \sum_n n f_{11}^{(n)} \qquad \left[\because \mu_{jj} = \sum_n n f_{jj}^{(n)}\right]$$

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$\mu_{11} = 1 \cdot f_{11}^{(1)} + 2 \cdot f_{11}^{(2)} = 1 \cdot 0 + 2 \cdot 1 = 2 < \infty$

The state 1 is non-null persistent

Also the state of chain are periodic each with period 2.

Also $p_{11}^{(2n)} \to \frac{t}{\mu_{11}} = \frac{2}{2} = 1 \ \forall n$

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Lemma:

Basis Limit theorem of renewable theorem.

Proof:

Let $\{f_n\}$ be a sequence such that $f_n \geq 0$, $\sum f_n = 1$ and $(t \geq 1)$ be the greatest common divisor (G.C.D) of those n for which $f_n > 0$

~~Let $\{u_n\}$ be another sequence $\ni u_0 = 1$~~

and $u_n = \sum_{r=1}^{n} f_r u_{n-r} \ (n \geq 1)$

Then $\lim_{n \to \infty} u_{nt} = t/\mu$

where $\mu = \sum_{n=1}^{\infty} n f_n$

The limit being zero when $\mu = \infty$ and $\lim_{N \to \infty} u_N = 0$ whenever N is not divisible by t.

Theorem:

If state j is persistent non-null then as $n \to \infty$

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i) $p_{jj}^{(nt)} \longrightarrow t/\mu_{jj}$ when state j is periodic with period t and

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ii) $p_{jj}^{(n)} \longrightarrow \dfrac{1}{\mu_{jj}}$ when state j is periodic

In case state j is persistent null (whether periodic or a periodic) then $p_{jj} \to 0$ as $n \to \infty$.

proof:

By first entrance theorem we have

$$p_{jk}^{(n)} = \sum_{r=0}^{n} f_{jk}^{(r)} \, p_{kk}^{(n-r)} \quad \text{put } k=j$$

$$p_{jj}^{(n)} = \sum_{r=0}^{n} f_{jj}^{(r)} \, p_{jj}^{(n-r)}$$

where

$$f_{jk}^{(1)} = p_{jk}, \quad f_{jk}^{(0)} = 0, \quad p_{kk}^{(0)} = 1$$

$$p_{jj}^{(n)} = f_{jj}^{(0)} \, p_{jj}^{(n)} + \sum_{r=1}^{n} f_{jj}^{(r)} \, p_{jj}^{(n-r)}$$

$$= \sum_{r=1}^{n} f_{jj}^{(r)} \, p_{jj}^{(n-r)} \longrightarrow (1) \quad \left[\because f_{jj}^{(0)} = 0\right]$$

Let state j be persistent

Then $\mu_{jj} = \sum_{n} n f_{jj}^{(n)}$ is defined

In the above lemma, we may put $f_{jj}^{(n)}$ for f_n, $p_{jj}^{(n)}$ for p_n and μ_{jj} for μ in the above lemma, then we get

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$$\lim_{n\to\infty} P_{jj}^{(nt)} = \frac{t}{\mu_{jj}} \longrightarrow (2)$$

(16)

$$\left[\because \lim_{n\to\infty} \mu_{nt} = t/\mu\right]$$

when state j is periodic with period t

when state j is aperiodic

ie) when t=1 then equation (2)

$$\Rightarrow \lim_{n\to\infty} P_{jj}^{(nt)} = \frac{1}{\mu_{jj}} \longrightarrow (3)$$

In case state j is persistent null the $\mu_{jj} = \infty$.

$\therefore$ Equation (3) become $\lim_{n\to\infty} P_{jj}^{(n)} \to 0$.

Note:

i) If j is persistent non-null. Then $\lim_{n\to\infty} P_{jj}^{(n)} > 0$.

ii) If j is persistent null or transient then $\lim_{n\to\infty} P_{jj}^{(n)} \to 0$.

Problems:-

1) Consider a Markov chain with transition matrix

JAN/AN 1-3

$$P = \begin{array}{c|cccc} & 1 & 2 & 3 & 4 \\ \hline 1 & & & & \\ 2 & & & & \\ 3 & & & & \\ 4 & & & & \end{array}$$

$$P = \begin{array}{cc} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1/4 & 1/8 & 1/8 & 1/2 \end{pmatrix} \end{array}$$

Find all state are ergodic.

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soln.

The chain is irreducible consider state 4 we have (1)

$$f_{jk} = P_{jk}$$

$$f_{44}^{(1)} = P_{44} = 1/2 > 0$$

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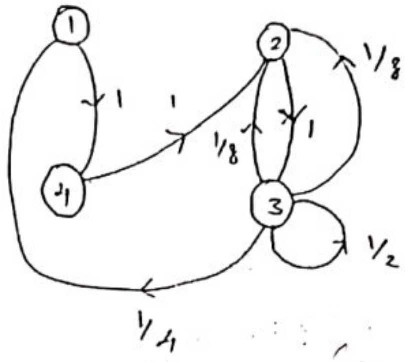

$\Rightarrow$ state is periodic

and $f_{44}^{(2)} = 1/8$

starting from (4) and reaching (4) by two stages.

ie) $[4 \to 2$ and $2 \to 4 = 1/8 \times 1 = 1/8]$

$$f_{44}^{(3)} = \frac{1}{8} \quad [\text{ie } 4 \to 3 \text{ and } 3 \to 4 = 1/8 \times 0 = 0]$$

$$f_{44}^{(4)} = \frac{1}{4} \quad [\text{ie } 4 \to 4 \text{ and } 4 \to 4 = 1/2 \times 1/2 = 1/4]$$

$$f_{44}^{(n)} = 0 \quad n > 4$$

$$F_{44} = f_{44}^{(1)} + f_{44}^{(2)} + f_{44}^{(3)} + f_{44}^{(4)}$$

$$F_{44} = \frac{1}{2} + \frac{1}{8} + \frac{1}{8} + \frac{1}{4} = 1$$

$$\mu_{jj} = \sum_n n\, f_{jj}^{(n)}$$

and $\mu_{44} = 1 \cdot 1/2 + 2 \cdot 1/8 + 3 \cdot 1/8 + 4 \cdot 1/4$

$$= \frac{17}{8} < \infty$$

The state 4 is ergodic

Hence all states are ergodic.

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2) Let $\{X_n, n \geq 0\}$ be a Markov chain with state space $S = \{1, 2, 3, 4\}$ and transition matrix.

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$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1/3 & 2/3 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix} \end{array}$$

soln:

$$f_{jk} = P_{jk} \qquad [\because j = k]$$

$$f_{jj} = P_{jj} \qquad j = 1, 2, 3, 4$$

$$f_{33}^{(1)} = P_{33} = \frac{1}{2}$$

$$f_{33}^{(2)} = f_{33}^{(3)} = \cdots\cdots 0.$$

[ie) $3 \to 2$ and $2 \to 3 = 0$ $\quad 3 \to 3$ and $3 \to 3 = 0$]

so that $F_{33} = 1/2$

$$F_{33} = 1/2 < 1$$

state 3 is transient

Again (1)

$$f_{44} = P_{44} = 1/2 \qquad f_{44}^{(n)} = 0, \quad n \geq 2$$

So that, $F_{44} = 1/2$

$$F_{44} = 1/2 < 1$$

State 4 is transient

Now,

$$f_{11}^{(1)} = 1/3, \quad f_{11}^{(2)} = 2/3 \text{ and}$$

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$F_{11} = \frac{1}{3} + \frac{2}{3} = 1$ (19)

state 1 is persistent.

since $\mu_{jj} = \sum_n n f_{jj}^{(n)}$ for $j = 1$

$\mu_{11} = 1 \cdot f_{11}^{(1)} + 2 \cdot f_{11}^{(2)}$

$= 1 \cdot 1/3 + 2 \cdot 2/3$

$= 1/3 + 4/3$

$\mu_{11} = 5/3$

$\mu_{11} = 5/3 < \infty$

state 1 is non-null persistent

Again

$P_{11} = 1/3 > 0$

So state 1 is periodic

state 1 is exogodic

For state 2

$f_{22}^{(1)} = 0, \quad f_{22}^{(2)} = 1 \cdot 2/3 = 2/3$

$f_{22}^{(3)} = 1 \cdot 1/3 \cdot 2/3 = 2/9$

$f_{22}^{(4)} = 1 \cdot \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right) = \frac{2}{27} \cdots\cdots$

$\vdots$

$f_{22}^{(n)} = 1 \cdot (1/3)^{n-2} \frac{2}{3} \quad n \geq 2$

$\mu_{jk} = \sum_n n f_{jk}^{(n)}$

$\mu_{jj} = \sum_n n f_{jj}^{(n)}$

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For $j = 2$

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$$\mu_{22} = \sum_n n f_{22}^{(n)}$$

$$= 1 \cdot f_{22}^{(1)} + 2 \cdot f_{22}^{(2)} + \cdots + n f_{22}^{(n)} + \cdots$$

$$= 1 \cdot 0 + 2 \cdot \frac{2}{3} + 3 \cdot \frac{1}{3} \cdot \frac{2}{3} + 4 \left(\frac{1}{3}\right)^2 (2/3) + \cdots$$

$$= \frac{2}{3} \cdot 2 + 3 \cdot \frac{1}{3} \cdot \frac{2}{3} + 4 (1/3)^2 (2/3) + \cdots$$

$$= \frac{2}{3} \left[2 + 3 \cdot 1/3 + 4 (1/3)^2 + \cdots \right]$$

Multiple and divide by 3

$$= \frac{2}{3} \left[3 \cdot \frac{2}{3} + 3 \cdot 3 \left(\frac{1}{3}\right)^2 + 3 \cdot 4 (1/3)^3 + \cdots \right]$$

$$= \cancel{3} \cdot 2/\cancel{3} \left[2 \cdot 1/3 + 3 (1/3)^2 + 4 (1/3)^3 + \cdots \right]$$

$$= 2 \left[2 \cdot 1/3 + 3 (1/3)^2 + 4 (1/3)^3 + \cdots \right]$$

$$= 2 \left[(1 - 1/3)^{-2} - 1 \right]$$

$$= 2 \left[(2/3)^{-2} - 1 \right]$$

$$= 2 \left[(3/2)^2 - 1 \right]$$

$$= 2 (9/4 - 1)$$

$$= \cancel{2} (5/\cancel{4}_{2})$$

$$\mu_{22} = 5/2$$

$$F_{jj} = \sum_n f_{jj}^{(n)}$$

For $j = 2$

$$F_{22} = \sum_n f_{22}^{(n)}$$

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$$= f_{22}^{(1)} + f_{22}^{(2)} + f_{22}^{(3)} + f_{22}^{(4)} + \cdots$$

$$= 0 + 1 \cdot 2/3 + 1 \cdot (1/3)(2/3) + 1\,(1/3)^2(2/3)$$

$$= \sum_{k=2}^{\infty} (1/3)^{k-2}(2/3)$$

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$F_{22} = 1$

state 2 is persistent

since $\mu_{22} = 5/2$ is finite

state 2 is non-null persistent

$d_i = G.C.D\{m : p_{ii}^{(m)} > 0\}$

$$\left[\because d_i = G.C.D\{m : p_{ii}^{(m)} > 0\} = G.C.D\{m : f_{ii}^{(m)} > 0\}\right]$$

$\Rightarrow f_{22}^{(1)}, f_{22}^{(2)}, f_{22}^{(3)} \ldots$ which are ≥ 0

$\Rightarrow 0, 2/3, 2/9 \ldots$ which are ≥ 0

and $d_i = 1 > 0$

$\Rightarrow$ The state 2 is periodic

Thus state 2 is non-null persistent and a periodic.

$\Rightarrow$ state 2 is ergodic.

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Theorem:

If state k is persistent null, then for every j

$$\lim_{n\to\infty} P_{jk}^{(n)} \to 0$$

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and if state k is aperiodic, persistent non-null then

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$$\lim_{n\to\infty} P_{jk}^{(n)} \to \frac{F_{jk}}{\mu_{kk}}$$

Proof:

By First entrance theorem we have

$$P_{jk}^{(n)} = \sum_{r=0}^{n} f_{jk}^{(r)} P_{kk}^{(n-r)}$$

where,

$$f_{jk}^{(0)} = 0, \quad f_{jk}^{(1)} = P_{jk}; \quad P_{kk}^{(0)} = 1$$

$$\therefore P_{jk}^{(n)} = \sum_{r=1}^{n} f_{jk}^{(r)} P_{kk}^{(n-r)}$$

Let $n > m$

Then $P_{jk}^{(n)} = \sum_{r=1}^{m} f_{jk}^{(r)} P_{kk}^{(n-r)} + \sum_{r=m+1}^{n} f_{jk}^{(r)} P_{kk}^{(n-r)}$

$$P_{jk}^{(n)} \le \sum_{r=1}^{m} f_{jk}^{(r)} P_{kk}^{(n-r)} + \sum_{r=m+1}^{n} f_{jk}^{(r)} \to (1)$$

[$\because$ state k is persistent null

$P_{kk}^{(n-r)} \to 0$ as $n \to \infty$]

Further since $\sum_{m=1}^{\infty} f_{jk}^{(m)} < \infty$, $\sum_{r=m+1}^{n} f_{jk}^{(r)} \to 0$

as $n, m \to \infty$

by result (1)

$\Rightarrow P_{jk}^{(n)} \to 0$ as $n \to \infty$

From (1) $\Rightarrow$

$$P_{jk}^{(n)} - \sum_{r=1}^{m} f_{jk}^{(r)} P_{kk}^{(n-r)} \le \sum_{r=m+1}^{n} f_{jk}^{(r)} \to (2)$$

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Since state k is aperiodic persistent and non-null

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By the above theorem

$$p_{kk}^{(n-r)} \to \frac{1}{\mu_{kk}} \text{ as } n \to \infty$$

Also from equation (2) we get

$$\sum_{r=m+1}^{n} f_{jk}^{(r)} \longrightarrow 0 \text{ as } n, m \to \infty$$

by result (2).

$$\Rightarrow p_{jk}^{(n)} - \sum_{r=1}^{m} f_{jk}^{(r)} \leq 0$$

$$\Rightarrow p_{jk}^{(n)} \longrightarrow \sum_{r=1}^{m} f_{jk}^{(r)} p_{kk}^{(n-r)}$$

$$\Rightarrow p_{jk}^{(n)} \longrightarrow \frac{F_{jk}}{\mu_{kk}} \text{ as } n, m \to \infty$$

$$\therefore F_{jk} = \sum_{n=1}^{\infty} f_{jk}^{(n)}$$

Hence proved.

Theorem:

In an irreducible chain, all the states are of the same type. Thus are either all transient, all persistent null or all persistent non-null. All the states are aperiodic and in the latter case they all have the same period.

Proof:

Since the chain is irreducible. Every state can be reached from every other state if i, j are any two states Then i can be reached from j and j from i 24

ie) $P_{ij}^{(N)} = a > 0$ for some $N \geq 1$ and

$P_{ij}^{(M)} = b > 0$ for some $M \geq 1$ $\Big\} \rightarrow (1)$

we have,

$$P_{jk}^{(n+m)} = P_{jk}^{(m+n)} = \sum_{r} P_{jr}^{(m)} P_{rk}^{(n)}$$

$$\geq P_{jr}^{(m)} P_{rk}^{(n)} \text{ for each } r \rightarrow (2)$$

Hence,

$$P_{ii}^{(n+N+M)} \geq P_{ij}^{(N)} P_{jj}^{(n)} P_{ji}^{(M)} = ab\, P_{jj}^{(n)} \rightarrow (3) \text{ [by equ (1)]}$$

And

$$P_{jj}^{(n+N+M)} = P_{ji}^{(M)} P_{ii}^{(n)} P_{ij}^{(N)} = ab\, P_{ii}^{(n)} \rightarrow (4) \text{ [by equ(1)]}$$

From (3) and (4)

It is clear that the two series $\sum_{n} P_{ii}^{(n)}$ and $\sum_{n} P_{jj}^{(n)}$ converges or diverges together.

Thus the two states i and j are either both transient or both persistent.

Suppose that i is persistent null then $P_{ii}^{(n)} \rightarrow 0$ as $n \rightarrow \infty$

$\Rightarrow$ The state j is also persistent null ie) Both the states are persistent null.

Suppose that i is persistent non-null and has period t, then $P_{ii}^{(n)} > 0$ whenever n is a multiple of t.

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Now

$$P_{ii}^{(N+M)} \geq P_{ij}^{(N)} P_{ji}^{(M)} = ab > 0$$ (25)

So that $(N+M)$ is a multiple of t

From equation (4)

$$P_{ij}^{(N+M+n)} \geq ab\, P_{ii}^{(n)} > 0 \quad (a>0,\ b>0,\ P_{ii}^{(n)} > 0)$$

$\Rightarrow (N+M+n)$ is a multiple of t and so t is a period of state j also.

Corollary :

In a finite irreducible Markov chain all states are non-null persistent.

proof :

Let $S = \{1, 2, \dots k\}$ be the state space of the chain and P be its TPM. suppose if possible that state ~~1 is null~~ persistent. Thus all other states are null persistent This implies that

$$\lim_{n \to \infty} P_{ij}^{(n)} = 0 \quad \forall\, j \in S$$

Now

$$\sum_{j \in S} P_{ij}^{(n)} = 1 \quad \forall\, n$$

Thus since S is finite. we are lead to a contradiction. Hence all states must be non-null persistent.

Determination of Higher Transition probabilities:

Let P be the T.P.M of order $m \times m$

The n-step transition probability $p_{ij}^{(n)}$

ii) The elements of the matrix $P^{(n)}$ can be obtained as follows.

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step : 1

calculate eigen values $t_1, t_2, \ldots t_n$ using the equation

$$|P - tI| = 0$$

where I is the unit matrix of order m.

step : 2

For each eigen value t_i calculate right eigen vector x_i using the relation

$$(P - t_i I)\, x_i = 0 \quad \text{where } x_i = \begin{pmatrix} x_{i1} \\ x_{i2} \\ \vdots \\ x_{im} \end{pmatrix}$$

similarly for each eigen value t_i calculate the left eigen vector y_i using the relation

$$y_i'(P - t_i I) = 0 \quad \text{where } y_i = \begin{pmatrix} y_{i1} \\ y_{i2} \\ \vdots \\ y_{im} \end{pmatrix}$$

step - 3 :

calculate $y_i' x_i$ and $c_i = \dfrac{1}{y_i' x_i}$ and check $\sum_{i=1}^{m} c_i = 1$

step - 4 :

calculate $x_i y_i'$

step - 5 :

To find $P^{(n)} = \sum_{i=1}^{m} t_i^{(n)} c_i x_i y_i'$

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Problem :-

1) Consider the 2 state Markov chain with TPM $P = \begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix}$. where $0 < a, b < 1$

Find $\lim_{n \to \infty} P^n$.

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Soln:

Step- 1 :

calculation of eigen values (say t_1, t_2)

The characteristic equation is given by

$|P - t_i I| = 0$.

$$|P - t_i I| = 0 \Rightarrow \begin{vmatrix} 1-a-t & a \\ b & 1-b-t \end{vmatrix} = 0$$

ie) $(1-a-t)(1-b-t) - ab = 0$

$\Rightarrow 1-b-t-a+ab+at-t+bt+t^2-ab = 0$

$\Rightarrow 1-a-b+t(a+b-2)+t^2 = 0$

$\Rightarrow t^2+(a+b-2)t+(1-a-b) = 0$

$\Rightarrow t = 1, (1-a-b)$

The eigen values are $t_1 = 1, t_2 = 1-a-b$

Step - 2 :

calculation of right eigen vector x_1 and x_2 where $x_1 = \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix}$, $x_2 = \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix}$ corresponding t_1, t_2

we have $(P - t_1 I)x_1 = 0$ when $t_1 = 1$

ie) $(P - 1 \cdot I)\begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = 0$

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$$\Rightarrow \left[\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right] \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = 0$$

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$$\Rightarrow \begin{pmatrix} -a & a \\ b & -b \end{pmatrix} \begin{pmatrix} x_{11} \\ x_{12} \end{pmatrix} = 0$$

$$\Rightarrow -a x_{11} + a x_{12} = 0 \text{ and}$$

$$b x_{11} - b x_{12} = 0$$

ie) $-x_{11} + x_{12} = 0$

$$x_{11} - x_{12} = 0$$

$$x_{11} = x_{12}$$

Taking x_{11} = co-efficient of x_{12}, x_{12} = co-eff of x_{11}

set $x_{11} = 1$ then $x_{12} = 1$ $\therefore x_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

Now $t_2 = 1-a-b$ where $|t_2| < 1$

we have $(P - t_2 I) x_2 = 0$

ie) $\left[\begin{pmatrix} 1-a & a \\ b & 1-b \end{pmatrix} - \begin{pmatrix} 1-a-b & 0 \\ 0 & 1-a-b \end{pmatrix}\right] \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = 0$

$$\Rightarrow \begin{pmatrix} b & a \\ b & a \end{pmatrix} \begin{pmatrix} x_{21} \\ x_{22} \end{pmatrix} = 0$$

$$\Rightarrow b x_{21} + a x_{22} = 0$$

$$b x_{21} + a x_{22} = 0$$

$$b x_{21} = -a x_{22}$$

$$-b/a \, x_{21} = x_{22}$$

set $x_{21} = 1$ then $x_{22} = -b/a$

$$\therefore x_2 = \begin{pmatrix} 1 \\ -b/a \end{pmatrix}$$

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step - 3 :

calculation of left eigen vector (29)

corresponding to t_1, t_2

we have $Y_1'(P - t_1 I) = 0$ where $Y_1 = \begin{pmatrix} y_{11} \\ y_{12} \end{pmatrix}$

$Y_2'(P - t_2 I) = 0$ where $Y_2 = \begin{pmatrix} y_{21} \\ y_{22} \end{pmatrix}$

when $t_1 = 1$ we get

$$\begin{pmatrix} y_{11} & y_{12} \end{pmatrix} \begin{pmatrix} -a & a \\ b & -b \end{pmatrix} = 0$$

$$-ay_{11} + by_{12} = 0$$

$$ay_{11} - by_{12} = 0$$

$$ay_{11} = by_{12}$$

$$a/b \; y_{11} = y_{12}$$

set $y_{11} = 1$ then $y_2 = a/b$ $\therefore Y_1 = \begin{pmatrix} 1 \\ a/b \end{pmatrix}$

$$\Rightarrow Y_1' = \begin{pmatrix} 1 & a/b \end{pmatrix}$$

when $t_2 = 1-a-b$ we get

$$\begin{pmatrix} y_{21} & y_{22} \end{pmatrix} \begin{pmatrix} b & a \\ b & a \end{pmatrix} = 0$$

$$by_{21} + by_{22} = 0$$

$$ay_{21} + by_{22} = 0$$

$$\Rightarrow y_{21} + y_{22} = 0$$

$$y_{21} = -y_{22}$$

set $y_{21} = 1$ then $y_{22} = -1$

$$Y_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$Y_2' = \begin{pmatrix} 1 , & -1 \end{pmatrix}$$

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step - 4 :

calculation of y_i', x_i

$$y_1' x_1 = (1 \quad a/b) \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 1 + a/b$$

$$y_2' x_2 = (1 \quad -1) \begin{pmatrix} 1 \\ -b/a \end{pmatrix} = 1 + b/a$$

$$c_1 = \frac{1}{y_1' x_1} = \frac{1}{1+a/b} = \frac{b}{a+b}$$

$$c_2 = \frac{1}{y_2' x_2} = \frac{1}{1+b/a} = \frac{a}{a+b}$$

$$\sum_{i=1}^{2} c_i = c_1 + c_2$$

$$= \frac{b}{a+b} + \frac{a}{a+b} = \frac{a+b}{a+b} = 1$$

$$\sum_{i=1}^{2} c_i = 1$$

step - 5 :

calculation x_i, y_i'

$$x_1 y_1' = \begin{pmatrix} 1 \\ 1 \end{pmatrix} (1 \quad a/b) = \begin{pmatrix} 1 & a/b \\ 1 & a/b \end{pmatrix}$$

$$x_2 y_2' = \begin{pmatrix} 1 \\ -b/a \end{pmatrix} (1 \quad -1) = \begin{pmatrix} 1 & -1 \\ -b/a & b/a \end{pmatrix}$$

step - 6 :

calculation of p^n

$$p^n = \sum_{i=1}^{2} t_i^n c_i x_i y_i'$$

$$= t_1^n c_1 x_1 y_1' + t_2^n c_2 x_2 y_2'$$

$$= (1)^n \left(\frac{b}{a+b}\right) \begin{pmatrix} 1 & a/b \\ 1 & a/b \end{pmatrix} + (1-a-b)^n \left(\frac{a}{a+b}\right) \begin{pmatrix} 1 & -1 \\ -b/a & b/a \end{pmatrix}$$

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$$P^n = \frac{1}{a+b}\begin{pmatrix} b & a \\ b & a \end{pmatrix} + \frac{(1-a-b)^n}{a+b}\begin{pmatrix} a & -a \\ -b & b \end{pmatrix}$$

$$\lim_{n\to\infty} P^n = \begin{pmatrix} b & a \\ b & a \end{pmatrix}\left(\frac{1}{a+b}\right)$$

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$[\because |t_2| < 1 \Rightarrow |1-a-b| < 1 . t_2^n \text{ as } t \to 0]$

ie) $$\lim_{n\to\infty} P^n = \begin{pmatrix} \frac{b}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{a}{a+b} \end{pmatrix}$$

$$\therefore \lim_{n\to\infty} P_{i1} = \frac{b}{a+b} \quad i = 1, 2$$

$$\lim_{n\to\infty} P_{i2} = \frac{a}{a+b} \quad i = 1, 2$$

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2) consider the three state Markov chain with T.P.M

VENKATESAN

$$P = \begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.4 \\ 0.1 & 0.5 & 0.4 \end{pmatrix}$$

Find $P_{ij}^{(n)}$.

soln:-

calculation of eigen values t_1, t_2, t_3

The characteristic equation is given by

$|P - t_1 I| = 0$

$$\begin{vmatrix} 0.5-t & 0.3 & 0.2 \\ 0.2 & 0.4-t & 0.4 \\ 0.1 & 0.5 & 0.4-t \end{vmatrix} = 0$$

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$$\Rightarrow (0.5-t)\left[(0.4-t)^2 - 0.20\right] - 0.3\left[0.08 - 0.2t - 0.04\right] + 0.2\left[0.1 - 0.04 + 0.1t\right] = 0$$

$$\Rightarrow (0.5-t)\left[0.16 + t^2 - 0.8t - 0.20\right] - 0.3\left[0.04 - 0.2t\right] + 0.2\left[0.06 + 0.1t\right] = 0$$

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$$\Rightarrow 0.5t^2 - 0.4t - 0.02 - t^3 + 0.08t^2 + 0.04t - 0.012 + 0.06t + 0.012 + 0.02t = 0$$

$$\Rightarrow -t^3 + 1.3t^2 - 0.28t - 0.02 = 0$$

$$\Rightarrow t^3 - 1.3t^2 + 0.28t + 0.02 = 0$$

$$\Rightarrow (t-1)(t^2 - 0.3t - 0.02) = 0$$

$$\Rightarrow t = 1 \text{ (or) } t = \frac{0.3 \pm \sqrt{0.17}}{2}$$

$$= 0.3561 \text{ (or) } -0.0562$$

Here $t = 1$ $|t_2|$ and $|t_3| < 1$

In this case the chain is aperiodic.
In this case p^n tends to a matrix with all rows equal to $c_1 y_1'$

$\Rightarrow (c_1 y_{11}, c_1 y_{12}, c_1 y_{13})$ $\lim p_{ij}^{(n)}$ exists

This can be determined by using the left eigen vector say y_1' corresponds to $t_1 = 1$

The left eigen vectors

$$y_1' = (y_{11} \quad y_{12} \quad y_{13})$$

corresponding to $t_1 = 1$ is given by

$$y_1'(P - t_1 I) = 0$$

$$\Rightarrow (y_{11} \quad y_{12} \quad y_{13})(P - I) = 0$$

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$$\Rightarrow (y_{11} \quad y_{12} \quad y_{13}) \left[\begin{pmatrix} 0.5 & 0.3 & 0.2 \\ 0.2 & 0.4 & 0.4 \\ 0.1 & 0.5 & 0.4 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right] = 0$$

$$\Rightarrow (y_{11} \quad y_{12} \quad y_{13}) \begin{pmatrix} -0.5 & 0.3 & 0.2 \\ 0.2 & -0.6 & 0.4 \\ 0.1 & 0.5 & -0.6 \end{pmatrix} = 0$$

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$$\Rightarrow -0.5 y_{11} + 0.2 y_{12} + 0.1 y_{13} = 0$$

$$0.3 y_{11} - 0.6 y_{12} + 0.5 y_{13} = 0$$

$$0.2 y_{11} + 0.4 y_{12} - 0.6 y_{13} = 0$$

on solving $y_1' = (y_{11} \quad y_{12} \quad y_{13})$

$$= (0.16 \quad 0.28 \quad 0.24)$$

$$c_1 = \frac{1}{\sum_{j=1}^{3} y_{ij}} = \frac{1}{y_{11} + y_{12} + y_{13}} = \frac{1}{0.68}$$

Hence $P_{ij}^{(n)} \longrightarrow c_1 y_{ij} \quad (j = 1, 2, 3)$

$$P_{11} \rightarrow c_1 y_{11} = \frac{1}{0.68} \times 0.16 = 0.2353$$

$$P_{12} \rightarrow c_1 y_{12} = \frac{1}{0.68} \times 0.28 = 0.4118$$

$$P_{13} \rightarrow c_1 y_{13} = \frac{1}{0.68} \times 0.24 = 0.3529.$$

Example - 3:

Consider a 3 state Markov chain with TPM

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

soln :-

The characteristic equation is given by $|P - tI| = 0$

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$$\Rightarrow \begin{vmatrix} -t & 1 & 0 \\ 0 & -t & 1 \\ 1 & 0 & -t \end{vmatrix} = 0$$

$$\Rightarrow -t \cdot t^2 + 1 = 0$$

$$t^3 - 1 = 0$$

$$\Rightarrow (t-1)(t^2+t+1) = 0$$

$$t = 1, \quad t = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2}$$

The eigen values are $t_1 = 1$, $t_2 = \frac{-1+i\sqrt{3}}{2} = \omega$ and $t_3 = \frac{-1-i\sqrt{3}}{2} = \omega^2$

ω and ω^2 are two imaginary cube roots of unity As $|\omega| = |\omega^2| = 1$. The matrix is not primitive.

$\therefore$ The chain is periodic with period 3 ($\because P^3 = I$).

Stability of a Markov system:

Limiting behaviour

Finite irreducible chains:

Given the bases the matrix given in appending A stability of a Markov chain limiting behaviour finite irreducible chains stationary distribution.

Definition : Stationary distribution

Consider a Markov chain with transition probabilities p_{jk} and T.P.M $P = (p_{jk})$. A probability distribution $\{v_j\}$ is called stationary (or invariant) for the given chain if

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$$V_k = \sum_j v_j \, P_{jk} \longrightarrow (1)$$

Such that

$$v_j \geq 0, \; \sum_j v_j = 1$$

Again we write

$$V_k = \sum_j v_j P_{jk} = \sum_j \left\{ \sum_i v_i P_{ij} \right\} P_{jk}$$

$$= \sum_i v_i \sum_j P_{ij} P_{jk}$$

$$= \sum_i v_i \, P_{ik}^{(2)}$$

Hence in general $\left[\because P_{jk}^{(2)} = \sum_r P_{jr} P_{rk}\right]$

$$V_k = \sum_i v_i \, P_{ik}^{(n)} \quad n \geq 1$$

As $n \to \infty$ $P_{jk}^{(n)}$ tends to a limit v_k independent of the initial state j.

ie) Independent of the initial state

From the result (1) we get

$$v' = v'P \text{ (in the matrix form)}$$

where $v' = v_1, v_2, \ldots \, v_k$ and $\sum v_k = 1$

$$v'P - v' = 0 \Rightarrow v'(P - I) = 0$$

1) Consider the Markov chain Based on 3 states with TPM given as follows

$$\begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} \quad \text{Find } v_1, v_2, v_3$$

Soln;-

Here the chain is irreducible and states are aperiodic persistent non-null. The equation

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$V_k = \sum_j v_j P_{jk}$, $k = 1, 2, 3$ can be written as

$v'(P-I) = 0$ where P is the given matrix

$v' = (v_1 \, v_2 \, v_3)$

Also we have $\sum v_k = 1 \Rightarrow v_1 + v_2 + v_3 = 1$

Now $v'(P-I) = 0$

$$\Rightarrow (v_1 \; v_2 \; v_3) \begin{pmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 0$$

$$\Rightarrow (v_1 \; v_2 \; v_3) \begin{pmatrix} -1 & 2/3 & 1/3 \\ 1/2 & -1 & 1/2 \\ 1/2 & 1/2 & -1 \end{pmatrix} = 0$$

$$\Rightarrow -v_1 + 1/2\, v_2 + 1/2\, v_3 = 0 \longrightarrow (1)$$

$$2/3\, v_1 - v_2 + 1/2\, v_3 = 0 \longrightarrow (2)$$

$$1/3\, v_1 + 1/2\, v_2 - v_3 = 0 \longrightarrow (3)$$

$$v_1 + v_2 + v_3 = 1 \longrightarrow (4)$$

$(1) - (3) \Rightarrow -4/3\, v_1 + 3/2\, v_2 = 0 \longrightarrow (5)$

$(2) + (4) \Rightarrow (-)\, 5/3\, v_1 \overset{(-)}{+} 3/2\, v_2 = 1 \longrightarrow (6)$

$$-3 v_1 = -1$$

$$v_1 = 1/3$$

From equation (5)

$$3/2\, v_3 = 4/3\, v_1$$

$$v_3 = 2/3 \cdot 4/3 \cdot 1/3$$

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$$v_3 = \frac{8}{27}$$

From equation (4) ⇒

$$v_2 = 1 - v_1 - v_3$$

$$= 1 - \frac{8}{27} - \frac{1}{3} = \frac{27-8-9}{27} = \frac{10}{27}$$

$$v_2 = \frac{10}{27}$$

$$v_1 = 1/3, \quad v_2 = 10/27, \quad v_3 = 8/27.$$

we have $v_k = \lim_{n\to\infty} P_{jk}^{(n)}$

$$v_1 = \lim_{n\to\infty} P_{j1}^{(n)}$$

$$v_2 = \lim_{n\to\infty} P_{j2}^{(n)}$$

$$v_3 = \lim_{n\to\infty} P_{j3}^{(n)}$$

$$\lim_{n\to\infty} P^n = \begin{pmatrix} v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 \\ v_1 & v_2 & v_3 \end{pmatrix} = \begin{pmatrix} 1/3 & 10/27 & 8/27 \\ 1/3 & 10/27 & 8/27 \\ 1/3 & 10/27 & 8/27 \end{pmatrix}$$

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Definition:

Ergodicity :

$\lim_{n\to\alpha} P_{jk}^{n} \to v_k$

The property of limiting distribution of $\lim_{n\to\infty} P_{jk}^{(n)} \to v_k$ independent of the initial state j ((or) p^n tends to a matrix with identical rows) is known as ergodicity and corresponding markov chain is called ergodic. When such limits exists the probabilities

settle down and become ...
system then shows some long run regularity properties.

Theorem:

If the state j is persistent then for every state k then can be reached from state j $F_{jk} = 1$.

proof:

Let a_k be the probability that starting from state j the state k is reached without previously returning to state j.

Let F_{jk} denote the Pr that system will always reach the state j itself.

$(1 - F_{jk})$ denote the Pr, that the system starting from state j will never come back to original state j. Hence the Pr of compound event that the system starting from state j reaching the state k and never come back to the original state j is $a_k(1 - F_{jk})$.

If there are some other states, say r, s.... then we get similar terms

$a_r(1 - F_{rj})$ $a_s(1 - F_{sj})$.

Thus the Pr, a that starting from

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state j the system never returns to state j is given by,

$$Q = a_k(1-F_{jk}) + a_r(1-F_{rj}) + a_s(1-F_{sj}) + \cdots$$

But since the state j is persistent $F_{jj}=1$ and the Pr of never returning to state j is $1-F_{jj}=0$. Thus $Q=0$. This implies that each term is zero.

so that $F_{jk}=1$

Theorem :

ERGODIC THEOREM

For a finite irreducible, aperiodic chain with TPM $P=(P_{jk})$ the limits

$$v_k = \lim_{n\to\infty} P_{jk}^{(n)}$$

exists and are the independent of the initial state j. The limits v_k are such that $v_k \geq 0$ $\Sigma v_k = 1$. ie) the limits v_k define a Pr distribution.

Further more the limits probability distribution $\{v_k\}$ is identical with the stationary distribution for the given chain so that

$$v_k = \sum_j v_j P_{jk}, \quad \Sigma v_k = 1 \longrightarrow (2)$$

writing $v' = (v_1, v_2, \cdots v_k)$ $\Sigma v_k = 1$ the relation in equation (2) may also be written as $v' = v'P$

$$\Rightarrow v'(P-1) = 0$$

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proof :

Since the states are aperiodic, persistent non-null, for each pair of j, k

$\lim_{n \to \infty} P_{jk}^{(n)}$ exists and is equal to $\frac{F_{jk}}{\mu_{kk}}$

by theorem 3.4

"If state k is persistent null, then for every j

$$\lim_{n \to \infty} P_{jk}^{(n)} \longrightarrow 0$$

and if state k is aperiodic, persistent non-null then

$$\lim_{n \to \infty} P_{jk}^{(n)} \rightarrow \frac{F_{jk}}{\mu_{kk}}$$

~~Again from the theorem,~~ since the state k is persistent, $F_{jk} = 1$

$$\therefore v_k = \lim_{n \to \infty} P_{jk}^{(n)} = \frac{1}{\mu_{kk}} > 0 \longrightarrow (3)$$

and is independent of initial state j.

To prove :

$$v_k = \sum_j v_j P_{jk} \text{ and } \sum v_k = 1$$

by chapmon kolomogorov equation

$$P_{jk}^{(n+m)} = \sum_i P_{ji}^{(n)} P_{ik}^{(m)} \text{ and}$$

$$v_k = \lim_{n \to \infty} P_{jk}^{(n+m)} = \lim_{n \to \infty} \sum_i P_{ij}^{(n)} P_{ik}^{(m)}$$

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UNIT - III

poisson process :

Here we shall consider same stochastic process in continuous time with discrete state space one such process is poisson process consider a random event E such as

i) Incoming telephone calls (at a switch board)

ii) Arrival of customers for series (at a counter)

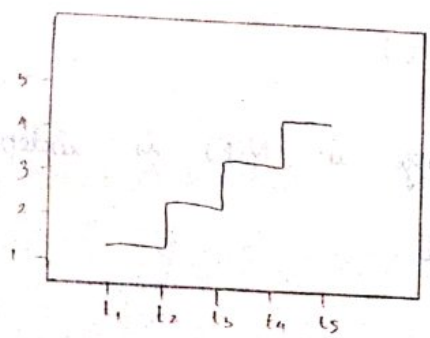
iii) occurence of accidents (at a certain place)

Let us consider the total number $N(t)$ of occurances of the event E is on interval of duration t.

i.e. If we start from an initial approach $t=0$ $N(t)$ will denote the number of occurance upto the approach 't'.

Ex:

If an event actually occurs at in start of $t_1, t_2 \ldots$ then $N(t)$ jumps abraply from 0 to 1 at $t=t_1$ from 1 to 2 at $t=t_2 \ldots$ the situation can be represented graphically.



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$N(t)$ assume the value n.

i.e. $P_n(t) = P_r\{N(t) = n\}$

This P_r is a function of the time 't'

Since the only possible values of n, are $n = 0, 1, \ldots \infty$

$$\sum_{n=0}^{\infty} P_n(t) = 1$$

Thus $\{P_n(t)\}$ represent the P_r distribution of the random variable $N(t)$ for every value of 't' the family of random variable $\{N(t), t \geq 0\}$ is a stochastic process here the time is continuous the state space of $N(t)$ is discrete and integral value and the process is integral value

Definition:

A stochastic process $\{X(t)\}$ with integral valued state space such that as 't' increases the commutative count can only increases is called a counting (or) point process.

postulates of poission process

i) Independence:

$N(t)$ is independent of the number of occurences (of the event E) in an interval period to the interval $(0, t)$

i, Future change in $N(t)$ is independent of the past change

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ii) Homogenity on time :

$P_n(t)$ depends only on the length 't' of the interval and is independent of where their interval is situated

ie, $P_n(t)$ gives the P_r of the number of occurences (of E) in the interval (t_1, t_1+t) $\forall$ t_1.

iii) Regularity :

In an interval of infinite symbol of length 'h' The Pr of exactly one occurences is $\lambda h + o(h)$ and that of more than one occurance is of $o(h)$. $o(h)$ is used as a symbol to denote a been of h which tends to 0 mor rapidly then, h.

ie, As $h \to 0$, $\frac{o(h)}{h} \to 0$

In other words if the interval between 't' and t+h is of short duration h.

Then $P_1(h) = \lambda h + o(h)$

$$\sum_{K=2}^{\infty} P_K(h) = o(h) \qquad \left\{ \begin{array}{l} P_1(t) = \lambda h + o(h) \\ P_2(t) = o(h) \end{array} \right\}$$

Since $\sum_{n=0}^{\infty} P_n(h) = 1$

$$P_0(t) = 1 - \sum_{n=1}^{\infty} P_n(t)$$

$$= 1 - P_1(t) + P_2(t) + \cdots$$

It follows that $P_0(h) = 1 - P_1(h) - P_2(h) - P_3(h)$

$$P_0(h) = 1 - [\lambda h + o(h)] - o(h) - o(h).$$

$$= 1 - \lambda h + o(h)$$

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Theorem :

under the postulates I, II, III , $N(t)$ follows poission distribution with mean λt

i.e, $P_n(t)$ is given by the poission law

$$P_n(t) = \frac{e^{-\lambda t} (\lambda t)^n}{n!} \quad n = 0,1,2 \ldots$$

proof :

consider $P_n(t+h)$ for $n \geq 0$

'n' events by epoch $(t+h)$ can happen in the following mutually exclusive ways

$A_1, A_2, A_3 \ldots A_{n+1}$ for $n \geq 1$

A_1 : n events by epoch t and no events b/w t and t+h we have

$$P_r(A_1) = P_r\{N(t) = n\} . P_r\{N(h) = 0 / N(t) = n\}$$

$$= P_n(t) . P_0(h)$$

$$= P_n(t)(1 - \lambda h) + o(h)$$

A_2 : (n-1) events by epoch t and 1 event between t and t+h.

$$P_r(A_2) = P_r\{N(t) = n-1\} \; P_r\{N(h) = 1 / N(t) = n-1\}$$

$$= P_{n-1}(t) . P_1(h)$$

$$= P_{n-1}(t) \lambda h + o(h)$$

For $n \geq 2$

A_3 : (n-2) events by epoch t and 2 events b/w t and t+h

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We have $P_r(A_3) = P_r\{N(t)=n-2\} \cdot P_r\{N(h)=2 / N(t)=n-2\}$

$$P_r(A_3) = P_{n-2}(t) \cdot P_2(h) = o(h)$$

And soon for $P_r(A_4), P_r(A_5) \ldots$ are this have

$$\cancel{\sum_{k=2}^{n} P_r(A_{k+1})} \leq \sum_{k=2}^{n} P_k(h) = o(h)$$

and so,

$$P_n(t+h) = P_n(t) P_0(h) + P_{n-1}(t) P_1(h) + P_{n-2}(t) P_2(h) + \cdots$$

$$P_n(t+h) = P_n(t)(1-\lambda h) + P_{n-1}(t)(\lambda h) + o(h)$$

$$P_n(t+h) = P_n(t) - \lambda h(P_n(t)) + P_{n-1}(t)\lambda h + o(h)$$

$\div h$

$$\frac{P_n(t+h) - P_n(t)}{h} = -\lambda P_n(t) + P_{n-1}(t)\lambda + \frac{o(h)}{h}$$

In the lim as $h \to 0$

$$P_n'(t) = -\lambda [P_n(t) - P_{n-1}(t)] \quad n \geq 1 \longrightarrow (1)$$

put $n = 0$

$$P_0'(t) = -\lambda P_0(t)$$

Initial Condition:

Suppose the process start at the time 0 (or) at the origin of epoch of measurement so that $N(0) = 0$.

ie, $P_0(0) = 1$, $P_n(0) = 0$ for $n \neq 0 \longrightarrow (3)$

The differential difference eqn (1) differential eqn (2) together with (3) completely specify the system

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Now to prove.

$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!} \quad n = 0, 1, 2 \ldots$$

proof:

Consider the Pr generating function

$$P(s,t) = \sum_{n=0}^{\infty} P_n(t)\, s^n$$

$$P(s,t) = P_0(t)s^0 + P_1(t)s^1 + P_2(t)s^2 + \ldots$$

$$P(s,0) = P_0(0)s^0 + P_1(0)s^1 + P_2(0)s^2 + \ldots$$

$$P(s,0) = 1$$

put $n = n-1$, $s^0 = 1$

$$P(s,t) = \sum_{n-1=0}^{\infty} P_{n-1}(t)\, s^{n-1}$$

$$= \sum_{n=1}^{\infty} P_{n-1}(t)\, s^n s^{-1}$$

$$= \sum_{n=1}^{\infty} P_{n-1}(t)\, s^n/s$$

$$sP(s,t) = \sum_{n=1}^{\infty} P_{n-1}(t)\, s^n \rightarrow (*) \quad \partial/\partial t\, P(s,t) = \sum_{n=0}^{\infty} \partial/\partial t\, P_n(t) s^n$$

$$= \sum_{n=0}^{\infty} P_n'(t)\, s^n \quad n = 0, 1, 2 \ldots$$

$$\partial/\partial t\, P(s,t) = P_0'(t)s^0 + \sum_{n=1}^{\infty} P_n'(t)\, s^n \rightarrow (1)$$

$$\partial/\partial t\, P(s,t) - P_0'(t)s^0 = \sum_{n=1}^{\infty} P_n'(t)\, s^n \rightarrow (2)$$

$$P(s,t) = \sum_{n=0}^{\infty} P_n(t)\, s^n$$

$$= P_0(t)s^0 + \sum_{n=1}^{\infty} P_n(t)\, s^n$$

$$P(s,t) = P_0(t) + \sum_{n=1}^{\infty} P_n(t)\, s^n$$

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$$p(s,t) - P_0(t) = \sum_{n=1}^{\infty} P_n(t)\, s^n \rightarrow (3)$$

$$\text{eqn } (1) \Rightarrow \quad P_n'(t) = -[\lambda P_n(t) + P_{n-1}(t)\lambda]$$

This equation 1- s^n adding & multiple over $n = 0, 1 \ldots \infty$

$$\sum_{n=1}^{\infty} P_n'(t)\, s^n = -\lambda \left[\sum_{n=1}^{\infty} P_n(t)\, s^n - \sum_{n=1}^{\infty} P_{n-1}(t)\, s^n \right]$$

$$\partial/\partial t\ p(s,t) - P_0'(t)$$

$$\cancel{\sum_{n=1}^{\infty} \partial/\partial t\, [P_0(t)]\, s^n} = -\lambda [p(s,t) - P_0(t) - s\,p(s,t)]$$

$$\partial/\partial t\ p(s,t) + \lambda P_0(t) = -\lambda p(s,t) + \lambda P_0(t) + \lambda s\, p(s,t)$$

$$\partial/\partial t\ p(s,t) + \lambda \cancel{P_0(t)} = -\lambda p(s,t) + \lambda \cancel{P_0(t)} + \lambda s\, p(s,t)$$

$$\partial/\partial t\ p(s,t) = p(s,t)\, [\lambda(s-1)]$$

$$\partial/\partial t\ p(s,t) - \lambda(s-1)\, p(s,t) = 0$$

$$y e^{\int p\, dx} = \int Q\, e^{\int p\, dx} . dx + c$$

$$dy/dx + Py = Q$$

$$P = -\lambda(s-1) \quad , \quad Q = 0$$

$$p(s,t)\, e^{-\int \lambda(s-1)\, dt} = \int 0\, e^{-\lambda(s-1)\, dt} . dt + c$$

$$p(s,t)\, e^{-\lambda(s-1)t} = c$$

put $t = 0$

$$p(s,0)\, e^{-\lambda(s-1)0} = c \Rightarrow c = 1$$

$$p(s,t)\, e^{-\lambda(s-1)t} = \phi$$

$$p(s,t) = \frac{1}{e^{-\lambda(s-1)t}} = e^{\lambda(s-1)t}$$

$$= e^{\lambda s t} . e^{-\lambda t}$$

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$$= e^{-\lambda t} \cdot e^{\lambda s t}$$

$$\sum_{n=0}^{\infty} P_n(t)\, s^n = e^{-\lambda t}\left[1 + \frac{\lambda t s}{1!} + \frac{(\lambda t s)^2}{2!} + \cdots\right]$$

$$= e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda s t)^n}{n!}$$

$$\cancel{\sum_{n=0}^{\infty}} P_n(t)\, \cancel{s^n} = \cancel{\sum_{n=0}^{\infty}} \frac{e^{-\lambda t}(\lambda t)^n}{n!} \cancel{s^n}$$

$$P_n(t) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$$

properties of poission process:

1. Additive property:

Sum of two independent poisson process is an poission process.

$$P_n(t) = P_r\{N(t) = n\} = \frac{e^{-(\lambda_1+\lambda_2)t}\,[(\lambda_1+\lambda_2)t]^n}{n!}$$

proof:

Let $N_1(t)$ and $N_2(t)$ be a poission process with parameters λ_1, λ_2 respectively.

Let $N(t) = N_1(t) + N_2(t)$

Then the probability generating function of $N_i(t)$, $i = 1, 2, \ldots$

$$E\{s^{N_i(t)}\} = E\{s^{N_1(t)+N_2(t)}\}$$

and because of independence of $N_1(t)$ and $N_2(t)$

$$E\{s^{N(t)}\} = E\{s^{N_1(t)}\}\, E\{s^{N_2(t)}\}$$

$$= e^{\lambda_1(s-1)t} \cdot e^{\lambda_2(s-1)t}$$

$$= e^{(\lambda_1+\lambda_2)(s-1)t}$$

$$= e^{\lambda(s-1)t}$$

$$P_r\{N(t) = n\} = \sum_{r=0}^{\infty} P_r\{N(t) = r\} \cdot P_r\{N(t) = n - r\}$$

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$$= \sum_{r=0}^{n} \frac{e^{-\lambda_1(t)} (\lambda_1 t)^r}{r!} \cdot \frac{e^{-\lambda_2(t)} (\lambda_2 t)^{n-r}}{(n-r)!}$$

$$= \sum_{r=0}^{n} \frac{e^{-(\lambda_1+\lambda_2)t} (\lambda_1 t)^r (\lambda_2 t)^{n-r}}{r!\,(n-r)!}$$

$$= \sum_{r=0}^{n} \frac{e^{-(\lambda_1+\lambda_2)t} (\lambda_1)^r (t)^r (\lambda_2)^{n-r} (t)^{n-r}}{r!\,(n-r)!}$$

$$= \sum_{r=0}^{n} \frac{e^{-(\lambda_1+\lambda_2)t} (\lambda_1)^r (\lambda_2)^{n-r} (t^n)}{r!\,(n-r)!}$$

$$= e^{-(\lambda_1+\lambda_2)t} (t^n) \sum_{r=0}^{n} \frac{(\lambda_1)^r (\lambda_2)^{n-r}}{r!\,(n-r)!}$$

put $r = 0, 1, 2 \ldots$

$$= e^{-(\lambda_1+\lambda_2)t} (t^n) \left[\frac{(\lambda_1)^0 (\lambda_2)^{n-0}}{0!\,(n-0)!} + \frac{(\lambda_1)^1 (\lambda_2)^{n-1}}{1!\,(n-1)!} + \frac{(\lambda_1)^2 (\lambda_2)^{n-2}}{2!\,(n-2)!} + \cdots \right]$$

$$= e^{-(\lambda_1+\lambda_2)t} (t^n) \left[\frac{(\lambda_2)^n}{n!} + \frac{\lambda_1 (\lambda_2)^{n-1}}{(n-1)!} + \frac{(\lambda_1)^2 (\lambda_2)^{n-2}}{2!\,(n-2)!} + \cdots \right]$$

$$= e^{-(\lambda_1+\lambda_2)t} t^n \left[\frac{(\lambda_2)^n}{n!} + \frac{n(\lambda_1)(\lambda_2)^{n-1}}{n(n-1)!} + \frac{n(n-1)(\lambda_1)^2 (\lambda_2)^{n-2}}{2!\, n(n-1)(n-2)!} + \cdots \right]$$

$$= e^{-(\lambda_1+\lambda_2)t} t^n \left[\frac{(\lambda_2)^n}{n!} + \frac{n(\lambda_1)(\lambda_2)^{n-1}}{n!} + \frac{n(n-1)(\lambda_1)^2 (\lambda_2)^{n-2}}{2!\, n!} + \cdots \right]$$

$$= e^{-(\lambda_1+\lambda_2)t} t^n \left[(\lambda_2)^n + \frac{nc_1 \lambda_1 (\lambda_2)^{n-1}}{1!} + \frac{nc_2 (\lambda_1)^2 (\lambda_2)^{n-2}}{2!} + \cdots \right]$$

[Bionomial formula]

$$(x+a)^n = x^n + nc_1 x^{n-1} a + nc_2 x^{n-2} a^2 + \cdots$$

$$= \frac{e^{-(\lambda_1+\lambda_2)t} t^n}{n!} [\lambda_2 + \lambda_1]^n$$

$$Pr\{N(t) = n\} = \frac{e^{-(\lambda_1+\lambda_2)t} [(\lambda_2+\lambda_1)t]^n}{n!}$$

Then $N(t)$ is a poission process with parameter $\lambda_1+\lambda_2$.

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2 Difference of two independent poission process is not a poission process :

The probability distribution of $N(t) = N_1(t) - N_2(t)$ is given by

$$P_r\{N(t)=n\} = e^{-(\lambda_1+\lambda_2)t} (\lambda_1/\lambda_2)^{n/2} I_n(2t\sqrt{\lambda_1\lambda_2})$$

where $n = 0, \pm 1, \pm 2 \ldots$

$$I_n(x) = \sum_{r=0}^{\infty} \frac{(x/2)^{2r+n}}{r! \sqrt{n+r+1}}$$

is the modified bessel function of order $n (\geq +1)$

proof :

The probability generating function of $N(t)$ is

$$N(t) = N_1(t) - N_2(t)$$

$$E(N(t)) = E\left(s^{N_1(t) - N_2(t)}\right)$$

$$= E\left(s^{N_1(t)}\right) . E\left(s^{-N_2(t)}\right)$$

$$E(N(t)) = E\left(s^{N_1(t)}\right) E\left(1/s^{N_2(t)}\right)$$

$$E\left(s^{N(t)}\right) = e^{\lambda(s-1)t}$$

$$= e^{\lambda_1(s-1)t} . e^{\lambda_2(1/s-1)t}$$

$$= e^{\lambda_1 st} . e^{-\lambda_1 t} . e^{\lambda_2 1/s\, t} e^{-\lambda_2 t}$$

$$E\{N(t)\} = e^{-(\lambda_1+\lambda_2)t} e^{(\lambda_1 s + \lambda_2 1/s)t}$$

$$P_r\{N(t)=n\} = \sum_{r=0}^{\infty} P_r\{N(t) = n+r\} . P_r\{N(t) = r\}$$

$P_r\{N(t)=n\}$ can also be obtained directly as follows

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda_1 t}(\lambda_1 t)^{n+r}}{(n+r)!} . \frac{e^{-\lambda_2 t}(\lambda_2 t)^r}{r!}$$

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$$= \sum_{r=0}^{\infty} e^{-(\lambda_1+\lambda_2)t} \frac{(\lambda_1 t)^{n+r} (\lambda_2 t)^r}{r!\,(n+r)!}$$

put $n+r = \frac{2r+n}{2} + \frac{n}{2}$

$r = \frac{2r+n}{2} - n/2$

$$= \sum_{r=0}^{\infty} \frac{e^{-(\lambda_1+\lambda_2)t} (\lambda_1 t)^{\frac{2r+n}{2}+n/2} (\lambda_2 t)^{\frac{2r+n}{2}-\frac{n}{2}}}{r!\,(n+r)!}$$

$[\because n! = \sqrt{n+1}, \quad (n+r)! = \sqrt{n+r+1}]$

$$= e^{-(\lambda_1+\lambda_2)t} \sum_{r=0}^{\infty} \frac{(\lambda_1 t)^{\frac{2r+n}{2}} (\lambda_1 t)^{n/2} (\lambda_2 t)^{\frac{2r+n}{2}} . (\lambda_2 t)^{-n/2}}{r!\,\sqrt{n+r+1}}$$

$$= e^{-(\lambda_1+\lambda_2)t} \sum_{r=0}^{\infty} \frac{(\lambda_1)^{\frac{2r+n}{2}} (t)^{\frac{2r+n}{2}} (\lambda_1)^{n/2} (t^{n/2}).(\lambda_2)^{\frac{2r+n}{2}} (t)^{\frac{2r+n}{2}} (\lambda_2)^{-n/2} (t)^{-n/2}}{r!\,\sqrt{n+r+1}}$$

$$= e^{-(\lambda_1+\lambda_2)t} \sum_{r=0}^{\infty} \frac{(\lambda_1)^{\frac{2r+n}{2}} (\lambda_2)^{\frac{2r+n}{2}} (t)^{\frac{2r+n}{2}+\frac{2r+n}{2}+n/2-n/2} (\lambda_1/\lambda_2)^{n/2}}{r!\,\sqrt{n+r+1}}$$

$$= e^{-(\lambda_1+\lambda_2)t} \sum_{r=0}^{\infty} \frac{(\sqrt{\lambda_1})^{2r+n} (\sqrt{\lambda_2})^{2r+n}\, t^{\frac{2r+n+2r-n}{2}} (\lambda_1/\lambda_2)^{n/2}}{r!\,\sqrt{n+r+1}}$$

$$= e^{-(\lambda_1+\lambda_2)t} \sum_{r=0}^{\infty} \frac{(t)^{2r+n} (\sqrt{\lambda_1\lambda_2})^{2r+n} (\lambda_1/\lambda_2)^{n/2}}{r!\,\sqrt{n+r+1}}$$

$$= e^{-(\lambda_1+\lambda_2)t} \sum_{r=0}^{\infty} \frac{(t\sqrt{\lambda_1\lambda_2})^{2r+n} (\lambda_1/\lambda_2)^{n/2}}{r!\,\sqrt{n+r+1}}$$

$$\Pr\{N(t)=n\} = e^{-(\lambda_1+\lambda_2)t} (\lambda_1/\lambda_2)^{n/2} I_{(n)}(2t\sqrt{\lambda_1\lambda_2})$$

By using modified Bessel function of order $(n \geq 1)$

$$\text{ie, } I_n(x) = \sum_{r=0}^{\infty} \frac{(x/2)^{2r+n}}{r!\,\sqrt{n+r+1}}$$

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$$I_n(x)\,(2t\sqrt{\lambda_1\lambda_2}) = \sum_{r=0}^{\infty} \frac{\left(\frac{2t\sqrt{\lambda_1\lambda_2}}{2}\right)^{2r+n}}{r!\,\sqrt{n+r+1}}$$

The difference of two independent poission process is not a poission process.

3. Decomposition of a poission process:

A random selection from a poission process yields a poission process suppose that N(t), the no. of occurrances of an event E is an interval of length t is a poission process with parameter λ. suppose also that each occurance of E has a constant p of being recored and that the recording of an occurance is independent of that of other occurances and also of N(t).

If M(t) is the no. of ocurrances recorded is an interval of length then M(t) is also a poission process with parameter λp.

proof:

The event $\{M(t)=n\}$ can happen in the following mutually exclusive ways.

A_r: E occurs $(n+r)$ times by epoch t and exactly n out of $(n+r)$ occurrancs. are recorded probability of epoch occurrancs recorded being p. $(r=0,1,2\ldots)$

we've

$P_r\{A_r\} = P_r\{$E occurs $(n+r)$ times by epoch t$\}$.

$P_r\{$n ocurrances are recorded given that the no: of ocurrances is $n+r\}$.

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{n+r}}{(n+r)!} \cdot (n+r)C_n p^n q^{n+r-n}$$ [Binomial process $nC_x p^x q^{n-x}$, $n = n+r$, $x = n$, $(n+r)C_n p^n q^{n+r-n}$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{n+r}}{(n+r)!} (n+r)C_n p^n q^r$$

$$= \sum_{r=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{n+r}}{(n+r)!} \binom{n+r}{n} p^n q^r$$ [$nC_r = \binom{n}{r}$]

$n = n+r$, $r = n$

$$= e^{-\lambda t} \sum_{r=0}^{\infty} \frac{(\lambda t)^n (\lambda t)^r}{(n+r)!} \binom{n+r}{n} p^n q^r$$ $(n+r)C_n = \binom{n+r}{n}$

$$nC_r = \frac{n!}{(n-r)!\,r!}$$

$$= e^{-\lambda t} \sum_{r=0}^{\infty} \frac{(\lambda p t)^n (\lambda q t)^r}{(n+r)!} \frac{(n+r)!}{(n+r-n)!\,n!}$$ $\boxed{r = n}$]

$$= e^{-\lambda t} \frac{(\lambda p t)^n}{n!} \sum_{r=0}^{\infty} \frac{(\lambda q t)^r}{r!} \quad [r = 0, 1, 2 \ldots]$$

$$= e^{-\lambda t} \frac{(\lambda p t)^n}{n!} \left[\lambda q t + \frac{\lambda q t}{1!} + \frac{(\lambda q t)^2}{2!} + \ldots\right]$$

$$= e^{-\lambda t} \frac{(\lambda p t)^n}{n!} e^{\lambda q t}$$

$$= \frac{e^{-\lambda t + \lambda q t} (\lambda p t)^n}{n!} = \frac{e^{-\lambda t(1-q)} (\lambda p t)^n}{n!}$$

$$P_r A_r = \frac{e^{-\lambda t p} (\lambda p t)^n}{n!}$$

5. poission process and Binomial distribution.

If $N(t)$ is a poission process and $s < t$ then,

$$P_r\{N(s)\} = \{K / N(t) = n\} = \binom{n}{k} \left(\frac{s}{k}\right)^t (1 - s/t)^{n-k}$$

proof:

$$P_r\{N(s) = k / N(t) = n\} = P_r\left\{\frac{N(s) = k,\ N(t) = n}{P_r\{N(t) = n}\right\}$$

$$= \frac{P_r\{N(s) = k,\ N(t-s) = n-k\}}{P_r\{N(t) = n\}}$$

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$$= \frac{P_r\{N(s)=k\} \cdot P_r\{N(t-s)=n-k\}}{P_r\{N(t)=n\}}$$

$$= \frac{\dfrac{e^{-\lambda s}(\lambda s)^k}{k!} \cdot \dfrac{e^{-\lambda(t-s)}(\lambda(t-s))^{n-k}}{(n-k)!}}{\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$= \frac{\dfrac{e^{-\lambda s}\lambda^k s^k e^{-\lambda t} e^{\lambda s} \lambda^{n-k}(t-s)^{n-k}}{k!\,(n-k)!}}{\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$= \frac{\dfrac{\lambda^k s^k e^{-\lambda t} \lambda^{n-k}(t-s)^{n-k}}{k!\,(n-k)!}}{\dfrac{e^{-\lambda t}(\lambda t)^n}{n!}}$$

$$= \frac{\lambda^k s^k e^{-\lambda t} \lambda^{(n-k)}(t-s)^{n-k}\, n!}{e^{-\lambda t}(\lambda t)^n\, k!\,(n-k)!}$$

$$= \frac{\cancel{\lambda^k} s^k \cancel{e^{-\lambda t}} \cancel{\lambda^n} \cancel{\lambda^{-k}} (t-s)^{n-k} n!}{\cancel{e^{-\lambda t}} \cancel{\lambda^n} t^n\, k!\,(n-k)!}$$

$$= \frac{s^k (t-s)^{n-k}\, n!}{k!\,(n-k)!\, t^n}$$

$$= \frac{t^k s^k (t-s)^{n-k}\, n!}{t^k\, k!\,(n-k)!\, t^n} \qquad (\times \text{ \& } \div \text{ by } t^k)$$

$$= \left(\frac{s}{t}\right)^k \left(\frac{(t-s)^{n-k}}{t^{n-k}}\right)\left(\frac{n!}{k!\,(n-k)!}\right)$$

$$= \left(\frac{s}{t}\right)^k \left(\frac{t-s}{s}\right)^{n-k} \binom{n}{k}$$

$$P_r = \left(\frac{s}{t}\right)^k \left(1-\frac{s}{t}\right)^{n-k} \binom{n}{k}$$

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6. If $\{N(t)\}$ is a poission process then the (correlation) auto correlation co-efficient b/w $N(t)$ and $N(t+s)$ is $\left(\frac{t}{t+s}\right)^{1/2}$.

proof:

Let λ be the parameter of the process

$$\text{correlation } (t, t+s) = \frac{\text{cov}(t, t+s)}{\sqrt{\text{var } N(t)}\sqrt{\text{var } N(t+s)}}$$

Then

$$E\{N(t)\} = \lambda t \ ; \ \text{var}\{N(t)\} = \lambda t \ , \ \text{var}\{N(t+s)\} = \lambda(t+s)$$

$$\text{var } N(t) = E\{N^2(t)\} - [E\{N(t)\}]^2$$

$$\lambda t = E\{N^2(t)\} - (\lambda t)^2$$

$$\lambda t + (\lambda t)^2 = E\{N^2(t)\}$$

$$E\{N^2(t)\} = \lambda t + (\lambda t)^2$$

Since $N(t)$ and $\{N(t+s)\} - N(t)\}$ are independent $\{N(t), t \geq 0\}$ being a poission process

$$\text{cov}(t, t+s) = E\{N(t).N(t+s)\} - E\{N(t)\}E\{N(t+s)\}$$

$$E\{N(t), N(t+s)\} = E\{\{N(t)\}.[N(t+s) - N(t)] + N(t)]\}$$

$$= E\{N(t)[N(t+s) - N(t)] + N(t)]\}$$

$$= E\{N(t)N(t) + N(t)[N(t+s)] - N(t)\}$$

$$= E\{N(t)N(t)\} + E\{N(t)[N(t+s) - N(t)]\}$$

$$= E\{N^2(t)\} + E\{N(t)\}.E\{N(t+s) - N(t)\}$$

$$= E\{N^2(t)\} + E\{N(t)\}[E\{N(t+s)\} - E\{N(t)\}]$$

$$= [\lambda t + (\lambda t)^2] + \lambda t[\lambda(t+s) - \lambda t]$$

$$= \lambda t + \lambda^2 t^2 + \lambda t[\lambda t + \lambda s - \lambda t]$$

$$= \lambda t + \lambda^2 t^2 + \lambda t(\lambda s)$$

$$= \lambda t + \lambda^2 t^2 + \lambda^2 ts$$

$$E\{\{N(t)\} . N(t+s)\} = \lambda t + \lambda^2 t^2 + \lambda^2 ts$$

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Thus the auto co-variance b/w $N(t)$ and $N(t+s)$ is given by,

$$Cov(t, t+s) = E\{N(t).N(t+s)\} - E\{N(t)\}\,E\{N(t+s)\}$$

$$= \{\lambda t + \lambda^2 t^2 + \lambda^2 ts - (\lambda t)\,\lambda(t+s)\}$$

$$= \lambda t + \lambda^2 t^2 + \lambda^2 t s - (\lambda t)(\lambda t + \lambda s)$$

$$= \lambda t + \cancel{\lambda^2 t^2} + \cancel{\lambda^2 ts} - \cancel{\lambda^2 t^2} - \cancel{\lambda^2 ts}$$

$$Cov(t, t+s) = \lambda t.$$

The auto correlation then,

$$Cov\ r(t, t+s) = \frac{Cov(t, t+s)}{\sqrt{var\ N(t)}\sqrt{var\ N(t+s)}}$$

$$= \frac{\lambda t}{\sqrt{\lambda t}\sqrt{\lambda(t+s)}}$$

$$= \frac{\lambda t}{\sqrt{\lambda^2 t(t+s)}}$$

$$= \frac{\cancel{\lambda} t}{\cancel{\lambda}\sqrt{t(t+s)}}$$

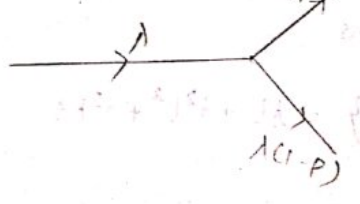
$$= \frac{t}{t^{1/2}(t+s)^{1/2}}$$

$$= \frac{t^{1/2}}{(t+s)^{1/2}}$$

$$Cov\ r(t, t+s) = \left(\frac{t}{t+s}\right)^{1/2}$$

* Continuous of property : 3

The number $M_1(t)$ of occurances Not recorded is also a poission process with parameter $\lambda q = \lambda(1-P)$ and $M(t)$ & $M_1(t)$ and independent.



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As an example suppose that the birth occur in accordance with a poission process with parameter λ. If the probability that an individual born is male is P, then the male births form a poission process with parameter λP and the female birth form an independent poission process with parameter $\lambda(1-P)$.

More generally a poission process $\{N(t)\}$ with parameter λ may be decomposed into r streams of poission process. If $P_1, P_2 \dots P_r$ are the probability of the process decomposing into r independent streams such that

$$P_1 + P_2 + \dots + P_r = 1$$

Then the poission process is decomposed into 'r' independent poission process with parameter $\lambda P_1, \lambda P_2 \dots \lambda P_r$.

poission process and Related distribution

Interarrival Time :-

With a poission process $\{N(t), t \geq 0\}$ where $N(t)$ denotes the number of occurances of an event E by epoch t. there is associated a random variable the interval X b/w two successive occurances of E. we proceed to show that X has a -ve exponential distribution

Theorem :-

Negative exponential distribution.

The interval b/w two successive occurances of a poission process $\{N(t), t \geq 0\}$ having parameter λ has a negative exponential distribution with mean $1/\lambda$.

proof:

Let X be the random variable representing the

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Interval b/w 2 successive occurances of $\{N(t), t \geq 0\}$

and let.

$P_r\{X \leq x\} = F(x)$ be its distribution function.

Let us denote two successive events by E_i and E_{i+1} and suppose that E_i occured at the instant t_0 then,

$P_r\{X > x\} = P_r\{E_{i+1}$ did not occur in $(t_i, t_{i+x})\}$
{given that E_i occured at the instant t_i}

$= P_r\{E_{i+1}$ does not occur in $[t_i, t_{i+x} / N(t) = i\}$
(because of the postulate of independence).

$= P_r\{$no occurance takes place in an interval (t_i, t_{i+x}) of length $\{x / N(t_i) = i\}\}$.

$$P_r\{X > x\} = P_r\{N(x) = 0 / N(t_i) = i\}$$

$$= P_0(x)$$

$$= \frac{e^{-\lambda x}(\lambda x)^0}{0!} \qquad [\because P_n(t) = P_r\{N(t) = n\}]$$

$$P_r\{X > x\} = e^{-\lambda x} \quad x > 0.$$

Since i is arbitary we have for the interval x b/w any two successive occurances.

$$F(x) = P_r\{X \leq x\}$$

$$= 1 - P_r\{x > x\}$$

$$F(x) = 1 - e^{-\lambda x}$$

The density function is.

$$f(x) = F'(x) = 0 - e^{-\lambda x}(-\lambda)$$

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0.$$

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Theorem :

If the interval b/w successive occurances of an event E are independently distributed with a common exponential distribution with mean $1/\lambda$ then the event E form a poission process with mean λt.

proof :

Let Z_n denote the interval b/w $(n-1)^{th}$ and n^{th} occurances of a process $\{N(t), t \geq 0\}$ and let the sequence $Z_1, Z_2 \ldots$ be independently and identically distributed random variables have -ve exponential distribution with mean $1/\lambda$.

The sum $W_n = Z_1 + Z_2 + \cdots + Z_n$ is the waiting time upto the n^{th} occurance

ie, the time from the origin to the n^{th} subsequente occurances ie, W_n has a Gamma distribution with parameter λ, n.

The pdf $g(x)$ and the distribution function F_{W_n} are given respectively by.

$$g(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{\Gamma(n)} \quad x > 0 \qquad F_{W_n}(t) = P_r\{W_n < t\} = \int_0^t g(x)\,dx$$

The events $\{N(t) < n\}$ and $\{W_n = Z_1 + \ldots + Z_n > t\}$ are equivalent

$$[\because F(x) = P_r\{X \leq x\}]$$

Hence the distribution functions of $F_{N(t)}$ and F_{W_n} satisfy the relation.

$$F_{W_n}(t) = P_r\{W_n \leq t\}$$
$$= 1 - P_r\{W_n > t\}$$
$$= 1 - P_r\{N(t) < n\}$$
$$= 1 - P_r\{N(t) \leq n-1\}$$

$$F_{W_n}(t) = 1 - F_{N(t)}\, n-1$$

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Hence the dist Func of $N(t)$ is given by

$$F_{N(t)}(n-1) = 1 - F_{W_n}(t)$$

$$= 1 - \int_0^t g(x)\,dx$$

$$= 1 - \int_0^t \frac{\lambda^n x^{n-1} e^{-\lambda x}}{\Gamma n}\,dx$$

put $\lambda x = y$

$\lambda dx = dy$, $dx = dy/\lambda$

put $x = 0$, $y = 0$

$x = t$, $y = \lambda t$

$$F_{N(t)}(n-1) = 1 - \int_0^{\lambda t} \frac{\lambda^n x^{n-1} e^{-y}}{\Gamma n} . dy/\lambda.$$

$$= 1 - \int_0^{\lambda t} \frac{\lambda^{n-1} x^{n-1} e^{-y}}{\Gamma n} . dy$$

$$= 1 - \frac{1}{\Gamma n} \int_0^{\lambda t} (\lambda x)^{n-1} e^{-y} . dy$$

$$= 1 - \frac{1}{\Gamma n} \int_0^{\lambda t} (y)^{n-1} e^{-y}\,dy$$

$$= \frac{\Gamma n - \int_0^{\lambda t} y^{n-1} e^{-y} . dy}{n.}$$

$$\Gamma n\, F_{N(t)}(n-1) = \Gamma n - \int_0^{\lambda t} y^{n-1} e^{-y} . dy - \int_{\lambda t}^{\infty} y^{n-1} e^{-y} dy + \int_{\lambda t}^{\infty} y^{n-1} e^{-y} dy.$$

$$= \Gamma n - \int_0^{\infty} y^{n-1} e^{-y} dy + \int_{\lambda t}^{\infty} y^{n-1} e^{-y} dy \quad \left[(+) \text{ and } (-) \int_{\lambda t}^{\infty} y^{n-1} e^{-y} dy \right]$$

$$= \Gamma n - \Gamma n + \int_{\lambda t}^{\infty} y^{n-1} e^{-y} dy \quad \left[\because \int_0^{\infty} y^{n-1} e^{-y} . dy = \Gamma n \right]$$

$$\Gamma n\, F_{N(t)}(n-1) = \int_{\lambda t}^{\infty} y^{n-1} e^{-y} dy$$

$$F_{N(t)}(n-1) = \frac{1}{\Gamma n} \int_{\lambda t}^{\infty} y^{n-1} e^{-y} dy$$

$$F_{N(t)}(n-1) = \int_{\lambda t}^{\infty} \frac{y^{n-1} e^{-y} dy}{(n-1)!}$$

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Integrating by parts we've

$$F_{N(t)}(n-1) = \sum_{j=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^j}{j!}$$

Then the probability law of $N(t)$ is

$$P_n(t) = P_r\{N(t)=n\} = F_{N(t)}(n) - F_{N(t)}(n-1)$$

$$= \sum_{j=0}^{n} \frac{e^{-\lambda t}(\lambda t)^j}{j!} - \sum_{j=0}^{n-1} \frac{e^{-\lambda t}(\lambda t)^j}{j!}$$

$$= \frac{e^{-\lambda t}(\lambda t)^n}{n!}, \quad n=0,1,2---$$

Thus the process $\{N(t), t\geq 0\}$ is a poission process with mean λt.

Theorem:

Given that only one occurance of a poission process $N(t)$ has occured by epoch t, Then the distribution of the time interval γ in $[0,T]$ in which is occured is uniform in $[0,T]$.

ie, $P_r\{t<\gamma\leq t+dt\ (N(t)=1)\} = \frac{dt}{T}, \quad 0<t<T$

proof:

$$P_r\{t<\gamma\leq t+dt\} = \lambda e^{-\lambda t} dt$$

$$P_r\{N(t)=1\} = e^{-\lambda t}(\lambda T)$$

$$P_r\{N(T)=1/\gamma=t\} = e^{-\lambda}(T-t)$$

The last one being the P_r that there was no occurance in the interval of length $(T-t)$

$$P_r\{t<\gamma<t+dt/N(T)=1\} = \frac{P_r\{t<\gamma<t+dt\ (N(T)=1)\}}{P_r\{N(T)=1\}}$$

$$= \frac{P_r\{t<\gamma\leq t+dt\}.P\{N(T)=1/\gamma=t\}}{P_r(N(T)=1)}$$

$$= \frac{\lambda e^{-\lambda t}dt}{\lambda T.e^{-\lambda T}} . e^{-\lambda(T-t)} = \frac{dt}{T}.$$

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Remark :

It may be said that a poission process distribution at random over the infinite interval $[0, \infty]$ in the same way as the uniform distribution distributes point at random over a finite interval $[a, b]$

Further interesting properties of poission process :

we consider here some interesting propoilies which are used in study of various , phenomena.

we've show that the interval $X = t_{i+1} - t_i$ b/w two successive occurance E_i , E_{i+1} $(i \geq 1)$ of a poission process with parameter λ has an exponential distribution with mean $1/\lambda$.

(i) For a poission process with parameter λ the interval of time X upto the 1^{st} occurance also follows an exponential dist with $1/\lambda$ for

$$P_r\{X > x\} = P_r\{N(x) = 0\} = e^{-\lambda x} \quad (x > 0)$$

Definition : Random modification of X :-

i) Suppose that the interval X is measured from an arbilary instant of time $t_i + r$ (r arbitary) in the interval (t_i, t_{i+1}) and not just the instant t_i of the occurance of E_i and Y is the interval upto the occurence of E_{i+1} measured from $t_i + r$.

i.e. $Y = t_{i+1} - (t_i + r)$

Y is called random modification of X (or) residual time of X.

It follows that

ii) If X is exponential then its random modification Y has also exponential dist with some mean

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iii) for a poission process with parameter λ. The interval upto the ocurance of the next event measured from any start of time (since the previous instant of ocuurance) and is a random variable having exponential dist with mean $1/\lambda$.

EX: Suppose that the r.v $N(t)$ denotes the No. of fish caught by an angular in $[0,t]$ under certain "ideal conditions" such as

i) The No. of fish available is very large

ii) The angular starts in no better chance of eating fish than other.

iii) The No. of fish likely to nibblesal one particular instant is the same as at another instant.

The process $\{N(t), t \geq 0\}$ may be considered as P. P the interval upto the first catch as also the interval b/w the two successive catches has the same exponential dist, so also is the time upto the next epoch which is independent of the elapsed time.

Ex:

How Geometric distribution is associated poission process and geometric distribution (or).

Consider two independent series of events E & F occuring in accordence with poission process with mean at and bt respectively the number N of occurence of E b/w two successive occurence of F has a geometric dist.

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proof :

The interval b/w two successive occurence of F has the density $f(x) = b e^{-bx}$

And therefore $P_r\{N=K\}$ the P_r that K occurence of E take place during an arbitary intervals (b/w two successive occurence of F) is given by.

$$P_n(t) = P_r\{N(t)=n\}$$

$$= \int_0^\infty P_n(t) f(t)\, dt$$

$$P_r\{N=K\} = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^K}{K!} b e^{-bt} dt$$

$$= \frac{e^{-\lambda t} b}{K!} \int_0^\infty (\lambda)^K (t)^K e^{-bt} . dt$$

$$= \frac{b(\lambda)^K}{K!} \int_0^\infty (t)^K e^{-\lambda t} e^{-bt} dt \qquad [\because \lambda = a]$$

$$= \frac{ba^K}{K!} \int_0^\infty (t)^K e^{-at} . e^{-bt} . dt$$

$$= \frac{ba^K}{K!} \int_0^\infty e^{-(a+b)t} (t)^K . dt$$

$$= \frac{ba^K}{K!} \int_0^\infty e^{-(a+b)t} t^{K+1-1} . dt$$

$$= \frac{ba^K}{K!} \frac{\Gamma K+1}{(a+b)^{K+1}} \qquad \Gamma K+1 = K!$$

$$= \frac{ba^K}{\cancel{K!}} . \frac{\cancel{K!}}{(a+b)^{K+1}}$$

$$= \frac{ba^K}{(a+b)^{K+1}} = \frac{a^K b}{(a+b)^K (a+b)}$$

$$P_n(t) = \left(\frac{a}{a+b}\right)^K \left(\frac{b}{a+b}\right)$$

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Example:

Suppose that E and F occur independently and in accordance with P. P with parameter a & b respectively

The interval b/w two consecutive occurence of F is the sum of two independent exponential dist X this the density $f(x) = b^2 x e^{-bx}, x>0.$

Soln:

The pr that K - occurance of E take place b/w every second occurences of F is given by

$$P_n(E) = P_r\{N(t)=K\} = \int_0^\infty P_K(t) f(t) dt$$

$$= \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^K}{K!} b^2 x e^{-bx} . dt$$

$$= \int_0^\infty \frac{e^{-at}(at)^K}{K!} b^2 t e^{-bt} dt \qquad \left[\because \lambda = a \atop x = t\right]$$

$$= \int_0^\infty \frac{e^{-at} a^K t^K}{K!} b^2 t e^{-bt} dt$$

$$= \frac{a^K b^2}{K!} \int_0^\infty e^{-(a+b)t} t^{K+1} . dt$$

$$= \frac{a^K b^2}{K!} \int_0^\infty e^{-(a+b)t} t^{K+1+1-1} . dt$$

$$= \frac{a^K b^2}{K!} \int_0^\infty e^{-(a+b)t} t^{(K+2)-1} . dt$$

$$= \frac{b^2 a^K}{K!} \frac{\Gamma K+2}{(a+b)^{K+2}}$$

$$P_r\{N=K\} = \frac{a^K b^2}{K!} \frac{(K+1)!}{(a+b)^{K+2}}$$

$$= \frac{a^K b^2}{K!} \frac{K!(K+1)}{(a+b)^{K+2}} = \frac{a^K b^2 (K+1)}{(a+b)^K (a+b)^2}$$

$$= (a/a+b)^K (b/a+b)^2 (K+1)$$

$$= \left(\frac{a}{a+b}\right)^K \left(\frac{b}{a+b}\right)^2 \binom{K+1}{K}$$

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V. Mathivathani, M.Sc.

Elementary
central limit theorem.

UNIT - IV

Renewal process:

In the classical poission process when the interval b/w the successive occurrence are independently and identically distributed with negative exponential distribution suppose that F_i is a sequence of events E ∋ : the intervals b/w the successive occurrence of E on distributed not necessarily negative exponential. we have then a certain generalization of a classical poission process. The corresponding process is called renewal process.

Renewal process in discrete time:

consider a sequence of repeated trails possible outcomes E_j, $j = 1, 2, \ldots$ the trails need not be independent we assume that the trails can be repeated infinitely. suppose that we are interested in a certain outcome in a trail (or) a pattern of outcomes in a number of trail we denote this event by E^* whenever E^* occur we say that a renewal has occured. If it occur at n^{th} trail we say that occur at trail no. n once a renewal process occurs at n^{th} trail. Trails are counted hence fourth from scratch.

Renewal period processes:

The interval b/w occurrence of successive renewals (two successive occurrence

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of the pattern E^*) is called a renewal period of the process.

$f_n = P_r \{ E^*$ occurs for the 1^{st} time at the n^{th} trail $\}$

$P_n = P_r \{ E^*$ occurs at n the trail (not neccesarily for the 1^{st} time) $\}$

Define, $f_0 = 0$, $P_0 = 1$

$$F(s) = \sum_{n=0}^{\infty} f_n s^n, \quad P(s) = \sum_{n=0}^{\infty} P_n s^n$$

Now, $f^* = \Sigma f_n = F(1)$ is the P_r. That the renewal E^* occur at some trail in a long sequence of trails.

we have $f^* \leq 1$. when $f^* = 1$. Then $\{f_n\}$ is a proper probability distributing representating the distribution of length of a renewal period T.

ie) $P\{T = n\} = f_n$

However $\{P_n\}$ is not a P_r distribution the renewal formed as persistant when $f^* = 1$ and transient when $f^* < 1$.

Relation b/w F(s) and P(s):

The event that E^* occurs at the n^{th} trail may be compound event such that E^* occurs for the 1^{st} time at the r^{th} ($r < n$) trail and again at the later trail number n. ie) in subsequence $n-r$ trails and thus

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$$P_n = \sum_{r=1}^{n} f_r P_{n-r} \ , n \geq 1 \ \ (1^{st} \text{ entrance theorem}) \longrightarrow (1)$$

The R.H.S in a convolution relation $\{f_n\} * \{R_n\}$ b/w two subsequences.

Multiplying both sides by s^n, $n = 1, 2, \ldots\ldots$ in equation (1)

$$\sum_{n=1}^{\infty} P_n s^n = \sum_{n=1}^{\infty} \sum_{r=1}^{n} f_r P_{n-r} s^n$$

$$= \sum_{n=1}^{\infty} \sum_{r=1}^{n} f_r P_{n-r} s^{n-r+r} \quad ((+) \ \& \ (-) \text{ by } r)$$

$$= \sum_{n=1}^{\infty} \sum_{r=1}^{n} f_r P_{n-r} s^{n-r} s^r$$

$$-1+1+\sum_{n=1}^{\infty} P_n s^n = \sum_{n=1}^{\infty} \left[f_1 P_{n-1} s^{n-1} s + f_2 P_{n-2} s^{n-2} s^2 + \cdots\cdots + f_n P_{n-n} s^{n-n} s^n \right].$$

$$-1+P_0 s^0 + \sum_{n=1}^{\infty} P_n s^n = \left[f_1 s \sum_{n=1}^{\infty} P_{n-1} s^{n-1} + f_2 s^2 \sum_{n=1}^{\infty} P_{n-2} s^2 + \cdots + f_n s^n \sum_{n=1}^{\infty} P_{n-n} s^{n-n} \right]$$

$$-1+\sum_{n=0}^{\infty} P_n s^n = f_1(s) P(s) + f_2 s^2 P(s) + \cdots\cdots + f_n s^n P(s)$$

that

$$-1+P(s) = P(s)\left[f_1(s)^1 + f_2 s^2 + \cdots\cdots + f_n s^n \right]$$

$$= P(s)\left[0 + f_1 s^1 + f_2 s^2 + \cdots + f_n s^n \right]$$

$$= P(s)\left[f_1 s^1 + f_2 s^2 + \cdots\cdots + f_n s^n \right]$$

$$P(s) - 1 = P(s) F(s) \longrightarrow (2) \quad \left[\because f_0 s^0 \ \ f_0 = 0 \right]$$

$$F(s) = \frac{P(s) - 1}{P(s)} \longrightarrow (3)$$

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$$P(s) - 1 = P(s)\,F(s)$$

$$P(s) - P(s)\,F(s) = 1$$

$$P(s)[1 - F(s)] = 1$$

$$P(s) = \frac{1}{1 - F(s)}$$

Case (i) :

$$P(s) = \frac{1}{1 - F(s)}$$

put $s = 1$

$$\sum p_n = P(1) = \frac{1}{1 - F(1)} < 1$$

$\sum p_n = P(1)$ is convergent. E^* is transient

ie) $\sum p_n = P(1)$ is finite.

case (ii) :

$$P(1) = \frac{1}{1 - F(1)} > 1$$

$P(1)$ is divergent

E^* is persistent

$\sum p_n$ is divergent.

Definition :

The renewal event E^* is said to be periodic if $\exists$ an integer $m (>1) \ni :$ E^* can occur only at trails numbered $m, 2m, \ldots$ The greatest m with this property is said to be the period of E^*.

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It is said to be a aperiodic if no such m exists.

The sequence $\{a_n\}$ is said to be periodic with period $m(>1)$ if $a_n = 0$ for $n \neq km$, $k = 1, 2, \ldots$ and m is the greatest integer with the property.

Definition :

For a persistent and aperiodic renewal event $F'(1) = \Sigma n f_n = E(T)$ is the mean recurrence time. ie) mean time b/w 2 consecutives renewals or mean waiting time b/w two consecutives renewals. $F'(1)$ may be finite (or) infinite.

Renewal interval :

The renewal interval T has the probability mass function $P_r\{T = n\} = f_n$.

T is a proper random variable. when $\Sigma f_n = F(1) = 1$ with mean recurrence time $\Sigma n f_n = F'(1)$. T is also called the waiting time for the occurrence of the renewals E^*.

The generality function of T is $F(s) = \Sigma f_n s^n$ The probability $f_n^{(2)}$ that E^* occurs for the 2nd time at nth trail is given by

$$f_n^{(2)} = \sum_{k=1}^{n-1} f_k f_{n-k}$$

Similarly the probability f_n^* that E^* occurs for the rth time at nth trail is given by

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$$f_n^{(r)} = \sum_{k=1}^{n-1} f_k \, f_{n-k}^{(r)}$$

Thus $\{f_n^{(r)}\}$ gives the Pr distribution of

$$T^{(r)} = T_1 + \cdots\cdots + T_r$$

where T_i are identically independent random variable distribution as T. The generating function of $\{f_n^{(r)}\}$ is given by

$$F^r(s) = \sum_n f_n^{(r)} s^n$$

$$= [F(s)]^r$$

putting $s=1$ we get,

$\sum_n f_n^{(r)}$ is the Pr that E^* occurs atleast r times if the process is continued indefinitely.

It follows that

Pr $\{E^*$ occurs exactly r times if the process is continued indefinitely$\}$

$$= (f^*)^r - (f^*)^{r+1}$$

$$= (f^*)^r [1 - f^*]$$

Generalised form

Delayed Recurrent event:

So for we have assumed that the renewal upto the 1st occurrences of the renewal event E^* has the same distribution as $\{f_n\}$ the recurrence interval b/w successive occurrence of E^*. In many situation is ...

the recurrence interval $\{f_n\}$ upto the 1st occurrence of E^* has a distribution from $\{f_n\}$.

The 1st occurrence of E^* is called a delayed recurrent event while the subsequence occurrence are ordinary recurrent events.

Notation :-

For this situations denote

$v_n = P_r \{ E^*$ occur at nth trail $\}$.

suppose that the 1st occurrence of E^* happens at trail numbers k (with P_r b_k) and then a renewal occurrence of E^* occurs at the subsequent $(n-k)$ trails (with probability P_{n-k}). Thus we get

$$v_n = b_n P_0 + b_{n-1} P_1 + b_{n-2} P_2 + \cdots + b_0 P_n \longrightarrow (1)$$

$$v_n = b_n + b_{n-1} P_1 + b_{n-2} P_2 + \cdots + b_0 P_n$$

$$v_n = \{b_n\} * \{P_n\}$$

Denoting $v(s) = \Sigma v_n s^n$, $P(s) = \Sigma P_n s^n$

$B(s) = \Sigma b_n s^n$ we can write

$$V(s) = B(s) P(s) \qquad \left[\because P(s) = \frac{1}{1-F(s)} \right]$$

$$v(s) = B(s) \frac{1}{1-F(s)}$$

$$v(s) = \frac{B(s)}{1-F(s)}$$

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Theorem :

In $p_n \to \alpha$ then $v_n \to \alpha b$ where $b = \Sigma b_n = B(1)$. In $p_n \to \beta\ (< \infty)$ then $\Sigma v_n \to b\beta$.

proof :

Denote $r_k = P_r\{1^{st}$ renewal period $> k\}$

$$r_k = b_{k+1} + b_{k+2} + \cdots \longrightarrow (1)$$

we can choose k sufficiently large such that $r_k < \varepsilon$ and $p_n \le 1$.

From (1) we get

$$v_n = b_n p_0 + b_{n-1} p_1 + b_{n-2} p_2 + \cdots + b_0 p_n$$

$$\Rightarrow b_0 p_n + b_1 p_{n-1} + \cdots + b_k p_{n-k} \le v_n \le b_0 p_n + b_1 p_{n-1} + \cdots + b_k p_{n-k} + b_{k+1} p_{n-(k+1)} + \cdots + b_n$$

$b_n p_0 + b_{n-1} p_1, \ldots b_0 p_n$

$$\cancel{= b_0 p_n + b_1 p_{n-1} + \cdots + b_k p_{n-k} + (b_{k+1} p_{n-(k+1)}} + \cdots + b_n)$$

$$= b_0 p_n + b_1 p_{n-1} + \cdots + b_k p_{n-k} + \{b_{k+1} + b_{k+2} + \cdots + b_n\}$$

$$b_0 p_n + b_1 p_{n-1} + \cdots + b_k p_{n-k} \le v_n \le b_0 p_n + \cdots + b_k p_{n-k} + r_k \longrightarrow (2)$$

consider,

As $p_n \to \alpha$

$$b_0 p_n + b_1 p_{n-1} + \cdots + b_k p_{n-k} = (b_0 + b_1 + \cdots + b_k)\alpha$$
$$= (b - r_k)\alpha$$
$$= b\alpha - r_k \alpha$$
$$> b\alpha - \epsilon\alpha$$
$$b\alpha - 2\varepsilon$$

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consider,

$$b_0 p_n + b_1 p_{n-1} + \cdots + b_k p_{n-k} + r_k = (b - r_k)\alpha + r_k$$
$$= b\alpha - r_k\alpha + r_k$$
$$= b\alpha + r_k(1-\alpha)$$
$$< b\alpha + \epsilon(1-\alpha)$$
$$< b\alpha + 2\epsilon \rightarrow (4)$$

So from (2) we get,

$$b\alpha - 2\epsilon \le v_n \le b\alpha + 2\epsilon$$

Making and sufficiently small we get,

$$\lim_{n\to\infty} v_n \Rightarrow b_\alpha$$

From $v(s) = B(s)\,p(s)$ we get

put $s = 1$

$$\sum v_n = v(1) = B(1)\,p(1)$$

$v(s) = B(s)\,p(s)$

$$= b \sum p_n$$
$$= b\beta \qquad (\because \sum p_n \rightarrow \beta)$$

which gives $\sum v_n \rightarrow b\beta$

Corollary :

If E^* is persistent then $v_n \rightarrow b/\mu$

proof :

$$v_n = \lim_{s\to 1-0} (1-s)\,v(s)$$
$$= \lim_{s\to 1-0} (1-s)\frac{B(s)}{1-F(s)}$$
$$= (1-s)\frac{B(1)}{1-F(1)} \qquad \left[\because v(s) = \frac{B(s)}{1-F(s)}\right]$$

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$$= B(1) \frac{1-s}{1-F(1)}$$

$$= B(1) \frac{-s}{-F(1)}$$

$$= B(1) \frac{(-1)}{-F(1)} \quad (\because \text{by 'L' hospital's rule})$$

$$= \frac{B(1)}{F(1)} = \frac{B(1)}{F'(1)} = \frac{b}{\Sigma n f_n}$$

$$v_n = \frac{b}{\mu} \qquad \left[\because F(1) = \Sigma f_n = \mu \quad \therefore F'(1) = \Sigma n f_n = \mu\right]$$

Also implise that when E^* is persistent b/μ.

Renewal theory in discrete time :

we shall now consider a result of greater generality (not necessary connected with the stochastic behaviour of recurrent events).

proof :

Suppose that $\{f_n, n=0,1,2,\dots\}$ $\{b_n, n=0,1,2,\dots\}$ are two sequences of real number $\ni$:

$$f_n \geq 0, \quad f = \Sigma f_n < \infty$$

$$b_n \geq 0, \quad b = \Sigma b_n < \infty$$

Define a new sequence $\{v_n, n=0,1,2,\dots\}$ by the convolution relation

$$v_n = b_n + v_{n-1} f_1 + v_{n-2} f_2 + \cdots + v_0 f_n$$

$$v_n = b_n + \sum_{r=1}^{n} f_r v_{n-r}$$

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The above define v_n uniquely interms of $\{b_n\}$ and $\{s_n\}$. Interms of their generating function we get

$$V(s) = B(s) + F(s)\,V(s)$$

$$V(s) - F(s)\,V(s) = B(s)$$

$$V(s)\,[1 - F(s)] = B(s)$$

$$V(s) = \frac{B(s)}{1 - F(s)}$$

$F(s)$ and $B(s)$ converges atleast for $0 \le s \le 1$ and if $F(s) \le 1$ then $V(s)$ is a power series in s.

~~Then~~ the $\{f_n\}$ is periodic if $\exists$ an integer $m \ni : f_n = 0$ except for $n = km$.

Renewal theorem:

Suppose that the relation $b_n \ge 0$, $b = \Sigma b_n < \infty$ hold and $\{f_n\}$ is not periodic

$b = \Sigma b_n < \infty$

a) If $f < 1$ then $V_n \to 0$ and $\Sigma V_n = \dfrac{b}{1-f}$

b) If $f = 1$ then $v_n \to b/\mu$

Proof:

a) Suppose that $\{f_n, n = 0, 1, 2, \dots\}$ $\{b_n, n = 0, 1, \dots\}$ are two sequences of real number $\ni$

$$f_n \ge 0, \quad f = \Sigma f_n < \infty$$

$$b_n \ge 0, \quad b = \Sigma b_n < \infty$$

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Define a new sequence $\{v_n, n=0,1,2\ldots\}$ by the convolution relation

$$v_n = b_n + v_{n-1} f_1 + v_{n-2} f_2 + \cdots + v_0 f_n$$

$$v_n = b_n + \sum_{r=1}^{n} f_r v_{n-r}$$

The above define v_n uniquely interms of $\{b_n\}$ and $\{s_n\}$. Interms of their generating function. we get,

$$V(s) = B(s) + F(s) V(s)$$

$$V(s) - F(s) V(s) = B(s)$$

$$V(s)[1 - F(s)] = B(s)$$

$$V(s) = \frac{B(s)}{[1 - F(s)]}$$

$F(s)$ and $B(s)$ converges atleast for $0 \leq s \leq 1$ and if $F(s) < 1$, then $V(s)$ is a power series in s.

The $\{f_n\}$ is periodic if $\exists$ an integer $m \ni : f_n = 0$ except for $n = km$

we know that,

$$B(1) = \Sigma b_n$$

$$V(s) = \frac{B(s)}{1 - F(s)} \text{ ; put } s = 1$$

$$F(1) = \Sigma f_n$$

$$\Sigma v_n = V(1)$$

when $f < 1$, $V(1) = \dfrac{B(1)}{1 - F(1)}$

$$V(s) = \sum_{n=0}^{\infty} v_n s^n$$

$$B(s) = \Sigma b_n s^n$$

(or) $\Sigma v_n = \dfrac{b}{1-f}$

$$F(s) = \Sigma f_n s^n.$$

Σv_n is convergent this implies $v_n \to 0$

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b) If $f = 1$ then $v_n \to b/\mu$

$$v_n = \lim_{s \to 1-0} (1-s) V(s)$$

$$= \lim_{s \to 1-0} (1-s) \frac{B(s)}{1-F(s)}$$

$$= (1-s) \frac{B(1)}{1-F(1)}$$

$$= B(1) \frac{(1-s)}{1-F(1)}$$

$$= B(1) \frac{(-s)}{-F(1)}$$

$$= B(1) \frac{(-1)}{-F(1)}$$

$$= \frac{B(1)}{F'(1)}$$

$$= \frac{b}{\sum n f_n} \quad \left[\because F'(1) = \sum n f_n = \mu \quad \therefore F(1) = \sum f_n = \mu\right]$$

$$v_n = \frac{b}{\mu}$$

Note - 1:

The above result has great importance In a large number of applications in stochastic process one can find that probabilities connected with a process satisfy a relation of convolution type as given in

$$v_n = b_n + \sum_{r=1}^{n} f_r v_{n-r}$$

The renewal theorem that enables us to get limit that,

2) For rigorous proofs and other details.

Renewal process in continuous time:

Let $\{X_n, n = 1, 2, \ldots\}$ be a sequence of non-negative independent random variables. Assume that $P_r\{X_n = 0\} < 1$ and that the random variables are identically distributed and are continuous with a distribution function $F(\cdot)$

Since X_n is non-negative, it follows that $E\{X_n\}$ exists and let us denote

$$E(X_n) = \int_0^\infty x \, dF(x)$$

$$= \mu$$

where μ may be finite, whenever $\mu \to \infty$, $\frac{1}{\mu}$ shall be independent as 0.

Let $S_0 = 0$

$$S_n = X_1 + X_2 + \cdots + X_n, \; n \geq 1 \text{ and}$$

Let $F_n(x) = P_r\{S_n \leq x\}$ be distribution function of S_n, $n \geq 1$

$F_0(x) = 1$, if $x \geq 0$ and $F_0(x) = 0$ if $x < 0$.

Definition :

Define the random variable

$$N(t) = \sup\{n ; S_n \leq t\}$$

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The process of $\{N(t), t \geq 0\}$ is called a renewal process with distribution F [(or) generated or induced by P]

Remark :

1) It is also customary to say that the sequence of random variable $\{S_n\}, n = 1, 2, \ldots$ constitutes a renewal process with distribution F.

2) If for some n, $S_n = t$ then a renewal is said to occur at t, S_n gives the time (epoch) of the n^{th} renewal and is called n^{th} renewal epoch.

3) The random variables $N(t)$ gives the number of renewal occurring $[0, t]$. The random variable X_n gives the inter arrival time (or waiting time) b/w $(n-1)^{th}$ and n^{th} renewals. The inter arrival times are independently and identically distributed. when they have common exponential distributed distribution we get P.P as a particular case.

Definition :

The renewals where X_i and identically independent random variables are called palm flow of events. where X_i are identically independent exponential the renewals are called ordinary (or) poission flow of events.

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Simple Egs :

1) The simplest Eg of renewal process is life time distribution of a component such as an electric bulb which either works or fails completely. Suppose that the deletion of the failure of a bulb and its replacement by a new bulb take place instaneously and suppose that the lifetime of bulbs are IID random variables with distributions F. We then have a renewal process with distribution F.

2) Consider a stage in an industrial process relating production of a certain component in batches. Immediates on completion of production of a batch, that of another batch is under taken. Suppose that the times taken the produce successive batches are IID random variables with distribution F. We get a renewal process with distribution F.

Renewal function and Renewal density:

The function $M(t) = E\{N(t)\}$ is called the renewal function of the process with distribution F. It is clear that

$$\{N(t) \geq n\} \text{ iff } \{S_n \leq t\} \text{ (or)}$$

$$\{N(t) \geq n\} \iff \{S_n \leq t\}$$

Equivalently $\{N(t) < n\} \iff \{S_n > t\}$

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Theorem :

The distribution of $N(t)$ is given by $P_n(t) = P_r\{N(t) = n\} = F_n(t) - F_{n+1}(t)$ and the expected numbers of renewals by

$$M(t) = \sum_{n=1}^{\infty} F_n(t)$$

Proof :

we have

$$P_n(t) = P_r\{N(t)=n\} = P_r\{N(t) \geq n\} - P_r\{N(t) \geq n+1\}$$

$$= P_r\{S_n \leq t\} - P_r\{S_{n+1} \leq t\}$$

$$P_n(t) = F_n(t) - F_{n+1}(t)$$

Again

$$M(t) = E\{N(t)\} = \sum_{n=0}^{\infty} n \cdot P_n(t)$$

$$= \sum_{n=0}^{\infty} n\left[F_n(t) - F_{n+1}(t)\right]$$

$$= 0 + \left[F_1(t) - F_2(t)\right] + 2\left[F_2(t) - F_3(t)\right] + 3\left[F_3(t) - F_4(t)\right] + \cdots$$

$$= F_1(t) - F_2(t) + 2F_2(t) - 2F_3(t) + 3F_3(t) - 3F_4(t) + \cdots$$

$$= F_1(t) + F_2(t) + F_3(t) + \cdots$$

$$M(t) = \sum_{n=1}^{\infty} F_n(t)$$

$$M(t) = \sum_{n=1}^{\infty} P_r\{S_n \leq t\}$$

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Laplace transform of distribution function:

Statement :

Let x be the non-negative random variable with distribution function $F(\cdot)$. Given by $F(x) = P(X \leq x)$ and $f(x)$ be its p.d.f. The Laplace transform of $f(x)$ is denoted by $f^*(s)$ and is given by

$$f^*(s) = \int_0^\infty e^{-sx} f(x)\, dx \quad (or) = \int_0^\infty e^{-sx}$$

Proof :

$$\text{Now } f^*(s) = \int_0^\infty e^{-sx} f(x)\, dx \rightarrow (1)$$

$$= E[e^{-sx}]$$

Differentiate w.r to (1) 's' under integral sign

$$\frac{d}{ds} f^*(s) = \int_0^\infty e^{-sx} (-x) f(x)\, dx$$

$$= -\int_0^\infty x e^{-sx} f(x)\, dx \longrightarrow (2)$$

$$\frac{d}{ds} f^*(s)/_{s=0} = -\int_0^\infty x f(x)\, dx$$

$$= -E(x) \longrightarrow (3)$$

$$\text{Thus, } E(x) = \text{Mean} = \mu_1' = \left[\frac{-d}{ds} f^*(s)\right]_{s=0}$$

Differentiate w.r to "s" (2) under integral sign we get,

$$\frac{d^2}{\quad} f^*(s) = \int^\infty x^2 e^{-sx} f(x)\, dx \rightarrow (4)$$

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$$= (-1)^2 \int_0 x^2 e^{-sx} f(x)\, dx$$

$$\frac{d^2}{ds^2} f^*(s) \Big|_{s=0} = (-1)^2 \int_0^{\infty} x^2 f(x)\, dx$$

$$= (-1)^2 E(x^2)$$

Thus $E(x^2) = \left[\frac{d^2}{ds^2} f^2(s)\right]$

In general

$$\frac{d^n}{ds^n} f^*(s) = (-1)^n \int_0^{\infty} x^n e^{-sx} f(x)\, dx$$

From (2), (4)

Also $E(x^n) = \left[(-1)^n \frac{d^n}{ds^n} f^*(s)\right]$

$\therefore$ variance $\sigma^2 = E(x^2) - [E(x)]^2$ (or)

$$\mu^2 = \mu_2' - (\mu_1')^2$$

Now $\sigma^2 = \left[\frac{d^2}{ds^2} f^*(s)\right]_{s=0} - \left[\frac{d}{ds} f^*(s)\right]^2$

Let x be the non-negative random variable with distribution function $F(\cdot)$ and p.d.f $f(x)$. Then Laplace transform of $f(x)$ is given by

$$f^*(s) = \int_0^{\infty} e^{-sx} f(x)\, dx$$

The Laplace transform $F(x)$ is given by

$$F^*(s) = \int_0^{\infty} e^{-sx} f(x)\, dx$$

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Let $u = F(x)$ and $dv = e^{-sx} dx$

$$v = \frac{e^{-sx}}{x}$$

$$F^*(s) = \left[F(x) \cdot \frac{e^{-sx}}{-s}\right]_0^\infty + \int_0^\infty \frac{e^{-sx}}{-s} \, d F(x)$$

$$= 1/s \int_0^\infty e^{-sx} \, d F(x) \qquad \left[\because \int_0^\infty e^{-sx} d F(x)\right]$$

$$= 1/s \, f^*(s)$$

Relation b/w $M^*(s)$ and $F^*(s)$:

By definition

$$M(t) = E\{N(t)\} = \sum_{n=1}^\infty F_n(t) \rightarrow (1)$$

$$M(s) = \sum_{n=1}^\infty F_n(s) \longrightarrow (2) \qquad \left[\because L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)\right]$$

Taking Laplace transform on both sides on (2) we get,

$$M^*(s) = \sum_{n=1}^\infty F_n^*(s) = \sum_{n=1}^\infty \frac{f_n^*(s)}{s}$$

when $f_n^*(s)$ is the linear independent of sum.

$\left[\because F_n^*(s) = \frac{1}{s} f_n^*(s)\right]$ of a independent random variable.

$$\therefore M^*(s) = \sum_{n=1}^\infty \left[\frac{f^*(s)}{s}\right]^n \text{(by additive property)}$$

[∴ The linear transform of a sum of finite independent random variable is equal to their product of that Laplace transform]

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$$M^*(s) = \frac{1}{s}\sum_{n=1}^{\infty} f_n^*(s) \quad \sum_{n=1}^{\infty}[F^*(s)]$$

$$= \frac{1}{s}\left[F^*(s) + [F^*(s)]^2 + \cdots\right]$$

$$= \frac{F^*(s)}{s}\left[1 + F^*(s) + \cdots\right]$$

$$(1-x)^{-1} = 1 + x + x^2 + \cdots$$

$$= \frac{F^*(s)}{s}\left[1 - F^*(s)\right]^{-1}$$

$$M^*(s) = \frac{F^*(s)}{s[1-F^*(s)]} \longrightarrow (3)$$

$$sM^*(s)[1-F^*(s)] = F^*(s)$$

$$sM^*(s) - sM^*(s)F^*(s) = F^*(s)$$

$$sM^*(s) = F^*(s) + sM^*(s)F^*(s)$$

$$sM^*(s) = F^*(s)[1 + sM^*(s)]$$

$$\frac{sM^*(s)}{1+sM^*(s)} = F^*(s)$$

$$F^*(s) = \frac{sM^*(s)}{[1+sM^*(s)]} \longrightarrow (4)$$

From (3) and (4), $M(t)$ and $F(t)$ can be determined uniquely one from the other.

Renewal Density :

The derivative $m(t)$ of $M(t)$ ie) $M'(t) = m(t)$ is called the renewal density. The function $M(t)$ specifies the Mean number of renewals to be expected

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in a narrow interval near t $m(t)$ is not a p.d.f

$$m^*(s) = L.T\{M(t)\} = s\,M^*(s) \text{ then}$$

$$m^*(s) = \frac{f^*(s)}{1-f^*(s)}$$

Eg: 2(a)

Find the Laplace transition of renewal function of process with gamma distribution

Let x_n have gamma distribution having density

$$f(x) = \frac{a^k x^{k-1} e^{-ax}}{(k-1)!}, \quad x \geq 0$$

$$= 0 \quad , \text{ otherwise.}$$

Find (i) $f^*(s)$, $M^*(s)$

(ii) Find the distribution of $S_n = x_1 + x_2 + \cdots + x_n$ and hence find $P_n(t)$

(iii) when $k=1$ show that the renewal process will become a poission process.

proof:

i) The L.T of $f(x)$ is given by,

$$f^*(s) = \int_0^\infty e^{-sx} f(x)\,dx$$

$$= \int_0^\infty e^{-sx} \frac{a^k x^{k-1} e^{-ax}}{(k-1)!}\,dx$$

$$= \frac{a^k}{(k-1)!} \int e^{-(s+a)x} x^{k-1}\,dx$$

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$$= \frac{a^k}{(k-1)!} \cdot \frac{\Gamma k}{(s+a)^k} \quad \left[\because \int_0^\infty e^{-ax} x^{k-1}\, dx = \frac{\Gamma n}{a^n}\right]$$

$$f^*(s) = \frac{a^k}{\cancel{(k-1)!}} \cdot \frac{\cancel{(k-1)!}}{(s+a)^k}$$

$$f^*(s) = \frac{a^k}{(s+a)^k} \quad \left[\because \Gamma k = (k-1)!\right]$$

we know that,

$$M^*(s) = \frac{f^*(s)}{s\left[1 - f^*(s)\right]}$$

$$= \frac{\frac{a^k}{(s+a)^k}}{s\left[1 - \frac{a^k}{(s+a)^k}\right]}$$

$$= \frac{\frac{a^k}{(s+a)^k}}{s\left[\frac{(s+a)^k - a^k}{(s+a)^k}\right]}$$

$$= \frac{a^k}{s\cancel{(s+a)^k}} \cdot \frac{\cancel{(s+a)^k}}{(s+a)^k - a^k}$$

$$M^*(s) = \frac{a^k}{s\left[(s+a)^k - a^k\right]}$$

ii) Given that

$S_n = x_1 + x_2 + \cdots + x_n$ where each x_i are IID random variable with p.d.f f^* we know that the linear transformation of a sum of independent random variables

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is equal to the product of their Linear independent.

using this the L.T of random variable s_n is

$$\left[\left(\frac{a}{s+a}\right)^k\right]^n = \left(\frac{a}{s+a}\right)^{nk}$$

Hence the corresponding density function random variable s_n is given by

$$f_n(x) = \frac{a^{nk} x^{nk-1} e^{-ax}}{(nk-1)!}, \quad x \geq 0$$

The distribution function of s_n is given by

$$F_n(x) = P(s_n \leq x) = \int_{-\infty}^{x} f_n(y)\,dy$$

$$= \int_0^x \frac{a^{nk} y^{nk-1}}{(nk-1)!}\,dy$$

$$\Rightarrow \frac{a^{nk}}{(nk-1)!}\int_0^\infty y^{nk-1} e^{-ay}\,dy$$

$$\left[\because u = y^{nk-1},\ dv = e^{-ay}dy\right]$$

$$F_n(x) = 1 - e^{-ax}\sum_{r=0}^{nk-1}\frac{(ax)^r}{r!}, \quad n \geq 1.$$

iii) $P_n(t) = F_n(t) - F_{n+1}(t)$

$$= \left[1 - e^{-at}\sum_{n=0}^{nk-1}\frac{(at)^r}{r!}\right] - \left[1 - e^{-at}\sum_{n=0}^{(n+1)k-1}\frac{(at)^r}{r!}\right]$$

$$= e^{-at}\sum_{r=nk}^{(n+k)k-1}\frac{(at)^r}{r!}$$

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when $k=1$, $P_n(t) = e^{-at}\cdot\frac{(at)^n}{n!}$

which is the p.d.f and p.p and

$$f^*(s) = \frac{a}{s+a}$$

when $k=1$ $M^*(s) = \frac{a}{s[(s+a)-a]}$

$$= \frac{a}{s^2+sa-sa}$$

$$= \frac{a}{s^2}$$

$$= L^{-1}\{M^*(s)\}$$

$$= L^{-1}(a/s^2)$$

$$= aL^{-1}(1/s^2)$$

$$M^*(s) = at$$

Hyper exponential distribution:

Let x_n be have density $f(t) = pae^{-at}+(1-p)be^{-bt}$ $0 \le p \le 2$, $a > b > 0$.

Find i) $f^*(s)$, $M^*(s)$

ii) when $p=1$ (or) $p=0$

show that the process reduces to poission process.

Soln:

$$f^*(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$= \int_0^\infty e^{-st}\left[pae^{-at}+(1-p)be^{-bt}\right]dt$$

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$$= \int_0^\infty e^{-st}\, pa\, e^{-at}\, dt + \int_0^\infty (1-p)\, b e^{-bt} e^{-st}\, dt$$

$$= ap \int_0^\infty e^{-st} e^{-at}\, dt + \int_0^\infty (1-p)\, b e^{-bt} e^{-st}\, dt$$

$$= ap \int_0^\infty e^{-t(s+a)}\, dt + b(1-p) \int_0^\infty e^{-t(s+b)}\, dt$$

$$= ap \left[\frac{e^{-t(s+a)}}{-(s+a)} \right]_0^\infty + b(1-p) \left[\frac{e^{-t(s+b)}}{-(s+b)} \right]_0^\infty$$

$$= ap \left[\frac{e^{\infty}}{s+a} - \frac{e^{0}}{-(s+a)} \right] + b(1-p) \left[\frac{e^{\infty}}{-(s+b)} - \frac{e^{0}}{-s+b} \right]$$

$$= ap \left[0 + \frac{1}{s+a} \right] + b(1-p) \left[0 + \frac{1}{s+b} \right]$$

$$f^*(s) = \frac{ap}{s+a} + \frac{b(1-p)}{s+b}$$

$$M^*(s) = \frac{f^*(s)}{s[1 - f^*(s)]}$$

$$= \frac{ap/s+a + b(1-p)/s+b}{s\left[1 - \left(\frac{ap}{s+a} + \frac{b(1-p)}{s+b}\right)\right]}$$

$$= \frac{\dfrac{ap(s+b) + b(1-p)(s+a)}{(s+a)(s+b)}}{s\left[1 - \dfrac{[ap(s+b) + b(1-p)(s+a)]}{(s+a)(s+b)}\right]}$$

$$= \frac{\dfrac{ap(s+b) + b(1-p)(s+a)}{\cancel{(s+a)(s+b)}}}{s[(s+a)(s+b) - ap(s+b) - b(1-p)(s+a)]}$$

$$= \frac{ap(s+b) + b(1-p)(s+a)}{s\left[(s+a)(s+b) - ap(s+b) - b(1-p)(s+a)\right]}$$

$$= \frac{aps + apb + (b-bp)(s+a)}{s\left[s^2 + sb + sa + ab - aps - apb - (b-bp)(s+a)\right]}$$

$$= \frac{aps + \cancel{apb} + bs - bps + ab - \cancel{bpa}}{s\left[s^2 + \cancel{sb} + sa + \cancel{ab} - aps - \cancel{apb} - \cancel{bs} - \cancel{ba} + bps + \cancel{bpa}\right]}$$

$$= \frac{ab + aps + bs - bps}{s\left[s^2 + as - aps + bsp\right]}$$

$$= \frac{ab + aps + bs - bps}{s^3 + as^2 - aps^2 + bps^2}$$

$$= \frac{ab + s\left[ap + b - bp\right]}{s^2\left[s + a - ap + bp\right]}$$

$$M^*(s) = \frac{ab + s\left[ap + (1-p)b\right]}{s^2\left[s + (1-p)a + bp\right]}$$

put $A = pa + (1-p)b$; $B = (1-p)a + bp$

Then $M^*(s) = \frac{ab + sA}{s^2(s+B)} = \frac{As + ab}{s^2(s+B)}$

$$= \frac{As}{s^2(s+B)} + \frac{ab}{s^2(s+B)}$$

$$= \frac{A}{s(s+B)} + \frac{ab}{s^2(s+B)}$$

By partial function

cosider $\frac{A}{s(s+B)} = \frac{l}{s} + \frac{m}{s+B}$

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$\Rightarrow A = l(s+B) + ms$

put $s = 0$

$A = l(B)$

$A = Bl$

$\boxed{l = A/B}$

$s = -B$

$A = l(-B+B) + m(-B)$

$A = -mB$

$\boxed{m = -A/B}$

$$\frac{A}{s(s+B)} = \frac{A}{Bs} - \frac{A}{B(s+B)}$$

$$\frac{A}{s(s+B)} = \frac{A}{B}\left[\frac{1}{s} - \frac{1}{s+B}\right] \rightarrow (1)$$

so, $\dfrac{A}{s^2(s+B)} = \dfrac{l}{s} + \dfrac{m}{s^2} + \dfrac{n}{s+B}$

$1 = ls(s+B) + m(s+B) + ns^2$

$1 = ls^2 + lsB + ms + mB + ns^2$

$1 = (l+n)s^2 + s(lB+m) + mB \rightarrow (2)$

put $s=0$ in $mB = 1$

equation (2) $m = 1/B$

put $lB + m = 0$

$lB + 1/B = 0$

$$\frac{lB^2+1}{B} = 0$$

$lB^2 + 1 = 0$

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$$lB^2 = -1$$
$$l = \frac{-1}{B^2}$$

put $l+n = 0$

$$n = -l.$$
$$n = 1/B^2$$

$$\therefore \frac{ab}{s^2(s+B)} = ab\left\{\frac{-1}{B^2 s} + \frac{1}{Bs^2} + \frac{1}{B^2(s+B)}\right\}$$

$$\frac{ab}{s^2(s+B)} = \frac{ab}{B}\left\{\frac{1}{s^2} - \frac{1}{B}\left(\frac{1}{s} - \frac{1}{s+B}\right)\right\} \rightarrow (3)$$

From (1) and (3)

$$M^*(s) = \frac{A}{B}\left(\frac{1}{s} - \frac{1}{s+B}\right) + \frac{ab}{B}\left[\frac{1}{s^2} - \frac{1}{B}\left(\frac{1}{s} - \frac{1}{s+B}\right)\right]$$

$$L\{M^*(s)\} = \frac{A}{B}\left(\frac{1}{s} - \frac{1}{s+B}\right) + \frac{ab}{B}\left[\frac{1}{s^2} - \frac{1}{B}\left(\frac{1}{s} - \frac{1}{s+B}\right)\right]$$

$$m(t) = L^{-1}\{M^*(s)\}$$

$$= L^{-1}\frac{A}{B}\left(\frac{1}{s} - \frac{1}{s+B}\right) + \frac{ab}{B}L^{-1}\left[\frac{1}{s^2} - \frac{1}{B}\left(\frac{1}{s} - \frac{1}{s+B}\right)\right]$$

$$= \frac{A}{B}\left\{L^{-1}(1/s) - L^{-1}\left(\frac{1}{s+B}\right)\right\} + \frac{ab}{B}\left\{L^{-1}\left(\frac{1}{s^2}\right) - \frac{1}{B}L^{-1}(1/s) + L^{-1}(1/B)\right\}$$

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Note :

Formula :

$$L(1) = \frac{1}{s} \quad ; \quad L(t^{n-1}) = \frac{(n-1)!}{s^n}$$

$$L(e^{at}) = \frac{1}{s-a}$$

$$= \frac{A}{B}(1-e^{-Bt}) + \frac{abt}{B} - \frac{ab}{B^2}(1-e^{-Bt})$$

$$= \frac{abt}{B} + \left(\frac{A}{B} - \frac{ab}{B^2}\right)(1-e^{-Bt})$$

$$= \frac{A}{B}(1-e^{-Bt}) + \frac{ab}{B}\left(t - \frac{1}{B}(1-e^{-Bt})\right)$$

$$= c(1-e^{-Bt}) + \frac{ab}{B}t$$

Thus $M(t) = \frac{abt}{B} + c(1-e^{-Bt}) \quad \because c = \frac{A}{B} - \frac{ab}{B^2}$

Substitute for A and B we get,

$$c = \frac{pa+(1-p)b}{(1-p)a+pb} - \frac{ab}{[(1-p)a+pb]^2}$$

$$= \frac{1}{(1-p)a+pb}\left[\left\{pa+(1-p)b\right\} - \frac{ab}{(1-p)a+bp}\right]$$

$$= \frac{1}{(1-p)a+pb}\left[\frac{pa+(1-p)b\cdot[(1-p)a+bp]-ab}{(1-p)a+pb}\right]$$

$$= \frac{1}{(1-p)a+pb}\left[\frac{p(1-p)a^2+p^2ab+(1-p^2)ab+p(1-b)b^2-ab}{(1-p)a+bp}\right]$$

$$= \frac{1}{B^2}\left[p(1-p)(a^2+b^2)+ab[p^2+(1-p)^2]-ab\right]$$

$$= \frac{1}{B^2}\left[p(1-p)(a^2+b^2)+ab[2p(p-1)+1]-ab\right]$$

$$= \frac{1}{B^2}\left[p(1-p)(a^2+b^2)+p(p-1)2ab+ab-ab\right]$$

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$$= \frac{1}{B^2}\left[p(1-p)(a^2+b^2-2ab)\right]$$

$$= \frac{1}{B^2}\left[p(1-p)(a-b)^2\right]$$

$$c = \frac{p(1-p)(a-b)^2}{B^2} \qquad (\because a>b)$$

Markovian case:

when $p=1$, $A = pa + (1-p)b$

$$= (1)(a) + (1-1)b$$

$$A = a$$

$$B = (1-p)a + pb$$

$$= b$$

$$B = b$$

put $p=1$, $c = \frac{p(1-p)(a-b)^2}{B^2} = \frac{0}{b} = 0$

Then $M(t) = \frac{abt}{B} + c(1-e^{-Bt})$

$$M(t) = \frac{abt}{B} + 0 = \frac{abt}{b} = at$$

$$M(t) = at$$

Hence the distribution of x_n, reduces non -ve exponential ($p=1$, $f(t) = ae^{t}$ and then $c=0$).

ie) The 2nd item in $M(t)$ vanishes. so that we get $M(t) = at$.

$M(t) = E\{N(t)\} = at$ which is the mean p. p. Thus when $p=1$, the result is true.

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(1) = Stochastic process in Queuing

Queue discipline :-

24/48 The queue displine indicates the way in which the units form a queue and are served. The usual of discipline is first come first served (FCFS) or First in first out (FIFO).

Other rules such as last come first served (or) random ordering before service are also adopted.

The system may be have a single channel (or) a number of parallel channels for service.

Note :

The mean arrival rate usually denoted by λ is the mean no. of arrivals per unit time its reciprocal is the mean of the inter arrival time distribution.

The mean service rate usually denoted by μ is the mean no. of units served per unit time its reciprocal being the mean service time.

Traffic Intensity :

channel $\rho = \lambda / \mu$

(In a single channel system the ratio $\rho = \lambda/\mu$ is called the "Traffic Intensity" where λ is the mean arrival, μ is the mean service rate.

Queuing process :- or families of r.v that arise in the study

The following random variable provide important measures of performance and effectiveness of a stochastic queing system.

i) The no's N(t) in the system at time t

ie) The number at time t waiting in the queue including those being served if any

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of the interval from the moment the service commences with arrival of an unit at an empty ~~connected~~ counter to the moment the server become free for the first time.

iii) The waiting time in the queue (system)

ie) the duration of time a unit has to spend in the queue (system) also the waiting time W_n of the n^{th} arrival.

iv) The virtual waiting time $w(t)$

ie) The interval of time a unit would have to wait in the queue ~~more~~ were it to arrive at instant t.

It is clear that $\{N(t), t \geq 0\}$ $\{w(t), t \geq 0\}$ $\{W_n, n \geq 0\}$ are stochastic process, the ~~fact~~ first two being the continuous time and the 3rd one being in discrete time.

Notation:

If consists of a three part A/B/C where the first and second symbols denote the inter arrival and service time distribution and the third denotes the no. of channels (or) serves. A and B usually take one of the following symbols

M = For exponential (~~Moskoviars~~ Markovian) distribution

E_K = For ~~along~~ Erlang - K - distribution

G = For arbitary (general) distribution

D = For fixed (Deterministic) interval.

Then by an M/G/1 system is meant as single channel queueing system having exponential inter arrival time distribution and arbitary (general) service time distribution.

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that the system has a limited holding capacity k.

steady state distribution:

N(t) the no. in the system at time t and its probability distribution denoted by,

$$P_n(t) = P_r\{N(t) = n / N(0) = 0\}$$

are both time dependent. further in many practical situations, one needs to know the behaviour in steady state ie) the system reaches an equilibrium state after being in operation for a pretty long time.

It is easier and convenient ~~connecniet~~ to determine $P_n = \lim P_n(t)$ as $t \to \infty$. provided the limit exists. when the limit exists, it is said that the system has reached ~~has~~ equilibrium or ~~as~~ steady state.

some general relationship in Queuing process:

1) The most important one is $L = \lambda w$, where λ is the arrival rate. L is a expected no. of units in the system and w is the expected of waiting time in the system in steady state. This is the relation & known as Little's formula.

2) Denote the expected no. in the queue and expected waiting time in the queue in the steady state by L_Q and w_Q respectively. The similar formula $L_Q = \lambda w_Q$.

3) The relation which hold for G/G/1 queue in steady state is,

$$\lambda = (1 - P_0)\mu \quad (\text{or}) \quad P_0 = 1 - \rho$$

For a GI/G/1 queue ...

No. of ideal servers is $k(1-\rho)$.

NOV-2009 2m Little formula, $L = \lambda W$:

A-2017 NOV-2011 NOV-2013 The average value λ and W exists and are finite and one essential condition is that the system is in steady state (or) in equilibrium.

Since, W is the average waiting time of a unit, the average rate of departure (for all the aggregate of units) is L/W.

Now, since the system is in equilibrium this average departure rate must be equal to the arrival, arrival rate which is λ. Thus $\lambda = L/W$.

N-2015 N-2017 5m NOV-2019 Queuing model $M/M/1$: steady state behaviour

The single server model gives assumes poission input and exponential service time with FCFS Queue discipline. Such a queue is also known as the simple queue (or) poission queue (or) simple Markovian queue.

The arrivals occur in accordance with a poission process with intensity identity λ. (i.e) The probability that of an arrival occurs in an infinitesimal interval of length h is $\lambda h + o(h)$ while that of more than one arrival is $o(h)$. This is equivalent to the statement that the distribution of the inter arrival times is exponential with parameter λ, having probability density function that,

$$a(t) = \lambda e^{-\lambda t}$$

$a(t) = \lambda e^{-\lambda t}$

The distribution of the service times is exponential with parameter μ having p.d.f completed is in an interval of infinitesimal length h is $\mu h + o(h)$ which the P_r of More than one completion of services is $o(h)$.

It is assumed that inter arrival times as well as service time are stochastically independent. we shall call such a queue on $M|M|1$ queue with parameter λ, μ (for a simple queue with parameter λ, μ).

The Mean inter arrival time is $1/\lambda$ and the mean service time is $1/\mu$.

The ratio $\rho = \lambda/\mu$ is the traffic intensity. (also equal to load factor or utilization factor) Let $N(t)$ be the no. in the system at instant $t (\geq 0)$. Then $\{N(t), t \geq 0\}$ is a Markov process in continuous time with Denumerable of states $\{0, 1, 2, \ldots\}$

Here transition take place only to ~~the~~ two neighbouring states. This is a type of birth and death process.

The arrivals here can be through of as "birth" and the service completions as "death". the birth and death process with $\lambda_n = \lambda$; $n \geq 0$: $\mu_n = \mu$ $n \geq 1$ $\mu_0 = 0$ is also known as, immigration - emmigration process.

Let $P_r\{N(t) = n / N(0)\} = P_n(t)$; $n \geq 0\}$ using the same P_r arguments

$P_0'(t) = -\lambda P_0(t) + \mu P_1(t)$

$P_n'(t) = -(\lambda+\mu) P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t)$

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t) \longrightarrow (1)$$

$$P_n'(t) = -(\lambda + \mu) P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t), \quad n \geq 1. \rightarrow (2)$$

$P_n(t) . P_r\{N(t) = n / N(0) = 0\}$

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the "steady state" then as $t \to \infty$ $P_n(t) \to P_n$ be a independent t.

The equation of steady state probability $P_r\{N=n\} = P_n$ can be obtained by putting $P_n'(t)=0$ twice of $t=0$

Replacing $P_n(t)$ in (1) and (2) we get,

$$(1) \Rightarrow 0 = -\lambda P_0 + \mu P_1 \rightarrow (3)$$

$$0 = -(\mu+\lambda) P_n + \lambda P_{n-1} + \mu P_{n+1} ; n=1,2,3 \ldots \rightarrow (4)$$

The above equation (3) and (4) are called balance (or) equilibrium (or) conservation follow). Equations

The same equation for an markovian queuing system in steady state can also be obtained from the following principle.

N-2010 2m

Rate equality principle :-

A-2017 N-2011

In the steady state no. of build up occurs at any state as such the two rates

ie) The rate of flow out follow of an arbitary state k and the rate of flow into the same state k $(k=0,1,\ldots)$ are equal.

In equilibrium (ie) in steady state) the rate of flow out of state 0 equals to λP_0 and the rate of flow into state 0 equals to μP_1.

$$\therefore \lambda P_0 = \mu P_1$$

which is the equation (3), Again for $k \neq 0$ the rate of flow out of state k is $(\lambda+\mu) P_k$ and

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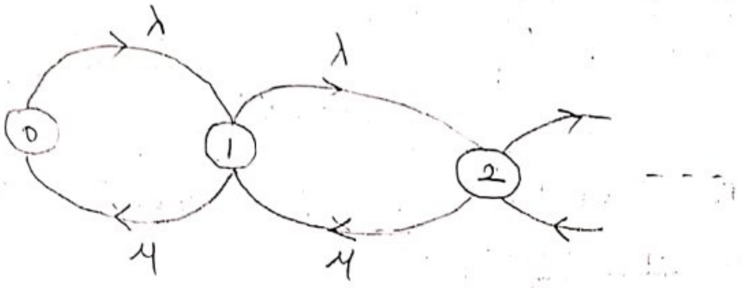
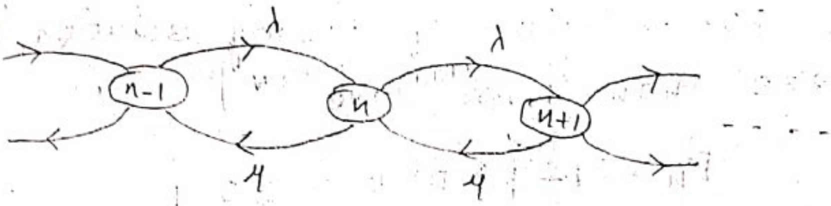
diagram is given as, $(\lambda+\mu)P_k = \lambda P_{k-1} + \mu P_{k+1}$





NOV-2010 M A-2018 5M

Steady state soln (M/M/1) :-

Dividing by μ in (3) and (4) Replacing $\rho = \lambda/\mu$ we get,

$-\lambda P_0 + \mu P_1 = 0$
$-(\lambda+\mu) P_n + \lambda P_{n-1} + \mu P_{n+1} = 0$

$(3) \Rightarrow -\lambda P_0 + \mu P_1 = 0$

$(4) \Rightarrow -(\lambda+\mu) P_n + \lambda P_{n-1} + \mu P_{n+1} = 0$

$\frac{(3)}{\mu} \Rightarrow -\frac{\lambda}{\mu} P_0 + \frac{\mu}{\mu} P_1 = 0$ $\quad \lambda/\mu = \rho$

$\Rightarrow -\rho P_0 + P_1 = 0$

$\frac{(4)}{\mu} \Rightarrow \left(-\frac{\lambda}{\mu} + \frac{\mu}{\mu}\right) P_n + \frac{\lambda}{\mu} P_{n-1} + \frac{\mu}{\mu} P_{n+1} = 0$

$\Rightarrow -(\rho+1) P_n + \rho P_{n-1} + P_{n+1} = 0$

The solution is given by,

$P_n = P_0 \rho^n$ $\quad P_n = P_0 \rho^n$ $\quad P_n = P_0 \rho^n$

since P_n is the probability distribution,

$$\sum_{n=0}^{\infty} P_n = 1$$

$\Rightarrow \sum^{\infty} P_0 \rho^n = 1 \Rightarrow P_0 \sum^{\infty} \rho^n = 1$

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$$\Rightarrow P_0 (1-p)^{-1} = 1$$

$$P_0 = \frac{1}{(1-p)^{-1}} \Rightarrow P_0 = (1-p)$$

The infinite Geometric service converges the sum $\frac{1}{1-p}$ iff ~~px~~, $p<1$

For existence of steady solution p must be Less than 1 we have

$$P_n = (1-p)\, p^n, \quad n \geq 0$$

The distribution $\{P_n\}$ of the random variable N the no. of the in the system in steady state is geometric.

The probability that the system is empty.

$$P_0 = 1-p$$

And we have,

$$E\{N\} = \frac{p}{1-p} = \frac{\lambda/\mu}{1-\lambda/\mu}$$

$$E\{N\} = \frac{\lambda/\mu}{\mu-\lambda/\mu} = \frac{\lambda}{\mu-\lambda}$$

and $var\{N\} = \frac{p}{(1-p)^2} = \frac{\lambda/\mu}{(1-\lambda/\mu)^2}$

$$= \frac{\lambda/\mu}{\frac{(\mu-\lambda)^2}{\mu^2}} = \frac{\lambda}{(\mu-\lambda)^2/\mu}$$

$$var\{N\} = \frac{\lambda\mu}{(\mu-\lambda)^2}$$

N-2010 5m

Waiting time in the queue :-

10m AP 2017 N-2011

We assume that the queue discipline is FIFO FCFS

Let, W_q denote the random variable the time spent speed in waiting in the queue (or 'queuing ti...

of the test unit there is no unit the system the probability for which is $P_0 = 1 - P$.

$$\omega_q = S_n, \left(\text{where } S_n = v_1' + v_2 + \cdots + v_n\right)$$

where v_1' being the residual service time of the customer being served and $v_2, \ldots v_n$ being the services time of the units waiting in the queue at the instant of the arrival of the test unit.

Now, v_1' is the residual of an exponential distribution with mean $1/\mu$ is again all an exponential distribution with the same mean.

Since the services time for each unit is independent and identically distribution. Therefore its probability is density function is given by $\mu e^{-\mu x}$, $x \geq 0$

where μ is the mean service rates.

$\therefore v_2, v_3, \ldots v_n$ also follows exponential distribution with "Gamma distribution having p.d.f"

$$\frac{\mu^n x^{n-1} e^{-\mu x}}{\Gamma n} \quad (x \geq 0)$$

$\frac{\mu^n x^{n-1} e^{-\mu x}}{\Gamma n} dx = P_r$

Hence, we have for $x > 0$

$$W_q(x)\, dx = P_r\{x \leq w_q \leq x + dx\}$$

$$= \sum_{n=1}^{\infty} P_r\{x \leq w_q < x + dx\} \mid \text{Test unit}$$

Finds x - is the system $\} \times P_r\{$Test unit finds n-units in the system$\}$.

$$W_q(x)\, dx = \sum^{\infty} \{\mu^n x^{n-1} e^{-\mu x}, \; (1-P) P^n$$

$$= \sum_{n=1}^{\infty} \frac{\mu^n x^{n-1} e^{-\mu x}}{(n-1)!} dx \quad p^n (1-p)$$

$$= \mu p (1-p) \sum_{n=1}^{\infty} \left[\frac{(\mu p x)^{n-1}}{(n-1)!} e^{-\mu x} \right] dx \quad e^{\mu p x}$$

$$= \mu p (1-p) e^{-\mu x} e^{\mu p x}$$

$$w_q^{(x)} dx = 4 p (1-p) e^{-4(1-p)x} dx \; ; \; x > 0$$

The p.d.f of distribution of w_q is given by,

$$w_q^{(x)} = P_0 = 1-p, \; x = 0$$

$$= 4 p (1-p) e^{-4(1-p)x}, \; x > 0$$

The Laplace transformation,

$$w_q^{*(\alpha)} = P_0 + \int_0^{\infty} e^{-\alpha x} w_q^{(x)} dx$$

$$w_q^{(x)} = (1-p) + 4 p (1-p) \int_0^{\infty} e^{-[\alpha + 4(1-p)]x} dx$$

$$= (1-p) + 4 p (1-p) \left[\frac{e^{-(\alpha + 4 - 4p)x}}{-(\alpha + 4 - 4p)} \right]_0^{\infty}$$

$$= (1-p) + \left[1 + 4p \left(\frac{1}{\alpha + 4 - 4p} \right) \right]$$

$$w_q^{(x)} = 1-p \frac{(\alpha + 4)}{\alpha + 4 - 4p}$$

The distribution function of $f(x)$ of w_q is given by,

$$f(x) = 1-p, \; x = 0 \text{ and}$$

$$f(x) = \int_0^x w_q(t) dt = 1 - \int_x^{\infty} w_q(t) dt$$

$$= 1 - \int_x^{\infty} 4 p (1-p) e^{-4(1-p)x} dx.$$

$$= 1 - 4p(1-p) \int_x^{\infty} e^{-4(1-p)x} dx$$

$$= 1 - 4p(1-p) \left[\frac{e^{-4(1-p)x}}{-4(1-p)} \right]^{\infty}$$

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$$f(x) = \overline{1-p}\, e^{-\mu(1-p)x}$$

The above gives the (conditional) distribution of the waiting time of a test unit the probability that the unit has to wait is $1-p$ and that is has to wait "a" is "p" we have

$$E(W_Q) = -\frac{d}{d\alpha}\left[\frac{W_q(\alpha)}{\alpha}\right]_{\alpha=0} \qquad -d/d\alpha\left[W_q(\alpha)\right]_{\alpha=0}$$

$$= -\frac{d}{d\alpha}\left[\frac{(1-p)(\alpha+\mu)}{\alpha-\lambda+\mu}\right]_{\alpha=0}$$

$\lambda = p\mu$

$$= -(1-p)\frac{d}{d\alpha}\left(\frac{\alpha+\mu}{\alpha-\lambda+\mu}\right)_{\alpha=0}$$

$$= -(1-p)\left(\frac{-\lambda}{(\alpha-\lambda+\mu)^2}\right)_{\alpha=0}$$

$$E(W_Q) = \frac{\lambda(1-p)}{(\mu-\lambda)^2}$$

$$E(W_Q^2) = \frac{d^2}{d\alpha^2}\left[W_q(\alpha)\right]_{\alpha=0}$$

$$= \frac{d^2}{d\alpha^2}\left[\frac{(1-p)(\alpha+\mu)}{\alpha-\lambda+\mu}\right]_{\alpha=0}$$

$$= (1-p)\frac{d^2}{d\alpha^2}\left[\frac{\alpha+\mu}{\alpha-\lambda+\mu}\right]_{\alpha=0}$$

$$= (1-p)\frac{d^2}{d\alpha^2}\left(\frac{1}{\alpha-\lambda+\mu}\right)_{\alpha=0}$$

$$= (1-p)\lambda\frac{d}{d\alpha}\left[\frac{-1}{(\alpha-\lambda+\mu)^2}\right]_{\alpha=0}$$

$$= (1-p)\lambda\left[\frac{2}{(\alpha-\lambda+\mu)^3}\right]_{\alpha=0}$$

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$$= \frac{2\lambda(1-p)}{(\mu-\lambda)^3} = \frac{2\lambda(1-p)}{\mu^3(1-\lambda/\mu)^3}$$

$$= \frac{2\lambda(1-p)}{\mu^3(1-p)^3}$$

$$E(W_Q^2) = \frac{2(\lambda/\mu)}{\mu^2(1-p)^2} = \frac{2p}{\mu^2(1-p)^2}$$

$$\text{var}(W_Q) = E(W_Q^2) - [E(W_Q)]^2$$

$$= \frac{2p}{\mu^2(1-p^2)} - \left[\frac{p}{\mu(1-p)}\right]^2$$

$$\text{var}(W_Q) = \frac{p}{\mu^2(1-p)^2}(2-p)$$

Now, the p.d.f of d-conditional distribution of the waiting time in the queue is denoted by

$$W_Q\, f_c^{(x)}\, dx = P_r\{x \leq W_Q < x + dx\}$$

$$= P_r\{x \leq W_Q < x + dx \,/\, \text{the test with also wait}\}$$

$$= \frac{\mu p(1-p)\, e^{-\mu(1-p)x}}{p}\, dx$$

$$= \mu(1-p)\, e^{-\mu(1-p)x}\, dx \,;\; x > 0$$

$$\left[\because f_c^{(x)}\, dx = \frac{f(x)\, dx}{P(W_Q > 0)}\right]$$

$$\left[\because P(W_Q > 0) = 1 - P(W_Q = 0) = 1-(1-p) = 1-1+p = p\right]$$

$\therefore$ The distribution of $(W_Q$ is exponential with mean $\frac{1}{\mu(1-p)}$ while that of W_Q is modified exponential

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5m 13 Nov 2011 Apr 2011

time (or) sojourn times :-

The random variable time spend in the system [(or) sojourn time] by a unit includes the service time of the unit besides its queuing time disvariance is given by $w_s = w_Q + s$. where w_Q is the queuing time and s is the service time and its distribution on be obtained by convolution.

The test unit has to wait in the system even if the system is empty $(n=0)$ the time spend being equal to n is service time.

Given that there are $(n \geq 0)$ units in the system the waiting time is,

$w_s = s_{n+1}$ where $s_{n+1} = v_1' + v_2 + \cdots + v_{n-1}, v_{n+1}$ being its own service time.

Thus, the p.d.f $w_s(x)$ of w_s for $(x > 0)$ is given by,

$$w_s\,dx = P_r\{x \leq w_s < x + dx\}.$$

$$= \sum_{n=0}^{\infty} P_r\{x \leq s_{n+1} < x + dx \mid \text{g'n that there are n units in the system}\}\, P\{N = n\}$$

$$= \sum_{n=0}^{\infty} \left\{ \frac{\mu^{n+1} x^n e^{-\mu x}}{\Gamma_{n+1}} \right\} \{(1-p)\, p^n\}$$

$$= (1-p)\, e^{-\mu x} \sum_{n=0}^{\infty} p^n \left\{ \frac{\mu^{n+1} x^n}{n!} \right\}$$

$$= (1-p)\, e^{-\mu x} \left[\mu + \frac{\mu^2 p x}{1!} + \frac{\mu^3 p^2 x^2}{2!} + \cdots\right]$$

$$= (1-p)\, e^{-\mu x} \mu \left[1 + \frac{\mu p x}{1!} + \frac{\mu^2 p^2 x^2}{2!} + \cdots\right]$$

$$= (1-p)\, e^{-\mu x}\, \mu\, e^{\mu p x}$$

$$= \mu(1-p)\, e^{-\mu x(1-p)}$$

$$\mu(1-p)\,p = \mu(1-p)\,x\,dx$$

$$= \mu(1-p)\, e^{-\mu x(1-p)}$$

Thus, w_s has an exponential distribution with mean $\{\,?\,1/\mu(1-p)\} = \frac{1}{\mu - \lambda}$ and variance

$\{\frac{1}{\quad}$ $?\ 1/\mu(1-p)^2$

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$$f(x) = P_r\{w_s \le x\}$$

$$= \int_0^x w_s(x)\,dx$$

$$= \int_0^x \mu(1-\rho)\,e^{-\mu(1-\rho)x}\,dx$$

$$f(x) = 1 - \int_x^\infty \mu(1-\rho)\,e^{-\mu(1-\rho)x}\,dx$$

$$= 1 - \mu(1-\rho)\int_x^\infty e^{-\mu(1-\rho)x}\,dx$$

$$= 1 - \mu(1-\rho)\left[\frac{e^{-\mu(1-\rho)x}}{-\mu(1-\rho)}\right]_x^\infty$$

$$= 1 - e^{-\mu(1-\rho)x}, \quad x > 0.$$

Note:

S_{n+1} is the random variable sum of $(n+1)$ exponential variable.

2m Nov-2009 Nov-2011 10m Ap-2018 2m

Transient behaviour of $m|m|1$ model :-

We have now to solve the equations,

$$P_0'(t) = -\lambda P_0(t) + \mu P_1(t) \longrightarrow (1)$$

$$P_n'(t) = -(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t) \longrightarrow (2)$$

In transient state a no. of methods have been part forward for their solution.

Difference equation technique :

We assume that the no. of units of the instant $t=0$ is 0,

Let $P_n(t) = P_r\{N(t) = n \mid N(0) = 0\}$, $P_0 = 1$ and $P_0 = 0$, $n \neq 0$

Let $f_n(s)$ be the Laplace transform of $P_n(t)$

ie) $L\{P_n(t)\} = f_n(s) \longrightarrow (1)$

and $L\{P_n'(t)\} = sL\{P_n(t) - P_n(0)\} \longrightarrow (2)$

$$(2) \Rightarrow P_n'(t) = -(\lambda+\mu) P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t)$$

$$L\{P_n'(t)\} = L[-(\lambda+\mu)P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t)]$$

$$= -(\lambda+\mu) L\, P_n(t) + \lambda L(P_{n-1}(t)) + \mu L(P_{n+1}(t))$$

using (1) and (2) we get,

$$sL\{P_n(t)\} - P_n(0) = -(\lambda+\mu) f_n(s) + \lambda f_{n-1}(s) + \mu f_{n+1}(s)$$

$$sL\{P_n(t)\} = -(\lambda+\mu) f_n(s) + \lambda f_{n-1}(s) + \mu f_{n+1}(s)$$

$$s f_n(s) = -(\lambda+\mu) f_n(s) + \lambda f_{n-1}(s) + \mu f_{n+1}(s)$$

$$(s+\lambda+\mu) f_n(s) = -\lambda f_{n-1}(s) + \mu f_{n+1}(s), \; n \geq 0 \longrightarrow (3)$$

Taking laplace transform on both sides of (1) we get,

$$(1) \Rightarrow P_0'(t) = -\lambda P_0(t) + \mu P_1(t)$$

$$L\{P_0'(t)\} = L\{-\lambda P_0(t) + \mu P_1(t)\}$$

$$L\{P_0'(t)\} = -\lambda L(P_0(t)) + \mu L(P_1(t))$$

$$sL\{P_0(t)\} - P_0(0) = -\lambda f_0(s) + \mu f_1(s)$$

$$s f_0(s) - 1 = -\lambda f_0(s) + \mu f_1(s)$$

$$s f_0(s) + \lambda f_0(s) = 1 + \mu f_1(s)$$

$$\Rightarrow (s+\lambda) f_0(s) = 1 + \mu f_1(s) \longrightarrow (4)$$

$$\frac{(3)}{s+\lambda+\mu} \Rightarrow f_n(s) = \frac{\lambda f_{n-1}(s)}{s+\lambda+\mu} + \frac{\mu f_{n+1}(s)}{s+\lambda+\mu}$$

[$\because$ from (3)]

$$\Rightarrow \left(\frac{\mu}{s+\lambda+\mu}\right) f_{n+1} - f_n(s) + \frac{\lambda}{s+\lambda-\mu} f_{n+1}(s) = 0$$

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$$\left(\frac{\mu}{s+\lambda+\mu}\right)x^2 - x + \left(\frac{\lambda}{s+\lambda+\mu}\right) = 0$$

$$\Rightarrow \mu x^2 - x(s+\lambda+\mu) + \lambda = 0 \longrightarrow (6)$$

If α, β are the roots of this equation (6) then the solution of (5) in,

$$f_n(s) = A\alpha^n + B\beta^n, \quad n \geq 1$$

where A and B are functions of s (not of "n")

Now we have to find the value of A and B.

Consider (6),

$$\mu x^2 - x(s+\lambda+\mu) + \lambda = 0$$

Here, $a = \mu$, $b = -(s+\lambda+\mu)$, $c = \lambda$

Now,

$$k(s) = k = (b^2 - 4ac)^{1/2}$$

$$= \left[(s+\lambda+\mu)^2 - 4\mu\lambda\right]^{1/2}$$

$$\therefore x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$x = \frac{(s+\lambda+\mu) \pm \sqrt{(s+\lambda+\mu)^2 - 4\mu\lambda}}{2\mu}$$

$$= \frac{s+\lambda+\mu \pm k}{2\mu}$$

$$\alpha = \frac{s+\lambda+\mu+k}{2\mu} \longrightarrow (7) \text{ and}$$

$$\beta = \frac{s+\lambda+\mu-k}{2\mu} \longrightarrow (8)$$

Here $\alpha = \alpha(s)$, $\beta = \beta(s)$

It can be seen that for all $Re(s) > 0$ and λ, μ are real and +ve

Here $|\alpha| > 1$ and $|\beta| < 1$

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$$\sum_{n=0}^{\infty} L(P_n(t)) = \sum_{n=0}^{\infty} f_n(s)$$

$$L\left[\sum_{n=0}^{\infty} P_n(t)\right] = \sum_{n=0}^{\infty} f_n(s)$$

$$L(1) = \sum_{n=0}^{\infty} f_n(s) \Rightarrow 1/s = \sum_{n=0}^{\infty} f_n(s) \longrightarrow (9)$$

Now,

$$f_n(s) = A\alpha^n + B\beta^n, \quad n \geq 1$$

$$\sum_{n=0}^{\infty} f_n(s) = \sum_{n=0}^{\infty} (A\alpha^n + B\beta^n)$$

since $|\alpha| > 1$ $f_n(s)$ convergent only when $A = 0$

$$f_n(s) = B\beta^n, \quad n \geq 1$$

choose, B such that this hold for $n=0$ also

$$f_n(s) = B\beta^n$$

$$n = 0 \Rightarrow f_0(s) = B \qquad [\because \beta^0 = 1]$$

$$\therefore f_n(s) = A\alpha^n + B\beta^n$$

$$\Rightarrow f_n(s) = f_0(s)\beta^n \longrightarrow (10) \qquad n \geq 0$$

Taking summation on both sides, $\left[\because A = 0,\ B = f_0(s)\right]$

$$\sum_{n=0}^{\infty} f_n(s) = \sum_{n=0}^{\infty} f_0(s)\beta^n$$

$$1/s = f_0(s)\sum_{n=0}^{\infty} \beta^n \quad [\text{by } (10)]$$

$$= f_0(s)(\beta^0 + \beta^1 + \beta^2 + \cdots\cdots)$$

$$= f_0(s)(1-\beta)^{-1}$$

$$1/s = \frac{f_0(s)}{1-\beta} \Rightarrow f_0(s) = \frac{1-\beta}{s}$$

using this in (10) we get,

$$(10) \Rightarrow f_n(s) = f_0(s)\beta^n$$

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From (4) $\Rightarrow f_n(s) = (1 - \beta/s)\, \beta^n$

$$\sum_{r=n}^{\infty} f_r(s) = \sum_{r=n}^{\infty} \left(\frac{1-\beta}{s}\right) \beta^n$$

$$= \frac{1-\beta}{s} \left(\beta^n + \beta^{n+1} + \cdots\cdots\right)$$

$$= \frac{1-\beta}{s} \beta^n \left(1 + \beta^1 + \beta^2 + \cdots\right)$$

$$\sum_{r=n}^{\infty} f_r(s) = \frac{1-\beta}{s} \beta^n (1-\beta)^{-1}$$

$$\sum_{r=n}^{\infty} f_r(s) = \frac{\beta^n}{s} \longrightarrow (12)$$

From (6) $\Rightarrow \mu x^2 = (s + \mu + \lambda) x + \lambda = 0$. the product of the roots $\alpha\beta = \lambda/\mu$

$$\beta = \lambda/\mu\alpha$$

(12) $\Rightarrow \sum_{r=n}^{\infty} f_r(s) = \frac{1}{s} (\lambda/\mu\alpha)^n$

$f(s)\Big(\sum_{r=n}^{\infty} f_r(s) = (\lambda/\mu)^n \frac{\alpha^{-n}}{s} = L(A_n(t))\Big)$

where, $A_n(t) = P_r\{N(t) \geq n\}$

$$= P_r(t) + P_{n+1}(t) + P_{n+2}(t) + \cdots\cdots$$

$$L(A_n(t)) = L(P_n(t)) + L(P_{n+1}(t)) + \cdots\cdots$$

$$= f_n(s) + f_{n+1}(s) + \cdots\cdots$$

$$L(A_n(t)) = \sum_{r=n}^{\infty} (f_r(s))$$

$$L(A_n(t)) = (\lambda/\mu)^n \left(\frac{\alpha^{-n}}{s}\right) \longrightarrow (13)$$

$$A_n(t) = L^{-1}\left\{(\lambda/\mu)^n \frac{\alpha^{-n}}{s}\right\}$$

$$= (\lambda/\mu)^n \alpha^{-n} L^{-1}(1/s)$$

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The Laplace transform of,

$$a_n(t) = (\lambda/\mu)^{n/2}\, e^{-(\lambda+\mu)t}\,\{ n/t\; I_n(2t\sqrt{\lambda\mu})\}$$

is,

$$(\lambda/\mu)^{n/2}\left\{\frac{2\sqrt{\lambda\mu}}{(s+\lambda+\mu)+\sqrt{(s+\lambda+\mu)^2-4\lambda\mu}}\right\}^n$$

ie) $$L(a_n(t)) = (\lambda/\mu)^{n/2}\left\{\frac{2\sqrt{\lambda\mu}}{(s+\lambda+\mu)+\sqrt{(s+\lambda+\mu)^2-4\lambda\mu}}\right\}^n$$

$$L[a_n(t)] = \frac{\lambda^{n/2}}{\mu^{n/2}}\left\{\frac{2^n\cdot\lambda^{n/2}\cdot\mu^{n/2}}{(s+\lambda+\mu)+\sqrt{(s+\lambda+\mu)^2-(4\lambda\mu)^n}}\right\}^n$$

$$L[a_n(t)] = \frac{\lambda^{n/2}\,2^n\,\lambda^{n/2}\,\mu^{n/2}}{\mu^{n/2}\{(s+\lambda+\mu)+\sqrt{(s+\lambda+\mu)^2-(4\lambda\mu)^n}\}^n}$$

$$L[a_n(t)] = \frac{2^n\lambda^n}{\{(s+\lambda+\mu)+\sqrt{(s+\lambda+\mu)^2-(4\lambda\mu)}\}^n}$$

(x) and (÷) by μ^n we get,

$$L(a_n(t)) = \frac{\lambda^n}{\mu^n}\,\frac{2^n\cdot\mu^n}{\{(s+\lambda+\mu)+\sqrt{(s+\lambda+\mu)^2-4\lambda\mu}\}^n}$$

$$= \left(\frac{\lambda}{\mu}\right)^n \frac{(2\mu)^n}{\{(s+\lambda+\mu)+\sqrt{(s+\lambda+\mu)^2-4\lambda\mu}\}^n}$$

$$= \left(\frac{\lambda}{\mu}\right)^n\left\{\frac{1}{\frac{(s+\lambda+\mu)+\sqrt{(s+\lambda+\mu)^2-4\lambda\mu}}{2\mu}}\right\}^n$$

$$\left[\because \sqrt{(s+\lambda+\mu)^2-4\lambda\mu} = k\right]$$

$$L[a_n(t)] = \left(\frac{\lambda}{\mu}\right)^n\left\{\frac{1}{\frac{s+\lambda+\mu+k}{2\mu}}\right\}^n$$

$$= (\lambda/\mu)^n (1/\alpha)^n \quad \{\text{by } (7)\}$$

$$L[a_n(t)] = (\lambda/\mu)^n\ \alpha^{-n}$$

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$$\left(L\left[\int_0^t f(x)\,dx\right] = \frac{f(s)}{s} \right)$$

Here,

$$L\left[\int_0^t a_n(u)\,du\right] = \frac{(\lambda/\mu)^n \alpha^{-n}}{s} \longrightarrow (15)$$

$$(13) \Rightarrow L(A_n(t)) = (\lambda/\mu)^n \left(\frac{\alpha^{-n}}{s}\right)$$

From (13) and (15) we get,

$$L[A_n(t)] = L\left[\int_0^t a_n(u)\,du\right]$$

$$\Rightarrow A_n(t) = \int_0^t a_n(u)\,du$$

$$\Rightarrow \frac{d}{dt} A_n(t) = \frac{d}{dt}\left(\int_0^t a_n(u)\,du\right)$$

$$\frac{d}{dt}(A_n(t)) = a_n(t)$$

Thus the derivation of $A_n(t)$ is $a_n(t)$

$$(8) \Rightarrow \beta(s) = \beta = \frac{s+\lambda+\mu-k}{2\mu}$$

where, $k = \left[(s+\lambda+\mu)^2 - 4\lambda\mu\right]^{1/2}$

$$\lim_{s\to 0} \beta(s) = \lim_{s\to 0} \frac{(s+\lambda+\mu) - \sqrt{(s+\lambda+\mu)^2 - 4\lambda\mu}}{2\mu}$$

$$= \frac{(\lambda+\mu) - \sqrt{(\lambda+\mu)^2 - 4\lambda\mu}}{2\mu}$$

$$= \frac{(\lambda+\mu) - \sqrt{\lambda^2+\mu^2+2\lambda\mu-4\lambda\mu}}{2\mu}$$

$$\lim_{s\to 0} \beta(s) = \begin{cases} \dfrac{\lambda+\mu - \sqrt{(\lambda-\mu)^2}}{2\mu} & \text{when } \lambda \geq \mu \\[2ex] \dfrac{\lambda+\mu - \sqrt{(\lambda-\mu)^2}}{2\mu} & \text{when } \lambda > \mu \end{cases}$$

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