a unique density, etc.

Moreover, for a continuous or ideal fluid we can define a fluid particle as the fluid contained within an infinitesimal volume whose size is so small that it may be regarded as a geometrical point.

We conclude this introductory section by mentioning briefly the natures of the different types of force which are called into play in moving fluids. Suppose two fluid particles, moving at different velocities, have a common boundary. Then across the boundary there will be interchange of momentum. The normal transport of molecules across the boundary will lead to a direct or normal force. In the case of a viscous fluid there is friction between the particles:

This will manifest itself in the form of equal and opposite tangential or shearing forces on each particle at the common boundary. In the case of inviscid fluids, however, there is no friction and consequently there are no tangential or shearing forces. All real fluids exhibit viscosity but in many cases, such as arise when the rates of variation of fluid velocity with distances are small, viscous effects may be ignored.

2.2 Velocity of a Fluid at a point

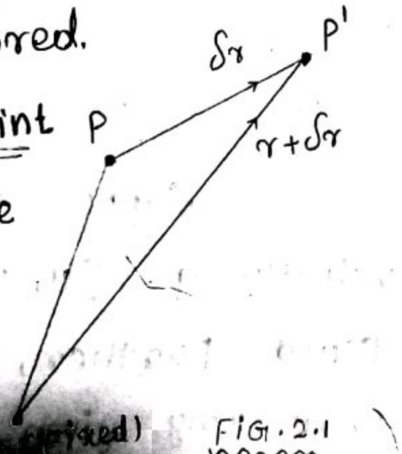

FIG. 2.1

At time t a fluid particle is at P where $\overline{OP} \equiv r$ and at time [illegible] (fixed)

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reached P'

where $\overline{OP'} \equiv r + \delta r$. Then in the interval δt the movement of the particle is $\overline{PP'} \equiv \delta r$ and so the particle velocity q at P is

$$q = \lim_{\delta t \to 0} \left(\frac{\delta r}{\delta t} \right) = \frac{dr}{dt}$$

assuming such a limit to exist uniquely.

This assumption is reasonable if we postulate that the fluid is continuous. Clearly q is in general dependent on both r and t so that we may write

$$q = q(r, t)$$

Alternatively, if P has Cartesian coordinates (x, y, z) relative to a fixed tri-reactangular coordinate frame through O, then we may write

$$q = q(x, y, z, t).$$

Let us further suppose that $[u, v, w]$ are the cartesian components of q in this frame. Then

$$q = ui + vj + wk.$$

Moreover, since $r = xi + yj + zk$

$$q = \frac{dx}{dt} i + \frac{dy}{dt} j + \frac{dz}{dt} k,$$

So that $u = \dfrac{dx}{dt}$, $v = \dfrac{dy}{dt}$, $w = \dfrac{dz}{dt}$.

2.3 Streamlines and Pathlines; Steady and Unsteady Flows:

Suppose that at any time t we know the velocity $q = [u, v, w]$ at each point $P(x, y, z)$ of the fluid. Further, suppose that at each such point at any given instant t we can draw a curve $\mathcal{C}$

in the fluid such that the direction of the tangent at P to $\mathscr{C}$ coincides with the direction of q at P. Then $\mathscr{C}$ is termed a streamline.

It follows that the streamlines are the solutions of the differential equations.

4

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \rightarrow (1)$$

These eqns yield a double infinity of solns. Knowing q at successive points of a streamline at any instant t enables the streamline to be approximated by straight-line segments. Thus in Fig 2.2 if $q_1, q_2, q_3 \ldots$ denote the velocities at neighbouring points $P_1, P_2, P_3 \ldots$ of the streamline, then small straight segments $\overline{P_1P_2}$, $\overline{P_2P_3}$, $\overline{P_3P_4}$, ... are drawn in the directions $q_1, q_2, q_3 \ldots$ respectively.

(when the motion is steady so that the pattern of flow does not vary with time, the paths of the fluid particles

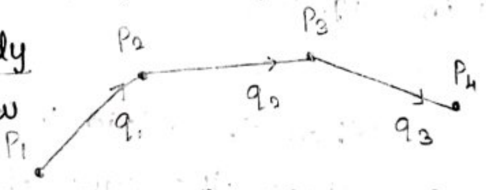


FIG. 2.2

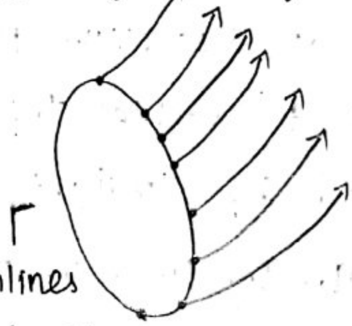
coincide with the streamlines. In unsteady motion, however, the flow pattern varies with time and the paths of the particles do not coincide with the streamlines, though the streamline through any point P does touch the pathline through P. The pathlines are the solutions of the diff. eqns.)

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = w \rightarrow (2)$$

These equations have a triply - infinite set of Sol

A time exposure may be used to photograph the Streamlines for steady flow, but a snapshot must be used for unsteady flow. (5)

In the case of a liquid the particles are illuminated by a suspension of aluminium dust in the liquid. Alternatively the fluid particles may be rendered visible by means of a few crystals of potassium permanganate. For a gas smoke streams may be used.



If we draw the streamlines through every point of a closed curve Γ in the fluid we obtain a stream tube, as in Fig. 2.3.

FIG 2.3

2.4 The Velocity Potential :

When the fluid velocity at time t is $q = [u, v, w]$ in cartesians, the eqns of the streamlines at that instant are $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \rightarrow (1)$

These curves cut the surfaces

$$u\,dx + v\,dy + w\,dz = 0 \rightarrow (2)$$

orthogonally. Now suppose that at the considered instant t, we can find a scalar function $\phi(x, y, z, t)$, uniform throughout the entire field of flow and such that $-d\phi = u\,dx + v\,dy + w\,dz. \rightarrow (3)$

Then the expression on the R.H.S of (3) in an

$$u = -\frac{\partial\phi}{\partial x}, \quad v = -\frac{\partial\phi}{\partial y}, \quad w = -\frac{\partial\phi}{\partial z} \Big\} \rightarrow (4) \quad (6)$$

(or) $\therefore q = -\nabla\phi$

ϕ is termed the velocity potential. By the elements

Chapter 1 (Sec 1.18)

The necessary and sufficient condition for (4) to hold is

$$\text{curl } q = 0 \rightarrow (5)$$

The surfaces $\phi(x, y, z, t) = \text{constant} \rightarrow (6)$

are called equipotentials. eqns (1) and (2) show that at all points of the field of flow the equipotentials are cut orthogonally by the streamlines.

The negative sign in the eqn. $q = -\nabla\phi$ is a convention.

It ensures that flow takes place from the higher to the lower potentials, but some writers adopt the opposite convention.

When (5) holds, the flow is said to be of the potential kind. It is also said to be irrotational for reasons which will appear later in this chapter (Sec 2.11)

For such flow the field of q is conservative, and q is a lamellar vector.

Example:-

1) At the point in an incompressible fluid having Spherical polar co-ordinates (r, θ, ψ), the velocity components are $[2Mr^{-3}\cos\theta, Mr^{-3}\sin\theta, 0]$, where M is a constant. Show that the velocity is of ~~the~~ potential kind. Find the velocity potential and the equations of the streamlines.

Soln:

(7)

Taking,

$$ds = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\varphi\,\hat{\varphi}.$$

$$q = 2Mr^{-3}\cos\theta\,\hat{r} + Mr^{-3}\sin\theta\,\hat{\theta},$$

we know that $x = r\cos\varphi\sin\theta$

$y = r\sin\varphi\sin\theta$

$z = r\cos\theta$

we obtain,

$$\text{curl } q = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ u & rv & r\sin\theta w \end{vmatrix}$$

$$\text{curl} q = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{r} & r\hat{\theta} & r\sin\theta\,\hat{\varphi} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ 2Mr^{-3}\cos\theta & Mr^{-2}\sin\theta & 0 \end{vmatrix}$$

$$= \frac{1}{r^2\sin\theta}\left[\hat{r}\left(\frac{\partial}{\partial\theta}(0) - \frac{\partial}{\partial\varphi}Mr^{-2}\sin\theta\right) - r\hat{\theta}\left[\frac{\partial}{\partial r}(0) - \frac{\partial}{\partial\varphi}2Mr^{-3}\cos\theta\right] + r\sin\theta\,\hat{\varphi}\left(\frac{\partial}{\partial r}(Mr^{-2}\sin\theta) - \frac{\partial}{\partial\theta}2Mr^{-3}\cos\theta\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\hat{r}(0) - r\hat{\theta}(0) + r\sin\theta\,\hat{\varphi}\left(-2Mr^{-3}\sin\theta + 2Mr^{-3}\sin\theta\right)\right]$$

$$= \frac{r\sin\theta}{r^2\sin\theta}\left[0 - 0 + \hat{\varphi}(0)\right]$$

$$= \frac{1}{r}(0) = 0$$

Thus the flow is of the potential kind.

Let $\phi(r, \theta, \psi)$ be the appropriate velocity potential.

Then

polar coordinates

$$\frac{-\partial\phi}{\partial r} = 2Mr^{-3}\cos\theta, \quad \frac{-\partial\phi}{r\partial\theta} = Mr^{-3}\sin\theta, \quad \frac{r\phi}{r\sin\theta\,\partial\psi} = 0 \quad (8)$$

$$d\phi = \frac{\partial\phi}{\partial r}dr + \frac{\partial\phi}{r\phi}d\theta + \frac{\partial\phi}{\partial\psi}d\psi$$

$$d\phi = -(2Mr^{-3}\cos\theta)\,dr - (Mr^{-2}\sin\theta)\,d\theta + 0\,d\psi$$

Integrating on both sides we get,

$$\int d\phi = -\int 2Mr^{-3}\cos\theta\,dr - \int Mr^{-2}\sin\theta\,d\theta$$

$$\phi = -2M\cos\theta\int r^{-3}dr - Mr^{-2}\int\sin\theta\,d\theta$$

$$= -2M\cos\theta\left[\frac{r^{-3+1}}{-3+1}\right] + Mr^{-2}\cos\theta(1)$$

$$= -2M\cos\theta\,\frac{r^{-2}}{-2} + Mr^{-2}\cos\theta$$

$$= Mr^{-2}\cos\theta + Mr^{-2}\cos\theta = 2Mr^{-2}\cos\theta$$

$$\phi = 2Mr^{-2}\cos\theta.$$

The streamlines are given by,

$$\frac{dr}{2Mr^{-3}\cos\theta} = \frac{r\,d\theta}{Mr^{-3}\sin\theta} = \frac{r\sin\theta\,d\psi}{0},$$

(or)

$$d\psi = 0$$

$$\frac{dr}{2Mr^{-3}\cos\theta} = \frac{r\,d\theta}{Mr^{-3}\sin\theta}$$

$$\frac{1}{r}dr = 2\,\frac{\cos\theta}{\sin\theta}\,d\theta$$

$$2 \cot\theta \, d\theta = (1/r)\, dr \quad (9)$$

Integrating the eqn. of the streamlines are,

$$\int d\varphi = \int 0$$

$$\varphi = \text{constant}$$

$$\int 2 \cot\theta \, d\theta = \int 1/r \, dr$$

$$2 \int \frac{\cos\theta}{\sin\theta} d\theta = \log r + A$$

$$2 \log \sin\theta = \log r + A$$

$$\log \sin^2\theta = \log r + A$$

$$e^{\log \sin^2\theta} = e^{\log r} + A$$

$$e^{\log r} = e^{\log \sin^2\theta} + A$$

$$r = A \sin^2\theta.$$

$$\begin{cases} \varphi = \text{constant} \\ r = A \sin^2\theta \end{cases}$$

The equation φ = constant. Show that the streamlines lie in planes which pass through the axis of symmetry $\theta = 0$.

2.5 The vorticity Vector.

We now consider flows for which curl $q \neq 0$.

The vector

$$\zeta = \nabla \wedge q \rightarrow (1)$$

is called the vorticity vector. The necessary and sufficient condition for potential flow may be expressed by $\zeta = 0$. A vortex line is a curve drawn in the fluid such that the tangent to it at every point is in the direction of the vorticity vector ζ. If the cartesian components of ζ are $[\zeta_1, \zeta_2, \zeta_3]$, then the equations of the vortex lines are given by.

$$\frac{dx}{\zeta_1} = \frac{dy}{\zeta_2} = \frac{dz}{\zeta_3} \qquad (10)$$

In general these do not coincide with the streamlines.

As in Sec. 2.3, Fig. 2.3, we can draw the vortex lines through all points of a closed curve Γ to form a vortex tube. Let δS_1, δS_2 be two sections of a vortex tube and let n_1, n_2 be the unit normals to these sections drawn outwards from the fluid between them. Also, let δS be the curved surface of the vortex tube, $\Delta S = \delta S_1 + \delta S + \delta S_2 =$ total surface area of element, and $\Delta v =$ total volume which ΔS contains. Then

$$\int_{\Delta S} n \cdot \zeta \, dS = \int_{\Delta v} \nabla \cdot \zeta \, dv = 0$$

Since $\nabla \cdot \zeta = \nabla \cdot (\nabla \wedge q) = 0$. Thus

$$\int_{\delta S_1} n \cdot \zeta \, dS + \int_{\delta S} n \cdot \zeta \, dS + \int_{\delta S_2} n \cdot \zeta \, dS = 0.$$

At each point of δS, $n \cdot \zeta = 0$, since ζ is tangential to the curved surface. Thus, to the first order, the last equation gives (11)

$$(n_1 \cdot \zeta_1)\,\delta S_1 + (n_2 \cdot \zeta_2)\,\delta S_2 = 0. \rightarrow (3)$$

Eqn. (3) shows that $(n \cdot \zeta)\,\delta S$ is constant over every section δS of the vortex tube. Its value is called the strength of the vortex tube. A vortex tube whose strength is unity is called a unit vortex tube.

Now suppose S is any closed surface containing a volume v. Then

$$\int_S n \cdot \zeta \, ds = \int_v \nabla \cdot \zeta \, dv = 0 \rightarrow (4)$$

Eqn. (4) shows that the total strength of vortex tubes emerging from S is equal to that entering S. This means that vortex lines and tubes cannot originate or terminate at internal points in a fluid. They can only form closed curves or terminate on boundaries. In the case of smoke rings, the vortex lines form closed curves. On the other hand, the vortex lines in a whirlpool terminate on the boundary of the fluid.

If C is a closed curve drawn in a moving fluid and if S denotes an area contained by C, then the circulation Γ of the fluid velocity q is defined to be,

$$\Gamma = \oint q \cdot ds = \int_S n \cdot \xi \, dS$$

Since ξ = curl q. For potential flow the circulation round any closed circuit is clearly zero. We shall later obtain wider conditions under which Γ is constant (Section 3.12). (12)

2.6 Local and Particle Rates of change.

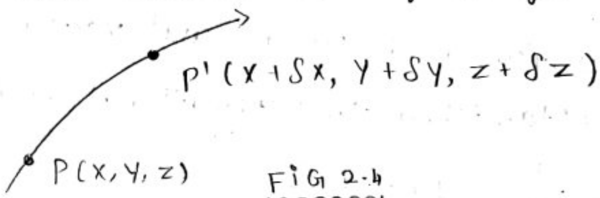


FIG 2.4

Suppose a particle of fluid moves from $P(x,y,z)$ at time t to $P'(x+\delta x,\, y+\delta y,\, z+\delta z)$ at time $t+\delta t$. Let $f(x,y,z,t)$ be a scalar function associated with some property of the fluid (e.g. the density). Then in the motion of the particle from P to P' the total change of f is given by

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial z}\delta z + \frac{\partial f}{\partial t}\delta t.$$

Thus the total rate of change of f at the point P at time t in the motion of the particle is

$$\frac{df}{dt} = \lim_{\delta t \to 0}\left(\frac{\delta f}{\delta t}\right) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial t},$$

$$= u\frac{\partial f}{\partial x} + v\frac{\partial f}{\partial y} + w\frac{\partial f}{\partial z} + \frac{\partial f}{\partial t},$$

if $q = [u, v, w]$ is the velocity of the fluid particle at P. Thus we may write

$$\frac{df}{dt} = q \cdot \nabla f + \frac{\partial f}{\partial t} \rightarrow (1)$$

Similarly, for a vector function $F(x, y, z, t)$ associated with some property of the fluid (e.g its velocity) we can show that

$$\frac{dF}{dt} = q \cdot \nabla F + \frac{\partial F}{\partial t} \rightarrow (2)$$

Thus for both scalar and vector functions we have established the operational equivalence.

$$\frac{d}{dt} \equiv q \cdot \nabla + \frac{\partial}{\partial t} \rightarrow (3)$$

applicable to both scalar and vector functions of position and time, provided that these functions are associated with properties of the moving fluid.

In obtaining (1) and (2) we are considering the total change in f or F when the fluid particle moves from $P(x, y, z)$ to $P'(x+\delta x, y+\delta y, z+\delta z)$ in time δt. Thus $\frac{df}{dt}, \frac{dF}{dt}$ are total time differentiations following the fluid particle and are called the particle rates of change. On the other hand the partial time derivatives $\frac{\partial f}{\partial t}, \frac{\partial F}{\partial t}$ are only the time rates of change at the point $P(x, y, z)$ considered fixed in space : they are the local rates of change. It follows that $q \cdot \nabla f$ or $q \cdot \nabla F$ represents the rate of change due solely to the motion of the particle along its path. This point may also be seen by denoting the arc length of the path by s and pp' by δs. Then if $\overline{PP'} = \delta s \hat{s}$, $q = q \delta \hat{s}$, where $q = |q|$, and so

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$$q \cdot \nabla f = q \delta \hat{s} \cdot \nabla f = q \frac{\partial f}{\partial s}$$

with similar results for the vector function F. Here we are using $\delta \hat{s} \cdot \nabla = \frac{\partial}{\partial s}$

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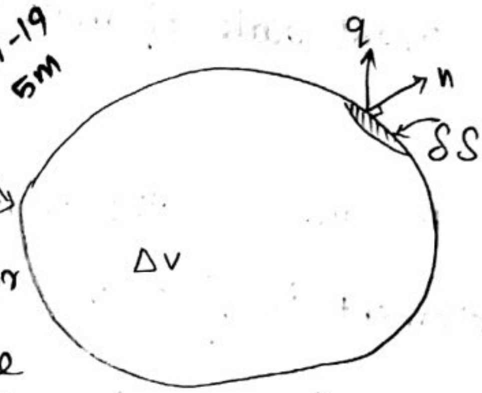

2.7 The equation of continuity.

4

When a region of a fluid contains neither sources nor sinks, that is to say when there are no inlets or outlets through which fluid can enter or leave the region, the amount of fluid within the region is conserved in accordance with the principle of conservation of matter. We now attempt to formulate this principle mathematically by means of the so-called equation of continuity.

Let ΔS be a closed surface drawn in the fluid and taken fixed in space. Suppose it contains a volume Δv of the fluid and let $\rho = \rho(x, y, z, t)$ be the fluid density (mass per unit volume) at any point (x, y, z) of the fluid in Δv at any time t.

Suppose n is the unit outward-drawn normal at any surface element δS of ΔS, where $\delta S \ll \Delta S$. Then if q is the fluid velocity at the element

δS, the normal component of q measured outwards from Δv is n.q. Thus 15

Rate of efflux of fluid mass per unit time across $\delta S = \rho\, n.q\, \delta S$.

Total rate of mass flow out of Δv across

$\Delta S = \int_{\Delta S} \rho\, n.q\, dS.$

Total rate of mass flow into $\Delta v = -\int_{\Delta S} n.(\rho q)\, dS$

$$= -\int_{\Delta v} \nabla.(\rho q)\, dv.$$

At time t, the mass of fluid within the element is $\int_{\Delta v} \rho\, dv$.

Local rate of mass increase within $\Delta v = \frac{-\partial}{\partial t}\int_{\Delta v} \rho\, dv$

$$= \int_{\Delta v} \frac{\partial \rho}{\partial t}\, dv$$

In the absence of sources and sinks within Δv, matter is not created or destroyed in this region so that

$$\int_{\Delta v} \frac{\partial \rho}{\partial t}\, dv = -\int_{\Delta v} \nabla.(\rho q)\, dv$$

(or)

$$\int_{\Delta v} \left\{ \frac{\partial \rho}{\partial t} + \nabla.(\rho q) \right\} dv = 0$$

This last relation is true for all volumes Δv if

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho q) = 0 \rightarrow (1)$$

eqn. (1) is the general eqn. of continuity which must always hold at any points of a fluid free from sources and sinks. Since

$$\nabla.(\rho q) = \rho\, \nabla.q + \nabla\rho.q,$$

other forms of (1) are:

$$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot q + q \cdot \nabla \rho = 0 \quad \rightarrow (1')$$

$\frac{d}{dt} = q \cdot \nabla + \frac{\partial}{\partial t}$ $$\frac{d\rho}{dt} + \rho \nabla \cdot q = 0 \quad \rightarrow (1'')$$

$\frac{d\rho}{dt} = q \cdot \nabla \rho + \frac{\partial \rho}{\partial t}$ $$\frac{d}{dt}(\log \rho) + \nabla \cdot q = 0 \quad \rightarrow (1''')$$

In the last two variants $\frac{d}{dt}$ denotes the differentiation following the fluid motion and the operational equivalence $\frac{d}{dt} = \frac{\partial}{\partial t} + q \cdot \nabla$ has been used.

In the special case of steady flow in which the pattern does not vary with time at any location, $\frac{\partial \rho}{\partial t} = 0$ and (1) gives

$$\nabla \cdot (\rho q) = 0 \quad \rightarrow (2)$$

For an incompressible fluid the density of any particle is invariable with time so that $\frac{d\rho}{dt} = 0$. In such a fluid there could be a variation of ρ from particle to particle as in the case of a non-homogeneous incompressible fluid. For a homogeneous and incompressible fluid ρ is constant throughout the entire fluid. In either case $(1'')$ shows that the equation of continuity is

$$\nabla \cdot q = 0 \quad \rightarrow (3)$$

From now onwards, unless otherwise stated, the term "incompressible fluid" will be taken to

Imply one which is not only incompressible but also homogeneous. If in addition to equation (3), the flow is of the potential kind, then there exists a velocity potential ϕ such that $q = -\nabla\phi$ and (3) becomes

(17)

$$\nabla^2\phi = 0 \rightarrow (4)$$

which is Laplace's equation.

Examples

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1) Test whether the motion specified by

$$q = \frac{k^2(xj - yi)}{x^2+y^2} \quad (k = \text{constant})$$ is a possible motion for an incompressible fluid. If so, determine the equations of the streamlines. Also test whether the motion is of the potential kind and if so determine the velocity potential.

$\frac{u}{v} = \frac{uv+vu}{v^2}$

Soln:

$$\nabla\cdot q = k^2\left\{-\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right\}$$

$$= k^2\left\{-\frac{(x^2+y^2)(1) + y(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)(1) - x(2y)}{(x^2+y^2)^2}\right\}$$

$$= k^2\left\{\frac{-\cancel{(x^2+y^2)} + 2xy + \cancel{(x^2+y^2)} - 2xy}{(x^2+y^2)^2}\right\}$$

$$= k^2(0)$$

$$\nabla\cdot q = 0$$

Thus the equation of continuity for an incompressible fluid is satisfied and so such a motion is possible

In the usual notation, the cartesian components of velocity are, (18)

$$u = \frac{-k^2 y}{x^2+y^2}, \quad v = \frac{k^2 x}{x^2+y^2}, \quad w = 0$$

and so the equations of the streamlines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\frac{dx}{\frac{-k^2 y}{x^2+y^2}} = \frac{dy}{\frac{k^2 x}{x^2+y^2}} = \frac{dz}{0} \Rightarrow \frac{dx}{\frac{-k^2 y}{x^2+y^2}} = \frac{dy}{\frac{k^2 x}{x^2+y^2}}$$

$$x\,dx = -y\,dy$$

$$x\,dx + y\,dy = 0$$

$$dz = 0$$

which are immediately integrable to the forms,

$$\int x\,dx = -\int y\,dy$$

$$\frac{x^2}{2} = \frac{-y^2}{2} + c \quad \Rightarrow x^2+y^2 = c \text{ (constant)}$$

$$\int dz = 0$$

$$z = c \text{ (constant)}$$

Thus the streamlines are circle whose centres are on the z-axis, their planes being perpendicular to this axis. Now,

$$\nabla \wedge q = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-k^2 y}{x^2+y^2} & \frac{k^2 x}{x^2+y^2} & 0 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}\left(\frac{\partial}{\partial x}\left(\frac{k^2 x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\frac{k^2 y}{x^2+y^2}\right)$$

$$= k^2\left\{\frac{(x^2+y^2)(1) - x(2x)}{(x^2+y^2)^2} + \frac{(x^2+y^2)(1) - y(2y)}{(x^2+y^2)^2}\right\}$$

$$= k^2\left\{\frac{x^2+y^2-2x^2+x^2+y^2-2y^2}{(x^2+y^2)^2}\right\} \quad (19)$$

$$= k^2\left\{\frac{2x^2-2x^2+2y^2-2y^2}{(x^2+y^2)^2}\right\} = k^2(0)$$

$\nabla \wedge q = 0$

Thus the flow is of the potential kind and we can find $\phi(x,y,z)$ such that $q = -\nabla\phi$, we have,

$$\frac{\partial\phi}{\partial x} = -u = \frac{k^2 y}{x^2+y^2}$$

$$\frac{\partial\phi}{\partial y} = -v = \frac{-k^2 x}{x^2+y^2}$$

$$\frac{\partial\phi}{\partial z} = -w = 0$$

The last relations shows that $\phi = \phi(x,y)$

The first gives on integration

$$\frac{\partial\phi}{\partial x} = \frac{k^2 y}{x^2+y^2} \qquad \int\frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \partial\phi = \int\frac{k^2 y}{x^2+y^2}\,\partial x \qquad \int\frac{dx}{1+x^2} = \tan^{-1}\frac{x}{1}$$

$$\phi = k^2\int\frac{y}{x^2+y^2}\,dx$$

$$= \frac{k^2 y}{y}\int\frac{dx}{x^2+y^2}$$

$$\phi = k^2 \tan^{-1}\left(\frac{x}{y}\right) + f(y)$$

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From this we find

$$\frac{\partial \phi}{\partial y} = \frac{-k^2 x}{x^2+y^2} + f'(y) \Rightarrow \frac{\partial \phi}{\partial y} = f'(y) - \frac{k^2 x}{x^2+y^2}$$

$$\Rightarrow \frac{\partial \phi}{\partial y} = f'(y) - \frac{\partial \phi}{\partial y}$$

$$f'(y) = 0$$

(or)

$$f(y) = \text{constant}$$

As the constant is immaterial, we may take

$$\phi(x, y) = k^2 \tan^{-1}(x/y)$$

The equipotentials are thus given by the planes $x = cy$ through the z-axis. They are appropriately intersected orthogonally by the streamlines.

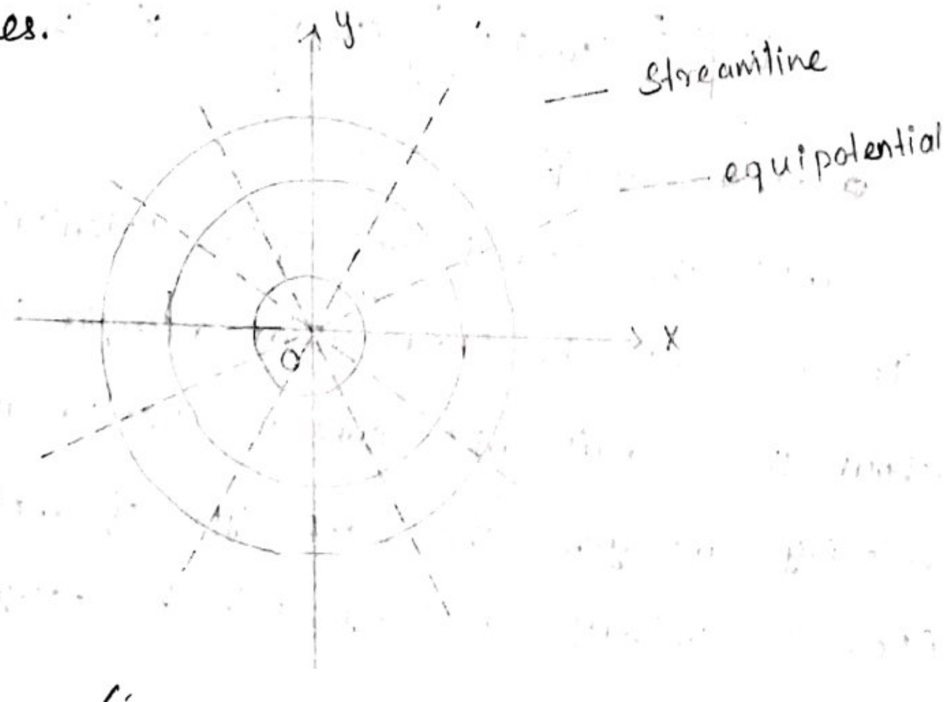


FIG. 2.6

2) For an incompressible fluid, $q = [-\omega y, \omega x, 0]$ (ω = const). Discuss the nature of the flow (4).

Soln:

21

5m

$$q = [-\omega y, \omega x, 0].$$

$$\nabla \cdot q = \left(\frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}\right)(-\omega y\,\vec{i} + \omega x\,\vec{j} + 0)$$

$$= \frac{\partial}{\partial x}(-\omega y) + \frac{\partial}{\partial y}(\omega x) + \frac{\partial}{\partial z}(0)$$

$$= 0 + 0 + 0$$

$$\nabla \cdot q = 0$$

So, that such a flow is possible

$$\nabla \wedge q = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -\omega y & \omega x & 0 \end{vmatrix}$$

$$= \vec{i}(0) - \vec{j}(0) + \vec{k}\left[\frac{\partial}{\partial x}(\omega x) + \frac{\partial}{\partial y}(\omega y)\right]$$

$$= \vec{k}\left[\frac{\partial}{\partial x}(\omega x) + \frac{\partial}{\partial y}(\omega y)\right] = k^{?}[\omega + \omega]$$

$$\nabla \wedge q = 2\omega\vec{k}$$

Thus the flow is not of the potential kind.

It can easily be shown that a rigid body rotating about the z-axis with constant vector angular velocity ωk gives the same type of motion.

(For the velocity at (x, y, z) in the body is $-\omega y\,\vec{i} + \omega x\,\vec{j}$)

The equations of the streamlines are

$$\frac{dx}{-\omega y} = \frac{dy}{\omega x} = \frac{dz}{0}$$

$$\frac{dx}{-\omega y} = \frac{dy}{\omega x} \Rightarrow \int x\,dx = -\int y\,dy$$

$$\frac{x^2}{2} = \frac{-y^2}{2} + c$$

$$x^2 + y^2 = c$$

$$\int dz = 0$$

$$z = c$$

ie) the streamlines are the circles

$$\begin{cases} x^2+y^2 = \text{constant} \\ x = \text{constant} \end{cases}$$

3) For a fluid moving in a fine tube of variable section A, prove from first principle that the equation of continuity is $A\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial s}(A\rho v) = 0$ where v is the speed at a point P of the fluid and s the length of the tube up to P. What does this become for steady incompressible flow? →

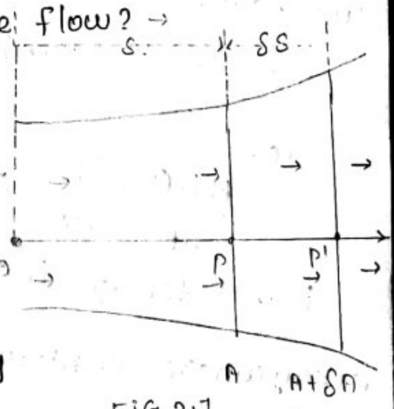


FIG 2.7

Let OPP' be the central streamline of the tube (not necessary straight) and let S, S+δS be the arc lengths OP, OP'. Let v be the fluid velocity at P and A the area of the section at P. Since the tube is of fine bore we may assume conditions are sensibly constant over the section A so that the rate of mass flux over A in the sense of s increasing is ρvA per unit time. At the neighbouring section A' through P the mass flux per unit time in the direction of s increasing is therefore

(23)

$$\rho vA + \delta s \frac{\partial}{\partial s}(\rho vA)$$

at the same instant of time t. Thus the net rate of flow of mass into the element between the sections A, $A+\delta A$ (considered fixed in space) is $-\delta s\left(\frac{\partial}{\partial s}\right)(\rho vA)$.

But at time t, the mass between the section is $\rho A \delta s$ and so its rate of increase is $\left(\frac{\partial}{\partial t}\right)(\rho A \delta s)$ or $\left(\frac{\partial \rho}{\partial t}\right) A \delta s$.

In the absence of sources and sinks, then we have,

$$-\delta s \frac{\partial}{\partial s}(\rho vA) = \frac{\partial \rho}{\partial t} A \delta s,$$

(or)

$$\frac{A \partial \rho}{\partial t} + \frac{\partial}{\partial s}(\rho vA) = 0$$

For steady incompressible flow, ρ is everywhere constant and the equation reduces to $\frac{d}{ds}(vA) = 0$ (or) vA = constant over every section

This last result means that the volume of fluid crossing every section per unit time is constant.

4) Liquid flows through a pipe whose surface is the surface of revolution of the curve $y = a + \frac{kx^2}{a}$ about the x-axis $(-a \le x \le a)$. If the liquid enters at the end $x = -a$ of the pipe with velocity V, show that the time taken by a liquid particle to traverse the entire length of the pipe from $x = -a$ to $x = +a$ is 24

$$\left\{ 2a / V(1+k)^2 \right\} \left(1 + \frac{2}{3}k + \frac{1}{5}k^2 \right) \quad P$$

[Assume that k is so small that the flow remains appreciable one-dimensional throughout].

Soln:

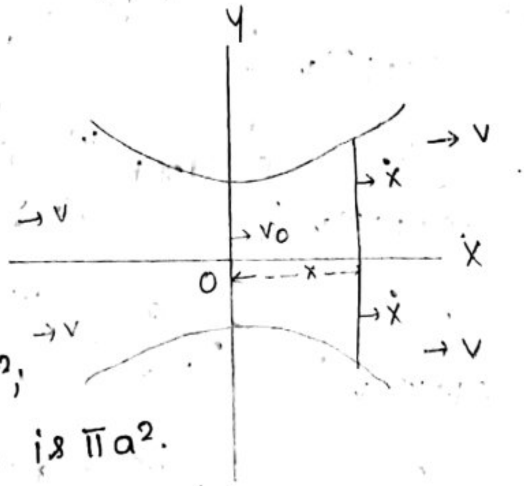


FIG. 2.8

Let v_0 be the velocity at the section $x = 0$. The area of the section $x = -a$ is $\pi a^2 (1+k)^2$; that of the section $x = 0$ is πa^2.

The area of the section distant x from 0 is $\pi \{ a + (kx^2/a) \}^2$ and so the equation of continuity (expressing equal rates of volumetric flow across the three sections) is

$$\pi a^2 (1+k)^2 v = \pi a^2 v_0 = \pi \left\{ a + \left(\frac{kx^2}{a} \right) \right\}^2 \dot{x}$$

$$\frac{dt}{dx} = \frac{\cancel{\pi} \left\{ a + \left(\frac{kx^2}{a} \right) \right\}^2}{\cancel{\pi} \left\{ a^2 (1+k)^2 v \right\}}$$

$$dt = \frac{\left\{a + \left(\frac{kx^2}{a}\right)\right\}^2}{a^2(1+k)^2 v}\,dx$$

$$= \frac{a^2\left(1+\frac{kx^2}{a^2}\right)^2 dx}{a^2(1+k)^2 v}$$

$$= \left(1+\frac{kx^2}{a^2}\right)^2 \frac{dx}{(1+k)^2 v}$$

Integrating on both sides we get,

$$\int dt = \frac{1}{(1+k)^2 v}\int_{-a}^{a}\left(1+\frac{kx^2}{a^2}\right)^2 dx$$

$$= \frac{2}{(1+k)^2 v}\int_{0}^{a}\left(1+\frac{k^2x^4}{a^4}+\frac{2kx^2}{a^2}\right) dx$$

$$= \frac{2}{(1+k)^2 v}\int_0^a dx + \frac{k^2}{a^4}\int_0^a x^4\,dx + \frac{2k}{a^2}\int_0^a x^2\,dx$$

$$= \frac{2}{(1+k)^2 v}\left\{[x]_0^a + \frac{k^2}{a^4}\left[\frac{x^5}{5}\right]_0^a + \frac{2k}{a^2}\left[\frac{x^3}{3}\right]\right\}$$

$$= \frac{2}{(1+k)^2 v}\left[a + \frac{k^2}{a^4}\left(\frac{a^5}{5}\right) + \frac{2k}{a^2}\left(\frac{a^3}{3}\right) - 0\right]$$

$$= \frac{2}{(1+k)^2 v}\left[a + \frac{k^2 a}{5} + \frac{2ka}{3}\right]$$

$$t = \left\{\frac{2a}{(1+k)^2 v}\right\}\left(1+\frac{2}{3}k+\frac{1}{5}k^2\right)$$

2.9 Acceleration of a Fluid.

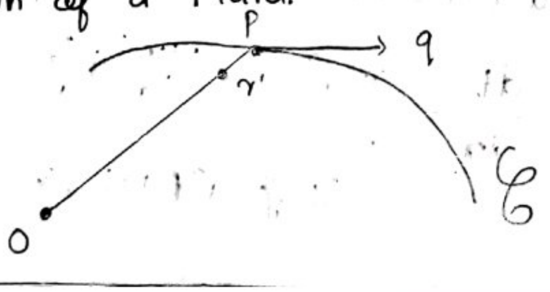


FIG 2.9

Figure 2.9 shows a fluid particle travelling along a curve $\mathscr{C}$. At time t its position p is specified by $\overline{OP} \equiv r$ and its velocity q along the tangent at P to $\mathscr{C}$ is in the direction of the particle's motion. Then the instantaneous acceleration f at P is

(26)

$$f = \frac{dq}{dt} = \frac{\partial q}{\partial t} + (q \cdot \nabla) q \quad \rightarrow (1)$$

Taking the cartesian components of q as $[u, v, w]$ this shows that the components of acceleration are

$$\left.\begin{aligned} \frac{du}{dt} &= \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \\ \frac{dv}{dt} &= \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \\ \frac{dw}{dt} &= \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \end{aligned}\right\} \rightarrow (2)$$

In tensor form, with coordinates x_i and velocity components u_i $(i = 1, 2, 3)$ the set of equation (2) could be written

$$\frac{du_i}{dt} = \frac{\partial u_i}{\partial t} + u_j u_{i,j} \rightarrow (3)$$

Reverting to the vectorial form (1), the term $(q \cdot \nabla) q$ may be develop in a form more suitable for certain purposes. Thus

$$(q \cdot \nabla) q = \sum \left\{ (q \cdot i) \frac{\partial q}{\partial x} \right\}$$

Since,

UNIT - IV

5.10 THE USE OF CONFORMAL TRANSFORMATION :-

Suppose z and t are two complex variables defined by $z = x+iy$, $t = \xi + i\eta$. (1)
where x, y, ξ, η are real variables.

Now suppose that z describes a certain curve $\mathcal{C}$ in the (x,y) plane and suppose that t is related to z by means of the transformation

$$t = g(z).$$

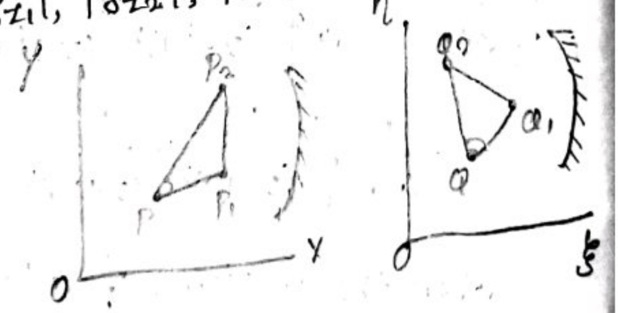
Let us now make the supposition that the fn $g(z)$ is analytic, and let P_1, P_2, P, be neighbouring points in the z-plane such that

$$\overline{OP} \equiv z, \quad \overline{OP_1} \equiv z+\delta z_1, \quad \overline{OP_2} \equiv z+\delta z_2.$$

suppose that under the given transformation $t = f(z)$, P, P_1, P_2 map into the points Q, Q_1, Q_2 in the t-plane, where

$$\overline{OQ} \equiv t, \quad \overline{OQ_1} = t+\delta t_1, \quad \overline{OQ_2} \equiv t+\delta t_2$$

It is assumed that $|\delta z_1|, |\delta z_2|, |\delta t_3|$ are small.



$$\frac{\delta t_1}{\delta z_1} = \frac{\delta t_2}{\delta z_2}.$$

or $$\frac{\delta t_1}{\delta t_2} = \frac{\delta z_1}{\delta z_2}$$

$$\left|\frac{\delta t_1}{\delta t_2}\right| = \left|\frac{\delta z_1}{\delta z_2}\right| \rightarrow (1)$$

and $$\arg \delta t_1 - \arg \delta t_2 = \arg \delta z_1 - \arg \delta z_2 \rightarrow (2)$$

$$\frac{QQ_1}{QQ_2} = \frac{PP_1}{PP_2} \rightarrow (1') \quad \angle Q_2QQ_1 = \angle P_2PP_1 \rightarrow (2')$$

The eqn. (1'), (2') show that the triangles Q_2QQ_1, P_2PP_1 are similar.

5.10.1 SOME HYDRODYNAMICAL ASPECTS OF CONFORMAL TRANSFORMATION :

The complex velocity potential for the z-plane be

$$w = f(z) = \phi + i\psi \quad (2)$$

$$w = \bar{f}(t) = \bar{\phi} + i\bar{\psi}$$

where $\bar{\phi} = \bar{\phi}(\xi, \eta)$, $\bar{\psi} = \bar{\psi}(\xi\, \eta)$ At corresponding points t, z, w takes the same values so that

$$\phi = \bar{\phi}, \quad \psi = \bar{\psi}$$

$$\frac{dw}{dt} = \frac{dw}{dz}\frac{dz}{dt} = \frac{1}{g'(z)}\frac{dw}{dz}$$

Hence $\left|\frac{dw}{dz}\right|^2 = \left|\frac{dw}{dt}\right|^2 \left|\frac{dt}{dz}\right|^2$

$$\delta T = \frac{1}{2}\rho \times \text{Area of triangle } PP_1P_2 \times (\text{Speed of } P)^2$$

$$= \frac{1}{2}\rho \cdot (\Delta PP_1P_2) \cdot \left|\frac{dw}{dz}\right|^2$$

$$= \frac{1}{2}\rho \cdot (\Delta PP_1P_2) \cdot \left|\frac{dw}{dt}\right|^2 \left|\frac{dt}{dz}\right|^2$$

$$= \frac{1}{2}\rho \cdot (\Delta QQ_1Q_2) \cdot \left|\frac{dw}{dt}\right|^2$$

Total K.E of liquid in z-plane

= Total K.E of liquid in t-plane.

$$\left|\frac{dw}{dz}\right| \neq \left|\frac{dw}{dt}\right|.$$

THEOREM : I

Under conformal transformation a uniform line source maps into another uniform line source of the same strength.

PROOF :

Let there be a uniform line source of strength m per unit length through the point $z = z_0$ and

Suppose the conformal transformation $t = g(z)$ is made from the z-to the t-plane so that the points $= z_0$ maps into the point $t = t_0$. Let $\mathscr{C}$ be a closed curve in the z-plane containing the point $z = z_0$ and suppose $\mathscr{C}$ maps into $\mathscr{C}'$ in the t - plane. Then $t = t_0$ lies within $\mathscr{C}'$. The complex potential w is the same for both systems and has the forms. (3)

$$w = \phi + i\psi \text{ for the z-plane}$$
$$= \phi' + i\psi' \text{ for the t-plane}$$

Thus $\phi = \phi'$, $\psi = \psi'$. Since ψ is the same at corresponding points of $\mathscr{C}$, $\mathscr{C}'$.

$$\oint_{\mathscr{C}} d\psi = \oint d\psi' \rightarrow (1)$$

Now in the z-plane $w = -m\log(z - z_0)$

$$dw = -m\,dz/(z - z_0).$$

Hence

$$\oint_{\mathscr{C}} dw = -m \oint_{\mathscr{C}} \frac{dz}{z - z_0} = -m \times 2\pi i$$

Since the integrand has a residue of 1 at $z = z_0$

But $dw = d\phi + i d\psi$, so that equating imaginary parts gives

$$\oint_{\mathscr{C}} d\psi = -2\pi m. \rightarrow (2)$$

Thus (1), (2) show that the same volume crosses unit thickness of $\mathscr{C}'$ per unit time which implies an equal line source of strength m per unit length through $t = t_0$.

THEOREM: Under conformal transformation a uniform line vortex maps into another uniform line vortex of the same strength.

PROOF:

Let there be a uniform line vortex of strength k per unit length through $z = z_0$ and suppose the conformal transformation $t = g(z)$ is made form the z-to the t-plane.

The complex potential is the same for both systems and has the forms

$$w = \phi + i\psi \text{ for the } z\text{-plane}$$

$$= \phi' + i\psi' \text{ for the } t\text{-plane} \quad (4)$$

Thus $\phi = \phi'$, $\psi = \psi'$

$$\oint_{C} d\phi = \oint_{C'} d\phi'$$

$$w = (ik/2\pi) \log(z - z_0)$$

$$\oint_{C} dw = \frac{ik}{2\pi} \oint \frac{dz}{z - z_0} = \frac{ik}{2\pi} \times 2\pi i = -k$$

equating real parts gives

$$-\oint_{\psi} d\phi = +k.$$

The line source through $z = z_0$ of strength k per length maps into an equal line source through $t = t_0$.

THEOREM III :

under conformal transformation a uniform line doublet maps into another uniform line doublet of different strength.

PROOF :

Let there be a uniform line doublet of strength μ per unit length through p where $z = z_0$ and suppose that under the conformal transformation $t = g(z)$. $\overline{pp'} \equiv \delta z$, $\mu = m|\delta z|$ and pp' the axis of the line doublet. QQ' if $\overline{QQ'} \equiv \delta t$, then $\delta t = g'(z)\, \delta z$, So that $|\delta t| = |g'(z)| \cdot |\delta t|$, $\arg \delta t = \arg g'(z) + \arg \delta z$

Hence the two line sources through Q, Q'

strength μ' where

$$\mu' = m'|St| = \mu|g(z_1)|.$$

The inclination of the axis of the line doublet to the real axis is increased by $\arg g'(z)$

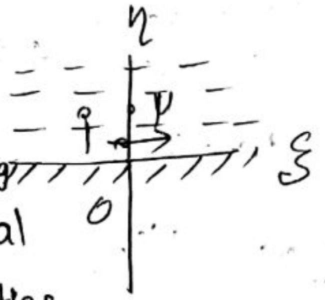
5.10.2 SOME WORKED EXAMPLES.

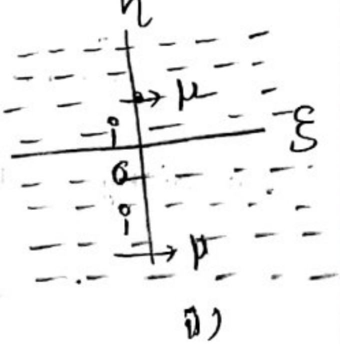
(5)

EXAMPLE 1.

A uniform line doublet of strength μ per unit length is situated at the point $t = i$ in the t-plane. its axis pointing in the direction of the +ve real axis

PROOF:





i) The physical model and ii) the corresponding image system with equal line doublets of strengths μ per unit length at the points where $t = \pm i$.

The complex potential is thus

$$w = \frac{\mu}{t-i} + \frac{\mu}{t+i} = \frac{2\mu t}{t^2+1} \quad Im(t) \geq 0$$

putting $t = e^z$, we find $w = \dfrac{2\mu e^z}{e^{2z}+1} = \mu \operatorname{sech} z.$

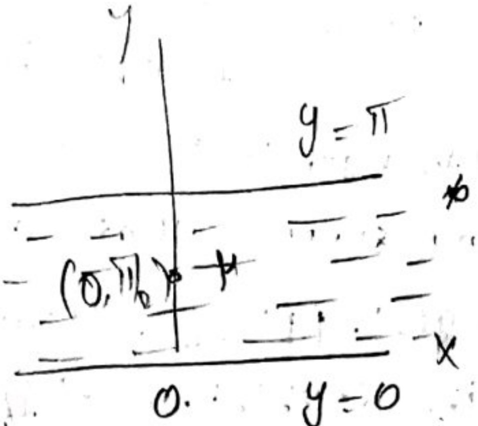
writing $t = \xi + i\eta$ we have $z = \log(\xi + i\eta)$

Thus the line $\eta = 0$ becomes

$$z = \log \xi = \log|\xi| + i \arg \xi,$$

So that $y = \arg \xi$,

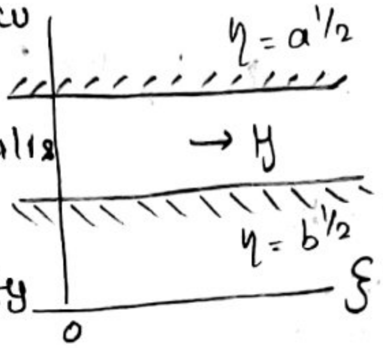
$$y = 0 \text{ for } \xi > 0$$
$$= \pi \text{ for } \xi < 0$$



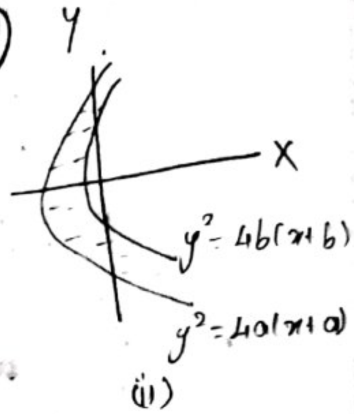
and the planes $y = 0$, $y = \pi$ when a uniform line doublet of strength μ per unit $(0, \frac{1}{2}\pi)$, EX: 2

An inviscid fluid of constant density ρ is in irrotational two-dimensional motion. Show that the fluid velocity $[u, v]$ is given in terms of a function $w(z)$ of a complex variable $z = x+iy$ by the relation $-u+iv = w'(z)$.

(i) shows the flow in the ζ-plane. The velocity potential is $-V\xi = -V\,\mathrm{Re}(\zeta)$.



(b)



The complex velocity potential is

$$w = -V\zeta = -V(\xi + i\eta) \quad (i)$$

Applying the transformation $\zeta = z^{1/2}$, we find

$$w = -Vz^{1/2} = -V(x+iy)^{1/2}$$

when $\eta = b^{1/2}$,

$$x+iy = (\xi + ib^{1/2})^2 = (\xi^2 - b) + 2ib^{1/2}\xi,$$

So that $x = \xi^2 - b$, $y = 2b^{1/2}\xi$

$$y^2 = 4b(x+b)$$

The parabola, vertex $(-b, 0)$ and focus $(0,0)$ $-\infty \leq \xi \leq \infty$

$$\eta = a^{1/2}, \quad y^2 = 4a(x+a)$$

Taking $z = re^{i\theta}$, $w = \phi + i\psi$

$$\phi + i\psi = -Vz^{1/2} = -Vr^{1/2}e^{i\theta/2}$$

$$\phi = -Vr^{1/2}\cos(\theta/2) \quad \psi = -Vr^{1/2}\sin(\theta/2)$$

$$\psi = c = \text{constant}$$

$$r = \frac{c^2}{\sin^2(\theta/2)} = \frac{2c^2}{1-\cos\theta}$$

$$\therefore 2c^2 = \sqrt{x^2+y^2} - x$$

$$y^2 = 4c^2(c^2 + x^2)$$

$$q^2 = \left(\frac{\partial \phi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)^2$$

$$= \left\{\frac{1}{2} v r^{-1/2} \cos(\theta/2)\right\}^2 + \left\{\frac{1}{2} v r^{-1/2} \sin(\theta/2)\right\}^2$$

$$q = \frac{1}{2} v r^{-1/2}$$

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$r = a, \quad q = \frac{1}{2} v a^{-1/2}$

$\frac{p}{\rho} * \frac{1}{2} q^2 = k$, by the Bernoulli's equation

$$\frac{p}{\rho} + \frac{1}{2} q^2 = k.$$

$$p = p_\infty - \frac{1}{8}\left(\frac{\rho v^2}{r}\right)$$

$\min r = b$

$$\min p = p_\infty - \frac{1}{8}\left(\frac{\rho v^2}{b}\right)$$

$$p_\infty = \frac{1}{8}\left(\frac{\rho v^2}{b}\right)$$

$$p = \frac{1}{8} \rho v^2 \left(\frac{1}{b} - \frac{1}{r}\right)$$

$|z| = a$

$$p = \frac{1}{8} \rho v^2 \left(\frac{1}{b} - \frac{1}{a}\right)$$

5.12 VORTXE ROWS :-

5.12.1 SINGLE INFINITE ROW OF LINE VORTICES :-

We First consider the case of a single infinite row of vortices each of strength k at the points

$$z = 0, \pm a, \pm 2a, \dots \pm na, \dots$$

The complex velocity potential due to the $(2n+1)$ vortices nearest the origin is

$$w_n = ik/2\pi \{\log z + \log(z-a) + \log(z-2a) + \dots + \log(z-na)$$
$$+ \log(z+a) + \log(z+2a) + \dots + \log(z+na)\}$$

$$= (ik/2\pi) \log\{z(z^2-a^2)(z^2-4a^2) \dots (z^2-n^2a^2)\}$$

$$= \frac{ik}{2\pi} \log\left\{ \frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{4a^2}\right)\cdots\left(1-\frac{z^2}{n^2a^2}\right)\right\} + \text{const}$$

$$w = \frac{ik}{2\pi} \log\left\{ \frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{4a^2}\right)\cdots\left(1-\frac{z^2}{n^2a^2}\right)\cdots\right\} \quad (8)$$

$$= (ik/2\pi) \log \sin(\pi z/a)$$

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{4\pi^2}\right)\cdots\left(1-\frac{x^2}{n^2\pi^2}\right)\cdots$$

The velocity components are found from

$$-u+iv = \frac{dw}{dz} = \left(\frac{ik}{2a}\right)\cot\left(\frac{\pi z}{a}\right)$$

$$= \frac{ik}{2a} \cdot \frac{\cos\{\pi(x+iy)/a\} \cdot \sin\{\pi(x-iy)/a\}}{\sin\{\pi(x+iy)/a\} \cdot \sin\{\pi(x-iy)/a\}}$$

$$u = \frac{-k}{2a} \cdot \frac{\operatorname{Sh}(2\pi y/a)}{\operatorname{ch}(2\pi y/a) - \cos(2\pi x)/a)}$$

$$v = \frac{k}{2a} \cdot \frac{\sin(2\pi x/a)}{\operatorname{ch}(2\pi y/a) - \cos(2\pi x/a)}$$

$$u = \mp (k/2a), \; v \to 0.$$

The line vortex through $\zeta = 0$ maps into equal line vortices through $z = 0, \pm a, \pm 2a, \ldots$ Also

$$w = \frac{ik}{2\pi} \log \zeta = \frac{ik}{2\pi} \log \sin\left(\frac{\pi z}{a}\right).$$

5.12.2 THE KARMAN VORTEX STREET:

The First row consists of line vortices of strength k at the points having cartesian coordinates $(na, \frac{1}{2}b)$ $(n = 0, \pm 1, \pm 2 \ldots)$

The second row consists of line vortices of strength $-k$ at the points $(\frac{1}{2}, 2n+1a, -\frac{1}{2}b)$ $(n = 0, 1, 2)$.

It is found to be

$$w = \frac{ik}{2\pi} \log \sin \frac{\pi}{a}\left(z - \frac{ib}{2}\right) - \frac{ik}{2\pi} \log \sin \frac{\pi}{a}\left(z - \frac{a}{2} + \frac{ib}{2}\right)$$

The velocity of the vortex at $z = \frac{1}{2}a - \frac{1}{2}ib$ is given by

$$-u + iv = \frac{ik}{2\pi} \frac{d}{dz}\left\{ \log \sin \frac{\pi}{a}\left(z - \frac{ib}{2}\right)\right\}_{(a-ib)/2}$$

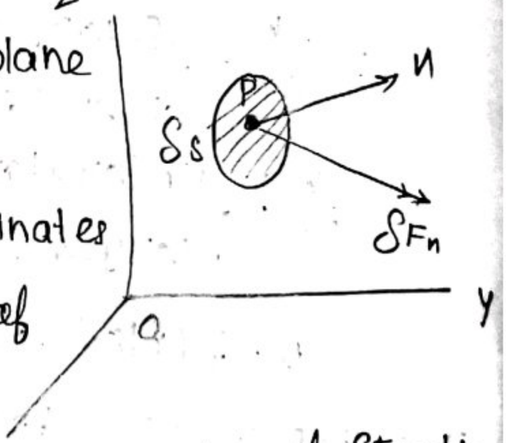
$$= -\frac{k}{2a} \, th\left(\frac{\pi b}{a}\right)$$

The +ve x-direction with velocity $(k/2a)\, th(\pi b/a)$

A karman vortex street is often realized when a flat plate moves broad. side through a liquid.

VISCOUS FLOW

8.1 Stress components in a Real Fluid.



Let δs be a small rigid plane area inserted at a point P in a viscous fluid. cartesian coordinates (x, y, z) are referred to a set of fixed axes OX, OY, OZ.

We suppose the cartesian components of δF_n to be so that

$$\delta F_n = \delta F_{nx} \hat{i} + \delta F_{ny} \hat{j} + \delta F_{nz} \hat{k}.$$

Then the components of stress parallel to the axes are defined to be P_{nx}, P_{ny}, P_{nz}, where

$$P_{nx} = \lim_{\delta s \to 0} \left(\frac{\delta F_{nx}}{\delta s}\right) = \frac{dF_{nx}}{ds},$$

$$P_{ny} = \lim_{\delta s \to 0} \left(\frac{\delta F_{ny}}{\delta s}\right) = \frac{dF_{ny}}{ds},$$

$$P_{nz} = \lim_{\delta s \to 0} \left(\frac{\delta F_{nz}}{\delta s}\right) = \frac{dF_{nz}}{ds}.$$

If we identify n in turn with the unit vectors $\bar{OX}, \bar{OY}, \bar{OZ}$ (i, j, k) we obtain the following three sets of stress components:

$$p_{xx},\ p_{xy},\ p_{xz}$$
$$p_{yx},\ p_{yy},\ p_{yz}.$$
$$p_{zx},\ p_{zy},\ p_{zz}$$

(10)

The diagonal elements p_{xx}, p_{yy}, p_{zz} of this array are called normal or direct stresses. The remaining six elements are called shearing stress. For an inviscid fluid

$$p_{xx} = p_{yy} = p_{zz} = -p;$$

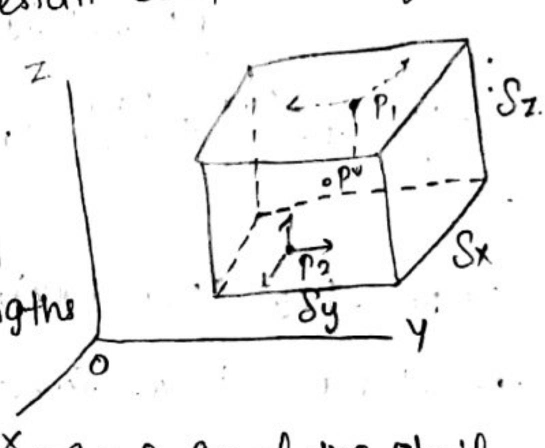
$$p_{xy} = p_{xz} = \cdots = 0.$$

The matrix,

$$\begin{bmatrix} p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p_{zz} \end{bmatrix}$$

is called the stress matrix. knowledge of its components enables us to calculate the total forces on any area at any chosen point. The quantities p_{ij} $(i, j = x, y, z)$ are called the components of the stress tensor whose matrix is of the above form. Clearly p_{ij} is a second order tensor.

8.2 Relations between cartesian components of stress.



The motion of a small rectangular parallelepiped of viscous fluid, its centre being $P(x, y, z)$ and its edges of lengths $\delta x, \delta y, \delta z$ parallel to axes cartesian axes. The mass $\rho\,\delta x\,\delta y\,\delta z$ of the fluid element remains constant and the element is presumed

UNIT - III

(1)

Some Two - Dimensional Flows :-

5.1 Meaning of Two - Dimensional Flow.

Suppose a fluid moves in such a way that at any given instant the flow pattern in a certain plane is the same as that in all other parallel plane within the fluid. Then the flow is said to be two dimensional. If we take any one of the parallel planes to be the plane $z=0$, then at any point in the fluid having cartesian coordinates (x, y, z), all physical quantities associated with the fluid are independent of z.

5.2 Use of cylindrical polar coordinates :

If the flow is irrotational, then the equation satisfied by the velocity potential ϕ at any point having cylindrical polar coordinates (R, θ, z) is

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \phi}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2 \phi}{\partial \theta^2}+\frac{\partial^2 \phi}{\partial z^2}=0$$

which is Laplace's equation in these coordinates, If the motion is two dimensional and the coordinate axes so chosen that all physical quantities associated with the flow are independent of z, then

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \phi}{\partial R}\right)+\frac{1}{R^2}\frac{\partial^2 \phi}{\partial \theta^2}=0 \rightarrow (1)$$

In order to obtain special solutions of this equation we try separating the variables by putting $\phi = f(R)\, g(\theta)$ to give

$$g(\theta)\frac{1}{R}\frac{d}{dR}\{Rf'(R)\}+\frac{1}{R^2}f(R)g''(\theta)=0$$

or $$\frac{R(d/dR)\{Rf'(R)\}}{f(R)}=-\frac{g''(\theta)}{g(\theta)} \rightarrow (2) \quad (2)$$

The L.H.S of (2) is a function of R only : the R.H.S is a function of θ only Thus each is constant. Let n^2 be the value of the constant. Then we obtain the ordinary differential equation.

$$R^2f''(R)+Rf'(R)-n^2f(R)=0 \rightarrow (3)$$

$$g''(\theta)+n^2g(\theta)=0 \rightarrow (4)$$

eqn. (3) is of the euler-homogeneous type and is Solved by putting $R=e^t$ when the equation reduces to

$$\frac{d^2f}{dt^2}-n^2f=0$$

giving the solutions

$$f=e^{\pm nt}$$

(or) $f(R)=R^{\pm n}\ (n\neq 0)$

eqn. (4) has the special solutions

$$g(\theta)=\begin{matrix}\cos\\ \sin\end{matrix}\, n\theta \quad (n\neq 0)$$

Thus special solutions to equation (1) are

$$\phi(R,\theta)=(A_nR^n+B_nR^{-n})(C_n\cos n\theta+E_n\sin n\theta)$$

A more general type of solution is obtained by superposition :

$$\phi(R,\theta)=\sum_n(A_nR^n+B_nR^{-n})(C_n\cos n\theta+E_n\sin n\theta)$$

This summation, over n, may be finite or infinite and need not be restricted to integral values of n. In the special case of $n=0$, rejected in the above development;

$$f = K_1 + K_2 t = K_1 + K_2 \log R$$ (3)

$$g = K_3 + K_4 \theta.$$

So that

$$\phi = (K_1 + K_2 \log R)(K_3 + K_4 \theta)$$

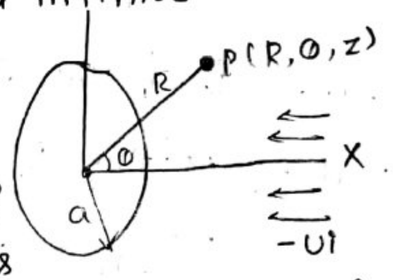
The particular case of $n=1$ yields the important special solutions.

$$\phi = R\cos\theta, \quad \phi = R\sin\theta, \quad \phi = \frac{1}{R}\cos\theta, \quad \phi = \frac{1}{R}\sin\theta$$

we now illustrate the use of these results.

Example 1:

Uniform flow past a fixed infinite circular cylinder. A uniform stream of undisturbed velocity $-Ui$ flowing past the fixed cylinder $R=a$. P is



a point in the fluid having cylindrical polar coordinates (R, θ, z) and cartesian coordinates (x, y, z) the common coordinate z being redundant as the flow is two dimensional. The velocity potential due to the uniform stream is

$$Ux = UR\cos\theta$$

This perturbation must be such as to satisfy Laplace's equation and to become vanishingly small for large R. The simplest harmonic function satisfying these requisites is clearly $R^{-1}\cos\theta$. Let us, then try to find A so that

$$\phi(R, \theta) = UR\cos\theta + AR^{-1}\cos\theta$$

is the required velocity potential. On the cylinder we require that $\frac{\partial \phi}{\partial R} = 0$ for all θ which gives at once $A = Ua^2$

Thus

$$\phi(R,\theta) = U\cos\theta\,(R + a^2R^{-1})$$ (4)

Hence the velocity components at P are

$$q_R = -\frac{\partial\phi}{\partial R} = -U\cos\theta\left\{1-\left(\frac{a}{R}\right)^2\right\}$$

$$q_\theta = -\frac{1}{R}\frac{\partial\phi}{\partial\theta} = U\sin\theta\left\{1+\left(\frac{a}{R}\right)^2\right\}$$

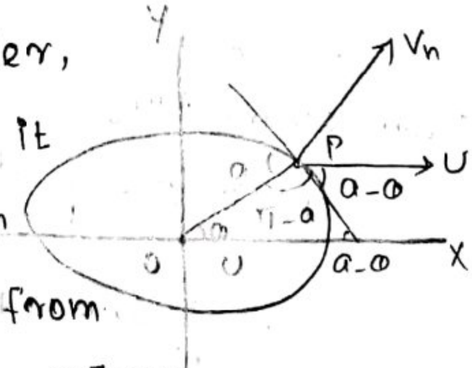
$$q_z = -\frac{\partial\phi}{\partial z} = 0$$

As $R\to\infty$, $q_R\to -U\cos\theta$, $q_\theta\to U\sin\theta$, appropriately

EX: 2.

A cylinder of infinite length and nearly circular section moves through an infinite volume of liquid with a velocity U at right-angles to its axis and in the direction of the x-axis. If its section is specified by the equation $R = a(1+\epsilon\cos n\theta)$, where ϵ is small, show that the approximate value of the velocity potential is

$$Ua\left\{\frac{a}{R}\cos\theta + \epsilon\left(\frac{a}{R}\right)^{n+1}\cos(n+1)\theta - \epsilon\left(\frac{a}{R}\right)^{n-1}\cos(n-1)\theta\right\}$$



A section of the cylinder, the tangent at a point on it making angles, α, $(\pi-\alpha)$ with the results vector drawn from O. At large radial distances R from Oz, the fluid velocity becomes vanishingly small.

Thus suitable harmonic functions are of the forms $R^{-k}\cos k\theta$, $R^{-k}\sin k\theta$ $(k=1,2,3\ldots)$ Let us try the solution

$$\phi(R,\theta) = \sum_{k=1}^{\infty} R^{-k}(A_k \cos k\theta + B_k \sin k\theta). \quad (5)$$

At $\theta = 0$ on the boundary $q_\theta = 0$, which is satisfied by taking $B_k = 0$ $(k = 1, 2 \ldots)$. Further the function

$$\phi(R,\theta) = \sum_{k=1}^{\infty} A_k R^{-k} \cos k\theta \quad \rightarrow (1)$$

is appropriately an even function of θ.

At any point (R, θ, z) in the fluid

$$q_R = -\frac{\partial\phi}{\partial R} = \sum_{k=1}^{\infty} k A_k R^{-(k+1)} \cos k\theta$$

$$q_\theta = -\frac{1}{R}\frac{\partial\phi}{\partial\theta} = \sum_{k=1}^{\infty} k A_k R^{-(k+1)} \sin k\theta$$

$$q_z = -\frac{\partial\phi}{\partial z} = 0.$$

Hence at the point P on the surface where

$R = a(1+\epsilon \cos n\theta)$,

$$q_R = \sum_{k=1}^{\infty} k A_k a^{-(k+1)} \cos k\theta (1+\epsilon \cos n\theta)^{-(k+1)}.$$

$$q_\theta = \sum_{k=1}^{\infty} k A_k a^{-(k+1)} \sin k\theta (1+\epsilon \cos n\theta)^{-(k+1)}$$

$$q_z = 0$$

From elementary differential geometry. at P

$$\cot(\pi - \alpha) = \frac{1}{R}\frac{dR}{d\theta} = \frac{d}{d\theta}(\log R)$$

$$= \frac{d}{d\theta}\log(1+\epsilon \cos n\theta)$$

So that $\cot\alpha = \dfrac{\epsilon n \sin n\theta}{1+\epsilon \cos n\theta}$

The normal component of velocity v_n of the cylinder at p is

$V_n = U\sin(\alpha-\theta) = U(\sin\alpha\cos\theta - \cos\alpha\sin\theta)$

$$= \frac{U\{\cos\theta(1+\epsilon\cos n\theta) - \sin\theta.\, \epsilon n \sin n\theta\}}{\{(1+\epsilon\cos n\theta)^2+\epsilon^2 n^2 \sin^2\theta\, n\theta\}^{1/2}}$$ (6)

$V_n = q_R \sin\alpha + q_\theta \cos\alpha$

$$= \frac{\sum_{k=1}^{\infty} k A_k a^{-(k+1)} (1+\epsilon\cos n\theta)^{-(k+1)} \{\cos k\theta(1+\epsilon\cos n\theta) + \sin k\theta.\, \epsilon n \sin n\theta\}}{\{(1+\epsilon\cos n\theta)^2+\epsilon^2 n^2 \sin^2 n\theta\}^{1/2}}$$

equating the two forms for V_n gives, on simplification,

$$\sum_{k=1}^{\infty} k A_k a^{-(k+1)} (1+\epsilon\cos n\theta)^{-(k+1)} \{\cos k\theta(1+\epsilon\cos n\theta) + \sin k\theta.\, \epsilon n \sin n\theta\}$$

$$= U\{\cos\theta(1+\epsilon\cos n\theta) - \sin\theta.\, \epsilon n \sin n\theta\}$$

this gives the correct to the first order in ϵ approximation.

equating coefficients of $\cos\theta$

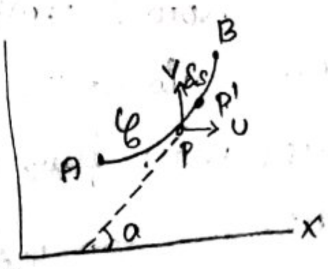
$$U = b_1/a^2$$

equating coefficients of $\cos(n-1)\theta$,

$$\frac{1}{2}U\epsilon + \frac{1}{2}U\epsilon n = (n+1)\frac{b_{n-1}}{a^{n+2}} - \frac{1}{2}(n+1)\epsilon\left(\frac{b_1}{a^2}\right).$$

So that $U_{n+1} = U\epsilon\, a^{n+2}$

All other b's are zero. Sub. for b_1, b_{n-1}, b_{n+1} yields the required result.



5.3 The stream Function.

$\mathcal{C}$ is an arc of a curve in the (x,y)-plane joining two points

A, B. Fluid is constrained to flow in two dimensional steady motion parallel to this plane. At $p(x, y)$ on the curve the velocity components are $[u, v]$. p' is the neighbouring point $(x+\delta x, y+\delta y)$ on $\mathcal{C}$ so that $pp' = \delta s$. (7)

If the tangent to $\mathcal{C}$ at p makes angle α with ox, the the velocity component at p along the normal from right to left as we travel from A to B is $v\cos\alpha - u\sin\alpha$. Hence the mass of fluid which crosses unit thinckness of the surface element though pp' normal to the plane of flow per unit time is

$$\rho(v\cos\alpha - u\sin\alpha)\delta s = \rho(v\delta x - u\delta y)$$

This mass flux is constant; denote it by $\delta\psi$. Then

$$\delta\psi = \rho(v\delta x - u\delta y)$$

(or) $$d\psi = \rho(v\,dx - u\,dy)$$

The total mass flux per unit thickness per unit time from right to left across the arc AB is

$$\psi_B - \psi_A = \int_A^B \rho(v\,dx - u\,dy) \rightarrow (2)$$

$$\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0$$

Thus

$$\frac{\partial\psi}{\partial x} = \rho v, \quad \frac{\partial\psi}{\partial y} = -\rho u \rightarrow (3)$$

Now suppose the curve $\mathcal{C}$ is a stream line. Then no fluid crosses in boundary. Thus since $\psi_P - \psi_A$ is the mass flux per unit time per unit thinkness of AP,

$$\psi_P - \psi_A = 0$$

or $$\psi = \text{const along } \mathcal{C}$$

It thus follows that the family of curves $\psi =$ const are the streamlines in the plane $z=0$. For this reason the function $\psi(x,y)$ is called the stream function. (8)

The equations (1), (2), (3) become

$$d\psi = v\,dx - u\,dy \rightarrow 1'$$

$$\psi B - \psi A = \int_A^B (v\,dx - u\,dy) \rightarrow 2'$$

$$\frac{\partial \psi}{\partial x} = v, \quad \frac{\partial \psi}{\partial y} = -u \rightarrow 3'$$

If we further suppose the flow to be irrotational, then there exists a velocity potential ϕ such that

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y} \rightarrow (4)$$

The eqn. of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \rightarrow (5)$$

The eqn. (3), (4), (5) now yield the following impo[illegible] results.

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \rightarrow (6)$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \rightarrow (7)$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \rightarrow (8)$$

$$\frac{\partial \phi}{\partial x}\frac{\partial \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y}\frac{\partial \phi}{\partial y} = 0 \rightarrow (9)$$

Thus for [illegible]onal incompressible two imension[illegible](x, y) $\psi(x,y)$ are harmonic unc[illegible] the families of curves $\phi =$ constant [illegible]tant intersect orthogonally. This last result follows from the fact that (9) expresses the perpendicularity of the vectors $\nabla\phi$, $\nabla\psi$.

5.4 The complex potential for Two-Dimensional, Irrotational Incompressible Flow.

(9)

The complex fn. $w = \phi + i\psi$ has far-reaching properties which facilitate solution of many types of two-dimensional flow problems.

The reader will find a fuller treatment in fn. of a complex variable by E.G Phillips

Suppose that $z = x + iy$ and that

$$w = f(z) = \phi(x,y) + i\psi(x,y)$$

where x, y, ϕ, ψ are all real and $i = \sqrt{(-1)}$.

we write $\phi = Re(w)$, $\psi = Im(w)$. The region specified by z the derivative $dw/dz = f'(z)$ is unique, then w is said to be analytic or regular throughout the region

By defn. the derivative of w at z is

$$\frac{dw}{dz} = f'(z) = \lim_{\delta x, \delta y \to 0} \left\{ \frac{\delta\phi + i\delta\psi}{\delta x + i\delta y} \right\}$$

$$\frac{dw}{dz} = \lim_{\delta z \to 0} \left\{ \frac{\delta\phi + i\delta\psi}{\delta x} \right\} = \frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x} \to (1)$$

If however we keep x constant so that $\delta x = 0$, the

$$\frac{dw}{dz} = \lim_{\delta y \to 0} \left\{ \frac{\delta\phi + i\delta\psi}{i\delta y} \right\} = -i\frac{\partial\phi}{\partial y} + \frac{\partial\psi}{\partial y} \to (2)$$

If w is an analytic fn. the two values of dw/dz expressed by (1) and (2) must be equal. Then, equating real and imaginary parts, we obtain at once

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}, \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}. \to (3)$$

The eqn. (3) are called the cauchy- Riemann equations. The fns ϕ, ψ are termed conjugat fn.

The meanings of velocity potential and stream function for two- dimensional, irrotational, incompressible, inviscid flow, then $w = \phi + i\psi$ is called the complex velocity potential. (10)

Also eqn. (1) above shows that

$$\frac{dw}{dz} = -u + iv.$$

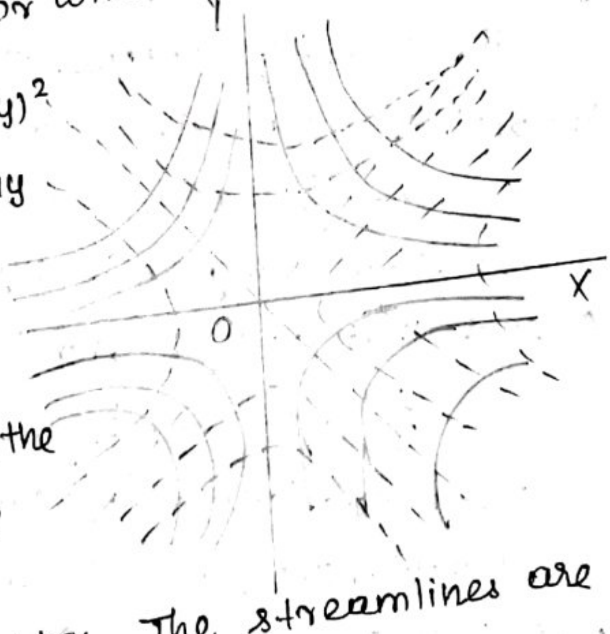
since $u = -\frac{\partial \phi}{\partial x} = \frac{-\partial \psi}{\partial y}$, $v = -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}$. Also we have

$$\left| \frac{dw}{dz} \right| = \sqrt{(u^2 + v^2)} = q.$$

giving the speed of the fluid at any point.

Example:

Discuss the flow for which $w = z^2$.



we have $\phi + i\psi = (x + iy)^2$

$= x^2 - y^2 + 2ixy$

Hence $\begin{cases} \phi = x^2 - y^2 \\ \psi = 2xy \end{cases}$

The equipotentials are the rectangular hyperbolae

$x^2 - y^2 = \text{const}$

having asymptotes $y = \pm x$. The streamlines are the rectangular hyperbolas

$xy = \text{const}.$

having asymptotes $x = 0, y = 0$. $\frac{dw}{dz} = 2z$ is zero only at the origin, this is the only stagnation

5.5 complex velocity potentials for standard Two-Dimensional Flows.

The more common types of two-dimensional Flows. (11)

i) uniform stream

ii) a line source and sink.

iii) a line doublet.

The flow is assumed to be inviscid, irrotational incompressible and two-dimensional.

5.5.1. UNIFORM STREAM.

i) We First consider the uniform stream having velocity $-Ui$. This gives rise to a velocity potential $\phi = Ux$. Clearly since $x = Re(z)$

$$w = Uz = U(x+iy).$$

The stream Fn. $\psi = Im(w) = Uy$, so that the lines $y = $ const. are the streamlines.

ii) Secondly, suppose the uniform stream is incident to the positive x-axis at angle α, so that

$$q = [-U\cos\alpha, -U\sin\alpha].$$

The flow being obviously irrotational, we can find ϕ such that $q = -\nabla\phi$ and hence

$$\frac{\partial\phi}{\partial x} = U\cos\alpha, \quad \frac{\partial\phi}{\partial y} = U\sin\alpha.$$

Thus $d\phi = (U\cos\alpha)dx + (U\sin\alpha)dy$, so that

$\phi(x,y) = U(x\cos\alpha + y\sin\alpha)$. in simplest form.

Also since $q = [-\partial\psi/\partial y, \partial\psi/\partial x]$

$$d\psi = (-U\sin\alpha)dx + (U\cos\alpha)dy$$

or $\psi = U(-x\sin\alpha + y\cos\alpha)$, in simplest form. Hence,

$$w = \phi + i\psi = U(xe^{-i\alpha} + iye^{-i\alpha}) = Uze^{-i\alpha}$$

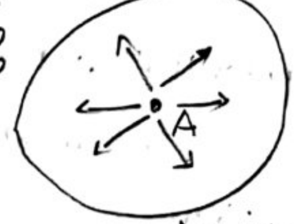
$$w'(z) = -u + iv = U\cos\alpha - iU\sin\alpha = Ue^{-i\alpha}.$$

So that $w(z) = Uze^{-i\alpha}$.

5.5.2 LINE SOURCES AND LINE SINKS : (12)

A is any point in the considered plane of flow and $\mathscr{C}$ is any closed curve surrounding it. Then the line being the same everywhere and parallel to the plane of flow. Then the line through A is called a line source. If fluid drains away through such a line and under the same conditions of symmetry, then the line is called a line sink.

$\mathscr{C}$



Suppose the infinite line through A is a source and that it emits fluid at the rate of $2\pi m\rho$ units of mass per unit length of the source per unit time.

The strength of the line source to be m. If we now take for $\mathscr{C}$ the circle centre A and radius r, then the speed of flow q is everywhere the same on $\mathscr{C}$ and the mass flux is now $2\pi r\rho q$ per unit length.

Thus $2\pi m\rho = 2\pi r\rho q$ or

$$q = m/r.$$

On such a circle the velocity potential is clearly $\phi(r)$, so that $-\frac{\partial\phi}{\partial r} = \frac{m}{r}$

(or) $\phi = -m\log r$. in simplest form.

The circle has polar coordinates (r,θ) w.r.t A and if $\overline{op} \equiv z = re^{i\theta}$, then since $\log r = \log|z|$, we see at once that the complex velocity potential of the line source through A of uniform strength m is

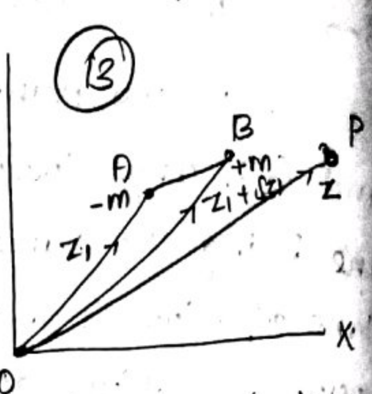
$$w = -m\log z = -m(\log r + i\theta).$$

The complex velocity potential

$$w = -m\log(z - z_1)$$

5.5.3 LINE DOUBLETS :

A uniform line source of strength $-m$ at A; one of strength $+m$ at B and the considered field point P, all in the same plane of flow, where



$\bar{OA} \equiv z_1,\ \bar{OB} \equiv z_1+\delta z_1\ \ \bar{OP} \equiv z$

Then $\bar{AP} \equiv z-z_1,\ \bar{BP} \equiv z-z_1-\delta z_1$. Hence the complex potential at P due to the two line sources at A, B is

$$w = m\log(z-z_1) - m\log(z-z_1-\delta z_1)$$

Now
$$\log(z-z_1-\delta z_1) = \log(z-z_1) - \delta z_1\left(\frac{\partial}{\partial z}\right)\left[\log(z-z_1)\right] + O\{(\delta z_1)^2\}$$
$$= \log(z-z_1) - \delta z_1(z-z_1)^{-1} + O\{(\delta z_1)^2\}$$

Thus if we assume that $|\delta z_1|$ is so small that $|\delta z_1|^2$, $|\delta z_1|^3, \ldots$ are negligible, then to the first order.

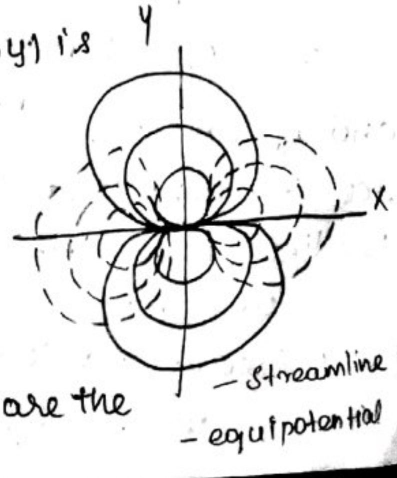
$$w = -m\delta z_1(z-z_1)^{-1} = m|\delta z_1|e^{i\alpha}(z-z_1)^{-1}$$

$\therefore$ A are said to comprise a line doublet of strength μ per unit length. $\bar{AB}$ gives the direction of the axis of the doublet. Thus in the limit, the complex potential at P is $\quad w = \mu e^{i\alpha}(z-z_1)^{-1}$.

EXAMPLE :

Discuss the flow due to a uniform line doublet at O of strength μ per unit length, its axis being along $\bar{OX}$.

The complex potential at $P(x,y)$ is



$$w = \frac{\mu}{z} = \frac{\mu(x-iy)}{x^2+y^2}$$

So that,
$$\begin{cases} \phi = \dfrac{\mu x}{x^2+y^2} \\ \psi = -\dfrac{\mu y}{x^2+y^2} \end{cases}$$

Thus the equipotentials, $\phi = \text{const}$, are the coaxial circles

$$x^2 + y^2 = 2k_1 x$$

and the streamlines, ψ = const, are the coaxial circles (14)

$$x^2 + y^2 = 2k_2 y$$

The first family have centres $(k_1, 0)$ and radii k_1. the second centres $(0, k_2)$ and radii k_2. shows some of each in sketch. The two families are mutually orthogonal

5.5.4. LINE VORTICES:

If for a two-dimensional flow $q = ui + vj$, where $u = u(x,y)$, $v = v(x,y)$ then the ~~red~~ vorticity vector ζ is given by. $\zeta = \nabla \wedge q = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) k$, Showing that in two-dimensional flow the vorticity vector is perpendicular to the plane of flow.

we now discuss the two-dimensional flow for which

$$w = (ik/2\pi) \log z,$$

where k is a real constant. put $z = re^{i\theta}$, we have

$$\begin{cases} \phi = -k\theta/2\pi \\ \psi = (k/2\pi) \log r \end{cases}$$

$$\begin{cases} q_r = -\frac{\partial \phi}{\partial r} = 0 \\ q_\theta = -\frac{1}{r} \frac{\partial \phi}{\partial \theta} = \frac{k}{2\pi r} \end{cases}$$

The circulation Γ round any closed curve $\mathcal{C}$ surrounding the origin and in the plane of flow is given by. $\Gamma = \oint_{\mathcal{C}} q \cdot ds$.

Taking $q = \hat{\theta}(k/2\pi r)$. $ds = dr\,\hat{r} + r\,d\theta\,\hat{\theta}$, we have $q \cdot ds = k/2\pi\, d\theta$

and so $\Gamma = \frac{k}{2\pi} \oint_{\mathcal{C}} d\theta = \frac{k}{2\pi} \times 2\pi = k.$

If $\mathcal{C}$ does not surround O, then Γ is easily shown to be zero. Hence we have shown that a two-dimensional distribution having a complex velocity potential $w = (ik/2\pi)\log z$ gives a circulation round any closed curve $\mathcal{C}$ in the plane of flow and enclosing the origin O of amount k. Also, round any other curve in the plane of flow which does not enclose O the circulation is zero. Further, the streamlines are the concentric circles $r =$ const. and the equipotentials the line $\theta =$ const, through O. Such a pattern is obtained in every plane perpendicular to the z-axis. By making the area contained within $\mathcal{C}$ vanishingly small we see that the direction producing such a flow must be uniform along the z-axis. Such a distribution along the z-axis is called a uniform line vortex of strength k.

The complex potential is easily seen to be

$$w = (ik/2\pi)\log(z - z_0)$$

$$q = \hat{\theta}(k/2\pi r), \quad ds = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + dz\,k.$$

So that $q \cdot ds = (k/2\pi)\,d\theta$, as before

$$w = (ik/2\pi)\log z,$$

$$w = (ik/2\pi)\log z, \quad |z| \geq 0$$

will give irrotational flow outside the cylinder $|z| = a$ of infinite length, the fluid being at rest at infinity and having a circulation k the cylinder

5.6) EX : 1

Find the equations of the streamlines due to uniform line sources of strength m through the points $A(-c,0)$, $B(c,0)$ and a uniform line sink of strength $2m$ through the origin. (16)

Let P be the point such that $\overline{OP} \equiv z = x+iy$.

Then $\overline{AP} \equiv z+c$, $\overline{BP} \equiv z-c$

and so the complex potential at p is

$$w = -m\log(z+c) - m\log(z-c) + 2m\log z$$

$$= m\log\left(\frac{z^2}{z^2-c^2}\right)$$

$$= m\log\left(\frac{x^2-y^2+2ixy}{x^2-y^2-c^2+2ixy}\right)$$

$$= m\log\left\{\frac{(x^2-y^2+2ixy)(x^2-y^2-c^2-2ixy)}{(x^2-y^2-c^2)^2+4x^2y^2}\right\}$$

$$= m\log\left\{\frac{(x^2+y^2)^2-c^2(x^2-y^2)-2ixyc^2}{(x^2-y^2-c^2)^2+4x^2y^2}\right\}$$

$$\therefore \psi = Im(w) = m\tan^{-1}\left\{\frac{-2xyc^2}{(x^2+y^2)^2-c^2(x^2-y^2)}\right\}$$

The streamlines are given by $\psi =$ const ie, by

$$-\frac{2xyc^2}{(x^2+y^2)^2-c^2(x^2-y^2)} = \text{const}$$

(or) $(x^2+y^2)^2 = c^2(x^2-y^2+kxy)$ ($k =$ const.)

EX: 2

Describe the irrotational motion of an incompressible liquid for which the complex potential is $w = ik\log z$.

$$w = ik \log z \Rightarrow z = ik \log(re^{i\theta})$$
$$= -k\theta + ik \log r.$$

Hence

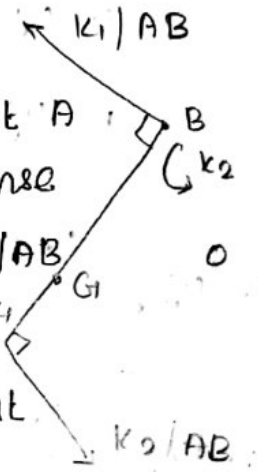
$$\phi = -k\theta, \; \psi = k \log r$$

This shows that the streamlines are concentric circles centred on O, the equipotentials being the radii through O cutting the circles orthogonally.

Also, at (r, θ) in the plane of flow the velocity components are

$$q_r = 0, \; q_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{k}{r}$$

The flow is due to a line vortex through O of strength k.



The line vortices at A, B. k_1 at A gives B a velocity k_1/AB in the sense shown. k_2 at B gives A a velocity k_2/AB in the sense shown. Associating masses k_1 at A, k_2 at B, we see that the total linear momentum of the system in the direction of the velocity of B is

$$k_1\left(-k_2/AB\right) + k_2\left(k_1/AB\right) = 0$$

Thus the centroid of the two masses does not move. The diagram shows that the velocity of B relative to A is $(k_1 + k_2)/AB$ so that the angular velocity of AB is $(k_1 + k_2)/(AB)^2$. Thus the line rotates with this angular velocity about G

Let O be the fixed origin of coordinates and suppose $\overline{OA} \equiv a$, $\overline{OB} \equiv b$. a and b being complex

numbers. If p be the point for which $\overline{OP} \equiv z$,
then $\overline{AP} \equiv z-a$, $\overline{BP} \equiv z-b$ and the complex potential at p is (18)

$$w = ik_1 \log(z-a) + ik_2 \log(z-b)$$

Thus $$\frac{dw}{dz} = \frac{ik_1}{z-a} + \frac{ik_2}{z-b} = \frac{i\{k_1(z-b)+k_2(z-a)\}}{(z-a)(z-b)}$$

If c denotes the centroid of k_2 at A, k_1 at B and if $\overline{OC} \equiv \bar{z}$, then

$$(k_1+k_2)\overline{CP} = k_2\overline{AP} + k_1\overline{BP}$$

(or) $$(k_1+k_2)(z-\bar{z}) = k_2(z-a) + k_1(z-b).$$

Hence $$\frac{dw}{dz} = \frac{i(k_1+k_2)(z-\bar{z})}{(z-a)(z-b)}$$

Thus the speed of p is

$$\left|\frac{dw}{dz}\right| = \frac{(k_1+k_2)|z-\bar{z}|}{|z-a|\,|z-b|} = \frac{(k_1+k_2)\,PC}{AP \cdot BP}$$

5.8. THE MILNE - THOMSON CIRCLE THEOREM:

Statement:

Let $f(z)$ be the complex velocity potential for a flow having no rigid boundaries and such that there are no singularities of flow within the circle $|z|=a$. Then, on introducing the solid circular cylinder $|z|=a$ into the flow, the new complex velocity potential is given by $w = f(z) + \bar{f}(a^2/z)$ for $|z| \geq a$.

Proof:

All the singularities of $f(z)$ occur in the region $|z|>a$ and so singularities of $f(a^2/z)$ lie in $|z|<a$. Hence the singularities of $\bar{f}(a^2/z)$ also lie in $|z|<a$.

Thus $f(z)$ and $f(z)+\bar{f}(a^2/z)$ both have the same singularities in the region $|z|>a$ and so both fns, considered as complex velocity potentials, may be ascribed to the same hydrodynamical distributions in the region $|z|>a$. (19)

On the circle $|z|=a$, we take $z=ae^{i\theta}$.

Then $a^2/z = ae^{-i\theta}$ and so

$$\omega = f(z) + \bar{f}(a^2/z) = f(ae^{i\theta}) + \bar{f}(ae^{-i\theta})$$

$$= f(ae^{i\theta}) + \overline{f(ae^{i\theta})}$$

Thus on $|z|=a$, ω is the sum of a complex quantity and its complex conjugate and is therefore a real number. Hence $\psi = \mathrm{Im}(\omega) = 0$ $|z|=a$. This shows that the circular boundary is a streamline across which no fluid flows. Hence $|z|=a$ is a possible boundary for the new flow and $\omega = f(z) + \bar{f}(a^2/z)$ is the appropriate complex velocity potential for the new flow.

5.8.1 SOME APPLICATIONS OF THE CIRCLE THEORM:

EXAMPLE : 1

Uniform flow past a stationary cylinder.

A uniform stream having velocity $-Ui$ gives rise to a complex potential Uz. Thus we take

$$f(z) = Uz, \quad \text{Then } \bar{f}(z) = Uz,$$

and so $\bar{f}(a^2/z) = Ua^2/z$.

The cylinder of circular section $|z|=a$ into the stream the complex potential for the region $|z|\geq a$ becomes

$$\omega = f(z) + \bar{f}(a^2/z) = U(z + a^2 z^{-1})$$

Taking $z = re^{i\theta}$ and eqn. real and Imaginary parts,

$$\phi = U\cos\theta(r + a^2 r^{-1})$$ (20)

$$\psi = U\sin\theta(r - a^2 r^{-1})$$

EXAMPLE : 2

Uniform stream at incidence α to $\vec{OX}$.

The complex potential for such a stream of velocity U is $Uze^{-i\alpha}$. Taking $f(z) = Uze^{-i\alpha}$

$$\bar{f}(z) = Uze^{i\alpha}$$

$$\bar{f}(a^2/z) = Ua^2/z\, e^{i\alpha}$$

Hence when the cylinder of section $|z| = a$ is introduced the complex potential in $|z| \geq a$ becomes

$$\omega = U\{ze^{-i\alpha} + (a^2/z)e^{i\alpha}\}.$$

Ex : 3

Image of a line source in a circular cylinder.

Suppose there is a uniform line source of strength m per unit length through the point $z = d$, where $d > a$. Then the complex potential at a point z is

$$f(z) = -m\log(z-d)$$

$$\bar{f}(z) = -m\log(z-d)$$

Then $\bar{f}(a^2/z) = -m\log\{a^2/z - d\}$.

The circular cylinder of section $|z| = a$, the complex velocity potential in the region $|z| \geq a$ becomes

$$\omega = -m\log(z-d) - m\log\{(a^2/z) - d\}$$

This may be written in the alternative form

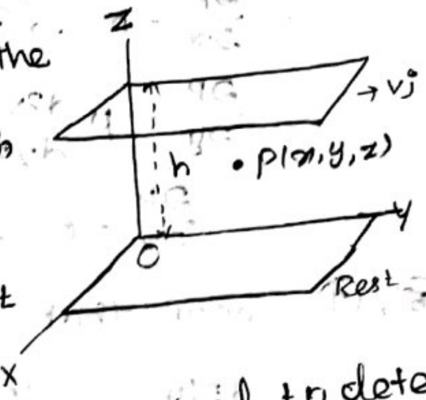
$$\omega = -m\log(z-d) - m\log\{z - (a^2/d)\} + m\log z + \text{const}$$

UNIT - V (1)

SOME SOLVABLE PROBLEMS in VISCOUS FLOW

8.10 SOME SOLVABLE PROBLEMS

8.10.1 STeady MOTION Between parallel planes.



The region $0 \le z \le h$ between the planes $z=0$, $z=h$ is filled with incompressible viscous fluid. The plane $z=0$ in held at rest and the plane $z=h$ moves with constant velocity vj. It is required to determine the nature of the flow when conditions are steady, assuming there is no slip between the fluid and either boundary, neglecting body forces.

Let $p(x,y,z)$ be any point within the fluid. Then the velocity q at p will be of the form

$$q = v(y,x)j \quad \to (1) \qquad \nabla \cdot q = 0 \text{ gives} \to (2)$$

and from (1), (2) we infer that

$$q = v(z)j \quad \to (3)$$

The Navier - stokes vector equation of motion may be taken in the form

$$\frac{\partial q}{\partial t} + (q \cdot \nabla)q = -\frac{1}{\rho}\nabla p + \nu \nabla^2 q \quad \to (4)$$

The form (4) with axes fixed in space. Since the flow is steady. $\frac{\partial q}{\partial t} = 0$. Also

$$(q \cdot \nabla)q = (v\,\partial/\partial y)\,v(z)j = 0$$

$$\nabla^2 q = v''(z)j$$

Hence (4) gives.

$$0 = -\frac{\partial p}{\partial x} i + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z} k) + \mu v''(z) j$$

equating coeff. of the unit vector

2

$$\frac{\partial p}{\partial x} = 0 \rightarrow (5)$$

$$\frac{\partial p}{\partial y} = \mu \frac{d^2 v}{dz^2} \rightarrow (6)$$

$$\frac{\partial p}{\partial z} = 0 \rightarrow (7)$$

(5), (7) show that $p = p(y)$. Hence (6) becomes

$$\frac{dp(y)}{dy} = \mu \frac{d^2 v(z)}{dz^2} \rightarrow (8)$$

$$\frac{dp(y)}{dy} = \frac{\mu d^2 u(z)}{dz^2} = -P$$

where $P > 0$ solving for v gives

$$v(z) = A + Bz - (P/2\mu) z^2 \rightarrow (9)$$

when $z = 0$, $v = 0$ and when $z = h$, $v = V$. Hence we find

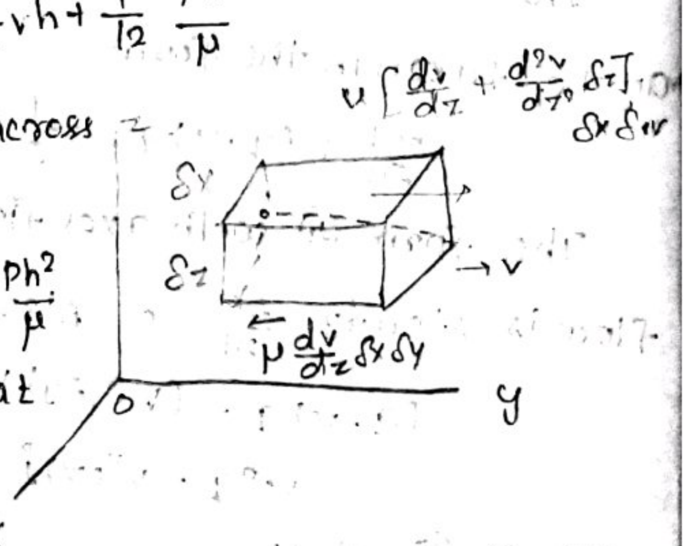
$$v(z) = \left(\frac{V}{h} + \frac{Ph}{2\mu}\right) z - \frac{P}{2\mu} z^2 \rightarrow (10)$$

The total flow per unit breadth across a plane perpendicular to oy is

$$\int_0^h v(z)\, dz = \frac{1}{2} V h + \frac{1}{12} \frac{Ph^3}{\mu}$$

and the mean velocity across such a section is

$$\frac{1}{h} \int_0^h v(z)\, dz = \frac{1}{2} V + \frac{1}{12} \frac{Ph^2}{\mu}$$



The tangential stress at any point $p(x, y, z)$ is

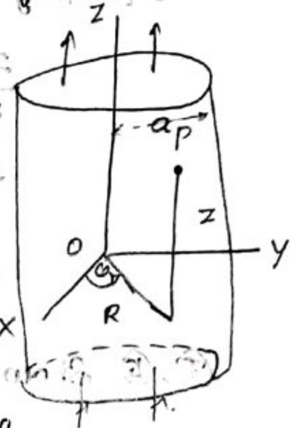
$$\mu \frac{dv}{dz} = \frac{V}{h} + \frac{Ph}{2\mu} - \frac{Pz}{\mu}$$

Thus the drag per unit area on the lower plane is $(V/h) + (Ph/2\mu)$ and that on the upper plane is $(V/h) - (Ph/2\mu)$ ③

$$\left\{ \mu(d^2v)dz^2) \quad \begin{array}{l} -\partial p/\partial x = 0 \\ -\partial p/\partial y = 0 \\ -\partial p/\partial z = 0 \end{array} \right.$$

as before.

8.10.2 Steady Flow Through Tube of uniform circular cross-section.



The steady flow of an inviscid incompressible fluid through a circular tube of radius a. Then continuity considerations applied to an annular-shaped element of radii R, $R+\delta R$ of the fluid indicate that the fluid velocity is of the form

$$q = \omega(R)k \to ①$$

Let us take the Navier-Stokes vector eqn. in the form

$$\frac{\partial q}{\partial t} + (q\cdot\nabla)q = -\frac{1}{\rho}\nabla p - \nu\nabla\wedge(\nabla\wedge q)$$

Then we have

$$\partial q/\partial t = 0$$

$$(q\cdot\nabla)q = (\omega\,\partial/\partial z)\cdot[\omega(R)k] = 0$$

$$\nabla p = \frac{\partial p}{\partial R}\hat{R} + \frac{1}{R}\frac{\partial p}{\partial \theta}\hat{\theta} + \frac{\partial p}{\partial z}k$$

$$\nabla\wedge q = \frac{1}{R}\begin{vmatrix} \hat{R} & R\hat{\theta} & k \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & 0 & \omega(R) \end{vmatrix} = -\omega'(R)\hat{\theta}$$

Hence.

$$\nabla\wedge(\nabla\wedge q) = \frac{1}{R}\begin{vmatrix} \hat{R} & R\hat{\theta} & k \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & -Rw'(R) & 0 \end{vmatrix} = -\frac{1}{R}\frac{d}{dR}(Rw')k$$

4

Thus the eqn. (2) becomes

$$0 = -\frac{1}{\rho}\left(\frac{\partial p}{\partial R}\hat{R} + \frac{\partial p}{R\partial\theta}\hat{\theta} + \frac{\partial p}{\partial z}k\right] + \frac{\nu}{R}\frac{d(Rw')}{dR}k$$

equating coeff of the unit vector gives

$$\frac{\partial p}{\partial R} = 0 \rightarrow (3)$$

$$\frac{\partial p}{\partial \theta} = 0 \rightarrow (4)$$

$$\frac{\partial p}{\partial z} = \frac{\mu}{R}\frac{d(Rw')}{dR} \rightarrow (5)$$

(3), (4) Show that $p = p(z)$. So that (5) becomes

$$\frac{dp(z)}{dz} = \frac{\mu}{R}\frac{d}{dR}[Rw'(R)] \rightarrow (6)$$

$$\frac{d}{dR}(Rw') = -\frac{PR}{\mu},$$

$$\frac{Rdw}{dR} = A - \frac{PR^2}{\mu}$$

$$\frac{dw}{dR} = \frac{A}{R} - \frac{1}{2}\frac{PR}{\mu}$$

$$w(R) = B + A\log R - \frac{1}{4}(PR^2/\mu) \rightarrow (7)$$

$$w(R) = \frac{1}{4}(P/\mu)(a^2 - R^2) \rightarrow (8)$$

The volume of fluid discharged over any section per unit time is

$$Q = \int_0^a w(R).\, 2\pi R\, dR = \frac{\pi P a^4}{8\mu}.$$

The tube distant l apart, then $P = P/l$.

8.10.3 Steady Flow b/w concentric Rotating cylinders.

The motion is steady the velocity at P will be of the form.

5

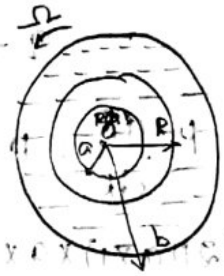
$$q = R\omega(R)\hat{\theta} \rightarrow (1)$$

The pressure at P will clearly be of the form

$$p = p(R) \rightarrow (2)$$

This time we take the Navier stokes vector eqn. in the form

$$\frac{\partial q}{\partial t} + \nabla\left(\frac{1}{2}q^2\right) - q\wedge(\nabla\wedge q) = -\frac{1}{\rho}\nabla p - \nu\nabla\wedge(\nabla\wedge q) \rightarrow (3)$$



$$\nabla\left(\frac{1}{2}q^2\right) = \hat{R}\frac{\partial}{\partial R}\left(\frac{1}{2}R^2\omega^2\right)$$

$$= \left\{R^2\omega\frac{d\omega}{dR} + R\omega^2\right\}\hat{R}$$

we have

$$\nabla\wedge q = \frac{1}{R}\begin{vmatrix} \hat{R} & R\hat{\theta} & k \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & R^2\omega & 0 \end{vmatrix} = \left(R\frac{d\omega}{dR} + 2\omega\right)k;$$

$$q\wedge(\nabla\wedge q) = R\omega\hat{\theta}\wedge\left(R\frac{d\omega}{dR} + 2\omega\right)k = R\omega\left(R\frac{d\omega}{dR} + 2\omega\right)\hat{R};$$

$$q\wedge(\nabla\wedge q) = R\omega\hat{\theta}\wedge\left(R\frac{d\omega}{dR}\right) + 2\omega)k = R\omega\left(R\frac{d\omega}{dR} + 2\omega\right)\hat{R}$$

$$\nabla\wedge(\nabla\wedge q) = \frac{1}{R}\begin{vmatrix} \hat{R} & R\hat{\theta} & k \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & 0 & R\frac{d\omega}{dR} + 2\omega \end{vmatrix} = -\left(R\frac{d^2\omega}{dR^2} + 3\frac{d\omega}{dR}\right)\hat{\theta}$$

Thus (3) becomes

$$-R\omega^2\hat{R} = -\frac{1}{\rho}\frac{dp}{dR}\hat{R} + \nu\left(R\frac{d^2\omega}{dR^2} + 3\frac{d\omega}{dR}\right)\hat{\theta}$$

equating coeff. of the unit vector gives.

$$\frac{1}{\rho}\frac{dR}{dR} = R\omega^2 \rightarrow (4)$$

$$R\frac{d^2\omega}{dR^2} + 3\frac{d\omega}{dR} = 0 \rightarrow (5) \qquad (6)$$

$$\frac{d}{dR}\left(\frac{R^3 d\omega}{dR}\right) = 0$$

$$\frac{d\omega}{dR} = \frac{A}{R^3}$$

$$\omega = B - (A/2R^2)$$

Taking $\omega = 0$ when $R = a$, $\omega = \Omega$ when $R = b$ we find

$$\omega(R) = \Omega b^2 (1 - a^2 R^{-2})(b^2 - a^2)^{-1} \rightarrow (6)$$

$$q = R\omega(R) = \Omega b^2 (R - a^2 R^{-1})(b^2 - a^2)^{-1}$$

$$\mu \frac{dR}{dR} = \mu \Omega b^2 (1 + a^2 R^{-2})(b^2 - a^2)^{-1}$$

$$(2\pi a \times 1) \times a \times \{2\mu \Omega b^2 (b^2 - a^2)^{-1}\} = 4\pi\mu a^2 b^2 \Omega (b^2 - a^2)^{-1}$$

This result forms the basis of measuring μ for some liquids using a rotation viscometer.

8.11 Steady viscous Flow in Tubes of uniform cross-Section.

The $\frac{dq}{dt} = 0$, $F = 0$, the Navier-Stokes vector equation becomes

$$(1/\rho)\nabla p - \nu \nabla^2 q = 0 \rightarrow (1)$$

$$\nabla p = \mu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) k \rightarrow (2)$$

equating coeff. of the unit vectors.

$$\partial p/\partial x = 0 \rightarrow (3)$$

$$\partial p/\partial y = 0 \rightarrow (4)$$

$$\partial p/\partial z = \mu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) \rightarrow (5)$$

eqns. (3), (4): Show $p = p(z)$ so that (5) gives.

$$\frac{dp(z)}{dz} = \mu \left(\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} \right) \rightarrow (6)$$

$$\frac{\partial^2 w(x,y)}{\partial x^2} + \frac{\partial^2 w(x,y)}{\partial y^2} = \frac{-p}{\mu} \rightarrow (7)$$

Subject to w vanishing on the walls of the tube

8.11.1 A UNIQUENESS THEOREM:

Statement:

If $\partial^2 w/\partial x^2 + \partial^2 w/\partial y^2 = F(x,y)$ at all points (x,y) of a region S in the plane Ox, Oy bounded by a closed curve C and if F is prescribed at each point (x,y) of S and w at each points of C. then any soln. $w = w(x,y)$ satisfying these conditions is unique.

PROOF:

Let $w = w_1(x,y)$, $w = w_2(x,y)$ be two solns. satisfying

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = f(x,y)$$

The $w_1 \equiv w_2$.

$$W(x,y) = w_1(x,y) - w_2(x,y)$$

Then $\partial^2 w/\partial x^2 + \partial^2 w/\partial y^2 = f - f = 0$ Also on C, $w = 0$.

Since $w_1 = w_2$ on C

Now consider

$$I = \iint_S \left\{ \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 \right\} dx\,dy = \iint_S \left(\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + W \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \right\} dx\,dy$$

$$= \iint_S \left\{ \frac{\partial}{\partial x}\left(W \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y}\left(w \frac{\partial w}{\partial y} \right) \right\} dx\,dy$$

$$= \oint_C \left(W \frac{dw}{dx}\, dy - W \frac{\partial w}{\partial y}\, dx \right) = 0 \quad \text{since } w = 0 \text{ on } C.$$

$$\frac{\partial w}{\partial x} = 0 = \frac{\partial w}{\partial y} \text{ at each point of } S.$$

The continuity of W that $W = 0$ throughout S.

This establishes the result.

8

8.11.2 Tube having uniform Elliptic cross-section

The tube has eqn. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

Then we must solve $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{p}{\mu}$

$w = 0$ on this section

The fn. $w = k\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$

$$k\left(-\frac{2}{a^2} - \frac{2}{b^2}\right) = -\frac{p}{\mu}$$

$$k = pa^2b^2 / \{2\mu(a^2+b^2)\}$$

$$w(x,y) = pa^2b^2(1 - x^2a^{-2} - y^2b^{-2})\{2\mu(a^2+b^2)\}^{-1}$$

$$Q = \int_S w(x,y)\, dS$$

$$x = \lambda a \cos\theta, \; y = \lambda b \sin\theta \; (0 \le \lambda \le 1)$$

where in fact $w = pa^2b^2(1-\lambda^2) / \{2\mu(a^2+b^2)\}$.

The area b/w this ellipse and the ellipse

$$x = (\lambda + \delta\lambda) a\cos\theta, \; y = (\lambda + \delta\lambda) b\sin\theta$$

$$\pi a^3 b^3 (1-\lambda^2)\lambda\delta\lambda) / \{\mu(a^2+b^2)\}.$$

The total volume discharged per unit time is

$$Q = \frac{\pi p a^3 b^3}{\mu(a^2+b^2)} \int_0^1 (1-\lambda^2)\lambda\, d\lambda = \frac{\pi p a^3 b^3}{4\mu(a^2+b^2)}$$

on taking $b = a$. We recover the formula for Q for the circular section of 8.10.2.

8.11.3 Tube having Equilateral Triangular cross-section.

The section of the tube is the equilateral triangle bounded by the lines (9)

$$x = a, \quad y = \pm 3^{-1/2} x.$$

Then if we take $w = k(x-a)(y^2 - \frac{1}{3}x^2)$

$w = 0$ the boundary. Sub. $\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{-p}{\mu}$

we find $k = 3p/(4\mu a)$

Thus $w = 3p(x-a)(y^2 - \frac{1}{3}x^2)/(4\mu a)$

The volume discharged per unit time is

$$Q = \int_S w\,ds = 2\int_0^a dx \int_0^{3^{-1/2}x} w\,dy = \frac{pa^4}{60 \cdot 3^{1/2} \mu}$$

8.11.4 Use of Harmonic functions.

It is sometimes possible to see a particular integral $w = w_1(x,y)$ for

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{-p}{\mu} \text{ in } S$$

one may be able to proceed by looking for a suitable harmonic fn. $w = w_2(x,y)$ satisfying Laplace's eqn.

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0 \text{ in } S$$

$$w = w_1 + k w_2.$$

The variables we easily show that typical harmonic function are

$$w(x,y) = \begin{pmatrix} \sinh \\ \cosh \end{pmatrix} nx \begin{pmatrix} \sin \\ \cos \end{pmatrix} ny \text{ or } e^{\pm nx} \begin{pmatrix} \sin \\ \cos \end{pmatrix} ny;$$

$$w(x,y) = \begin{pmatrix} \sin \\ \cos \end{pmatrix} nx \begin{pmatrix} \sinh \\ \cosh \end{pmatrix} ny \text{ or } e^{\pm ny} \begin{pmatrix} \sin \\ \cos \end{pmatrix} nx)$$

Such a procedure usually leads to Fourier series expansions. We illustrate this

EXAMPLE:

Steady flow through a channel of uniform rectangular cross - section. 10

Suppose the cross-section is bounded by the planes $x = \pm a$, $y = \pm b$. Then the problem is to solve

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = \frac{-P}{\mu} \quad (|x|<a, |y|<b)$$

Subject to $w=0$, on $x = \pm a$ and on $y = \pm b$

A particular integral satisfying $w=0$ on $x=\pm a$ is

$$w_1(x,y) = P(a^2-x^2)/2\mu$$

This does not, however vanish on $y=\pm b$.

$$w_2 = k_n \cos\frac{(2n+1)\pi x}{2a} \operatorname{ch}\frac{(2n+1)\pi y}{2a}.$$

where n is an integer. since $\cos\{(2n+1)\pi/2=0\}$

By superposition,

$$w(x,y) = \frac{P(a^2-x^2)}{2\mu} + \sum_{n=0}^{\infty}\left\{k_n \cos\frac{(2n+1)\pi x}{2a} \operatorname{ch}\frac{(2n+1)\pi y}{2a}\right\}$$

This requires that

$$-P/2\mu(a^2-x^2) = \sum_{n=0}^{\infty}\left[k_n \operatorname{ch}\left\{\frac{(2n+1)\pi b}{2a}\right\}\cos\left(\frac{(2n+1)\pi x}{2a}\right)\right]$$

Multiplying both sides by $\cos\{(2n+1)\pi x/2a\}$ and integrating from $x=-a$ to $x=a$, we find

$$k_n = \frac{(-1)^{n+1}32a^3}{\pi^3(2n+1)^3\operatorname{ch}\{(2n+1)\pi b/2a\}}\frac{P}{2\mu}$$

yielding the soln.

$$w(x,y) = \frac{Pa^2}{2\mu}\left\{1-\frac{x^2}{a^2}-\frac{32}{\pi^3}\sum_{n=0}^{\infty}\frac{(-1)^n \operatorname{ch}\{(2n+1)\pi y/2a\}\cos\{(2n+1)\pi x/2a\}}{(2n+1)^3\operatorname{ch}\{(2n+1)\pi b/2a\}}\right\}$$

8.12 Diffusion of vorticity.

The Navier-stokes vector eqn. for an incompressible viscous fluid in the form (1)

$$(\partial q/\partial t) + \nabla(\tfrac{1}{2}q^2) - q \wedge \zeta = F - \nabla \int \frac{dP}{\rho} + \nu \nabla^2 q.$$

If the body forces are conservative, so that $\nabla \wedge F = 0$.

on taking the curl of both sides of the eqn. we find

$$\nabla \wedge (\partial q/\partial t) - \nabla \wedge (q \wedge \zeta) = \nu \nabla \wedge (\nabla^2 q) \rightarrow (1)$$

$$\partial \zeta/\partial t + (q \cdot \nabla)\zeta - (\zeta \cdot \nabla) q = \nu \nabla^2 \zeta$$

$$\therefore d\zeta/dt = (\zeta \cdot \nabla) q + \nu \nabla^2 \zeta \rightarrow (2)$$

$$d\zeta/dt = \nu \nabla^2 \zeta \rightarrow (3)$$

The case of two-dimensional motion, if we take.

$$q = u(x,y)i + v(x,y)j$$

$$\zeta = \nabla \wedge q = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)k.$$

$$(\nabla \cdot \zeta) q = (\partial v/\partial x - \partial u/\partial y)(\partial/\partial z) q = 0.$$

In the special case when the motion is two-dimensional and in circles with centres on the z-axis at any point having cylindrical polar coordinates (R, θ, z) when the flow is unsteady

$$q = q(R,t)\hat{\theta},$$

$$\zeta = \frac{1}{R}\begin{vmatrix} \hat{R} & R\hat{\theta} & k \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ 0 & Rq & 0 \end{vmatrix} = \frac{1}{R}\frac{\partial}{\partial R}(Rq)k.$$

Since $\nabla = \hat{R}(\partial/\partial R) + \hat{\theta}(\partial/R\partial\theta) + k(\partial/\partial z)$,

$$(q \cdot \nabla)\zeta = k \cdot \frac{q}{R^2}\frac{\partial^2 (Rq)}{\partial\theta \partial R} = 0$$

Thus $d\zeta/dt = \partial\zeta/\partial t$

Since $\zeta = \zeta(R,t)k$, (12)

$$\frac{\partial \zeta}{\partial t} = \frac{\partial \zeta}{\partial t}k, \quad \nabla^2 \zeta = (\nabla^2 \zeta)k$$

Thus eqn. (3) assumes the form

$$\frac{\partial \zeta}{\partial t} = \nu\left(\frac{\partial^2 \zeta}{\partial R^2} + \frac{1}{R}\frac{\partial \zeta}{\partial R}\right) \rightarrow (4)$$

$$\zeta = \frac{k}{8\pi\nu t}\exp\left(-\frac{R^2}{4\nu t}\right) \rightarrow (5)$$

where k is a constant. This shows that ζ decays rapidly with time

8.13 Energy Dissipation due to viscosity

The rate of gain of kinetic energy at time t as we follow the particle is

$$\frac{d}{dt}\left\{\frac{1}{2}(\rho\delta v)q^2\right\} = \rho\delta v q\cdot\frac{dq}{dt}$$

Hence the total rate of gain of kinetic energy of the entire fluid is

$$\int_V \rho q\cdot\left(\frac{dq}{dt}\right)dv = \rho\int_V q\cdot\left(\frac{dq}{dt}\right)dv,$$

$$\frac{dq}{dt} = F - \nabla\int\frac{dp}{\rho} - \nu\nabla\wedge(\nabla\wedge q)$$

$$\therefore \omega = \mu\int_V q\cdot\{\nabla\wedge(\nabla\wedge q)\}dv$$

$$\omega = \mu\int_V(\nabla\wedge q)^2 dv - \mu\int_V \nabla\cdot(q\wedge \operatorname{curl} q)\,dv$$

$$= \mu\int_V(\nabla\wedge q)^2 dv - \mu\int_S n\cdot(q\wedge \operatorname{curl} q)\,ds$$

$$\omega = \mu\int_V \zeta^2 dv$$

8.14 Steady Flow past a fixed sphere:

The Navier - stokes vector eqn. in the usual notation, is (18)

$$\frac{\partial q}{\partial t} + q.\nabla q = F - \frac{1}{\rho}\nabla P - \nu \nabla \wedge (\nabla \wedge q)$$

Thus we shall assume the approximation

$$F - (1/\rho)\nabla P - \nu \nabla \wedge (\nabla \wedge q) = 0 \rightarrow (1)$$

for slow steady flow, where $q \rightarrow -U$ at infinity and $q = 0$ on the surface $r = a$.

$$\text{curl curl curl } q = 0 \rightarrow (2)$$

The eqn of continuity is $\nabla . q = 0$. This is satisfied by a vector function A where

$$q = \nabla \wedge A \rightarrow (3)$$

From (2), (3)

$$\text{curl curl curl } A = 0 \rightarrow (4)$$

$$= 3U - U = 2U.$$

Let us try to find solns of (4) in the form

$$A = r^{-n} U \wedge r \rightarrow (5)$$

we have.

$$\nabla \wedge A = \Sigma . \{ i \wedge \partial/\partial x) . (r^{-n} U \wedge r) \}.$$

$$= r^{-n} \{ \Sigma U - \Sigma (U.i) i \} - n r^{-n-2} r \wedge (U \wedge r)$$

$$= (2-n) r^{-n} U + n r^{n-2} (U.r).r.$$

Hence since $A = r^{-n} U \wedge r$.

$$\text{curl curl curl curl } A = n(3-n) \nabla \wedge [\nabla \wedge \{ r^{-n-2} (U \wedge r) \}]$$

$$= n(3-n)(n+2)(1-n) r^{-n-4} (U \wedge r).$$

$n = -2, 0, 1$ or 3. By superposition we may take as a more general solution.

$$A = (U \wedge r)\left(k_1 r^2 + k_2 + \frac{k_3}{r} + \frac{k_4}{r^3}\right) \rightarrow (6)$$

$$k_1 = 0,\ k_2 = \frac{1}{2} \rightarrow (7)$$

(14)

$$q = \nabla \wedge A$$

$$= \frac{1}{2}\nabla \wedge (U \wedge r) + k_3 \nabla \wedge (r^{-1} U \wedge r) + k_4 \nabla \wedge (r^{-3} U \wedge r)$$

$$= (1 + k_3 r^{-1} - k_4 r^{-3}) U + (k_3 r^{-3} + 3k_4 r^{-5})(U \cdot r) r \rightarrow (8)$$

on $r = a$, $q = 0$, everywhere, so that

$$1 + k_3 a^{-1} - k_4 a^{-3} = 0 = k_3 a^{-3} + 3k_4 a^{-5}$$

$$q = \left\{1 - \frac{3}{4}\frac{a}{r} - \frac{1}{4}\left(\frac{a}{r}\right)^3\right\} \cdot U + \frac{3}{4}\left(\frac{a^3}{r^3} - \frac{a}{r^3}\right)(U \cdot r) r \rightarrow (9)$$

and the vector potential is

$$A = \left(\frac{r}{2} - \frac{3}{4}\frac{a}{r} + \frac{1}{4}\frac{a^3}{r^3}\right)(U \wedge r) \rightarrow (10)$$

The vorticity vector is

$$\zeta = \nabla \wedge q = \nabla \wedge (q \wedge A)$$

$$\zeta = \frac{-3}{2} a r^{-3} U \wedge r \rightarrow (11)$$

$$\zeta = \frac{3}{2} a r^{-2} U \sin\theta \rightarrow (12)$$

$$W = \mu \int_V \zeta^2 \, dv.$$

where v is the entire volume of fluid outside the sphere, Thus.

$$W = \mu \int_{\theta=0}^{\pi} \int_{r=0}^{\infty} 2\pi r^2 \sin\theta \cdot \zeta^2 \, dr.$$

$$= \frac{9}{2} \mu \pi U^2 a^2 \int_a^{\infty} r^{-2} \, dr \cdot \int_0^{\pi} \sin^3\theta \cdot d\theta$$

$$= 6 \mu \pi U^2 a.$$

If D denotes the total drag on the sphere then $W = DU$. Thus.

$D = 6\pi\mu a U \rightarrow (13)$

In the special case when a solid sphere of radius a and density σ falls vertically through liquid of density $\rho (< \sigma)$, the viscous drag produces a terminal velocity U say. When this is attained we have equating the weight of the sphere to the upthrust plus drag,

$$\frac{4}{3}\pi a^3 \sigma g = \frac{4}{3}\pi a^3 \rho g + 6\pi a\mu U,$$

or $$U = \frac{2}{9}(\sigma - \rho)a^2 g/\mu. \rightarrow (14)$$

8.15 Dimensional Analysis, Reynolds Number

The Navier-Stokes eqn. of motion of a viscous incompressible fluid in the x-direction is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right] \rightarrow (1)$$

Suppose L, V, P denote a characteristic length, velocity and pressure respectively. Then the length, velocities and pressures in (1) may be expressed in terms of these standards. Thus we write

$$x = Lx', \; y = Ly', \; z = Lz' \rightarrow (2)$$

$$u = Vu', \; v = Vv', \; w = Vw' \rightarrow (3)$$

$$p = Pp' \rightarrow (4)$$

where all primed quantities are pure numbers having no dimensions. Then since L/V is the time,

$$\frac{\partial u}{\partial t} = \frac{\partial (Vu')}{\partial (LV^{-1}t')} = \frac{V^2}{L}\frac{\partial u'}{\partial t'},$$

$$u\frac{\partial u}{\partial x} = (Vu')\frac{\partial (Vu')}{\partial (Lux')} = \frac{V^2}{L}u'\frac{\partial u'}{\partial x'} \quad \text{etc}$$

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{1}{\rho}\frac{\partial (Pp')}{\partial (Lx')} = \frac{P}{\rho L}\frac{\partial p'}{\partial x'} \quad (16)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 (Vu')}{\partial (Lx')^2} = \frac{V}{L^2}\frac{\partial^2 u'}{\partial x'^2}$$

Sub these results into (1) and simplifying gives

$$\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'} + w\frac{\partial u'}{\partial z'} = \frac{LX}{V^2} - \frac{P}{\rho V^2}\cdot\frac{\partial p'}{\partial x'}$$

$$+ \frac{v}{VL}\left(\frac{\partial^2 u'}{\partial x'^2} + \frac{\partial^2 u'}{\partial y'^2} + \frac{\partial^2 u'}{\partial z'^2}\right) \rightarrow (5)$$

In eqn. (5) the L.H.S is entirely dimensionless. Hence the R.H.S must likewise be so. It follows that the three quantities

$$\frac{LX}{V^2}, \frac{P}{\rho V^2}, \frac{v}{VL}$$

must be dimensionless quantities. In order to produce a faithful model of a given incompressible viscous flow it, is essential to keep these three numbers constant.

The first number (LX/V^2) tells us how to scale the body forces.

The second number $(P/\rho V^2)$ ensures dynamical similarity in the two flows at points where viscosity is unimportant. Such points would occur at stations remote from the boundaries

UNIT - II

(1)

3.7 Discussion of the case of steady Motion under conservative Body forces. A-19 10M

Bernoulli's equation was derived for potential flows under conservative body forces. We now study cases that arise when the flow is no longer of the potential kind but is steady. The body forces are still assumed to be conservative. We shall see that in such conditions equations somewhat akin to Bernoulli's equation arise.

Euler's equation of motion may be written in the form

$$F - \frac{1}{\rho}\nabla P = \frac{\partial q}{\partial t} + \nabla\left(\frac{1}{2}q^2\right) - q \wedge \zeta,$$

where $\zeta = \text{curl}\, q$, the vorticity vector. If the forces are conservative, then $F = -\nabla\Omega$ So that when the flow is steady.

$$\Rightarrow -\nabla\Omega - \frac{1}{\rho}\nabla P = \frac{\partial q}{\partial t} + \nabla\left(\frac{1}{2}q^2\right) - q \wedge \zeta,$$

$$\Rightarrow \frac{\partial q}{\partial t} + \nabla\left(\frac{1}{2}q^2\right) - q\wedge\zeta + \nabla\Omega + \frac{1}{\rho}\nabla P = 0$$

$$\nabla\left(\frac{1}{2}q^2 + \Omega + \int\frac{dP}{\rho}\right) = q \wedge \zeta,$$

Scalar multiplying this equation through by dr, a time independent variation in the position vector r of the fluid particle, gives

$$dr\,\nabla\left(\frac{1}{2}q^2 + \Omega + \int\frac{dP}{\rho}\right) = dr \cdot q \wedge \zeta, \quad \rightarrow (1)$$

by the Bernoullis $dr \cdot \nabla\Omega = d\Omega$

$dr \cdot \nabla = d$. apply (1) eqn

$$d\,\nabla\left(\frac{1}{2}q^2 + \Omega + \int\frac{dP}{\rho}\right) = dr \cdot (q \wedge \zeta),$$

Case I : $q \wedge \zeta = 0$

This is realized when either

i) q and ζ are parallel, i.e, when the streamline and vortex lines coincide. For such motions, q is termed a Beltrami vector.

(or)

ii) When $\zeta = 0$, the condition for potential flow.

In both cases we have

$$d\left(\frac{1}{2}q^2 + \Omega + \int \frac{dp}{\rho}\right) = 0$$

at all times throughout the entire flow field. Then

$$\frac{1}{2}q^2 + \Omega + \int \frac{dp}{\rho} = \text{constant} \rightarrow (2)$$

throughout the entire field of flow. The constant is the same throughout the entire field since the differential dr in (1) is any arbitrary small variation of position vector r in the field.

Case II : $q \wedge \zeta \neq 0$.

Now $q \wedge \zeta$ is perpendicular to the vectors q, ζ. Hence if $dr \neq 0$, then $dr \cdot (q \wedge \zeta) = 0$ wherever dr lies in the plane of q, ζ. Thus if we take the variation dr in the surface containing both the streamlines and vortex lines, then (1) shows that

$$d\left(\frac{1}{2}q^2 + \Omega + \int \frac{dp}{\rho}\right) = 0$$

over such a surface, or

$$\frac{1}{2}q^2 + \Omega + \int \frac{dp}{\rho} = \text{Constant} \rightarrow (3)$$

Over a surface containing the streamlines and vortex lines. We note that the constant in (3) is the

$$\therefore \frac{\partial \bar{\phi}}{\partial r} = 0$$

Hence $\bar{\phi}$ is independent of r, so that the mean value of ϕ is the same over all spheres having the same centre and therefore is equal to its value at the centre. (4)

Theorem II :

If Σ is the solid boundary of a large spherical surface of radius R, containing fluid in motion and also enclosing one or more closed surfaces, then the mean value of ϕ on Σ is of the form

$$\bar{\phi} = \left(\frac{M}{R}\right) + C,$$

where M, C are constants, provided that the fluid extends to infinity and is at rest there.

4/20/1/20 Proof :

Suppose that the volume of fluid crossing each of the internal surfaces contained within Σ per unit time is a finite quantity $4\pi M$. The this is the total volume flux per unit time across Σ. Since the fluid velocity at any point of Σ is $-\frac{\partial \phi}{\partial R}$ radially outwards, the eqn. of continuity is

$$\int_{\Sigma} \left(-\frac{\partial \phi}{\partial R}\right) ds = 4\pi M$$

If ds subtends a solid angle $d\omega$ at the centre of Σ then $ds = R^2 d\omega$ and so

$$\frac{1}{4\pi} \int_{\Sigma} \frac{\partial \phi}{\partial R} d\omega = -\frac{M}{R^2}$$

$$\frac{1}{4\pi} \frac{\partial}{\partial R} \int_{\Sigma} \phi \, d\omega = -\frac{M}{R^2}$$

Integrating w.r.t R gives

$$\frac{1}{4\pi}\int_{\Sigma} \phi \, d\omega = \frac{M}{R} + c$$

(or) $$\frac{1}{4\pi R^2}\int_{\Sigma} \phi \, dS = \frac{M}{R} + c = \bar{\phi}$$

c is here independent of R. To show that it is an absolute constant, we must prove that it is independent of the coordinates of the centre of Σ. To this end, let the sphere be displaced a distance δx in any direction, keeping R constant. Then from the last equation, the attendant change in $\bar{\phi}$ is

$$\frac{\partial \bar{\phi}}{\partial x}\delta x = \frac{1}{4\pi R^2}\int_{\Sigma} \frac{\partial \phi}{\partial x}\delta x \, dS = \frac{\partial c}{\partial x}\delta x$$

But $\frac{\partial \phi}{\partial x} = 0$ on Σ when $R \to \infty$, since the liquid is at rest at infinity. Thus for large R, $\frac{\partial c}{\partial x} = 0$

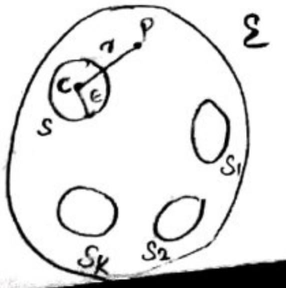
Hence $$\bar{\phi} = \frac{M}{R} + c$$

where, M, c are constant

We note that in the special case when the closed surfaces within Σ are all rigid, no flow can take place across them so that $M = 0$ and $\bar{\phi} = c$.

Theorem III :

With the notation of Theorem II if ϕ_P denotes the potential at any point P of the fluid, then $\phi_P \to c$ as $P \to \infty$



Proof :

Let $S_1, S_2 \ldots S_k$ be the surfaces contained within Σ.

Also let S be the sphere whose centre is C and radius ϵ. Then if r is the distance from C to any point P within the fluid,

$$\int \left\{ \phi \nabla^2 \left(\frac{1}{r}\right) - \frac{1}{r} \nabla^2 \phi \right\} dv = \int_{\text{Total surface}} \left\{ \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial \phi}{\partial n} \right\} \quad (6)$$

Using the symmetrical form of Green's theorem the unit normal at any point on the boundary being drawn outwards from the fluid. But $\nabla^2\phi = 0$, $\nabla^2\left(\frac{1}{r}\right) = 0$, So the last equation gives

$$\int_S \left\{ \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial \phi}{\partial n} \right\} ds + \int_\Sigma \left\{ \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) \frac{\partial \phi}{\partial n} \right\} ds$$

$$+ \sum_{m=1}^{K} \int_{S_m} \left\{ \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial \phi}{\partial n} \right\} ds = 0$$

on S,

$$\frac{\partial}{\partial n}\left(\frac{1}{r}\right) = \left\{ -\frac{\partial}{\partial r}\left(\frac{1}{r}\right) \right\}_{r=\epsilon} = +\frac{1}{\epsilon^2}, \quad \frac{\partial \phi}{\partial n} = -\frac{\partial \phi}{\partial r}$$

$$\int_S \left\{ \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial \phi}{\partial n} \right\} ds = \int_S \left\{ \phi\left(\frac{1}{\epsilon^2}\right) + \frac{1}{\epsilon}\frac{\partial \phi}{\partial r} \right\} ds$$

$$= \left\{ \phi\left(\frac{1}{\epsilon^2}\right) + \frac{1}{\epsilon}\frac{\partial \phi}{\partial r} \right\} 4\pi\epsilon^2, \text{ to 1st order}$$

$$\rightarrow 4\pi\phi_P \text{ as } \epsilon \to 0$$

on Σ

$$\frac{\partial}{\partial n}\left(\frac{1}{r}\right) = \left\{ \frac{\partial}{\partial r}\left(\frac{1}{r}\right) \right\}_{r=R} = -\frac{1}{R^2}, \quad \frac{\partial \phi}{\partial n} = \frac{\partial \phi}{\partial R}$$

$$\therefore \int_\Sigma \left\{ \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial \phi}{\partial n} \right\} ds = -\frac{1}{R^2}\int_\Sigma \phi \, ds - \frac{1}{R}\int \frac{\partial \phi}{\partial R} ds$$

$$= -\frac{1}{R^2}\int_\Sigma \left(\frac{M}{R} + c\right) ds + \frac{1}{R}\int_\Sigma \frac{M}{R^2} ds$$

$$= -\frac{M}{R^3}(4\pi R^2) - \frac{1}{R^2}(4\pi c R^2) + \frac{M}{R^3}(4\pi R^2)$$

$\rightarrow -4\pi c$ as $R \rightarrow \infty$

Thus,

$$4\pi(\phi_P - c) + \sum_{m=1}^{k} \int \left\{ \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial \phi}{\partial n} \right\} ds = 0$$

As $r \rightarrow \infty$, $\left(\frac{1}{r}\right) \rightarrow 0$ and $\left(\frac{\partial}{\partial n}\right)\left(\frac{1}{r}\right) \rightarrow 0$, so the last result shows that

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$$\phi_P \rightarrow c.$$

Theorem IV :

With the notation of Theorem III, if the fluid is at rest at infinity and if each surface S_m is rigid, then the kinetic energy of the moving fluid is

$$T = \frac{1}{2}\rho \int_V q^2 \, dv = \frac{1}{2}\rho \sum_{m=1}^{k} \int_{S_m} \phi \frac{\partial \phi}{\partial n} ds,$$

ds being drawn the normal at each surface elements outwards from the fluid.

Proof :

The K.E of a fluid particle of mass $\rho \delta v$ is

$$\frac{1}{2}\rho \delta v q^2 = \frac{1}{2}\rho (\nabla \phi)^2 dv.$$

Hence that of the entire fluid is

$$T = \frac{1}{2}\rho \int_V (\nabla \phi)^2 \, dv,$$

V denoting the total volume of fluid. Now

$$(\nabla \phi)^2 = (\nabla \phi)\cdot(\nabla \phi) + \phi \nabla^2 \phi$$

$$= \nabla \cdot (\phi \nabla \phi)$$

$$\therefore T = \frac{1}{2}\rho \int_V \nabla \cdot (\phi \nabla \phi) \, dv$$

$$= \frac{1}{2}\rho \int_{\substack{\text{Total} \\ \text{surface}}} n \cdot (\phi \nabla \phi) \, ds$$

$$= \frac{1}{2}\rho \int_{\Sigma} \phi \frac{\partial \phi}{\partial n} ds + \frac{1}{2}\rho \sum_{m=1}^{K} \int_{S_m} \phi \frac{\partial \phi}{\partial n} ds \rightarrow (1)$$

The eqn. of continuity for the entire region V is

$$\int_{\Sigma} \frac{\partial \phi}{\partial n} ds + \sum_{m=1}^{K} \int_{S_m} \frac{\partial \phi}{\partial n} ds = 0 \rightarrow (2)$$

(8)

Thus from (1), (2)

$$T = \frac{1}{2}\rho \int_{\Sigma} (\phi - c) \frac{\partial \phi}{\partial n} ds + \frac{1}{2}\rho \sum_{m=1}^{k} \int_{S_m} (\phi - c) \frac{\partial \phi}{\partial n} ds \rightarrow (3)$$

where c is any constant.

Taking c to be the value of ϕ when $P \to \infty$, we have,

on making $R \to \infty$,

$$T = \frac{1}{2}\rho \sum_{m=1}^{k} \int_{S_m} (\phi - c) \frac{\partial \phi}{\partial n} ds \rightarrow (4)$$

on each rigid surface S_m there is no flow, so that

$$\int_{S_m} \frac{\partial \phi}{\partial n} ds = 0$$

Thus.

$$T = \frac{1}{2}\rho \sum_{K=1}^{m} \int_{S_m} \phi \frac{\partial \phi}{\partial n} ds$$

Theorem V :

If the fluid of Theorem IV is at rest at infinitely and if either ϕ or $\frac{\partial \phi}{\partial n}$ is prescribed on each surface S_m, then ϕ is determined uniquely throughout V to within an arbitrary constant.

Proof :

From Theorem IV

$$\int_{V} (\nabla \phi)^2 dv = \sum_{K=1}^{m} \int_{S_m} \phi \frac{\partial \phi}{\partial n} ds \rightarrow (1)$$

whilst within V,

$$\nabla^2 \phi = 0 \rightarrow (2)$$

Suppose $\phi = \phi_1$, $\phi = \phi_2$ are two solutions of ②, subject

. ①. Write $\psi = \phi_1 - \phi_2$. Then

(9)

$$\nabla^2 \psi = \nabla^2 \phi_1 - \nabla^2 \phi_2 = 0 \text{ in } V.$$

Thus ψ is a harmonic function satisfying similar

nditions to ϕ_1, ϕ_2

Hence $$\int_V (\nabla\psi)^2 dv = \sum_{k=1}^{m} \int_{S_m} \psi \frac{\partial \psi}{\partial n} ds \rightarrow ③$$

on each S_m, either $\phi_1 = \phi_2$ (or) $\frac{\partial \phi_1}{\partial n} = \frac{\partial \phi_2}{\partial n}$.

ie) either $\psi = 0$ (or) $\frac{\partial \psi}{\partial n} = 0$

Thus in all cases, $\psi \frac{\partial \psi}{\partial n} = 0$ on each S_m so that the

H.S of ③ is zero. Thus

$$\int_V (\nabla\psi)^2 dv = 0 \rightarrow ④$$

Since $(\nabla\psi)^2 \geq 0$ the condition ④ can hold only if

$$\nabla \psi = 0 \text{ in } V \rightarrow ⑤$$

⑤ gives $\psi = \text{constant}$

or $\phi_1 - \phi_2 = \text{constant} \rightarrow ⑥$

The function ψ is continuous, so that if ϕ is

escribed on any one S_m, the constant is zero. If the

ocities are prescribed on each S_m, we cannot evaluate

constant.

Theorem VI :

If the fluid of Theorem V is in uniform motion

t infinity and if $\frac{\partial \phi}{\partial n}$ is prescribed on each surface

m, then ϕ is uniquely determined throughout V.

Proof :

Let V be the velocity of the fluid at infinity. Superpose a velocity $-V$ on the entire system so as to reduce the velocity of the fluid at infinity to rest. (10)

Then the conditions of Theorem V prevail in which the velocities at all points of each S_m are known in a fluid which is at rest at infinity. This leads to a unique value for ϕ at each point of V.

Theorems $\underline{V}$, $\underline{VI}$ are uniqueness theorems which are useful in finding solutions of $\nabla^2\phi = 0$ subject to prescribed boundary conditions.

3.9 Some Flows Involving Axial Symmetry.

Let $\phi(r, \theta, \psi)$ be the velocity potential at any point having spherical polar coordinates (r, θ, ψ) in a field of steady irrotational incompressible flow.

Then in this coordinatesystem, Laplace's equation $\nabla^2\phi = 0$ becomes

$$\sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) + \frac{1}{\sin\theta}\left(\frac{\partial^2 \phi}{\partial \psi^2}\right) = 0$$

when there is symmetry about the line $\theta = 0$, $\phi = \phi(r, \theta)$ and the equation becomes

$$\sin\theta \frac{\partial}{\partial r}\left(r^2 \frac{\partial \phi}{\partial r}\right) + \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \phi}{\partial \theta}\right) = 0 \rightarrow (1)$$

It is easily confirmed by direct substitution that special solutions of (1) are

$$\phi = r\cos\theta, \quad \phi = \left(\frac{1}{r^2}\right)\cos\theta$$

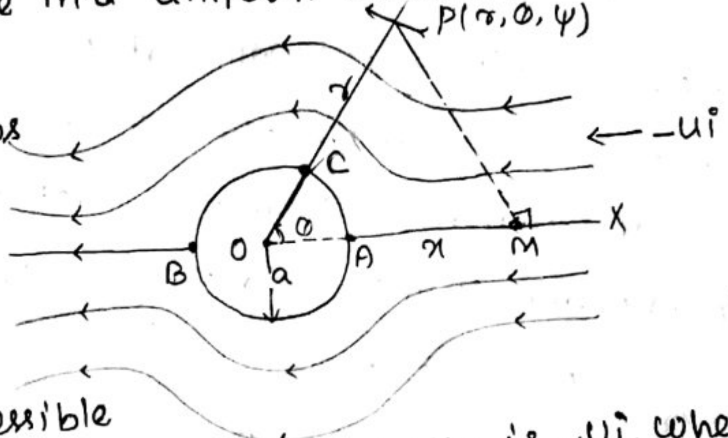
Thus a rather more general solution of (1) is

$$\phi(r,\theta) = (Ar + Br^{-2})\cos\theta \rightarrow (2)$$

we now investigate a number of flows with axial symmetry which have solutions of the form (2).

Example 1

1) Stationary sphere in a uniform stream.



The diagram shows a solid stationary sphere of radius a placed in a uniform stream of incompressible fluid for which the undisturbed velocity is $-U\vec{i}$, where U is a constant and $\vec{i}$ a constant unit vector in the direction $\overline{OX}$, the axis of symmetry. PM is ⊥r from P on OX and OM = x. The spherical polar coordinates of P are (r, θ, ψ) referred to OX as the line $\theta = 0$. Thus $x = r\cos\theta$.

In the absence of the sphere, the flow would be uniform and parallel to $\overline{XO}$, the velocity everywhere being $-U\vec{i}$ which is clearly $-\nabla(Ux)$. Hence the velocity potential due to such uniform flow would be

$$Ux = Ur\cos\theta.$$

The undisturbed potential, $Ur\cos\theta$ of the parallel flow has to be modified by a 'perturbation potential' due to the presence of the sphere. This perturbation must have the following properties:

i) It must satisfy Laplace's equation for the case of axial symmetry.

ii) It must tend to zero at large distances from the sphere, so as to recover the conditions of uniform parallel flow at infinity. A harmonic function satisfying these requirements is $\cos\theta/r^2$. Let us, then try the fun.

$$\phi(r,\theta) = Ur\cos\theta + Ar^{-2}\cos\theta \quad (r \geq a) \rightarrow (1)$$

(12)

for the velocity potential at $P(r,\theta,\psi)$ when the sphere is inserted in the stream. The constant A in (1) is determinable from the fact that there is no flow normal to the surface $r=a$. This means that

$\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = 0$, so that

$$(1) \Rightarrow U\cos\theta - \frac{2A}{r^3}\cos\theta = \frac{\partial\phi}{\partial r}$$

$$U\cos\theta - \frac{2A}{a^3}\cos\theta = 0 \qquad \therefore r=a$$

$$\Rightarrow U\cos\theta = \frac{2A}{a^3}\cos\theta$$

$$U = \frac{2A}{a^3}$$

$$a^3U = 2A$$

$$A = \frac{a^3U}{2}$$

Thus (1) becomes.

$$(1) \Rightarrow \phi(r,\theta) = Ur\cos\theta + \frac{1}{2}a^3Ur^{-2}\cos\theta$$

$$= U\cos\theta\left(r + \frac{1}{2}a^3r^{-2}\right) \quad (r\geq a) \rightarrow (2)$$

By theorem VI of Sec. 3.8 since $\frac{\partial\phi}{\partial n}$ on the sphere is correctly determined and since (2) gives the crt uniform velocity $-Ui$ at infinity, we infer that the form (2) is unique.

The velocity components at $P(r,\theta,\psi)$ for $r\geq a$, are given by.

$$q_r = -\frac{\partial\phi}{\partial r} = -\left[U\cos\theta - 2Ar^{-3}\cos\theta\right]$$

$$= -\left[U\cos\theta - \cancel{2}\left(\tfrac{1}{\cancel{2}}a^3U\right)r^{-3}\cos\theta\right]$$

$$= -U\cos\theta\left(1 - \frac{a^3}{r^3}\right)$$

$$q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = U\sin\theta\left(1 + \frac{1}{2}\frac{a^3}{r^3}\right)$$

$$q_\psi = -\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi} = 0$$

The stagnation points of the flow are those points, if any, where the velocity $q=0$. These are found from the equations. (13)

$$U\cos\theta(1-a^3r^{-3}) = 0 = U\sin\theta\left(1+\frac{1}{2}a^3r^{-3}\right)$$

which, since $(r \geq a)$ are consistently satisfied only by $r=a$, $\sin\theta=0$ i.e. by $r=a$, $\theta=0$ or π. These correspond to the points A, B in diagram. We can find the stagnation pressure P_0 at A by applying Bernoulli's equation along the streamline XA. Taking P_∞ to be the pressure at infinity where the velocity is $-Ui$, we have.

$$\frac{P_\infty}{\rho}+\frac{1}{2}U^2 = \frac{P_0}{\rho}+0 \Rightarrow \frac{P_\infty+\frac{1}{2}\rho U^2}{\cancel{\rho}} = \frac{P_0}{\cancel{\rho}}+0 \Rightarrow P_0 = P_\infty+\frac{1}{2}\rho U^2 \rightarrow (3)$$

The pressure at any point $C(a,\theta,\psi)$ on the surface of the sphere may be found. At such a point we have.

$$q_r=0,\ q_\theta=\frac{3}{2}U\sin\theta,\ q_\psi=0$$

Applying Bernoulli's eqn. along the streamline XAC if P denote the pressure at C then

$$\frac{P}{\rho}+\frac{1}{2}\left(\frac{3}{2}U\sin\theta\right)^2 = \frac{P_\infty}{\rho}+\frac{1}{2}U^2$$

$$\frac{P+\frac{1}{2}\rho\left(\frac{3}{2}U\sin\theta\right)^2}{\cancel{\rho}} = \frac{P_\infty+\frac{1}{2}U^2\rho}{\cancel{\rho}}$$

$$P = P_\infty+\frac{1}{2}U^2\rho-\frac{1}{2}\rho\left(\frac{9}{4}U^2\sin^2\theta\right)$$

$$= P_\infty+\frac{1}{2}\rho U^2\left(1-\frac{9}{4}\sin^2\theta\right) \rightarrow (4)$$

Now max. $\sin^2\theta=1$, so that (4) shows that

$$(4) \Rightarrow P = P_\infty+\frac{1}{2}\rho U^2\left(1-\frac{9}{4}\right) \Rightarrow P_\infty+\frac{1}{2}\rho U^2\left(\frac{4-9}{4}\right)$$

$$\Rightarrow P_\infty-\frac{5}{8}\rho U^2$$

$$\min.P = P_\infty-\frac{5}{8}\rho U^2$$

Thus the minimum pressure occurs on the equatorial line $r = a$, $\theta = \frac{1}{2}\pi$. when min. $p = 0$ we have

$$\Rightarrow 8P_\infty = 5\rho U^2 \Rightarrow \frac{8P_\infty}{5\rho} = U^2$$

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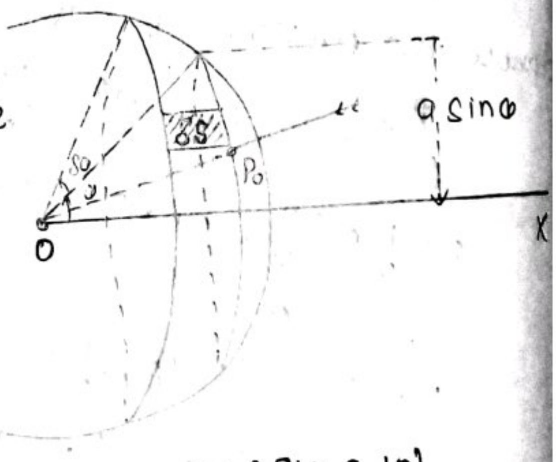
$$U = \sqrt{8P_\infty / 5\rho} \rightarrow (5)$$

This indicates that when the speed U of the undisturbed stream is increased, a point will be reached at which the pressure everywhere along the equatorial line $r = a$, $\theta = \frac{1}{2}\pi$ falls to zero. At this stage the fluid will tend to break away from the surface of the sphere along the equatorial line: cavitation is said to ensue.

(N.B. If we take account of the finite aqueous vapour pressure of the fluid at the given temperature of operation, cavitation will actually take place along the equatorial line when this aqueous vapour pressure is reached, before the pressure falls to zero)



Let us find the thrust on the hemisphere $r = a$, $0 \le \theta \le \frac{1}{2}\pi$. To this end let δS be a small element at P_0 of the zone bounded by the circles on $r = a$ at angular distances θ, $(\theta + \delta\theta)$ from OX (fig. 3.10).

Then the thrust on δS in $p\delta S$ along $\overline{P_0 O}$, where p is given by (4). From symmetry, the thrust on the zone bounded by the circles is along $\overline{XO}$. The component of thrust on δS along $\overline{XO}$ is $p\cos\theta\,\delta S$.

Hence the total on the zone is along $\overline{XO}$ and of amount

$$\int_{zone} p\cos\theta\, dS = p\cos\theta \int_{zone} dS = p\cos\theta\, 2\pi a^2 \sin\theta\, \delta\theta)$$

Since $p\cos\theta$ is to the first order, constant over the zone. Hence the total thrust on this hemisphere (15) $r=a$, $0 \le \theta \le \frac{1}{2}\pi$ is

$$2\pi a^2 \int_0^{\pi/2} \sin\theta\cos\theta\left(p_\infty + \frac{1}{2}\rho U^2 - \frac{9}{8}\rho U^2 \sin^2\theta\right) d\theta$$

$$= \pi a^2 \left(p_\infty - \frac{1}{16}\rho U^2\right).$$

(N.B The thrust on the entire sphere, obtained by integrating the same fun. from $\theta=0$ to π is easily found to be zero. This result can be generalized. The total thrust on a rigid body of any shape in a uniform stream is zero. This is called d'Alembert paradox.

The equation of the streamlines in spherical polars are given by

$$\frac{dr}{q_r} = r\frac{d\theta}{q_\theta} = r\sin\theta \frac{d\varphi}{q_\varphi}$$

or

$$\frac{dr}{-\cos\theta(1-a^3r^{-3})} = \frac{r\,d\theta}{\sin\theta\left(1+\frac{1}{2}a^3r^{-3}\right)} = \frac{r\sin\theta\, d\varphi}{0}$$

Thus

$$\begin{cases} d\varphi = 0 \\ 2\cot\theta\, d\theta = -\dfrac{2r^3+a^3}{r(r^3-a^3)}\, dr \end{cases}$$

The first of these integrates to give $\varphi = \text{const.}$ Showing that the streamlines lie in planes passing through the axis of symmetry OX. For the second eqn. we note that

$$2\cot\theta\, d\theta = \frac{-r^2(2r+a^3r^{-2})}{r^2(r^2-a^3r^{-1})}\, dr \Rightarrow \frac{2r+a^3r^{-2}}{r^2-a^3r^{-1}}\, dr,$$

So that $\log \sin^2\theta = \text{const}, -2\log(r^2 - a^3 r^{-1})$.

hus the stream lines are $\begin{cases} \psi = \text{const} \\ \sin^2\theta = cr/(r^3 - a^3) \end{cases}$

we note that on $r = a$, we must take $c = 0$ for

$\sin\theta$ to be finite

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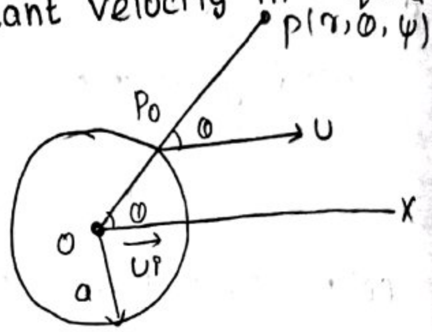
Example 2

2) Sphere moving with constant velocity in liquid which is otherwise at rest.



The diagram shows a solid Sphere, centre O radius a, moving with uniform velocity Ui in incompressible fluid of infinite extent which is at rest at infinity. Ox is the axis of symmetry and the direction of the unit vector i. As we require ϕ to be finite at infinity we take the velocity potential at $p(r, \theta, \psi)$ $(r \geq a)$ in the form.

$$\phi(r, \theta) = A r^{-2} \cos\theta \rightarrow (1)$$

which satisfies the axially symmetric form of Laplace's equation in spherical polar coordinates. From (1),

$$q_r = -\frac{\partial\phi}{\partial r} = \frac{2A\cos\theta}{r^3}$$

$$q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{A\sin\theta}{r^3}$$

$$q_\psi = -\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi} = 0$$

Hence at $P_0(a, \theta, \psi)$ on the surface,

$$q_r = 2Aa^{-3}\cos\theta$$

$$q_\theta = Aa^{-3}\sin\theta$$

$$q_\psi = 0$$

Now the velocity of P_0 is Ui. Hence $q_r = U\cos\theta$

$$\therefore U\cos\theta = 2Aa^{-3}\cos\theta$$

or
$$A = \frac{1}{2}Ua^3$$

(17)

Hence for $r >> a$

$$\phi(r,\theta) = \frac{1}{2}Ua^3 r^{-2}\cos\theta \rightarrow ②$$

To find the K.E of the fluid, we may use the result of Sec. 3.8, Them. IV. Then denoting by S the surface of the sphere, the K.E is given by.

$$T = \frac{1}{2}\rho \int_S \phi \frac{\partial\phi}{\partial n} ds = \frac{1}{2}\rho \int_S \left\{\phi\left(-\frac{\partial\phi}{\partial r}\right)\right\}_{r=a} ds$$

From ②,
$$\frac{\partial\phi}{\partial r} = \frac{1}{\cancel{2}}Ua^3 - \frac{\cancel{2}}{r^3}\cos\theta$$

$$= -\frac{Ua^3}{r^3}\cos\theta, \text{ so that}$$

$$\left\{\phi\left(-\frac{\partial\phi}{\partial r}\right)\right\}_{r=a} = \left(\frac{1}{2}Ua^3r^{-2}\cos\theta\right)\left(\frac{Ua^3}{r^3}\cos\theta\right)$$

$$= \frac{1}{2}U^2a^6r^{-5}\cos^2\theta \Rightarrow r = a$$

$$= \frac{1}{2}U^2a^6a^{-5}\cos^2\theta \Rightarrow \frac{1}{2}U^2a\cos^2\theta$$

$$T = \frac{1}{4}\rho a U^2 \int_S \cos^2\theta\, ds$$

$$= \frac{1}{4}\rho a U^2 \int_0^\pi \cos^2\theta \cdot 2\pi a^2 \sin\theta\, d\theta \Rightarrow \frac{1}{6}\rho a U^2 a^3 \left[-\cos^3\theta\right]_0^\pi$$

$$= \frac{1}{3}\rho\pi a^3 U^2 = \frac{1}{4}M'U^2,$$

where M' is the mass of fluid displaced. The total kinetic energy of the sphere and fluid is thus $\frac{1}{2}(M + \frac{1}{2}M')U^2$. The quantity $M + \frac{1}{2}M'$ is called the virtual mass of the sphere.

Example 3 : (18)

Accelerating Sphere moving in fluid at rest at infinity. A Sphere of centre O and radius a moves through an infinite liquid of constant density ρ at rest at infinity, O describing a straight line with velocity $V(t)$. If there are no body forces, show that the pressure p at points on the surface of the sphere in a plane perpendicular to the straight line at a distance x from O measured positively in the direction of V is Given

$$p = p_0 - \frac{5}{8}\rho V^2 + \frac{9}{8}\rho V^2\frac{x^2}{a^2} + \frac{1}{2}\rho x\frac{dV}{dt},$$

where p_0 is the pressure at infinity.

Deduce that the thrust on the sphere is $\frac{1}{2}M'(dV/dt)$, where M' is the mass of the liquid having the volume of the sphere.

Proof :

At $P(r, \theta, \psi)$ in the liquid, assume

$$\phi(r, \theta, t) = A(t)\, r^{-2}\cos\theta$$

which satisfies the spherical polar form of Laplace's eqn. and ensures conditions of zero velocity at infinity using the boundary condition.

$$\left(\frac{-\partial\phi}{\partial r}\right)_{r=a} = V(t)\cos\theta.$$

we find $\quad A(t) = \frac{1}{2}V(t)a^3$

Hence $\quad \phi(r, \theta, t) = \frac{1}{2}V(t)\,a^3 r^{-2}\cos\theta$

Bernoulli's equation for time-varying flow with no body forces is $\quad \frac{p}{\rho} + \frac{1}{2}q^2 - \frac{\partial\phi}{\partial t} = f(t).$

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where $\frac{\partial\phi}{\partial t}$ is to be evaluated subject to the coordinates of P being fixed At (r, θ, ψ) for $r \geq a$.

$$q_r = -\frac{\partial\phi}{\partial r} = va^3 r^{-3}\cos\theta.$$

(19)

$$q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{va^3}{2r^3}\sin\theta.$$

$$q_\psi = -\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi} = 0$$

Hence,

$$q^2 = \frac{1}{4}va^6 r^{-6}(4\cos^2\theta + \sin^2\theta) \quad v = v(t)$$

Now

$$\left(\frac{\partial\phi}{\partial t}\right)_{p\ \text{fixed}} = \frac{1}{2}a^3\frac{\partial}{\partial t}\left(\frac{v}{r^2}\cos\theta\right)_{p\ \text{fixed}}$$

$$= \frac{1}{2}a^3\left\{\frac{\cos\theta}{r^2}\frac{dv}{dt} + v\frac{\partial}{\partial t}\left(\frac{\cos\theta}{r^2}\right)\right\}_{p\ \text{fixed}}$$

since $\cos\theta/r^2 = (r.i)/r^3$,

$$\frac{\partial}{\partial t}\left(\frac{\cos\theta}{r^2}\right) = i\cdot\frac{\partial r}{\partial t}r^{-3} + (r\cdot i)(-3r^{-4})\frac{\partial r}{\partial t}$$

with p considered fixed since $\overline{PO} \equiv -r$,

$$\left(\frac{\partial}{\partial t}\right)(-r) = \text{vel. of } O = Vi,$$

and so

$$\frac{\partial r}{\partial t} = -Vi$$

Now from $r^2 = r^2$,

or

$$\left(\frac{\partial r}{\partial t}\right)_{p\ \text{fixed}} = \frac{1}{r}(-Vi\cdot r).$$

Thus

$$\frac{\partial}{\partial t}\left(\frac{\cos\theta}{r^2}\right) = \frac{V}{r^3}(-1+3\cos^2\theta)$$

and so

$$\left(\frac{\partial\phi}{\partial t}\right)_{p\ \text{fixed}} = \frac{a^3}{2r^3}\left(r\cos\theta\frac{dv}{dt} - V^2 + 3V^2\cos^2\theta\right) \rightarrow (3)$$

Thus (2) becomes.

$$\frac{p}{\rho} + \frac{V^2a^6}{4r^6}(4\cos^2\theta + \sin\theta) - \frac{a^3}{2r^3}\left(r\cos\theta\frac{dv}{dt} - V^2 + 3V^2\cos^2\theta\right)$$

$= f(t) = P_0/\rho$, using conditions at infinity.

Putting $r = a$, $x = a\cos\theta$ in this last result gives, on simplification.

(20)

$$P = P_0 - \frac{5}{8}\rho v^2 + \frac{9}{8}\rho v^2 \frac{x^2}{a^2} + \frac{1}{2}\rho x \frac{dv}{dt} \rightarrow (4)$$

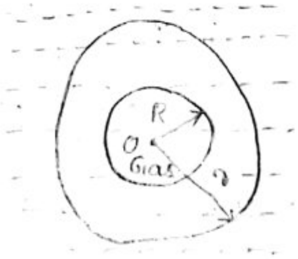
The thrust on a circular band on the sphere whose edges are at angular distances θ, $(\theta + \delta\theta)$ from ox is along $\bar{x}o$ and of magnitude $p\cos\theta$ | $(2\pi a \sin\theta, a\delta\theta) = -2\pi p x \delta x$. Thus the total thrust on the sphere is along $\bar{x}o$ and

$$= -2\pi \int_a^{-a} \left(P_0 - \frac{5}{8}\rho v^2 + \frac{9}{8}\rho v^2 \frac{x^2}{a^2} + \frac{1}{2}\rho x \frac{dv}{dt} \right) x\, dx$$

$$= \frac{2}{8}\pi\rho a^3 \frac{dv}{dt} = \frac{1}{2} M' \frac{dv}{dt}.$$

Example 4:

Underwater explosion giving spherical gas bubble.



An explosion takes place under water initially at rest. As a result, a spherical gas bubble of centre O is formed. During expansion the gas obeys the adiabatic law $pv^{\gamma} = $ const, where p is the pressure, v the volume of the gas and γ is a constant.

Since the water is incompressible, the eqn. of continuity applied to the two concentric surfaces is

$$4\pi R^2 \dot{R} = 4\pi r^2 \dot{r}$$

$$\dot{r} = R^2 \dot{R} / r^2 \rightarrow (1)$$

$\dot{r} = -\frac{\partial \phi}{\partial r}$, where $\phi(r, t)$ is the velocity potential

Thus we may take $\phi(r,t) = \frac{R^2\dot{R}}{r}$ → (2)

Bernoulli's eqn. at (r, θ, ψ) for unsteady incompressible flow under no body forces is

(21)

$$\frac{p}{\rho} + \frac{1}{2}\dot{r}^2 - \left(\frac{\partial\phi}{\partial t}\right)_r = f(t) \rightarrow (3)$$

assuming zero pressure at infinity (3) gives

$$\frac{p}{\rho} + \frac{R^4\dot{R}^2}{2r^4} - \frac{1}{r}\frac{\partial}{\partial t}(R^2\dot{R}) = 0.$$

$r = R$, $$\frac{p}{\rho} + \frac{1}{2}\dot{R}^2 - \frac{1}{R}(R^2\ddot{R} + 2R\dot{R}^2) = 0$$

or $$p = \rho\left(\frac{3}{2}\dot{R}^2 + R\ddot{R}\right) \rightarrow (4)$$

Let P be the pressure in the bubble at $t=0$ when $R = R_0$.

$$\frac{p}{P} = \left(\frac{R_0}{R}\right)^{3\gamma}$$

(4), (5)

$$\frac{3}{2}\dot{R}^2 + R\ddot{R} = \left(\frac{R_0}{R}\right)^{3\gamma}\frac{P}{\rho}. \rightarrow (6)$$

Now $\ddot{R} = \dot{R}(d\dot{R}/dR) = \left(\frac{d}{dR}\right)\left(\frac{1}{2}\dot{q}^2\right)$ so that (6) may be written.

$$\frac{d}{dR}(\dot{R}^2) + \frac{3}{R}\dot{R}^2 = \frac{2R_0^{3\gamma}P}{\gamma\rho R^3}$$

$$\frac{d}{dR}(R^3\dot{R}^2) = \frac{2\gamma R_0^3 P}{\rho R^{3\gamma-2}}$$

whence $$R^3\dot{R}^2 = \frac{2R_0^{3\gamma}}{3(\gamma-1)}\frac{P}{\rho}\left\{R_0^{-3(\gamma-1)} - R^{-3(\gamma-1)}\right\} \rightarrow (7)$$

$\gamma = 4/3$, $$\frac{dR}{dt} = \left[\frac{2P}{\rho}\left\{\left(\frac{R_0}{R}\right)^3 - \left(\frac{R_0}{R}\right)^4\right\}\right]^{1/2}$$

$$dt = \frac{R^2\, dR}{\{(2PR_0^3/\rho)(R-R_0)\}^{1/2}}$$

and sub. $R = R_0 + x$, we find on integration

$$t = (2\rho x / P R_0^3)^{1/2} \left(\frac{1}{5} x^2 + \frac{2}{3} R_0 + R_0^2 \right)$$

3.10 Some Special Two-dimensional Flows (22)

Suppose a fluid moves in such a way that its motions in all planes parallel to a given plane in the fluid have the same pattern at any considered instant of time. Such motion is said to be two-dimensional, since knowledge of the motion in any one such plane determines that throughout the body of fluid.

The equation of continuity in cartesians (x, y, z),

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \rightarrow (1)$$

and in cylindrical polars (R, θ, z) it is

$$\frac{\partial^2 \phi}{\partial R^2} + \frac{1}{R} \frac{\partial \phi}{\partial R} + \frac{1}{R^2} \frac{\partial \phi^2}{\partial \theta^2} = 0 \rightarrow (2)$$

The method of separation of variables. Here we content ourselves with the easily verifiable facts that $\phi = R\cos\theta$, $\phi = (\frac{1}{R}\cos\theta)$ are special solutions.

Hence since the equation is linear, a more general type of soln is

$$\phi = (AR + BR^{-1})\cos\theta \rightarrow (3)$$

Example.

A stationary infinite right circular solid cylinder of radius a is placed in a uniform stream its axis being perpendicular to the direction of flow

Let P have cylindrical polar coordinates ~~R,θ~~ (R, θ, z). The undisturbed velocity potential at p will be $Ux = UR\cos\theta$, so that when the cylinder is inserted we take, for $R \geq a$ (23)

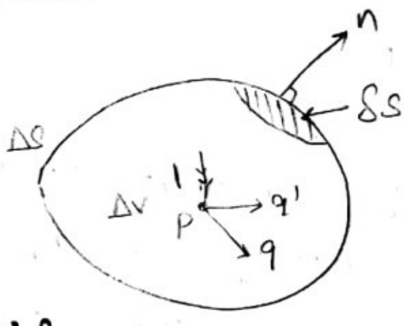
$$\phi(R,\theta) = UR\cos\theta + AR^{-1}\cos\theta \rightarrow (4)$$

the last term being a suitable perturbation potential which satifies (2). on $R = a$ we require $\frac{\partial\phi}{\partial R} = 0$, so that $A = Ua^2$

Thus $\phi(R,\theta) = U\cos\theta(R + a^2R^{-1}) \rightarrow (5)$

(5) analysis may now proceed in like manner to that of flow past a stationary sphere

3.11 Impulsive motion.



Suppose sudden velocity changes areproduced at the boundaries of an incompressible fluid or that impulsive forces are applied to its interior.

To put the matter on a quantitative basis, consider the incompressible fluid particle of volume ΔV bounded by a closed surface ΔS. If we denote the impulsive pressure at the element δS of ΔS, by P then since the impulse on a volume element δv of ΔV is $I\rho\,\delta v$ and that on δS is $-nP\delta S$, the total impulse applied to the fluid particle is

$$\int_{\Delta V} \rho I \, dv - \int_{\Delta S} nP\, dS = \int_{\Delta V} (\rho I - \nabla P)\, dv.$$

The momentum change is $\int_{\Delta V} \rho(q' - q)\, dv.$

Hence $$\int_{\Delta V} (\rho I - \nabla P)\, dv = \int_{\Delta V} \rho(q' - q)\, dv$$

Since Δv is an arbitrarily small volume, our usual assumption of fluid continuity yields the result

(24)

$$\rho I - \nabla p = \rho(q' - q) \rightarrow ①$$

at each point P of the fluid. when there are no externally applied impulses ① gives

$$-\nabla p = \rho(q' - q).$$

If the motion is irrotation and $q = -\nabla\phi$, $q' = -\nabla\phi'$

then $\quad p = \rho(\phi' - \phi)$

ignoring immaterial integration constants.

Example.

An incompressible fluid is contained within the region bounded by two concentric rigid spherical surfaces of radii R_1, R_2 $(R_2 > R_1)$. The fluid is initially at rest. If the inner surface in now given a sudden velocity Ui, where i is a constant unit vector, S.T the impulsive thrust on the outer surface is

$$2\pi\rho R_1^3 R_2^3 (R_2^3 - R_1^3)^{-1} Ui.$$

where ρ is the fluid density.

Soln:

For the motion set up assume that at the point having spherical polar coordinates (r, θ, ψ) in the fluid the velocity potential is

$$\phi(r, \theta) = (Ar + Br^{-2})\cos\theta \quad (R_1 \le r \le R_2) \rightarrow ①$$

on $r = R_2$, $\frac{\partial\phi}{\partial r} = 0$ so that

$$\frac{\partial\phi}{\partial r} = (A + (-2)r^{-3}B)\cos\theta$$

$$0 = (A - 2r^{-3}B)\cos\theta$$

$$= A - R_2^{-3}2B\cos\theta$$

on $r = R_1$, $\frac{\partial\phi}{\partial r} = -U\cos\theta$, so that

$$\frac{\partial \phi}{\partial r} = (A - 2r^{-3}) B \cos\theta$$

$$-U\cos\theta = (A - 2R_1^{-3}B)\cos\theta$$

(25)

$$A - 2R_1^{-3}B = -U \rightarrow (3)$$

(2), (3)

$$\begin{cases} A = \frac{1}{2} U R_1^3 R_2^3 (R_2^3 - R_1^3)^{-1} \\ B = U R_1^3 (R_2^3 - R_1^3)^{-1} \end{cases}$$

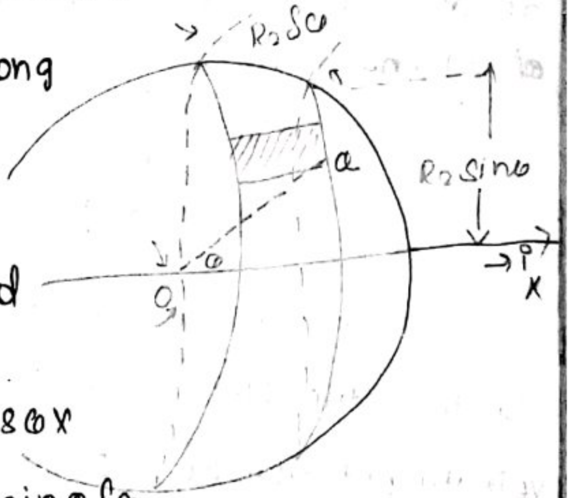
Hence

$$\phi(r,\theta) = U R_1^3 (R_2^3 - R_1^3)^{-1} \left[r + (R_2^3 / 2r^2) \right] \cos\theta \rightarrow (4)$$

$p(r,\theta) = \rho\phi$, $r = R_2$ is

$$p(R_2,\theta) = \frac{3}{2} \rho U R_1^3 R_2 (R_2^3 - R_1^3)^{-1} \cos\theta \rightarrow (5)$$

This has a component along i of $p(R_2,\theta)\cos\theta\,\delta S$.



The total thrust on the circular zone is along $\vec{OX}$ and its magnitude is

$$\int_{zone} p(R_2,\theta)\cos\theta\, dS = p(R_2,\theta)\cos\theta \times 2\pi R_2^2 \sin\theta\,\delta\theta$$

$$= 3\pi\rho U R_1^3 R_2^3 (R_2^3 - R_1^3)^{-1} \cos^2\theta \sin\theta\,\delta\theta$$

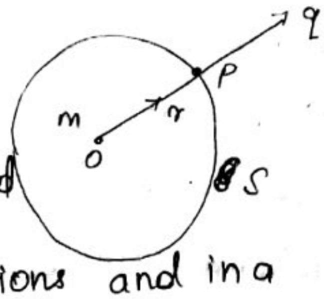
Hence the total thrust on the outer shell is along i and of magnitude

$$3\pi\rho U R_1^3 R_2^3 (R_2^3 - R_1^3)^{-1} \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta$$

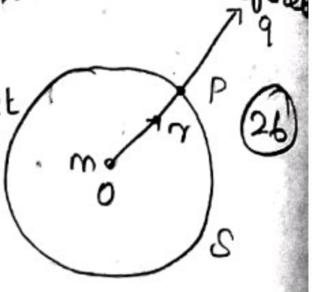
$$= 2\pi\rho U R_1^3 R_2^3 (R_2^3 - R_1^3)^{-1}$$

Some three-Dimensional Flows.

4.2. Sources, sinks and Doublets



Suppose at a point O in a fluid the flow is such that it is directed radially outwards from O in all directions and in a symmetrical manner. Then fluid enters the system through O which is termed a simple source. If at the volume entering per unit time is $4\pi m$, where m is a constant, then the strength of the source is defined to be m.



(26)

If however, the flow is such that fluid is directed radially inwards to O from all directions in a symmetrical manner, then fluid leaves the system at O which is termed a simple sink. A sink of strength m is a source of strength $-m$.

$$q = \frac{m}{r^2}$$

$$\mathbf{q} = \frac{m}{r^2}\hat{r}. \quad \rightarrow (1)$$

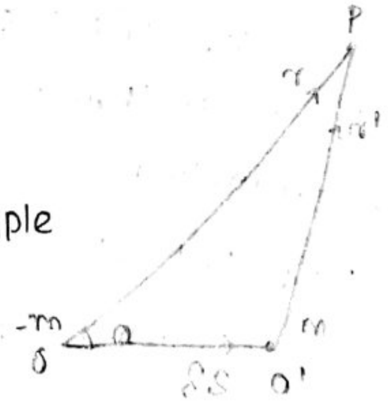
It is easily shown that curl $\mathbf{q} = 0$ (except at $r=0$) So that the flow is of the potential kind. Let ϕ be the velocity potential at P. From considerations of symmetry

$\phi = \phi(r)$ $\qquad \nabla\phi = \phi'(r)\hat{r}.$

Thus

$$\phi'(r) = -m/r^2$$

$$\phi(r) = m/r \quad \rightarrow (2)$$



Now suppose we have simple Sources of strengths $-m$ at O and m at O' in a fluid, where $\overline{OO'} = \delta s$. Let P be any other point in the fluid such that $\overline{OP} \equiv r$, $\overline{O'P} \equiv r'$, $r = |r|$, $r' = |r'|$, $\delta s = |\delta s|$.

Also let $\mu = m\delta s$. that $m \to \infty$, $\delta s \to 0$ in such a way that μ remains finite and constant. Then in the limit the two sources $\pm m$, of infinitely great magnitude and coincident at O, are said to constitute a three-dimensional doublet or dipole at O. The quantity μ is termed the moment or strength of the doublet. The quantity $\mu = \mu \hat{a}$, where $\hat{a}$ is the unit vector in the direction $\overline{OO'}$ is called the **vector moment** of the doublet. Also the direction of $\hat{a}$ is that of the axis of the doublet. (27)

The flow to be entirely due to $-m$ at O at m at O',

the velocity potential at P is

$$\phi = \frac{m}{r'} - \frac{m}{r} = \frac{m(r-r')}{rr'}$$

$$= \frac{m(r^2 - r'^2)}{rr'(r+r')} = \frac{m(r-r')\cdot(r+r')}{rr'(r+r')} = \frac{m\delta s.(r+r')}{rr'(r+r')} = \frac{\mu.(r+r')}{rr'(r+r')}$$

$$\phi = (\mu . r) r^{-3} \rightarrow (8)$$

with $\angle POO' = \theta$, other equivalent forms for ϕ are

$$\phi = (\mu . \hat{r}) r^{-2} = \mu r^{-2} \cos\theta \rightarrow (8')$$

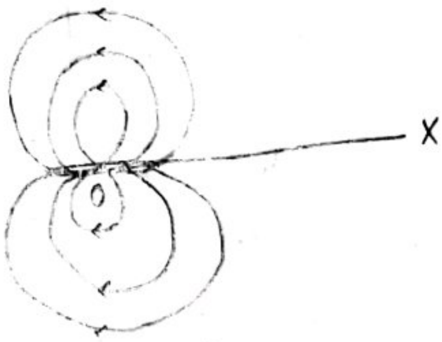
The velocity potential at P being $\phi(r,\theta) = (\mu\cos\theta)/r^2$

we find the velocity components at P are

$$q_r = -\frac{\partial\phi}{\partial r} = \frac{2\mu\cos\theta}{r^3}$$

$$q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{\mu\sin\theta}{r^3}$$

$$q_\psi = -\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\psi} = 0.$$



The equations of the streamlines are

$$\psi = \text{const}, \quad r = A\sin^2\theta.$$

Thus the streamlines lie in planes passing through the axis of the doublet.

The plane $\theta = \frac{1}{2}\pi$.

The form $\phi(r,\theta) = (\mu\cos\theta)/r^2$ shows that ϕ, at points other than O, must satisfy the axially symmetrical Spherical polar form of Laplace's equation