# UNIT I
# SOLUTION OF EQUATIONS AND EIGENVALUE PROBLEMS

***Solution of equation***

***Fixed point iteration: x=g(x) method***

***Newton's method***

***Solution of linear system by Gaussian elimination***

***Gauss – Jordon method***

***Gauss – Jacobi method***

***Gauss – Seidel method***

***Inverse of a matrix by Gauss Jordon method***

***Eigenvalue of a matrix by power method***

***Jacobi method for symmetric matrix.***

## Solution of Algebraic and Transcendental Equations

The equation of the form

$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + ....... + a_{n-1} x + a_n = 0$------------(A) is called *rational integral equation*

Here, $a_0 \neq 0$, n is a positive integer, $a_0$, $a_1$, $a_2$, ....., $a_n$ are constants.

The rational integral equation is classified into two parts

1. *Algebraic Equation*
2. *Transcendental Equation*

## Algebraic Equation

In this equation, f(x) is a polynomial purely in x as in (A)

Example: $x^3 - 3x + 1 = 0, x^4 + 2x^3 - 3x^2 + 2x + 1 = 0$

## Transcendental Equation

In this equation, f(x) contains some other functions such as trigonometric, logarithmic or exponential etc.,

Example: 3x-cosx-1=0, $x\log_{10}x - 1.2 = 0$.

## Properties

1. If f(a) and f(b) have opposite signs then one root of f(x)=0 lies between a and b
2. To find an equation whose roots are with opposites signs to those of the given equ. change x to (–x)
3. To find an equation whose roots are reciprocals of the roots of of the given equ. change x to $\left(\frac{1}{x}\right)$
4. Every equation of an odd degree has atleast one real root whose sign is opposite to that of its last term.
5. Every equation of an even degree with last term negative have atleast a pair of real roots one positive and other negative.

## Methods for solving Algebraic and Transcendental Equations

- Fixed point iteration: x = g(x) method (or) Method of successive approximation.
- Newton's method (or) Newton's Raphson method

## Fixed point iteration: x = g(x) method (or) Method of successive approximation

Let f(x) =0 be the given equation whose roots are to be determined.

## Steps for this method

1. Use the first property find 'a' and 'b' where the roots lies between.

2. Write the given equation in the form $x = \varphi(x)$ with the condition $|\varphi'(x)| < 1$
3. Let the initial approximation be $x_0$ which is lies in the interval (a, b)
4. Continue the process using $x_n = \varphi(x_{n-1})$
5. If the difference between the two consecutive values of $x_n$ is very small then we stop the process and that value is the root of the equation.

Convergence of iteration method

The iteration process converges quickly if $|\varphi'(x)| < 1$ where $x=\varphi(x)$ is the given equation. If $|\varphi'(x)| > 1$, $|x_n - \alpha|$ will become infinitely large and hence this process will not converge. The convergence is linear.

Example: Consider the equation $f(x) = x^3+x-1 = 0$

we can write $x=\varphi(x)$ in three types

1. $x=1-x^3$
2. $x = \frac{1}{1+x^2}$
3. $x = (1 - x)^{1/3}$

but we take the type which has the convergence property $|\varphi'(x)| < 1$

$f(x) = x^3+x-1$

f(0)=-ve and f(1)=+ve

hence the root lies between 0 and 1

Now consider **The equation (1.) $x=1-x^3$**

Here $\varphi(x)= 1-x^3$, $\varphi'(x)=-3x^2$

at $x=0.9$ $\varphi'(x)=-3(0.9)^2$

$\varphi'(x)=-3(0.81)=-2.81$

$\Rightarrow |\varphi'(x)| > 1$

$\Rightarrow$this equation $x=1-x^3$ will not converge

so the iteration will not work if we consider this equation

Now consider

**the equation** $x = \frac{1}{1+x^2}$

Here $\varphi'(x) = \frac{-2x}{(1+x^2)^2}$

at $x=0.9$

$\varphi'(x) = \frac{-2\times 0.9}{(1+0.9^2)^2} = \frac{-1.8}{3.2761} = 0.5494$

$\Rightarrow |\varphi'(x)| < 1$

$\Rightarrow$ the equation is converge

$\therefore$ we use this equation.

no need to consider the third type.

**Problems based on Fixed point iteration**

1. Find a real root of the equation $x^3+x^2-1=0$ by iteration method.

**Solution:**

Let f(x)= $x^3+x^2-1$

f(0)= -ve and f(1)=+ve

Hence a real root lies between 0 and 1.

Now can be written as $x = \frac{1}{\sqrt{1+x}} = \varphi(x)$

In this type only $|\varphi'(x)| < 1$ in (0, 1)

Let the initial approximation be $x_0=0.5$

$$x_1 = \varphi(x_0) = \frac{1}{\sqrt{0.5+1}} = 0.81649$$

$$x_2 = \varphi(x_1) = \frac{1}{\sqrt{0.81649+1}} = 0.74196$$

$$x_3 = \varphi(x_2) = \frac{1}{\sqrt{0.74196+1}} = 0.75767$$

$$x_4 = \varphi(x_3) = \frac{1}{\sqrt{0.75767+1}} = 0.75427$$

$$x_5 = \varphi(x_4) = \frac{1}{\sqrt{0.75427+1}} = 0.75500$$

$$x_6 = \varphi(x_5) = \frac{1}{\sqrt{0.75500+1}} = 0.75485$$

$$x_7 = \varphi(x_6) = \frac{1}{\sqrt{0.75485+1}} = 0.75488$$

Here the difference between $x_6$ and $x_7$ is very small.

therefore the root of the equation is **0.75488**

---

2. Find the real root of the equation cosx = 3x-1, using iteration method.

Solution:

Let f(x)= cosx - 3x-1

f(0)= +ve and f(1)=-ve

$\therefore$ A root lies between 0 and π/2

The given equation can be written as

$$x = \frac{1}{3}(1+\cos x) = \varphi(x)$$

$$\varphi'(x) = \frac{-\sin x}{3}$$

Clearly, $|\varphi'(x)| < 1$ in (0, π/2)

Let the initial approximation be $x_0=0$

$$x_1 = \varphi(x_0) = \frac{1}{3}(1+\cos 0) = 0.66667$$

$$x_2 = \varphi(x_1) = \frac{1}{3}(1+\cos 0.66667) = 0.59529$$

$$x_3 = \varphi(x_2) = \frac{1}{3}(1+\cos 0.59529) = 0.60933$$

$$x_4 = \varphi(x_3) = \frac{1}{3}(1+\cos 0.60933) = 0.60668$$

$$x_5 = \varphi(x_4) = \frac{1}{3}(1+\cos 0.60668) = 0.60718$$

$$x_6 = \varphi(x_5) = \frac{1}{3}(1+\cos 0.60718) = 0.60709$$

$$x_7 = \varphi(x_6) = \frac{1}{3}(1+\cos 0.60709) = 0.60710$$

$$x_8 = \varphi(x_7) = \frac{1}{3}(1+\cos 0.60710) = 0.60710$$

Since the values of $x_7$ and $x_8$ are equal, the required root is **0.60710**

-----------------------------------------------------------------------------------------3.

Find the negative root of the equation $x^3-2x+5=0$

Solution:

The given equation is $x^3-2x+5=0$ ----------------------------------------------(1)

we know that if α, β, γ are the roots of the equation (1), then the equation whose roots are -α, -β, - γ is $x^3+(-1)\,0\,x^2+(-1)^2(-2x)+(-1)^3 5=0$------------------------(2)

The negative root of the equation (1) is same as the positive root of the equation (2)

Let $f(x)= x^3-2x+5$

Now f(2)=-ve and f(3) =+ve

Hence the root lies between 2 and 3. Equation (2) can be written as $x=(2x+5)^{\frac{1}{3}}=\varphi(x)$

where $|\varphi'(x)|<1$ in (2, 3)

Let the initial approximation be $x_0=2$

Since the values of $x_6$ and $x_7$ are equal, the root is **2.09455**

Therefore the negative root of the given equation is **-2.09455**

-----------------------------------------------------------------------------------------

## Newton's method (or) Newton's Raphson method (Method of tangents)

Let f(x)=0 be the given equ. whose roots are to be determined.

**FORMULA:** $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

## Steps for this method

i. Use the first property find 'a' and 'b' where the roots lies between.
ii. The initial approximation $x_0$ is 'a' if $|f(a)|<|f(b)|$; the initial approximation $x_0$ is 'b' if $|f(b)|<|f(a)|$ in the interval (a, b).
iii. Use the formula and continue the process
iv. If the difference between the two consecutive values of $x_{n+1}$ is very small then we stop the process and that value is the root of the equ.

## Note

- The process will evidently fail if f'(x) = 0 in the neighbourhood of the root. In such cases Regula-Falsi method should be used.
- If we choose initial approximation $x_0$ close to the root then we get the root of the equ. very quickly.
- The order of convergence is two

## Condition for convergence of Newton's Raphson method

$$|f(x).f''(x)| < |f'(x)|^2$$

## Problems based on Newton's Method

1. Compute the real root of $x\log_{10}x=1.2$ correct to three decimal places using Newton's Raphson Method

Solution:

Let $f(x)= x\log_{10}x-1.2$

Now f(2)=-ve and f(3) =+ve

Hence the root lies between 2 and 3.

Let the initial approximation be $x_0=3$

$f(x)= x\log_{10}x-1.2$

$$f'(x)=\log_{10}x+x.\frac{1}{x}\log_{10}e \qquad \left[\because \frac{d}{dx}(\log_a x)=\frac{1}{x}\log_a e\right]$$

$$=\log_{10}x+0.4343 \qquad [\because \log_{10}e=0.4343]$$

| iteration | value of $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|
| **initial iteration $x_0$** | 3 |
| **1** | $x_1 = 3 - \frac{x_0 \log_{10} x_0 - 1.2}{\log_{10} x_0 + 0.4343} = 2.746$ |
| **2** | $x_2 = 2.746 - \frac{x_1 \log_{10} x_1 - 1.2}{\log_{10} x_1 + 0.4343} = 2.741$ |
| **3** | $x_3 = 2.741 - \frac{x_2 \log_{10} x_2 - 1.2}{\log_{10} x_2 + 0.4343} = 2.741$ |

Hence the real root of f(x)=0, correct to three decimal places is 2.741

-------------------------------------------------------------------------------------------2.

Evaluate $\sqrt{12}$ to four decimal places by Newton's Raphson Method

**Solution:**

Let x= $\sqrt{12} \Rightarrow x^2=12 \Rightarrow x^2-12 =0$

Let $f(x)= x^2-12$ and $f'(x)=2x$

Now f(3)=-ve and f(4) =+ve.Hence the root lies between 3 and 4. Here $|f(3)| < |f(4)|$ the root is nearer to 3. Therefore the initial approximation is $x_0=3$

| iteration | value of $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ |
|---|---|
| **initial iteration $x_0$** | 3 |
| **1** | $x_1 = 3 - \frac{f(3)}{f'(3)} = 3.5$ |
| **2** | $x_2 = 3.5 - \frac{f(3.5)}{f'(3.5)} = 3.4642$ |
| **3** | $x_3 = 3.4642 - \frac{f(3.4642)}{f'(3.4642)} = 3.4641$ |
| **4** | $x_4 = 3.4641 - \frac{f(3.4641)}{f'(3.4641)} = 3.4641$ |

Hence the value of $\sqrt{12}$ is **3.4641**

--------------------------------------------------------------------------------

## Solving Simultaneous Equations with two variables using Newton's Method

Let the simultaneous equations with two variables be f(x, y) =0 and g(x, y) =0

$x_1=x_0+h$ and $y_1=y_0+k$

$$h=\frac{-D_1}{D} \quad \text{and } k=\frac{D_2}{D}$$

$$where \quad D=\begin{vmatrix}(f_x)_0 & (f_y)_0 \\ (g_x)_0 & (g_y)_0\end{vmatrix}$$

$$D_1=\begin{vmatrix}f_0 & (f_y)_0 \\ g_0 & (g_y)_0\end{vmatrix}$$

$$D_2=\begin{vmatrix}(f_x)_0 & f_0 \\ (g_x)_0 & g_0\end{vmatrix}$$

## Problems

1. Find the solution of the equation $4x^2+2xy+y^2=30$ and $2x^2+3xy+y^2=3$ correct to 3 places of decimals, using Newton's Raphson method, given that $x_0=-3$ and $y_0=2$.

   Solution:

   Let $f(x, y) = 4x^2+2xy+y^2-30$ and $g(x, y) = 2x^2+3xy+y^2-3$

   $f_x=8x+2y$, $f_y=2x+2y$, $g_x=4x+3y$, $g_y=3x+2y$

| $x_0$ | $y_0$ | $(f_x)_0$ | $(f_y)_0$ | $(g_x)_0$ | $(g_y)_0$ | $f_0$ | $g_0$ |
|---|---|---|---|---|---|---|---|
| **-3** | **2** | **-20** | **-2** | **-6** | **-5** | **-2** | **1** |

$$D=\begin{vmatrix}(f_x)_0 & (f_y)_0 \\ (g_x)_0 & (g_y)_0\end{vmatrix}=\begin{vmatrix}-20 & -2 \\ -6 & -5\end{vmatrix}=88$$

$$D_1=\begin{vmatrix}f_0 & (f_y)_0 \\ g_0 & (g_y)_0\end{vmatrix}=\begin{vmatrix}-2 & -2 \\ 1 & -5\end{vmatrix}=12$$

$$D_2=\begin{vmatrix}(f_x)_0 & f_0 \\ (g_x)_0 & g_0\end{vmatrix}=\begin{vmatrix}-20 & -2 \\ -6 & 1\end{vmatrix}=-32$$

$$h=\frac{-D_1}{D}=\frac{-12}{88}=-0.1364$$

$$k=\frac{-D_2}{D}=\frac{32}{88}=0.3636$$

$x_1=x_0+h$ and $y_1=y_0+k$

$\Rightarrow$ $x_1$= -3.1364 and $y_1$=2.364

| $x_1$ | $y_1$ | $(f_x)_1$ | $(f_y)_1$ | $(g_x)_1$ | $(g_y)_1$ | $f_1$ | $g_1$ |
|---|---|---|---|---|---|---|---|
| **-3.1364** | **2.364** | **-20.360** | **-1.544** | **-5.452** | **-4.680** | **0.0995** | **0.0169** |

$$D = \begin{vmatrix} (f_x)_1 & (f_y)_1 \\ (g_x)_1 & (g_y)_1 \end{vmatrix} = 86.8669$$

$$D_1 = \begin{vmatrix} f_1 & (f_y)_1 \\ g_1 & (g_y)_1 \end{vmatrix} = -0.4395$$

$$D_2 = \begin{vmatrix} (f_x)_1 & f_1 \\ (g_x)_1 & g_1 \end{vmatrix} = 0.1984$$

$$h = \frac{-D_1}{D} = 0.0051$$

$$k = \frac{-D_2}{D} = -0.0023$$

$\Rightarrow$ $x_2$= -3.131 and $y_2$=2.362

| $x_2$ | $y_2$ | $(f_x)_2$ | $(f_y)_2$ | $(g_x)_2$ | $(g_y)_2$ | $f_2$ | $g_2$ |
|---|---|---|---|---|---|---|---|
| **-3.131** | **2.362** | **-20.324** | **-1.538** | **-5.438** | **-4.669** | **0.0008** | **-0.0009** |

$$D = \begin{vmatrix} (f_x)_2 & (f_y)_2 \\ (g_x)_2 & (g_y)_2 \end{vmatrix} = 86.5291$$

$$D_1 = \begin{vmatrix} f_2 & (f_y)_2 \\ g_2 & (g_y)_2 \end{vmatrix} = -0.0051$$

$$D_2 = \begin{vmatrix} (f_x)_2 & f_2 \\ (g_x)_2 & g_2 \end{vmatrix} = 0.0226$$

$$h = \frac{-D_1}{D} = 0.0001$$

$$k = \frac{-D_2}{D} = -0.0003$$

$\Rightarrow$ $x_3$= -3.1309 and $y_3$=2.3617.Since the two consecutive values of $x_2$, $x_3$ and $y_2$, $y_3$ are approximately equal, the correct solution can be taken as x= -3.1309 and y=2.3617.

-----------------------------------------------------------------------------------------------------------

# Method of False Position (or) Regula Falsi Method (or) Method of Chords

**FORMULA:** $x_1 = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$

- This is the first approximation to the actual root.
- Now if f(x1) and f(a) are of opposite signs, then the actual root lies between x1 and a.
- Replacing b by x1 and keeping a as it is we get the next approx. x2 to the actual root.
- Continuing this manner we get the real root.

## Note:

- The convergence of the root in this method is slower than Newton's Raphson Method

## Problems based on Regula Falsi Method

1. Sove the equation xtanx=-1 by Regula-Falsi method starting with $x_0$=2.5 and $x_1$ = 3.0 correct to 3 decimal places.
   Solution:
   Let f(x) = xtanx+1
   f(2.5)=-ve and f(3)=+ve
   Let us take a=3 and b=2.5

| iteration | a | b | $x = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$ | |
|---|---|---|---|---|
| 1 | 3 | 2.5 | $x_1 = \dfrac{3f(2.5) - 2.5f(3)}{f(2.5) - f(3)} = 2.8012$ | f(2.8012)=+ve |
| 2 | 2.8012 | 2.5 | $x_2 = \dfrac{2.8012\ f(2.5) - 2.5f(2.8012)}{f(2.5) - f(2.8012)} = 2.7984$ | f(2.7984)=+ve |
| 3 | 2.7984 | 2.5 | $x_3 = \dfrac{2.7984\ f(2.5) - 2.5f(2.7984)}{f(2.5) - f(2.7984)} = 2.7984$ | |

Since the two consecutive values of $x_2$ and $x_3$ are approximately equal, the required root of the equation f(x)=0 is 2.7984

------------------------------------------------------------------------------------------

2. Find the root of $xe^x = 3$ by Regula-Falsi method correct to 3 decimal places.
   Solution:
   Let f(x)= $xe^x - 3$
   f(1)= +ve and f(1.5) = -ve
   $\therefore$The root lies between 1 1nd 1.5
   Take a = 1 and b = 1.5

| iteration | a | b | $x=\frac{af(b)-bf(a)}{f(b)-f(a)}$ | |
|---|---|---|---|---|
| 1 | 1 | 1.5 | $x_1=\frac{1f(1.5)-1.5f(1)}{f(1.5)-f(1)}=1.035$ | f(1.035)=-ve |
| 2 | 1.035 | 1.5 | $x_2=\frac{1.035f(1.5)-1.5f(1.035)}{f(1.5)-f(1.035)}=1.045$ | f(1.045)=-ve |
| 3 | 1.045 | 1.5 | $x_3=\frac{1.045f(1.5)-1.5f(1.045)}{f(1.5)-f(1.045)}=1.048$ | f(1.045)=-ve |
| 4 | 1.048 | 1.5 | $x_4=\frac{1.048f(1.5)-1.5f(1.048)}{f(1.5)-f(1.048)}=1.048$ | |

Since the two consecutive values of $x_3$ and $x_4$ are equal, the required root of the equation f(x)=0 is 1.048

---------------------------------------------------------------------------------------

## Solutions of linear algebraic equations

A system of m linear equations (or a set of m simultaneous linear equations) in 'n' unknowns $x_1, x_2, ..., x_n$ is a set of equations of the form,

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n = b_m \end{array}\right\} \quad (1)$$

Where the coefficients of $x_1, x_2, ..., x_n$ and $b_1, b_2, ..., b_m$ are constants.

The left hand side members of (1) may be specified by the square array of the coefficients, known as the coefficient matrix.

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} \\ a_{21} & a_{22} \ldots\ldots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \ldots\ldots & a_{mn} \end{bmatrix}$$

Whereas the complete set may be specified by the rectangular array

$$M = \begin{bmatrix} a_{11} & a_{12} \ldots\ldots & a_{1n} & b_1 \\ a_{21} & a_{22} \ldots\ldots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \\ a_{m1} & a_{m2} \ldots\ldots & a_{mn} & b_m \end{bmatrix}$$

is known as the **augmented matrix.**

There are two methods to solve such a system by numerical methods.

- **Direct methods**
- **Iterative or indirect methods.**

**Gaussian elimination method, Gauss-Jordan method, belongs to Direct methods,**
**Gauss-Seidel iterative method and relaxation method belongs to iterative methods.**

## Back Substitution

Let A be a given square matrix of order 'n', b a given n-vector. We wish to solve the linear system.

$$Ax = b$$

For the unknown n-vector x. The solution vector x can be obtained without difficulty in case A is upper-triangular with all diagonal entries are non-zero. In that case the system has the form

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1,n-1}x_{n-1} + a_{1n}x_n &= b_1 \\ a_{22}x_2 + \ldots + a_{2,n-1}x_{n-1} + a_{2n}x_n &= b_2 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad &\vdots \\ a_{n-2,n-2}x_{n-2} + a_{n-2,n-1}x_{n-1} &= b_{n-2} \\ a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n + a_{n-2}x_n &= b_{n-1} \\ a_{nn}x_n &= b_n \end{aligned}\right\} \quad (1)$$

In particular, the last equation involves only $x_n$; hence, since $a_{nn} \neq 0$, we must have

$$x_n = \frac{b_n}{a_{nn}}$$

Since we now know, the second last equation

$$a_{n-1,n-1}x_{n-1} + a_{n-1,n}x_n = b_{n-1}$$

Involves only one unknown, namely, $x_{n-1}$.

As $a_{n-1,n-1} \neq 0$, it follows that

$$x_{n-1} = \frac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}}$$

With $x_n$ and $x_{n-1}$ now determined, the third from last equation

$$a_{n-2,n-2}x_{n-2} + a_{n-2,n-1}x_{n-1} + a_{n-2,n}x_n = b_{n-2}$$

Contains only one true unknown, namely, $x_{n-2}$. Once again, wince $a_{n-2,n-2} \neq 0$, we can solve for $x_{n-2}$.

$$x_{n-2} = \frac{b_{n-2} - a_{n-2,n-1}x_{n-1} - b_{n-1} - a_{n-2,n}x_n}{a_{n-2,n-2}}$$ and so on.

This process of determining the solution of (1) is called **Back substitution**.

## Gauss elimination method

Basically the most effective direct solution techniques, currently being used are applications of Gauss elimination, method which Gauss proposed over a century ago. In this method, the given system is transformed into an equivalent system with upper-triangular coefficient matrix i.e., a matrix in which all elements below the diagonal elements are zero which can be solved by back substitution.

## Note

- ✍ This method fails if the element in the top of the first column is zero. Therefore in this case we can interchange the rows so as to get the pivot element in the top of the first column.
- ✍ If we are not interested in the elimination of $x, y, z$ in a particular order, then we can choose at each stage the numerically largest coefficient of the entire coefficient matrix. This requires an interchange of equations and also an interchange of the position of the variables.

Problems based on Gauss Elimination Method

1. Solve 2x+y+4z=12; 8x-3y+2z=20; 4x+11y-z=33 by gauss elimination method

Solution:

The given equations are,

$$2x+y+4z=12$$
$$8x-3y+2z=20$$
$$4x+11y-z=33$$

above equations can be written as

$$\underbrace{\begin{bmatrix} 2 & 1 & 4 \\ 8 & -3 & 2 \\ 4 & 11 & -1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 12 \\ 20 \\ 33 \end{bmatrix}}_{B}$$

where, A-co efficient matrix; B-constants; X- unknown variables

The augmented matrix can be written as

$$[A,B]=\begin{bmatrix}2 & 1 & 4 & 12\\ 8 & -3 & 2 & 20\\ 4 & 11 & -1 & 33\end{bmatrix}$$

$$\sim\begin{bmatrix}2 & 1 & 4 & 12\\ 0 & 7 & 14 & 28\\ 4 & 11 & -1 & 33\end{bmatrix} \qquad R_2 \to 4R_1 - R_2$$

$$\sim\begin{bmatrix}2 & 1 & 4 & 12\\ 0 & 7 & 14 & 28\\ 0 & -9 & 9 & -9\end{bmatrix} \qquad R_3 \to 2R_1 - R_3$$

$$\sim\begin{bmatrix}2 & 1 & 4 & 12\\ 0 & 7 & 14 & 28\\ 0 & 0 & 189 & 189\end{bmatrix} \qquad R_3 \to 9R_1 + 7R_3$$

solutions are obtained from above matrix by back substitution method as

$2x+y+4z = 12 \longrightarrow (1)$

$7y+14z = 28 \longrightarrow (2)$

$189z = 189 \longrightarrow (3)$

from the above equations we get z=1, y=2, x=3

thus the solution of the equations are x=3 ; y= 2 ; z=1

---

2. Solve 3x+4y+5z=18; 2x-y+8z=13; 5x-2y+7z=20 by gauss elimination method

Solution:

The given equations are,

3x+4y+5z=18

2x-y+8z=13

5x-2y+7z=20

above equations can be written as

$$\begin{bmatrix}3 & 4 & 5\\ 2 & -1 & 8\\ 5 & -2 & 7\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}18\\ 13\\ 20\end{bmatrix}$$

A X B

where, A-co efficient matrix; B-constants; X- unknown variables

The augmented matrix can be written as

$$[A, B] = \begin{bmatrix} 3 & 4 & 5 & 18 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 4 & 5 & 18 \\ 0 & 11 & -14 & 3 \\ 0 & 26 & 4 & 30 \end{bmatrix} \qquad R_2 \to 2R_1 - 3R_2, R_3 \to 5R_1 - 3R_3$$

$$\sim \begin{bmatrix} 3 & 4 & 115 & 18 \\ 0 & 11 & -14 & -3 \\ 0 & 0 & -408 & -408 \end{bmatrix} \qquad R_3 \to 26R_2 - 11R_3$$

solutions are obtained from above matrix by back substitution method as

3x+4y +5z = 18 ⟶ (1)
11y -14z = -3 ⟶ (2)
-408z = -408 ⟶ (3)

from the above equations we get z=1, y=1, x=3
thus the solution of the equations are x=3 ; y=1 ; z=1

----------------------------------------------------------------------------------------------------

## Gauss – Jordan Method

This method is a modified from Gaussian elimination method. In this method, the coefficient matrix is reduced to a diagonal matrix (or even a unit matrix) rather than a triangular matrix as in the Gaussian method. Here the elimination of the unknowns is done not only in the equations below, but also in the equations above the leading diagonal. Here we get the solution without using the back substitution method since after completion of the Gauss – Jordan method the equations become

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \dots \\ \dots \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \dots \\ \dots \\ a_n \end{bmatrix}$$

**Note:**

- This method involves more computation than in the Gaussian method.
- In this method we can find the values of $x_1, x_2, \dots, x_n$ immediately without using back substitution.
- Iteration method is self-correcting method, since the error made in any computation is corrected in the subsequent iterations.

## Problems based on Gauss Jordan Method

1. Solve 3x+4y+5z=18; 2x-y+8z=13; 5x-2y+7z=20 by gauss elimination method

Solution:

The given equations are,

3x+4y+5z=18
2x-y+8z=13
5x-2y+7z=20

above equations can be written as

$$\begin{bmatrix} 3 & 4 & 5 \\ 2 & -1 & 8 \\ 5 & -2 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 18 \\ 13 \\ 20 \end{bmatrix}$$

A X B

where, A-co efficient matrix; B-constants; X- unknown variables

The augmented matrix can be written as

$$[A, B] = \begin{bmatrix} 3 & 4 & 5 & 18 \\ 2 & -1 & 8 & 13 \\ 5 & -2 & 7 & 20 \end{bmatrix}$$

$$\sim \begin{bmatrix} 3 & 4 & 5 & 18 \\ 0 & 11 & -14 & 3 \\ 0 & 26 & 4 & 30 \end{bmatrix} \quad R_2 \to 2R_1 - 3R_2, R_3 \to 5R_1 - 3R_3$$

$$\sim \begin{bmatrix} 33 & 0 & 111 & 210 \\ 0 & 11 & -14 & -3 \\ 0 & 0 & -408 & -408 \end{bmatrix} \quad R_1 \to 11R_1 - 4R_2, R_3 \to 26R_2 - 11R_3$$

$$\sim \begin{bmatrix} 13464 & 0 & 0 & 40392 \\ 0 & -4488 & 0 & -4488 \\ 0 & 0 & -408 & -408 \end{bmatrix} \quad R_1 \to 408R_1 + 111R_3, R_2 \to -408R_1 + 14R$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad R_1 \to R_1/13464, R_2 \to R_2/-4488, R_3 \to R_3/-408$$

without back substitution method we get z=1, y=1, x=3

Thus the solution of the equations are x=3 ; y=1 ; z=1

---

2. Solve 10x+y+z=12; 2x+10y+z=13; 2x+2y+10z=14 by gauss Jordan method

Solution:

The given equations are,

10x+y+z=12
2x+10y+z=13
2x+2y+10z=14

above equations can be written as

$$\underset{A}{\begin{bmatrix} 10 & 1 & 1 \\ 2 & 10 & 1 \\ 2 & 2 & 10 \end{bmatrix}} \underset{X}{\begin{bmatrix} x \\ y \\ z \end{bmatrix}} = \underset{B}{\begin{bmatrix} 12 \\ 13 \\ 14 \end{bmatrix}}$$

A-co efficient matrix
B-constants
X- unknown variables

The augmented matrix can be written as

$$[A, B] = \begin{bmatrix} 10 & 1 & 1 & 12 \\ 2 & 10 & 1 & 13 \\ 2 & 2 & 10 & 14 \end{bmatrix}$$

$$\sim \begin{bmatrix} 10 & 1 & 1 & 12 \\ 0 & -49 & -4 & -53 \\ 0 & -9 & -49 & -58 \end{bmatrix} \quad R_2 \to R_1 - 5R_2, R_3 \to R_1 - 5R_3$$

$$\sim \begin{bmatrix} 490 & 0 & 45 & 535 \\ 0 & -49 & -4 & -53 \\ 0 & 0 & 2365 & 2365 \end{bmatrix} \quad R_1 \to 49R_1 + R_2, R_3 \to 9R_2 - 49R_3$$

$$\sim \begin{bmatrix} 1158850 & 0 & 0 & 1158850 \\ 0 & -115885 & 0 & -115885 \\ 0 & 0 & 2365 & 2365 \end{bmatrix} \quad R_1 \to 2365R_1 - 45R_3, R_2 \to 2365R_2 + 4R_3$$ witho

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \quad R_1 \to R_1/1158850, R_2 \to R_2/-115885, R_3 \to R_3/2365$$

ut the back substitution x=1 ; y= 1 ; z=1

---

## Jacobi's (or Gauss – Jacobi's) iteration method

Let the system of simultaneous equations be

$$\left.\begin{aligned} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{aligned}\right\} \qquad (1)$$

This system of equations can also be written as

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \\ y &= \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \\ z &= \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \end{aligned}\right\} \qquad (2)$$

Let the first approximation be $x_0, y_0$ and $z_0$. Substituting $x_0, y_0$ and $z_0$ in (2) we get,

$$x_1 = \frac{1}{a_1}(d_1 - b_1 y_0 - c_1 z_0)$$

$$y_1 = \frac{1}{b_2}(d_2 - a_2 x_0 - c_2 z_0)$$

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_0 - b_3 y_0)$$

Substitution the values of $x_1, y_1$ and $z_1$ in (2) we get the second approximations $x_2, y_2$ and $z_2$ as given below

$$x_2 = \frac{1}{a_1}(d_1 - b_1 y_1 - c_1 z_1)$$

$$y_1 = \frac{1}{b_2}(d_2 - a_2 x_1 - c_2 z_1)$$

$$z_1 = \frac{1}{c_3}(d_3 - a_3 x_1 - b_3 y_1)$$

Substituting the values of $x_2, y_2$ and $z_2$ in (2) we get the third approximations $x_3, y_3$ and $z_3$.

This process may be repeated till the difference between two consecutive approximations is negligible.

## Problems based on Gauss Jacobi Method

1. Solve the following equations by Gauss Jacobi's iteration method, 20x+y-2z=17, 3x+20y-z=-18, 2x-3y+20z=25.

Solution:

The given equations are

20x+y-2z=17,

3x+20y-z=-18,

2x-3y+20z=25.

The equations can be written as,

$$x = \frac{1}{20}\left[17 - y + 2z\right]$$

$$y = \frac{1}{20}\left[-18 - 3x + z\right]$$

$$z = \frac{1}{20}\left[25 - 2x + 3y\right]$$

| Iteration | $x = \frac{1}{20}\left[17 - y + 2z\right]$ | $y = \frac{1}{20}\left[-18 - 3x + z\right]$ | $z = \frac{1}{20}\left[25 - 2x + 3y\right]$ |
|---|---|---|---|
| Initial | $x_0=0$ | $y_0=0$ | $z_0=0$ |
| 1 | $x_1 = \frac{1}{20}\left[17\right] = 0.85$ | $y_1 = \frac{1}{20}\left[-18\right] = -0.9$ | $z_1 = \frac{1}{20}\left[25\right] = 1.25$ |
| 2 | $x_2 = \frac{1}{20}\left[17 + 0.9 + 2.5\right]$ $= 1.02$ | $y_2 = \frac{1}{20}\left[-18 - 3(0.85) + 1.25\right]$ $= -0.965$ | $z_2 = \frac{1}{20}\left[25 - 1.7 - 2.7\right]$ $= 1.03$ |
| 3 | $x_3 = \frac{1}{20}\left[17 + 0.965 + 2(1.3)\right]$ $= 1.00125$ | $y_3 = \frac{1}{20}\left[-18 - 3(1.02) + 1.03\right]$ $= -1.0015$ | $z_3 = \frac{1}{20}\left[25 - 2(1.02) + 3(0.965)\right]$ $= 1.00325$ |
| 4 | $x_4 = \frac{1}{20}\left[17 + 1.0015 + 2(1.00325)\right]$ $= 1.0004$ | $y_4 = \frac{1}{20}\left[-18 - 3(1.00125) + 1.00325\right]$ $= -1.000025$ | $z_4 = \frac{1}{20}\left[25 - 2(1.00125) + 3(-1.0015)\right]$ $= 0.99965$ |

$x_3 \sim x_4$; $y_3 \sim y_4$ ; $z_3 \sim z_4$

Therefore the solution is x=1; y= -1; z=1

------------------------------------------------------------------------------------------------------------

2. Solve the following equations by Gauss Jacobi's method 9x+2y+4z=20; x+10y+4z=6; 2x-4y+10z=-15

Solution:

The given system of equations is

9x+2y+4z=20;

x+10y+4z=6;

2x-4y+10z=-15

The equations can be written as

$$x = \frac{1}{9}\left[20 - 2y - 4z\right]$$

$$y = \frac{1}{10}\left[6 - x - 4z\right]$$

$$z = \frac{1}{10}\left[-15 - 2x + 4y\right]$$

Let the initial values be $x_0=y_0=z_0=0$

| Iteration | $x = \frac{1}{9}\left(20 - 2y - 4z\right)$ | $y = \frac{1}{10}\left(6 - x - 4z\right)$ | $z = \frac{1}{10}\left(-15 - 2x + 4y\right)$ |
|---|---|---|---|
| Initial | $x_0=0$ | $y_0=0$ | $z_0=0$ |
| 1 | $x_1 = \frac{20}{9} = 2.222$ | $y_1 = \frac{6}{10} = 0.6$ | $z_1 = \frac{-15}{10} = -1.5$ |
| 2 | $x_2=2.7556$ | $y_2=0.9778$ | $z_2=-1.7044$ |
| 3 | $x_3=2.762$ | $y_3=1.0062$ | $z_3=-1.66$ |

$x_2 \sim x_3$; $y_2 \sim y_3$ ; $z_2 \sim z_3$

Therefore the solution is $x=2.8$; $y= 1$ ; $z=-1.7$

---

## Gauss – Seidal Iterative Method

Let the given system of equations be

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots\ldots + a_{1n}x_n = C_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots\ldots + a_{2n}x_n = C_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots\ldots + a_{3n}x_n = C_3$$

……………………………………………

……………………………………………

$$a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \ldots\ldots + a_{nn}x_n = C_n$$

Such system is often amenable to an iterative process in which the system is first rewritten in the form

$$x_1 = \frac{1}{a_{11}}\left(C_1 - a_{12}x_2 - a_{13}x_3 - \ldots\ldots - a_{1n}x_n\right) \qquad (1)$$

$$x_2 = \frac{1}{a_{22}}\left(C_2 - a_{22}x_2 - a_{23}x_3 - \ldots\ldots - a_{2n}x_n\right) \qquad (2)$$

$$x_3 = \frac{1}{a_{33}}\left(C_3 - a_{32}x_2 - a_{33}x_3 - \ldots\ldots - a_{3n}x_n\right) \qquad (3)$$

…………………………………………

$$x_n = \frac{1}{a_{nn}}\left(C_n - a_{n2}x_2 - a_{n3}x_3 - \ldots\ldots - a_{nn}x_n\right) \qquad (4)$$

First let us assume that $x_2 = x_3 = \ldots.. = x_n = 0$ in (1) and find $x_1$. Let it be $x_1^*$. Putting $x_1^*$ for $x_1$ and $x_3 = x_4 = \ldots.. = x_n = 0$ in (2) we get the value for $x_2$ and let it be $x_2^*$. Putting $x_1^*$ for $x_1$ and $x_2^*$ for $x_2$ and $x_3 = x_4 = \ldots.. = x_n = 0$ in (3) we get the value for $x_3$ and let it be $x_3^*$. In this way we can find the first approximate values for $x_1, x_2, \ldots, x_n$. Similarly we can find the better approximate value of $x_1, x_2, \ldots, x_n$ by using the relation

$$x_1^* = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2 - a_{13}x_3 - \ldots\ldots - a_{1n}x_n\right)$$

$$x_2^* = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^* - a_{23}x_3 - \ldots\ldots - a_{2n}x_n\right)$$

$$x_3^* = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^* - a_{32}x_3^* - \ldots\ldots - a_{3n}x_n\right)$$

$$x_n^* = \frac{1}{a_{nn}}\left(b_n - a_{n1}x_1^* - a_{n2}x_3^* - \ldots\ldots - a_{n,n-1}x_{n-1}^*\right)$$

**Note:**

- ✍ This method is very useful with less work for the given systems of equation whose augmented matrix have a large number of zero elements.
- ✍ We say a matrix is **diagonally dominant** if the numerical value of the leading diagonal element in each row is greater than or equal to the sum of the numerical values of the other elements in that row.
- ✍ For the Gauss – Seidal method to coverage quickly, the coefficient matrix must be diagonally dominant. If it is not so, we have to rearrange the equations in such a way that the coefficient matrix is diagonally dominant and then only we can apply Gauss – Seidal method.

## Problems based on Gauss Seidal Method

1. Solve x+y+54z=110, 27x+6y-z=85, 6x+15y+2z=72, by using Gauss Seidal method.

Solution:

The system of equations is

x+y+54z=110,
27x+6y-z=85,
6x+15y+2z=72,

The co-efficient matrix is

$$\begin{bmatrix} 1 & 1 & 54 \\ 27 & 6 & -1 \\ 6 & 15 & 2 \end{bmatrix} \begin{matrix} 1 \not> 1+54 \\ 6 \not> 27+1 \\ 2 \not> 6+15 \end{matrix}$$

$$\begin{bmatrix} 27 & 6 & -1 \\ 1 & 1 & 54 \\ 6 & 15 & 2 \end{bmatrix} \begin{matrix} 27 > 6+1 \\ 1 \not> 1+54 \\ 2 \not> 6+15 \end{matrix} \quad \text{Here } R_1 \Leftrightarrow R_2$$

$$\begin{bmatrix} 27 & 6 & -1 \\ 6 & 15 & 2 \\ 1 & 1 & 54 \end{bmatrix} \begin{matrix} 27 > 6+1 \\ 15 > 6+2 \\ 54 > 1+1 \end{matrix} \quad \text{Here } R_2 \Leftrightarrow R_3$$

Here the matrix is diagonally dominant

The diagonally dominant matrix is

$$\begin{bmatrix} 27 & 6 & -1 \\ 6 & 15 & 2 \\ 1 & 1 & 54 \end{bmatrix}$$

Thus the matrix is diagonally dominant, now the system of equations is

27x+6y-z=85,

6x+15y+2z=72,

x+y+54z=110.

The equations can be written as

$x = \frac{1}{27}(85-6y+z)$

$y = \frac{1}{15}(72-6x-2z)$

$z = \frac{1}{54}(110-x-y)$

The initial values be $x_0=y_0=z_0=0$.

| Iteration | $x = \frac{1}{27}(85-6y+z)$ | $y = \frac{1}{15}(72-6x-2z)$ | $z = \frac{1}{54}(110-x-y)$ |
|---|---|---|---|
| Initial | $x_0=0$ | $y_0=0$ | $z_0=0$ |
| 1 | $x_1=85/27=3.148$ | $y_1=\frac{1}{15}(72-18.888)$ $=3.5408$ | $z_1=\frac{1}{54}(110-3.148-3.541)$ $=1.9132$ |
| 2 | $x_2=\frac{1}{27}(85-1.2448+1.9132)$ $=2.432$ | $y_2=\frac{1}{15}(72-14.592-3.8264)$ $=3.572$ | $z_2=\frac{1}{54}(110-2.432-3.572)$ $=1.9259$ |
| 3 | $x_3=\frac{1}{27}(85-21.432+1.9258)$ $=2.4257$ | $y_3=\frac{1}{15}(72-14.5542-3.8516)$ $=3.5729$ | $z_3=\frac{1}{54}(110-2.4257-3.5729)$ $=1.9259$ |
| 4 | $x_4=\frac{1}{27}(85-21.4374+1.9259)$ $=2.4255$ | $y_4=\frac{1}{15}(72-14.553-3.8518)$ $=3.5730$ | $z_4=\frac{1}{54}(110-2.4255-3.5730)$ $=1.9259$ |
| 5 | $x_5=\frac{1}{27}(85-21.438+1.9259)$ $=2.4255$ | $y_5=\frac{1}{15}(72-14.553-3.8518)$ $=3.5730$ | $z_5=\frac{1}{54}(110-2.4255-3.5730)$ $=1.9259$ |

$x_4 \sim x_5$ ; $y_4 \sim y_5$ ; $z_4 \sim z_5$, thus the solution is **x=2.4255, y=3.5730, z=1.9259**.

---

2. Solve 8x-3y+2z=20; 6x+3y+12z=35; 4x+11y-z=33 by Gauss Seidal method

Solution:

The system of equations is

8x-3y+2z=20;

6x+3y+12z=35;

4x+11y-z=33

The co-efficient matrix is $\begin{bmatrix} 8 & -3 & 2 \\ 6 & 3 & 12 \\ 4 & 11 & -1 \end{bmatrix} \begin{matrix} 8 > 3+2 \\ 3 \not> 6+12 \\ 1 \not> 4+11 \end{matrix}$

Thus the matrix is not diagonally dominant

$$\begin{bmatrix} 8 & -3 & 2 \\ 4 & 11 & -1 \\ 6 & 3 & 12 \end{bmatrix} \begin{matrix} 8 > 3+2 \\ 11 > 4+1 \\ 12 > 6+3 \end{matrix}$$

Now the matrix is diagonally dominant

The system of equations is

8x-3y+2z=20

4x+11y-z=33

6x+3y+12z=35

The equations can be written as

$x=\frac{1}{8}$ (20+3y-2z)

$y=\frac{1}{11}$ (33-4x+z)

$z=\frac{1}{12}$ (35-6x-3y)

Let the initial values be $x_0=y_0=z_o=0$

| Iteration | $x=\frac{1}{8}(20+3y-2z)$ | $y=\frac{1}{11}(33-4x+z)$ | $z=\frac{1}{12}(35-6x-3y)$ |
|---|---|---|---|
| Initial | $x_o=0$ | $y_o=0$ | $z_o=0$ |
| 1 | $x_1=20/8=2.5$ | $y_1=\frac{1}{11}(33-10)=2.0909$ | $z_1=\frac{1}{12}(35-15-6)=11.439$ |
| 2 | $x_2=\frac{1}{8}(20+6-2.334)$ $=2.9583$ | $y_2=\frac{1}{11}(33-11.833+1.607)$ $=2.0758$ | $z_2=\frac{1}{12}(35-17.7498-6.2274)$ $=0.91875$ |
| 3 | $x_3=\frac{1}{8}(20+6.2274-1.8371)$ $=3.0260$ | $y_3=\frac{1}{11}(33-12.104+0.91857)$ $=1.9825$ | $z_3=\frac{1}{12}(35-18.156-5.9475)$ $=0.9077$ |
| 4 | $x_4=\frac{1}{8}(20-5.9475-1.8154)$ $=3.0165$ | $y_4=\frac{1}{11}(33-12.066+0.9077)$ $=1.9856$ | $z_4=\frac{1}{12}(35-18.0996-5.9568)$ $=0.9120$ |
| 5 | $x_5=\frac{1}{8}(20+5.9568-1.824)$ $=3.0166$ | $y_5=\frac{1}{11}(33-12.0664+0.9120)$ $=1.9859$ | $z_5=\frac{1}{12}(35-18.0996-5.9577)$ $=0.9120$ |

$x_4 \sim x_5$ ; $y_4 \sim y_5$ ; $z_4 \sim z_5$,

Therefore the solution **$x=3.0165; y=1.9856; z=0.9120$**

----------------------------------------------------------------------------------------------------

## Inverse of a Matrix

### Gauss Jordan Method

Let $A=\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ be the given matrix

Step 1:

Write the augmented matrix $[A/I]=\left[\begin{array}{ccc|ccc} a_1 & b_1 & c_1 & 1 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 1 & 0 \\ a_3 & b_3 & c_3 & 0 & 0 & 1 \end{array}\right]$

Step 2:

Use either row or column operations make the augmented matrix [A/I] as [I /$A^{-1}$].
Here $A^{-1}$ is the required inverse of the given matrix.

## Problems based on Inverse of a Matrix

1. Point the inverse of a matrix $\begin{bmatrix} 2 & 1 & 1 \\ 3 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix}$ by using Gauss-Jordan Method.

Solution:

$$[A/I] = \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 3 & 2 & 3 & 0 & 1 & 0 \\ 1 & 4 & 9 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 1 & 1 & 1 & 0 & 0 \\ 0 & -1 & -3 & 3 & -2 & 0 \\ 0 & -7 & -17 & 1 & 0 & -2 \end{array}\right] R_2 \to 3R_1 - 2R_2, R_3 \to R_1 - 2R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & -2 & 4 & -2 & 0 \\ 0 & -1 & -3 & 3 & -2 & 0 \\ 0 & 0 & -4 & 20 & -14 & 2 \end{array}\right] R_1 \to R_1 + R_2, R_3 \to 7R_2 - R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 4 & 0 & 0 & -12 & 10 & -2 \\ 0 & -4 & 0 & -48 & 34 & -6 \\ 0 & 0 & -4 & 20 & -14 & 2 \end{array}\right] R_1 \to 2R_1 - R_3, R_2 \to R_2 - 3R_3$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -3 & 2.5 & -0.5 \\ 0 & 1 & 0 & 12 & -8.5 & 1.5 \\ 0 & 0 & 1 & -5 & 3.5 & -0.5 \end{array}\right] R_1 \to \frac{R_1}{4}, R_2 \to \frac{R_2}{-4}, R_3 \to \frac{R_3}{-4}$$

$$\therefore A^{-1} = \begin{bmatrix} -3 & 2.5 & -0.5 \\ 12 & -8.5 & 1.5 \\ -5 & 3.5 & -0.5 \end{bmatrix}$$

---

2. Point the inverse of a matrix $\begin{bmatrix} 8 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix}$ by using Gauss-Jordan Method.

Solution:

$$[A/I]=\left[\begin{array}{ccc|ccc}8 & -4 & 0 & 1 & 0 & 0\\ -4 & 8 & -4 & 0 & 1 & 0\\ 0 & -4 & 8 & 0 & 0 & 1\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|ccc}8 & -4 & 0 & 1 & 0 & 0\\ 0 & 12 & -8 & 1 & 2 & 0\\ 0 & -4 & 8 & 0 & 0 & 1\end{array}\right] R_2 \to R_1 + 2R_2$$

$$\sim\left[\begin{array}{ccc|ccc}24 & 0 & -8 & 4 & 2 & 0\\ 0 & 12 & -8 & 1 & 2 & 0\\ 0 & 0 & 16 & 1 & 2 & 3\end{array}\right] R_1 \to 3R_1 + R_2, R_3 \to R_2 + 3R_3$$

$$\sim\left[\begin{array}{ccc|ccc}48 & 0 & 0 & 9 & 6 & 3\\ 0 & 24 & 0 & 3 & 6 & 3\\ 0 & 0 & 16 & 1 & 2 & 3\end{array}\right] R_1 \to 2R_1 + R_3, R_2 \to 2R_2 + R_3$$

$$\sim\left[\begin{array}{ccc|ccc}1 & 0 & 0 & \frac{3}{16} & \frac{1}{8} & \frac{1}{16}\\ 0 & 1 & 0 & \frac{1}{8} & \frac{1}{4} & \frac{1}{8}\\ 0 & 0 & 1 & \frac{1}{16} & \frac{1}{8} & \frac{3}{16}\end{array}\right] R_1 \to \frac{R_1}{48}, R_2 \to \frac{R_2}{24}, R_3 \to \frac{R_3}{16}$$

$$\therefore A^{-1}=\begin{bmatrix}\frac{3}{16} & \frac{1}{8} & \frac{1}{16}\\ \frac{1}{8} & \frac{1}{4} & \frac{1}{8}\\ \frac{1}{16} & \frac{1}{8} & \frac{3}{16}\end{bmatrix}$$

---

## Eigen values and Eigenvectors

Let A be any square matrix of order n. then for any scalar $\lambda$, we can form a matrix $(A-\lambda I)$ where I is the n$^{th}$ order unit matrix. The determinant of this matrix equated to zero is called the characteristic equation of A. i.e., the characteristic equation of the matrix A is $|A-\lambda_1|=0$. Clearly this a polynomial of degree n in $\lambda$ having n roots for $\lambda$, say $\lambda_1, \lambda_2, ..., \lambda_n$. These values are called eigenvalues of the given matrix A.

For each of these eigenvalues, the system of equations $(A-\lambda I)X=0$ has a non-trivial solution for the vector $X=\begin{pmatrix}x_1\\x_2\\\dots\\x_n\end{pmatrix}$. This solution $X=\begin{pmatrix}x_1\\x_2\\\dots\\x_n\end{pmatrix}$ is called a latent vector or eigen vector corresponding to the eigenvalue $\lambda$.

If A is of order n, then its characteristic equation is of n$^{th}$ degree. If n is large, it is very difficult to find the exact roots of the characteristic equation and hence the eigenvalues are difficult to find. But there are numerical methods available for such cases. We list below two such methods called

- Power method
- Jacobi's method

The second method can be applied only for symmetric matrices.

**Power method**

This method can be applied to find numerically the greatest eigenvalue of a square matrix (also called the dominant eigenvalue). The method is explained below.

Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of A and let $\lambda_1$ be the dominant eigenvalue.

i.e., $|\lambda_1| \ge |\lambda_2| \ge |\lambda_3| ...... \ge |\lambda_n|$

if the corresponding eigenvectors are $x_0, x_1, x_2, .... x_n$, then any arbitrary vector y can be written as $y = a_0x_0 + a_1x_1 + ... + a_nx_n$, since the eigenvectors are linearly independent. Now

$$A^k y = A^k (a_0x_0 + ... + a_nx_n)$$

$$= a_0\lambda_1^k x_0 + a_1\lambda_2^k x_1 + ... + a_n\lambda_n^k x_n \quad [\because A^k X = \lambda^k X]$$

$$= \lambda_1^k \left[ a_0x_0 + a_1\left(\frac{\lambda_2}{\lambda_1}\right)^k x_1 + ...... \right]$$

But $\left|\frac{\lambda_i}{\lambda_j}\right| < 1. (i = 2, ..., n)$. Hence $A^k y = \lambda_1^k a_0 x_0$ and $A^{k+1} y = \lambda_1^{k+1} a_0 x_0$.

Hence, if k is large, $\lambda_1 = \frac{A^{k+1}y}{A^k y}$ where the division is carried out in the corresponding components.

Here y is quite arbitrary. But generally we choose it as the vector having all its components ones.

**Note:**

- If $\lambda_1, \lambda_2, ..., \lambda_n$ are the eigenvalues of A, then the eigenvalue $\lambda_1$ is dominant if $|\lambda_1| > |\lambda_i|$ for $i = 2,3,\ldots,n$.
- The eigenvector $X = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix}$ corresponding to the eigenvalue $\lambda_1$ is called the dominant eigenvector.
- If the eigenvalues of A are -3,1,2, then -3 is dominant.
- If the eigen values of A are -4,1,4 then A has no dominant eigenvalue since $|-4| = |4|$.
- The power method will work satisfactorily only if A has a dominant eigenvalue.
- Eigen vector may be $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $(or)$ $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $(or)$ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ for 3 x 3 matrix.

## Problems based on **Eigen value of a matrix**

**1.** Using power method to find a dominant eigen value of a given matrix

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

Solution:

Let A= $\begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ and $x_0$= $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$Ax_0 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix}$$

$$Here \quad x_1 = \begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix}$$

$$Ax_1 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1/2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2.5 \\ -2 \\ 0.5 \end{bmatrix} = 2.5\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix}$$

$$Here \quad x_2 = \begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix}$$

$$Ax_2 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -0.8 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 2.8 \\ -2.8 \\ 1.2 \end{bmatrix} = 2.8\begin{bmatrix} 1 \\ -1 \\ 0.428 \end{bmatrix}$$

$$Here \quad x_3 = \begin{bmatrix} 1 \\ -1 \\ 0.428 \end{bmatrix}$$

$$Ax_3 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0.428 \end{bmatrix} = \begin{bmatrix} 3 \\ -3.428 \\ 1.856 \end{bmatrix} = -3.428\begin{bmatrix} -0.875 \\ 1 \\ -0.541 \end{bmatrix}$$

$$Here \quad x_4 = \begin{bmatrix} -0.875 \\ 1 \\ -0.541 \end{bmatrix}$$

$$Ax_4 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} -0.875 \\ 1 \\ -0.541 \end{bmatrix} = \begin{bmatrix} -2.75 \\ 3.416 \\ -2.082 \end{bmatrix} = 3.416\begin{bmatrix} -0.805 \\ 1 \\ -0.609 \end{bmatrix}$$

$$Here \quad x_5 = \begin{bmatrix} -0.805 \\ 1 \\ -0.609 \end{bmatrix}$$

$$Ax_5 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} -0.805 \\ 1 \\ -0.609 \end{bmatrix} = \begin{bmatrix} -2.61 \\ 3.414 \\ -2.3 \end{bmatrix} = 3.414\begin{bmatrix} -0.764 \\ 1 \\ -0.65 \end{bmatrix}$$

$$Here \quad x_6 = \begin{bmatrix} -0.764 \\ 1 \\ -0.65 \end{bmatrix}$$

$$Ax_6 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} -0.764 \\ 1 \\ -0.65 \end{bmatrix} = \begin{bmatrix} -2.528 \\ 3.414 \\ -2.3 \end{bmatrix} = 3.414\begin{bmatrix} -0.74 \\ 1 \\ -0.674 \end{bmatrix}$$

$$Here \quad x_7 = \begin{bmatrix} -0.74 \\ 1 \\ -0.674 \end{bmatrix}$$

$$Ax_7 = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}\begin{bmatrix} -0.74 \\ 1 \\ -0.674 \end{bmatrix} = \begin{bmatrix} -2.48 \\ 3.414 \\ -2.348 \end{bmatrix} = 3.414\begin{bmatrix} -0.726 \\ 1 \\ -0.68 \end{bmatrix}$$

Here $x_7 = x_8$ approximately.

$\therefore \lambda = 3.414$

Eigen value=3.414

Eigen vector= $\begin{bmatrix} -0.72 \\ 1 \\ -0.68 \end{bmatrix}$

---

## Steps for finding the smallest eigen value

- First obtained the largest eigen value $\lambda_1$ of the given matrix.
- Let $B=A-\lambda_1 I$. Let $\lambda$ be the largest eigen value of the matrix B then **the numerically smallest eigen value of A is $\lambda+\lambda_1$.**

2. Find the largest eigen value and the corresponding eigen vector of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$ and hence find the remaining eigen values.

Solution:

$$\textbf{Let } A=\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}$$

**Step 1**

To find the largest Eigen value of A

$$\text{Let } x_0=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$A\,x_0=\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}=\begin{pmatrix} 3 \\ 2 \\ 7 \end{pmatrix}=7\begin{pmatrix} 0.43 \\ 0.29 \\ 1 \end{pmatrix}$$

$$x_1=\begin{pmatrix} 0.43 \\ 0.29 \\ 1 \end{pmatrix}$$

$$A\,x_1=\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}\begin{pmatrix} 0.43 \\ 0.29 \\ 1 \end{pmatrix}=\begin{pmatrix} 4.01 \\ 0.84 \\ 7 \end{pmatrix}=7\begin{pmatrix} 0.57 \\ 0.12 \\ 1 \end{pmatrix}$$

$$x_2=\begin{pmatrix} 0.57 \\ 0.12 \\ 1 \end{pmatrix}$$

$$A\,x_2=\begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}\begin{pmatrix} 0.57 \\ 0.12 \\ 1 \end{pmatrix}=7\begin{pmatrix} 0.54 \\ 0.22 \\ 1 \end{pmatrix}$$

$$x_3=\begin{pmatrix} 0.54 \\ 0.22 \\ 1 \end{pmatrix}$$

$$A\,x_3 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}\begin{pmatrix} 0.54 \\ 0.22 \\ 1 \end{pmatrix} = 7\begin{pmatrix} 0.56 \\ 0.16 \\ 1 \end{pmatrix}$$

$$x_4 = \begin{pmatrix} 0.56 \\ 0.16 \\ 1 \end{pmatrix}$$

$$A\,x_4 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}\begin{pmatrix} 0.56 \\ 0.16 \\ 1 \end{pmatrix} = 7\begin{pmatrix} 0.556 \\ 0.19 \\ 1 \end{pmatrix}$$

$$x_5 = \begin{pmatrix} 0.556 \\ 0.19 \\ 1 \end{pmatrix}$$

$$A\,x_5 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & 2 \\ 0 & 0 & 7 \end{bmatrix}\begin{pmatrix} 0.556 \\ 0.19 \\ 1 \end{pmatrix} = 7\begin{pmatrix} 0.56 \\ 0.18 \\ 1 \end{pmatrix}$$

$$x_6 = \begin{pmatrix} 0.56 \\ 0.18 \\ 1 \end{pmatrix}$$

$x_5 = x_6$ approximately

$\therefore$ Eigen value $=\lambda=7$

Eigen vector $= \begin{pmatrix} 0.56 \\ 0.18 \\ 1 \end{pmatrix}$

**Step 2**

To find the largest Eigen value of B

$$B = A - \lambda I = \begin{bmatrix} -6 & 2 & 3 \\ 0 & -11 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Let $y_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$B\,y_0 = \begin{bmatrix} -6 & 2 & 3 \\ 0 & -11 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

Here the we get eigen vector is zero . But the eigen vector should be non- zero. So

we consider th value of $y_0$ be $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$B\,y_0 = \begin{bmatrix} -6 & 2 & 3 \\ 0 & -11 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ 0 \end{pmatrix} = -6\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$y_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Here $y_1 = y_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

The numerically largest eigen value of B = **-6** = $\lambda$

Numerically smallest eigen value of A = $\boldsymbol{\lambda + \lambda_1}$ **=** -6 + 7 =1= $\boldsymbol{\lambda_2}$

$\lambda_1 + \lambda_2 + \lambda_3$ = sum of the main diagonals of A = 1-4+7 =4

ie) 7 +1+ $\lambda_3$ = 4

$\Rightarrow \lambda_3 = -4$

The eigen values are 1, -4, 7.

-----------------------------------------------------------------------------------------

## UNIT- I

### PART-A

1. Under the conditions that $f(a)$ and $f(b)$ have opposite signs and $a<b$, find the approximation to the root of $f(x)=0$ by the method of false position.

Ans:- $x = \dfrac{af(b) - bf(a)}{f(b) - f(a)}$

---

2. Find the approx. value of the root of $f(x) = x^3 - 3x - 5 = 0$ by the method of false position, the root lying between 2 and 3.

Ans:- $a = 2 \quad f(2) = -3 \quad b = 3 \quad f(3) = 13$

$$x = \frac{2(13) - 3(-3)}{13+3} = 2.1875$$

---

3. State the criterion for convergence in Newton-Raphson method?

Ans:- Newton Raphson method converges if $|f(x) f''(x)| < [f'(x)]^2$ in the interval considered

---

4. Show that Newton Raphson Method has quadratic convergence. (or) Define that Newton Raphson method's order of convergence. What is the order of convergence of Newton's method?

Ans:-

Let $\varphi(x_n) = x_{n+1}$ be an iteration method for solving the equation $x = \varphi(x)$. If $\alpha$ is a root of the equation, then $x_n = \alpha + \epsilon_n$

$$\therefore x_{n+1} = \varphi(x_n) = \varphi(\alpha) + \epsilon_n \frac{\varphi'(\alpha)}{1!} + \epsilon_n^2 \frac{\varphi''(\alpha)}{2!} + \cdots$$

The power of $\epsilon_n$ in the first non vanishing term after $\varphi(\alpha)$ is called the order of convergence.

Order of convergence is two in Newton's method.

---

5. If an approx. value of the root of equation $x^x = 1000$ is 4.5, find a better approx. of root by Newton's method.

Ans: Taking log on bothsides of $x^x = 1000$.

$$f(x) = x\log_e^x - \log_e 1000 \Rightarrow f'(x) = 1 + \log_e^x.$$

Approx. is $x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 4 + \frac{0.0605}{1.6532} = 4.5366$

---

6. Write the Newton's formula to find the cube root of N.

Ans: $x = \sqrt[3]{N} \Rightarrow x^3 - N = 0 \Rightarrow f(x) = x^3 - N.\ f'(x) = 3x^2$

By Newton's Method $x_{n+1} = x_n - \frac{x_n^3 - N}{3x_n^2} = \frac{2x_n^3 + N}{3x_n^2}$.

---

7. Establish an iteration formula to find the reciprocal of a positive number N by Newton's Method.

Ans: $N = \frac{1}{x} \Rightarrow f(x) = -\frac{1}{x} + N$ and $f'(x) = \frac{1}{x^2}$

$$x_{n+1} = x_n - \frac{\left[N - \frac{1}{x_n}\right]}{+\frac{1}{x_n^2}} \Rightarrow x_{n+1} = x_n(2 - Nx_n)$$

---

8. What is the order of convergence of fixed point iteration $x = g(x)$ method?

Ans: The order of convergence is one

9. What is the sufficient condition for the convergence of $x = g(x)$ method.

Ans:- The sufficient condition for the convergence is $|\varphi'(x)| < 1$ for all $x$ in the interval I containing the root $x = \alpha$ of the equation $f(x) = 0$, which can be written as $x = \varphi(x)$.

10. How do you express the equation $x^3 + x^2 - 1 = 0$ for the positive root by iteration method.

Ans:- $x^2(x+1) = 1 \Rightarrow x = \frac{1}{\sqrt{x+1}} = \varphi(x)$.

11. Can we find a real root of the equation $x^3 + x^2 - 1 = 0$ in the interval $[0,1]$ by iteration method.

Ans: we can express $x^3 + x^2 - 1 = 0$ in the form $x = \frac{1}{\sqrt{x+1}} = \varphi(x) \Rightarrow \varphi'(x) = -\frac{1}{2(x+1)^{3/2}}$ and $|\varphi'(x)| < 1$ for all $x \in [0,1]$.

∴ A real root with suitable initial approximation $x_0$ can be found by iteration method.

12. Distinguish Gauss Elimination method and Gauss Jordan method.

Ans:-

| Gauss Elimination | Gauss Jordan |
| --- | --- |
| 1. Coefficient matrix A of the system reduces to upper triangular matrix | Coeff. matrix A of the system reduces into diagonal of unit matrix |
| 2. Back substitution process gives solution | Solution obtained directly |

13. State the principle involved in Gauss elimination method of solving a system of equations.

Ans: Augmented matrix (A, B) reduces into (U, k) and Solution is obtained from the equivalent upper triangular system of equation by back substitution.

14. Explain Gauss Jordan method to solve the System $Ax = B$.

Ans: Principle : Reduce augmented matrix (A, B) into [I, k] and obtain the solution directly, without back substitution.

15. By Gauss elimination, Solve $x+y=2$, $2x+3y=5$.

Ans:
$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 2 & 3 & 5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] \Rightarrow \begin{array}{l} x+y=2 \\ y=1 \end{array} \Rightarrow x=1$$

$\therefore$ solution $x=1$, $y=1$

16. Solve $3x+2y=4$, $2x-3y=7$ by Gauss Jordan method.

Ans:
$$\left[\begin{array}{cc|c} 3 & 2 & 4 \\ 2 & -3 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & -1 \end{array}\right] \Rightarrow x=2, y=-1$$

17. When will the Solution of $Ax=B$ by Gauss seidal converge quickly

Ans: The coefficient matrix of A is diagonally dominant

18. State the Condition for convergence of Gauss Seidal

Ans: If the matrix A is diagonally dominant

$\sum_{j=1}^{n} |a_{ij}| < |a_{ii}|$ for all i, the gauss-seidal method Converge, where $A = (a_{ij})$ the coeff. matrix of the given matrix $Ax = B$.

19. Check whether Gauss-Seidal method can be used to solve $2x-3y+20z=35$, $20x+y-2z=17$, $3x+20y-z=-18$ is less number of iteration? If possible solve them.

Ans:-

Coeff. matrix $\begin{pmatrix} 2 & -3 & 20 \\ 20 & 1 & -2 \\ 3 & 20 & -1 \end{pmatrix}$ $\begin{matrix} 2 \not> 3+20 \\ 1 \not> 20+2 \\ 1 \not> 3+20 \end{matrix}$

∴ The given system of equations is not diagonally dominant.

$\begin{pmatrix} 20 & 1 & -2 \\ 3 & 20 & -1 \\ 2 & -3 & 20 \end{pmatrix}$ $\begin{matrix} 20 > 1+2 \\ 20 > 3+1 \\ 20 > 2+3 \end{matrix}$

∴ Now the given system is diagonally dominant.

So the system becomes $20x+y-2z=17$, $3x+20y-z=-18$, $2x-3y+20z=35$.

---

20. Distinguish between direct and indirect method of solving a system of equation $Ax=B$.

Ans: Direct Method: Involve a certain amount of fixed computation

Indirect Method: The solution is obtained by successive approx. and the amount of computation depends on the degree of required accuracy.

---

21. State the basic principle involved for finding $A^{-1}$ by Gauss-Jordan.

Ans: Reduce the augmented matrix $(A/I)$ into $(I/x)$ then $x = A^{-1}$

---

22. How will you find the smallest eigen value of Square matrix A.

**Ans:** By power method, the largest eigen value of $A^{-1}$ can be found. Then smallest eigen values of A is the reciprocal of the largest eigen values of $A^{-1}$.

---

23. Find the Inverse of $\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ by Gauss-Jordan Method.

**Ans:**

$$(A/I) = \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 2 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix}$$

---

## PART-B

1. Find an approx. root of $x\log_{10}x - 1.2 = 0$ by false position method
2. $A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$ find the largest eigen value of in magnitude and its corresponding eigen vector
3. Find the root $4x - e^x = 0$ that lies between 2 & 3 by Newton's Method.
4. Apply Gauss-Seidal method to solve the following System of equation $20x + y - 2z = 17$, $3x + 20y - z = -18$, $2x - 3y + 20z = 25$
5. Find the Smallest positive root of the equation $xe^{-2x} = \frac{1}{2}\sin x$ correct to 3 decimal places using Newton's Raphson method.

6. Find all eigen value of the matrix $\begin{pmatrix} 2 & 1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$ by Jacobi method.

7. Gauss-Seidal Method, $2x+y+6z=9$, $8x+3y+2z=13$, $x+5y+z=7$

8. Find a root of ~~to~~ the equation ~~by~~ $\cos x = 3x-1$ by iteration method.

9. Gauss Jordan method find the inverse of

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{pmatrix}$$

10. Solve $10x+y+z=12$, $2x+10y+z=13$, $x+y+5z=7$ by Gauss Jordan method

11. Positive root of $xe^{x}=1$ correct to 4 decimal places using Regula-falsi method.

12. Find the numerically largest eigen value of $A = \begin{bmatrix} 1 & -3 & 2 \\ 4 & 4 & -1 \\ 6 & 3 & 5 \end{bmatrix}$ by power method.

— x —

# UNIT II
# INTERPOLATION AND APPROXIMATION

***Lagrangian Polynomials***

***Divided differences***

***Interpolating with a cubic spline***

***Newton's forward difference formula***

***Newton's backward difference formulA***

# LAGRANGIAN POLYNOMIALS

## Formula

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}y_1 +$$
$$\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)}y_3 +$$
$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)}y_4 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}y_5$$

## Problems based on Lagrange's Method

1. Using Lagrange's formula to calculate f(3) from the following table (A.U. N/D. 2007)

| x | 0 | 1 | 2 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| f(x) | 1 | 14 | 15 | 5 | 6 | 19 |

Solution:

| x | $x_0=0$ | $x_1=1$ | $x_2=2$ | $x_3=4$ | $x_4=5$ | $x_5=6$ |
|---|---|---|---|---|---|---|
| f(x) | $y_0=1$ | $y_1=14$ | $y_2=15$ | $y_3=5$ | $y_4=6$ | $y_5=19$ |

We know that Lagrange's formula is

$$y(x) = \frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}y_1 +$$
$$\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)}y_3 +$$
$$\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_5)}{(x_4-x_0)(x_4-x_1)(x_4-x_2)(x_4-x_3)(x_4-x_5)}y_4 + \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)}{(x_5-x_0)(x_5-x_1)(x_5-x_2)(x_5-x_3)(x_5-x_4)}y_5$$

$$y(x) = \frac{(x-1)(x-2)(x-4)(x-5)(x-6)}{(0-1)(0-2)(0-4)(0-5)(0-6)}(1) + \frac{(x-0)(x-2)(x-4)(x-5)(x-6)}{(1-0)(1-2)(1-4)(1-5)(1-6)}(14) +$$
$$\frac{(x-0)(x-1)(x-4)(x-5)(x-6)}{(2-0)(2-1)(2-4)(2-5)(2-6)}(15) + \frac{(x-0)(x-1)(x-2)(x-5)(x-6)}{(4-0)(4-1)(4-2)(4-5)(4-6)}(5) +$$
$$\frac{(x-0)(x-1)(x-2)(x-4)(x-6)}{(6-0)(6-1)(6-2)(6-4)(6-6)}(6) + \frac{(x-0)(x-1)(x-2)(x-4)(x-5)}{(6-0)(6-1)(6-2)(6-4)(6-5)}(19)$$

$$y(x)=\frac{(x-1)(x-2)(x-4)(x-5)(x-6)}{(-1)(-2)(-4)(-5)(-6)}+\frac{x(x-2)(x-4)(x-5)(x-6)}{(1)(-1)(-3)(-4)(-5)}(14)+$$
$$\frac{x(x-1)(x-4)(x-5)(x-6)}{(2)(1)(-2)(-3)(-4)}(15)+\frac{x(x-1)(x-2)(x-5)(x-6)}{(4)(3)(2)(-1)(-2)}(5)+$$
$$\frac{x(x-1)(x-2)(x-4)(x-6)}{(5)(4)(3)(1)(-1)}(6)+\frac{x(x-1)(x-2)(x-4)(x-5)}{(6)(5)(4)(2)(1)}(19)$$

put x=3 we get

$$y(3)=-\frac{(3-1)(3-2)(3-4)(3-5)(3-6)}{240}+\frac{3(3-2)(3-4)(3-5)(3-6)}{60}(14)-\frac{3(3-1)(3-4)(3-5)(3-6)}{48}(15)$$
$$+\frac{3(3-1)(3-2)(3-5)(3-6)}{48}(5)-\frac{3(3-1)(3-2)(3-4)(3-6)}{60}(6)+\frac{3(3-1)(3-2)(3-4)(3-5)(3-6)}{240}(19)$$

$$=-\frac{2(1)(-1)(-2)(-3)}{240}+\frac{3(1)(-1)(-2)(-3)}{60}(14)-\frac{2(1)(-1)(-2)(-3)}{48}(15)$$
$$+\frac{3(2)(1)(-2)(-3)}{48}(5)-\frac{3(2)(1)(-1)(-3)}{60}(6)+\frac{3(2)(1)(-1)(-2)}{240}(19)$$
$$=\frac{12}{240}-\frac{252}{60}+\frac{540}{48}+\frac{180}{48}-\frac{108}{60}+\frac{228}{240}$$
$$=\frac{12-1008+2700+900-432+228}{240}$$
$$=\frac{2400}{240}=10$$

Answer: **y(3) = 10**.

----------------------------------------------------------------------------------------------------

2. Using Lagrange's formula fit a polynomial to the data (A.U. N/D. 2006)

| *x* | *0* | *1* | *3* | *4* |
|---|---|---|---|---|
| **f(x)** | -12 | 0 | 6 | 12 |

Solution:

| x | $x_0=0$ | $x_1=1$ | $x_2=3$ | $x_3=4$ |
|---|---|---|---|---|
| f(x) | $y_0=-12$ | $y_1=0$ | $y_2=6$ | $y_3=12$ |

We know that Lagrange's formula is

$$y(x)=\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}y_1+$$
$$\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)}y_3$$

$$y(x)=\frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)}(-12)+\frac{(x-0)(x-1)(x-3)}{(1-0)(1-3)(1-4)}(0)$$

$$+\frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)}(6)+\frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)}(12)$$

$$y(x)=\frac{(x-1)(x-3)(x-4)}{(-1)(-3)(-4)}(-12)+\frac{(x)(x-1)(x-4)}{(3)(2)(-1)}(6)+\frac{(x)(x-1)(x-3)}{(4)(3)(1)}(12)$$

$$y(x)=(x-1)(x-3)(x-4)-x(x-1)(x-4)+(x)(x-1)(x-3)$$

$$=(x-1)[x^2-3x-4x+12-x^2+4x+x^2-3x]$$

$$=(x-1)[x^2-6x+12]$$

$$=x^3-6x^2+12x-x^2+6x-12$$

$$y(x)=x^3-7x^2+18x-12 \text{ ---------------------- } (1)$$

Substituting x=2 in (1), we get

$y(2) = 2^3-7(2^2)+18(2)-12 = 8 - 28 + 36 - 12 = 44 - 40 = 4$

Answer: y(2) = 4.

---

3. Using Lagrange's interpolation find the polynomial through (0, 0), (1, 1) and (2, 2) (A.U. M/J. 2007)

| ***x*** | ***0*** | ***1*** | ***2*** |
|---|---|---|---|
| **f(x)** | 0 | 1 | 2 |

***Solution:***

| x | $x_0=0$ | $x_1=1$ | $x_2=2$ |
|---|---|---|---|
| f(x) | $y_0=0$ | $y_1=1$ | $y_2=2$ |

We know that Lagrange's formula is

$$y(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}y_0+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}y_1+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}y_2$$

$$y(x)=\frac{(x-1)(x-2)}{(0-1)(0-2)}0+\frac{(x-0)(x-2)}{(1-0)(1-2)}1+\frac{(x-0)(x-1)}{(2-0)(2-1)}2$$

$$y(x)=\frac{(x)(x-2)}{(1)(-1)}1+\frac{(x)(x-1)}{(2)(1)}2$$

$$=-x(x-2)+x(x-1)$$

$$=-x^2+2x+x^2-x$$

$$y(x)=x$$

$\therefore$ The required polynomial is y = x

---

4. The following table gives certain corresponding values of x and $\log_{10}x$. Compute the value of $\log_{10}323.5$, by using Lagrange's formula

| x | 321.0 | 322.8 | 324.2 | 325.0 |
|---|---|---|---|---|
| f(x) | 2.50651 | 2.50893 | 2.51081 | 2.51188 |

Solution:

| x | $x_0$=321.0 | $x_1$=322.8 | $x_2$=324.2 | $x_3$=325.0 |
|---|---|---|---|---|
| f(x) | $y_0$= 2.50651 | $y_1$= 2.50893 | $y_2$= 2.51081 | $y_3$= 2.51188 |

$$y(x)=\frac{(x-x_1)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)(x_0-x_4)(x_0-x_5)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)(x-x_4)(x-x_5)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)(x_1-x_4)(x_1-x_5)}y_1+$$
$$\frac{(x-x_0)(x-x_1)(x-x_3)(x-x_4)(x-x_5)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)(x_2-x_4)(x_2-x_5)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)(x-x_4)(x-x_5)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)(x_3-x_4)(x_3-x_5)}y_3$$

Substituting these values in Lagrange's interpolation formula, we get,

$$f(323.5)=\frac{(323.5-322.8)(323.5-324.2)(323.5-325)}{(321-322.8)(321-324.2)(321-325)}(2.5061)+$$
$$\frac{(323.5-321)(323.5-324.2)(323.5-325)}{(322.8-321)(322.8-324.2)(322.8-325)}(2.50893)$$
$$+\frac{(323.5-321)(323.5-322.8)(323.5-325)}{(324.2-321)(324.2-322.8)(324.2-325)}(2.51081)+$$
$$\frac{(323.5-321)(323.5-322.8)(323.5-324.2)}{(325-321)(325-322.8)(325-324.2)}(2.51188)$$
$$=-0.07996+1.18794+1.83897-0.43708$$
$$f(323.5)=2.50987$$

---

## Inverse Interpolation

The process of finding a value of x for the corresponding value of y is called inverse interpolation.

$$x=\frac{(y-y_1)(y-y_2)\ldots(y-y_n)}{(y_0-y_1)(y_0-y_2)\ldots(y_0-y_n)}x_0$$
$$+\frac{(y-y_0)(y-y_2)\ldots(y-y_n)}{(y_1-y_0)(y_1-y_2)\ldots(y_1-y_n)}x_1$$
$$+\ldots+\frac{(y-y_0)(y-y_1)\ldots(y-y_{n-1})}{(y_n-0)(y_n-y_1)\ldots(y_n-y_{n-1})}x_n$$

## Problems based on Inverse Interpolation

1. Find the value of x when y=85, using Lagrange's formula from the following table.

| x | 2 | 5 | 8 | 14 |
|---|---|---|---|---|
| y | 94.8 | 87.9 | 81.3 | 68.7 |

**Solution:**

| x | $x_0=2$ | $x_1=5$ | $x_2=8$ | $x_3=14$ |
|---|---|---|---|---|
| y | $y_0=94.8$ | $y_1=87.9$ | $y_2=81.3$ | $y_3=68.7$ |

$$x=\frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}x_0+\frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}x_1+$$
$$\frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(x_2-y_1)(y_2-y_3)}x_2+\frac{(y-y_0)(y-x_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}x_3$$

Substituting the above values, we get,

$$x=\frac{(85-87.9)(85-81.3)(85-68.7)}{(94.8-87.9)(94.8-81.3)(94.8-68.7)}(2)+\frac{(85-94.8)(85-81.3)(85-68.7)}{(87.9-94.8)(87.9-81.3)(87.9-68.7)}(5)+$$
$$\frac{(85-94.8)(85-87.9)(85-68.7)}{(81.3-94.8)(81.3-87.9)(81.3-68.7)}(8)+\frac{(85-94.8)(85-87.9)(85-81.3)}{(68.7-94.8)(68.7-87.9)(68.7-81.3)}(14)$$

x = 0.1438778+3.3798011+3.3010599-0.2331532 = 6.3038

Therefore the value of x when y = 6.3038

-----------------------------------------------------------------------------------------------------

2. The following table gives the value of the elliptic integral $y(\theta)=\int_0^{\theta}\frac{d\theta}{\sqrt{1-\frac{1}{2}\sin^2\theta}}$ for certain values of θ. Find $\theta$ if y($\theta$) = 0.3887

| $\theta$ | 21° | 23° | 25° |
|---|---|---|---|
| y($\theta$) | 0.3706 | 0.4068 | 0.4433 |

Solution:

$$\theta = \frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}\theta_0 + \frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}\theta_1 +$$

$$\frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(x_2-y_1)(y_2-y_3)}\theta_2 + \frac{(y-y_0)(y-x_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}\theta_3$$

| $\theta$ | $\theta_1$=21° | $\theta_2$=23° | $\theta_3$=25° |
|---|---|---|---|
| y($\theta$) | $y_1$=0.3706 | $y_2$=0.4068 | $y_3$=0.4433 |

we have

$$\theta = \frac{(0.3887-0.4068)(0.3887-0.4433)}{(0.3706-0.4068)(0.3706-0.4433)}(21) + \frac{(0.3887-0.3706)(0.3887-0.4433)}{(0.4068-0.3706)(0.4068-0.4433)}(23) +$$

$$\frac{(0.3887-0.3706)(0.3887-0.4068)}{(0.3706-0.3706)(0.3706-0.4068)}(25)$$

$$\theta = 21.999^{o}$$

Therefore the value of θ such that y($\theta$) = 0.3887 is θ=21.999°

## DIVIDED DIFFERENCES

## Problems based on Newton's Divided Difference Formula

**2.2.1** Let the function $y = f(x)$ take the values $f(x_0), f(x_1), f(x_2), \ldots f(x_n)$ corresponding to the values $x_0, x_1, x_2, \ldots x_n$ for the argument x where $x_1 - x_0, x_2 - x_1, \ldots, x_n - x_{n-1}$ need not be necessarily equal.

The first divided of $f(x)$ for the argument $x_0, x_1$ is defined as

$$\therefore f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \qquad \ldots\text{(i)}$$

Similarly $f(x_1, x_2) = \dfrac{f(x_2) - f(x_1)}{x_2 - x_1}$

$f(x_2, x_3) = \dfrac{f(x_3) - f(x_2)}{x_3 - x_2}$ and so on.

Thus, for defining a first divided difference, we need the functional values corresponding to two arguments.

The second divided difference of f(x) for three arguments $x_0$, $x_1$, $x_2$ is defined as

$$\frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0} \qquad \ldots\text{(ii)}$$

Similarly $f(x_1, x_2, x_3) = \dfrac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$

The third divided difference of f(x) for the four arguments $x_0$, $x_1$, $x_2$, $x_3$ is defined as

$$f(x_0, x_1, x_2, x_3) = \frac{f(x_1, x_2, x_3) - f(x_0, x_1, x_2)}{x_3 - x_0} \qquad \ldots\text{(iii)}$$

The quantities in (i), (ii) and (iii) are called divided differences of orders 1, 2, 3 respectively.

1. Using Newton's Divided Difference formula, find the value of f(8) and f(5) given the following data.

| *x* | *4* | *5* | *7* | *10* | *11* | *13* |
|---|---|---|---|---|---|---|
| **f(x)** | 48 | 100 | 294 | 900 | 1210 | 2028 |

**Solution:**

| x | $x_0$=4 | $x_1$=5 | $x_2$=7 | $x_3$=10 | $x_4$=11 | $x_5$=13 |
|---|---|---|---|---|---|---|
| f(x) | $f(x_0)$=48 | $f(x_1)$=100 | $f(x_2)$=294 | $f(x_3)$=900 | $f(x_4)$=1210 | $f(x_5)$=2028 |

The divided difference table for the given data is given below.
Newton's divided difference formula,

$$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0,x_1,x_2,x_3) + \ldots\ldots\ldots\ldots\ldots$$

| x | f(x) | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ | $\Delta^4 f(x)$ |
|---|---|---|---|---|---|
| 4 | 48 | | | | |
| | | $\frac{100-48}{5-4}=52$ | | | |
| 5 | 100 | | $\frac{97-52}{7-4}=15$ | | |
| | | $\frac{294-100}{7-5}=97$ | | $\frac{100-48}{5-4}=52$ | |
| 7 | 294 | | $\frac{292-97}{10-5}=21$ | | 0 |
| | | $\frac{900-294}{10-7}=202$ | | $\frac{100-48}{5-4}=52$ | |
| 10 | 900 | | $\frac{310-202}{11-7}=27$ | | 0 |
| | | $\frac{1210-900}{11-10}=310$ | | $\frac{100-48}{5-4}=52$ | |
| 11 | 1210 | | $\frac{409-310}{13-10}=33$ | | |
| | | $\frac{2028-1210}{13-11}=409$ | | | |
| 13 | 2028 | | | | |

Using divided differences and the given data in (1),
f(x) = 48 + 52(x-4) + 15(x-4)(x-5) +(x-4)(x-5)(x-7)
when x=8,
f(8) = 48 + 208 +180 +12 =448
Therefore f(8) =448
when x=15,
f(15) = 48 + 572 +1650 +880 =3150
Therefore f(15) =3150

---

2. Use Newton's divided difference formula, to fit a polynomial to the data

and find y when x=1

| x | -1 | 0 | 2 | 3 |
|---|---|---|---|---|
| y | 8 | 3 | 1 | 12 |

Solution:

| x | $x_0$=-1 | $x_1$=0 | $x_2$=2 | $x_3$=3 |
|---|---|---|---|---|
| y | $y_0$=8 | $y_1$=3 | $y_2$=1 | $y_3$=12 |

The divided difference table for the given data as follows.

| **x** | **y** | **Δy** | **$\Delta^2$y** | **$\Delta^3$f(x)** |
|---|---|---|---|---|
| -1 | -8 | | | |
| | | $\frac{3+8}{0+1}=11$ | | |
| 0 | 3 | | $\frac{-1-11}{2+1}=-4$ | |
| | | $\frac{1-3}{2-0}=-1$ | | $\frac{4+4}{3+1}=2$ |
| 2 | 1 | | $\frac{11+1}{2+1}=4$ | |
| | | $\frac{12-1}{3-2}=11$ | | |
| 3 | 12 | | | |

By Newton's divided difference formula,

$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2)\ f(x_0,x_1,x_2,x_3)$
$+ \ldots\ldots\ldots\ldots\ldots..$

Using these we get,

$f(x) = -8+(x+1)11+(x+1)x(-4)+(x+1)x(x-2)2$
$= -8+11x+11-4x^2-4x+2x^3-2x^2-4x$
$= 2x^3-6x^2+3x+3$
$y = 2x^3-6x^2+3x+3$
$y(1) = 2-6+3+3 =2$

Answer: y(1)=2

----------------------------------------------------------------------------------------------------

## Interpolating with a Cubic Spline

### Definition: Cubic Spline

A cubic polynomial approximating the curve in every subinterval is called is called satisfying the following properties.

1. $F(x_i)=f_i$ for i = 0, 1, 2, ………, n.
2. On each interval $[x_{i-1}, x_i]$, $1 \le i \le n$, F(x) is a third degree polynomial.
3. F(x), F'(x) and F''(x) are continuous on the interval $[x_0, x_n]$.

The second derivatives at the end points of the given range are denoted as $M_0$ and $M_n$ respectively.

### Natural cubic spline

A cubic spline F(x) with end conditions $M_0=0$ and $M_n=0$ where $f''(x_i)=M_i$ in the interval $[x_0, x_n]$
i.e., $f''(x_0) = 0$ and $f''(x_n) = 0$ is called a natural cubic spline.

### Fitting a natural cubic spline for the given data:

In case of natural cubic spline the derivatives $f''(x_0) = 0$ and $f''(x_n) = 0$. i.e., $M_0=0$ and $M_n=0$
Suppose that the values of x are equally spaced with a spacing h.
The cubic spline approximation in the subinterval $(x_{i-1}, x_i)$ is given by

$$F_i(x) = \frac{1}{h}\left[\frac{(x_i-x)^3}{6}M_{i-1} + \frac{(x-x_{i-1})^3}{6}M_i\right] + \frac{x_i-x}{h}\left(y_{i-1} - \frac{h^2}{6}M_{i-1}\right) + \frac{x-x_{i-1}}{h}\left(y_i - \frac{h^2}{6}M_i\right)$$

$M_{i-1}+4M_i+M_{i+1} = \frac{6}{h^2}\,[y_{i-1}+y_{i+1}-2y_i]$ where i = 1, 2, ….., n- 1 with $M_0=0$ and $M_n=0$.

### Problems based on Cubic spline

1. Obtain the cubic spline approximation for the function y = f(x) from the following data, given that $y_0''=y_3''=0$.

| x | -1 | 0 | 1 | 2 |
|---|---|---|---|---|
| y | -1 | 1 | 3 | 35 |

Solution:

| x | $x_0=-1$ | $x_1=0$ | $x_2=1$ | $x_3=2$ |
|---|---|---|---|---|
| y | $y_0=-1$ | $y_1=1$ | $y_2=3$ | $y_3=35$ |

The values of x are equally spaced with h =1.
Therefore we have

$$M_{i-1}+4M_i+M_{i+1} = 6\,[y_{i-1}+y_{i+1}-2y_i] \text{ where } i = 1, 2, \ldots, n-1$$

Further $M_0=0$ and $M_3=0$

$\therefore\ M_0+4M_1+M_2 = 6\,(y_0+y_2-2y_1)$

$\Rightarrow 4M_1+M_2 = 0$ ---------------------------------------------------------------------------(1)

and

$M_1+4M_2+M_3 = 6\ (y_1+y_3-2y_2)$

$= 6(1-6+35) = 180$

$\therefore\ M_1+4M_2 = 180$ ----------------------------------------------------------------------(2)

Solving (1) and (2), $M_1=-12\ \ M_2 = 48$

The cubic spline in $(x_{i-1}, x_i)$ is given by

$$y=\left[\frac{(x_i-x)^3}{6}M_{i-1}+\frac{(x-x_{i-1})^3}{6}M_i\right]+\left(\frac{x_i-x}{1}\right)\left(y_{i-1}-\frac{1}{6}M_{i-1}\right)+\left(\frac{x-x_{i-1}}{1}\right)\left(y_i-\frac{1}{6}M_i\right) \text{---------------(3)}$$

where i = 1, 2, ….., n- 1.

In the interval $-1\le x\le 0$, i.e., $x_0\le x\le x_1 (i=1)$ the cubic spline is given by

$$y=\left[\frac{(x_1-x)^3}{6}M_0+\frac{(x-x_0)^3}{6}M_1\right]+\left(\frac{x_1-x}{1}\right)\left(y_0-\frac{1}{6}M_0\right)+\left(\frac{x-x_0}{1}\right)\left(y_1-\frac{1}{6}M_1\right)$$

$$\Rightarrow \quad y=\frac{1}{6}[(x+1)^3(-12)]+(-x)(-1)+(x+1)(1+2)$$

$= (-2)(x^3+3x^2+3x+1)+x+3x+3$

$y = -2x^3-6x^2-2x+1$

In the interval $0\le x\le 1$, i.e., $x_1\le x\le x_2 (i=2)$ the cubic spline is given by

$$y=\left[\frac{(x_2-x)^3}{6}M_1+\frac{(x-x_1)^3}{6}M_2\right]+\left(\frac{x_2-x}{1}\right)\left(y_1-\frac{1}{6}M_1\right)+\left(\frac{x-x_1}{1}\right)\left(y_2-\frac{1}{6}M_2\right)$$

$$\Rightarrow \quad y=\frac{1}{6}[(1-x)^3(-12)+x^3(48)]+(1-x)(1+2)+(x-0)(3-4)$$

$= (-2)(1-x)^3+8x^3+3-3x-x$

$\Rightarrow\quad y = 2x^3-6x^2+6x-2+8x^3+3-4x$

$\therefore\quad y = 10x^3-6x^2+2x+1$

In the interval $1\le x\le 2$, i.e., $x_2\le x\le x_3 (i=3)$ the cubic spline is given by

$$y=\left[\frac{(x_3-x)^3}{6}M_2+\frac{(x-x_2)^3}{6}M_3\right]+\left(\frac{x_3-x}{1}\right)\left(y_2-\frac{1}{6}M_2\right)+\left(\frac{x-x_2}{1}\right)\left(y_3-\frac{1}{6}M_3\right)$$

$$\Rightarrow \quad y=\frac{1}{6}[(2-x)^3(48)]+(2-x)(3-8)+(x-1)(35)$$

$= 8(8-12x+6x^2-x^3)+5x-10+35x-35$

$\Rightarrow\quad y = -8x^3+48x^2-56x+19$

$\therefore\quad y = -8x^3+48x^2-56x+19$

Hence the required cubic spline approximation for the given function is

$$y=\begin{cases}-2x^3-6x^2-2x+1 & \text{for } -1\le x\le 0\\ 10x^3-6x^2+2x+1 & \text{for } 0\le x\le 1\\ -8x^3+48x^2-56x+19 & \text{for } 1\le x\le 2\end{cases}$$

2. Obtain the natural cubic spline which agrees with y(x) at the set of data points given below:

| **x** | **2** | **3** | **4** |
|---|---|---|---|
| **y** | 11 | 49 | 123 |

Hence find y(2.5)

Solution:

| x | 2 | 3 | 4 |
|---|---|---|---|
| y | 11 | 49 | 123 |

The values of x are equally spaced with h =1.
Therefore we have

$$M_{i-1}+4M_i+M_{i+1}=6\,[y_{i-1}+y_{i+1}-2y_i] \text{ where } i = 1, 2, \ldots, n-1$$

Further $M_0=0$ and $M_2=0$

$\therefore\ M_0+4M_1+M_2=6\,(y_0+y_2-2y_1)$

$\Rightarrow 4M_1=6(11-98+123)$

$\therefore\ M_1= 54$

The cubic spline in $(x_{i-1}, x_i)$ is given by

$$y=\left[\frac{(x_i-x)^3}{6}M_{i-1}+\frac{(x-x_{i-1})^3}{6}M_i\right]+\left(\frac{x_i-x}{1}\right)\left(y_{i-1}-\frac{1}{6}M_{i-1}\right)+\left(\frac{x-x_{i-1}}{1}\right)\left(y_i-\frac{1}{6}M_i\right) \text{---------------(3)}$$

where i = 1, 2, ….., n- 1.

In the interval $2\le x\le 3$, i.e., $x_0\le x\le x_1 (i=1)$ the cubic spline is given by

$$y=\left[\frac{(x_1-x)^3}{6}M_0+\frac{(x-x_0)^3}{6}M_1\right]+\left(\frac{x_1-x}{1}\right)\left(y_0-\frac{1}{6}M_0\right)+\left(\frac{x-x_0}{1}\right)\left(y_1-\frac{1}{6}M_1\right)$$

$$\Rightarrow \quad y=\frac{1}{6}[(x-2)^3(54)]+(3-x)(11)+(x-2)(49-9)$$

$$= 9(x^3-6x^2+12x-8)+33-11x+40x-80$$

$$y = 9x^3-54x^2+137x-119$$

In the interval $3\le x\le 4$, i.e., $x_1\le x\le x_2 (i=2)$ the cubic spline is given by

$$y=\left[\frac{(x_2-x)^3}{6}M_1+\frac{(x-x_1)^3}{6}M_2\right]+\left(\frac{x_2-x}{1}\right)\left(y_1-\frac{1}{6}M_1\right)+\left(\frac{x-x_1}{1}\right)\left(y_2-\frac{1}{6}M_2\right)$$

$$\Rightarrow \quad y=\frac{1}{6}[(4-x)^3(54)]+(4-x)(40)+(x-3)(123)$$

$$= 9(64-48x+12x^2-x^3)+160-40x+123x-369$$

$$\Rightarrow \quad y = -9x^3+108x^2+349x+367$$

Hence the required cubic spline approximation for the given function is

$$y=\begin{cases} 9x^3-54x^2+137x-119 & \text{for } 2\le x\le 3 \\ -9x^3+108x^2-349x+367 & \text{for } 3\le x\le 4 \end{cases}$$

---

## Newton's Forward and Backward Difference Formulas

## Introduction:

If a function y=f(x) is not known explicitly the value of y can obtained when a set of values of $(x_i, y_i)$ i = 1, 2, 3, ….n are known by using the methods based on the principles of finite differences, provided the function y=f(x) is continuous. Here *the values of x being equally spaced*, i.e., $x_n = x_0 + nh,\ n = 0, 1, 2, \ldots, n$

## Forward Differences

If $y_0, y_1, y_2, \ldots, y_n$ denote the set of values of y, then the first forward differences of y = f(x) are defined by

$\Delta y_0 = y_1 - y_0$; $\Delta y_1 = y_2 - y_1$; …….; $\Delta y_{n-1} = y_n - y_{n-1}$

where Δ is called the forward difference operator.

## Forward Difference table

| **x** | **y** | Δ | $\Delta^2$ | $\Delta^3$ | $\Delta^4$ | $\Delta^5$ |
|---|---|---|---|---|---|---|
| $x_0$ | $y_0$ | | | | | |
| | | $\Delta y_0$ | | | | |
| $x_1$ | $y_1$ | | $\Delta^2 y_0$ | | | |
| | | $\Delta y_1$ | | $\Delta^3 y_0$ | | |
| $x_2$ | $y_2$ | | $\Delta^2 y_1$ | | $\Delta^4 y_0$ | |
| | | $\Delta y_2$ | | $\Delta^3 y_1$ | | $\Delta^5 \mathbf{y_0}$ |
| $x_3$ | $y_3$ | | $\Delta^2 y_2$ | | $\Delta^4 y_1$ | |
| | | $\Delta y_3$ | | $\Delta^3 y_2$ | | |
| $x_4$ | $y_4$ | | $\Delta^2 y_3$ | | | |
| | | $\Delta y_4$ | | | | |
| $x_5$ | $y_5$ | | | | | |

Formula

$$y(x_0 + nh) = y_0 + n\Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \ldots\ldots$$

## Problems based on Newton's forward interpolation formula

1. Using Newton's Forward interpolation formula, find f(1.5) from the following data

| x | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|
| f(x) | 858.3 | 869.6 | 880.9 | 829.3 | 903.6 |

Solution:

| x | $x_0$=0 | $x_1$=1 | $x_2$=2 | $x_4$=3 | $x_5$=4 |
|---|---|---|---|---|---|
| f(x) | 858.3 | 869.6 | 880.9 | 829.3 | 903.6 |

*Difference Table*

To find y for x = 1.5

| x | y | Δy | $\Delta^2$y | $\Delta^3$y | $\Delta^4$y |
|---|---|---|---|---|---|
| 0 | 858.3 | | | | |
| | | 869.6-858.3=11.3 | | | |
| 1 | 869.6 | | 11.3-11.3=0 | | |
| | | 880.9-869.6=11.3 | | 0.1-0=0.1 | |
| 2 | 880.9 | | 11.4-11.3=0.1 | | -0.2-0.1=-0.3 |
| | | 892.3-880.9=11.4 | | -0.1-0.1=-0.2 | |
| 3 | 892.3 | | 11.3-11.4=-0.1 | | |
| | | 903.6-892.3=11.3 | | | |
| 4 | 903.6 | | | | |

By Newton's forward interpolation formula,

$$y(x_0+nh)=y_0+n\Delta y_0+\frac{n(n-1)}{2!}\Delta^2 y_0+\frac{n(n-1)(n-2)}{3!}\Delta^3 y_0+\ldots\ldots$$

$$\Rightarrow y(1.5)=869.6+(0.5)(11.3)+\frac{(0.5)(0.5-1)(0.5-2)}{2}(0)+\frac{(0.5)(0.5-1)(0.5-2)}{2}(0.1)+\frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{6}(-0.3)$$

$$\Rightarrow y(1.5)=869.6+(0.5)(11.3)+\frac{(0.5)(0.5)(1.5)}{2}(0.1)+\frac{(0.5)(0.5)(1.5)(2.5)}{6}(0.3)$$

$$\Rightarrow y(1.5)=869.6+5.65+\frac{(0.0375)}{2}+\frac{(0.28125)}{6}$$

$$\Rightarrow y(1.5)=869.6+5.65+0.01875+0.46875$$

$$\Rightarrow y(1.5)=875.7375$$

---

2. Using Newton's forward interpolation, find the value of $\log_{10}^{\pi}$,given log3.141 = 0.4970679364 log3.142 = 0.4972061807 log3.143 = 0.4973443810 log3.144 = 0.49748253704 log3.145 = 0.4974825374

Solution

| x | y=logx | Δy | $\Delta^2$y | $\Delta^3$y | $\Delta^4$y |
|---|---|---|---|---|---|

| | | | | | |
|---|---|---|---|---|---|
| 0 | 0.4970679364 | | | | |
| | | $0.1382443 \times 10^{-3}$ | | | |
| 1 | 0.4972061807 | | 11.3-11.3=0 | | |
| | | $0.1382003 \times 10^{-3}$ | | 0.1-0=0.1 | |
| 2 | 0.4973443810 | | 11.4-11.3=0.1 | | -0.2-0.1=-0.3 |
| | | $0.1381564 \times 10^{-3}$ | | -0.1-0.1=-0.2 | |
| 3 | 0.49748253704 | | 11.3-11.4=-0.1 | | |
| | | $0.1381124 \times 10^{-3}$ | | | |
| 4 | 0.4974825374 | | | | |

Here $x_0=3.141$, $h = 0.001$, $y_0 = 0.4970679364$

The Newton's forward interpolation formula is

$x_0+nh=\pi=3.1415926536$

$$n=\frac{3.1415926536-3.141}{0.001}=0.5926536$$

$$y(x_0+nh)=y_0+n\Delta y_0+\frac{n(n-1)}{2!}\Delta^2 y_0+\frac{n(n-1)(n-2)}{3!}\Delta^3 y_0+\ldots\ldots$$

$$\Rightarrow y(\pi)=0.4970679364+(0.5926536)(0.000138244)+\frac{0.5926536(-0.4073464)(-0.440\times 10^{-7})}{2}$$

$$\Rightarrow y(\pi)=0.4970679364+0.0000819310+0.0000000053$$

$$\Rightarrow y(\pi)=0.4971498727$$

---

3. From the following data, estimate the no. of persons earning weekly wages between 60 and 70 rupees.

| Wages(in Rs.) | Below 40 | 40-60 | 60-80 | 80-100 | 100-120 |
|---|---|---|---|---|---|
| No.of person(in thousands) | 250 | 120 | 100 | 70 | 50 |

Solution

| x | $x_0=40$ | $x_1=60$ | $x_2=80$ | $x_4=100$ | $x_5=120$ |
|---|---|---|---|---|---|
| y | 250 | 250+120=370 | 370+100=470 | 470+70=540 | 540+50=590 |

Here $x_0=40$, $h = 20$, $y_0 = 250$

The Newton's forward interpolation formula is

$x_0+nh=70$

$$n=\frac{70-40}{20}=1.5$$

| **wages x** | **Frequency(y)** | **$\Delta y$** | **$\Delta^2 y$** | **$\Delta^3 y$** | **$\Delta^4 y$** |
|---|---|---|---|---|---|
| Below | 250 | | | | |

$y(x_0 +$

$\Rightarrow y(7$

$\Rightarrow y(7$

$\Rightarrow y(7$

| | | | | | |
|---|---|---|---|---|---|
| 40 | | | | | |
| | | 120 | | | |
| Below 60 | 370 | | -20 | | |
| | | 100 | | -10 | |
| Below 80 | 470 | | -30 | | 20 |
| | | 70 | | 10 | |
| Below 100 | 540 | | -20 | | |
| | | 50 | | | |
| Below 120 | 590 | | | | |

No. of persons whose weekly wages below 70 =423.5937

No. of persons whose weekly wages below 60 = 370

No. of persons whose weekly wages between 60 and 70 } = No. of persons whose weekly wages below 70- No. of persons whose weekly wages below 60

= 423.5937-370

----------------------------------------------------------------------------------------------------

## Newton's Backward Interpolation formula

The formula is used mainly to interpolate the values of y near the end of a set of values of y a short distance ahead (to the right) of y.

Formula:

$$y(x_n+nh)=y_n+n\nabla y_n+\frac{n(n+1)}{2!}\nabla^2 y_n+\frac{n(n+1)(n+2)}{3!}\nabla^3 y_n+\ldots\ldots$$

## Problems based on Newton's backward difference formula

1. The following data are taken from the steam table:

| Temp:°C | 140 | 150 | 160 | 170 | 180 |
|---|---|---|---|---|---|
| Pressure kgf/cm$^2$ | 3.685 | 4.854 | 6.302 | 8.076 | 10.225 |

Find the pressure at temperature t = 142°C and t = 175°C

Solution: We form the difference table:

| t | p | Δp | $\Delta^2$p | $\Delta^3$p | $\Delta^4$p |
|---|---|---|---|---|---|
| 140 | 3.685 | | | | |
| | | 1.169 | | | |
| 150 | 4.854 | | 0.279 | | |
| | | 1.448 | | 0.047 | |
| 160 | 6.302 | | 0.326 | | 0.002 |
| | | 1.774 | | 0.049 | |
| 170 | 8.076 | | 0.375 | | |
| | | 2.149 | | | |
| 180 | 10.225 | | | | |

$$+\frac{(0.2)(-0.8)(-1.8)}{6}(0.047)+\frac{(0.2)(-0.8)(-1.8)(-2.8)}{24}\times(0.002)$$

=3.685+0.2338-0.02332+0.002256-0.0000672

=3.897668

=3.898

$$P_4(=175^o)=P_4\left[180+\left(-\frac{1}{2}\right)\times 10\right],\ \textit{where}\ v=\frac{175-180}{10}=-0.5$$

$$=P_n+v\nabla P_n+\frac{v(v+1)}{2}\nabla^2 P_n+\ldots$$

$$=10.225+(-0.5)(2.149)+\frac{(-0.5)(0.5)}{2}(0.375)$$

$$+\frac{(-0.5)(0.5)(1.5)}{6}(0.049)+\frac{(-0.5)(0.5)(1.5)(2.5)}{24}(0.002)$$

=10.225-1.0745-0.0046875-0.0030625-0.000078125

=9.10048438=9.100

# Unit - II

## Part - A

1. State the Lagrange's formula to find $y(x)$ if three sets of values $(x_0, y_0), (x_1, y_1)$ and $(x_2, y_2)$ are given.

**Ans:-** $$y(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} y_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} y_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} y_2$$

---

2. Explain the inverse interpolation and the use of Lagrange's interpolation formula for inverse interpolation.

**Ans:-** Lagrange's interpolation formula is a relation between two variables $x$ and $y$ in which either $x$ or $y$ is taken as independent variable. Replacing $x$ by $y$ and $y$ by $x$ in Lagrange's formula, we can use the resulting formula for finding $x$ for a given $y$.

---

3. If $f(3)=5$ and $f(5)=3$ What is the form of $f(a)$ by Lagrange's formula?

**Ans:-** $$y = \frac{x-5}{3-5}(5) + \frac{x-3}{5-3}(3) \Rightarrow y = -\frac{5}{2}(x-5) + \frac{3}{2}(x-3)$$

$$\Rightarrow y = 8-x \Rightarrow f(x) = 8-x \Rightarrow f(a) = 8-a$$

---

4. State any two properties of divided difference.

**Ans:-** The divided difference are symmetrical in all their arguments.

The divided differences of sum or difference of two functions is equal to the sum or difference of the corresponding seperate divided differences.

5. Find the divided difference for the data

| $x$ | 2 | 5 | 10 |
|---|---|---|---|
| $y$ | 5 | 29 | 109 |

**Ans:-**

| $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ |
|---|---|---|---|
| 2 | 5 | | |
| | | $\frac{29-5}{3}=8$ | |
| 5 | 29 | | $\frac{16-8}{10-2}=1$ |
| | | $\frac{109-29}{10-5}=16$ | |
| 10 | 109 | | |

6. State the Newton's divided difference interpolation formula.

**Ans:-** $f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2) + \cdots + (x-x_0)(x-x_1)\cdots(x-x_{n-1}) f(x_0,x_1,x_2,\ldots,x_n)$

7. Using divided difference, show that $f(x,x) = f'(x)$ through limiting process.

**Ans:-** $f(x,x+h) = \frac{f(x+h)-f(x)}{x+h-x} = \frac{f(x+h)-f(x)}{h}$

Taking limit as $h\to 0$, $f(x,x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = f'(x)$

8. Show that $f(x_0,x_1) = f(x_1,x_0) = \frac{f(x_0)}{x_0-x_1} + \frac{f(x_1)}{x_1-x_0}$

**Ans:-** $f(x_0,x_1) = \frac{f(x_1)-f(x_0)}{x_1-x_0} = \frac{f(x_0)-f(x_1)}{x_0-x_1} = f(x_1,x_0)$

Again $f(x_0,x_1) = \frac{f(x_0)}{x_0-x_1} - \frac{f(x_1)}{x_0-x_1} = \frac{f(x_0)}{x_0-x_1} + \frac{f(x_1)}{x_1-x_0} = f(x_1,x_0)$

9. Fit a polynomial of least degree to the following data,

| $x$ | 1 | 2 | 4 |
|---|---|---|---|
| $y$ | 5 | 10 | 26 |

**Ans:-** $x_0 = 1$ $x_1 = 2$ $x_2 = 4$

$f(x_0) = 5$ $f(x_1) = 10$ $f(x_2) = 26$

| $x$ | $y$ | $\Delta f(x)$ | $\Delta^2 f(x)$ |
|---|---|---|---|
| 1 | 5 | | |
| | | 5 | |
| 2 | 10 | | 1 |
| | | 8 | |
| 4 | 26 | | |

By divided difference formula

$$y = 5 + (x-1)5 + (x-1)(x-2) = x^2 + 2x + 2.$$

---

10. If $f(x) = \frac{1}{x^2}$, find $f(a,b)$ and $f(2,3)$.

Ans:- $f(a) = \frac{1}{a^2}$, $f(b) = \frac{1}{b^2}$, $f(a,b) = \frac{f(b)-f(a)}{b-a} = \frac{\frac{1}{b^2} - \frac{1}{a^2}}{b-a}$

$$\Rightarrow f(a,b) = -\frac{(a+b)}{a^2 b^2} \quad \therefore f(2,3) = \frac{-5}{(4)(9)} = \frac{-5}{36}.$$

---

11. State the condition required for a natural cubic spline.

Ans:- A cubic spline $g(x)$ fits to each of the points is continuous and is continuous in slope and curvature such that $S_0 = g_0''(x_0) = 0$ and $S_n = g_{n-1}''(x_n) = 0$ is called a natural cubic spline.

---

12. What are the $n^{th}$ divided difference of polynomial of $n^{th}$ degree?

Ans:- The $n^{th}$ divided difference of an $n^{th}$ degree polynomial are Constant.

13. When will we use Newton's forward interpolation formula?

Ans:- Newton's forward interpolation formula is used when interpolation is required near the beginning of the table value and for extrapolation at a short distance from the initial value $x_0$.

13. State the Newton's backward interpolation is used to interpolate the values of $y = f(x)$ nearer to the end of a set of table values and for extrapolation closer to the right of $y_n$.

14. Given $f(0) = -1$, $f(1) = 1$ and $f(2) = 4$. Find the Newton interpolating polynomial

Ans:- $y_0 = -1$ $\quad \Delta y_0 = 1-(-1) = y_1 - y_0$ $\quad \Delta y_1 = 4-1 = 3 = y_2 - y_1$ $\quad \Delta^2 y_0 = 3-2 = 1$

$y_1 = 1$

$y_2 = 4$

$n = \frac{x - x_0}{h} = x$ $\qquad f(x) = -1 + 2x + \frac{x(x-1)}{2} = \frac{1}{2}(x^2 + 3x - 2)$

15. Find the sixth term in the sequence 8, 12, 19, 29, 42, ...

Ans:- $y_5 = E^5 y_0$

$= (1+\Delta)^5 y_0$

$= 1 + 5\Delta + 10\Delta^2 y_0$

$= 8 + 20 + 30 = 58.$

16. Given $f(0) = -1$, $f(1) = 1$ and $f(2) = 4$, find the roots of polynomial equation $f(x) = 0$.

Ans:-

| $x$ | $y$ | $\Delta y$ | $\Delta^2 y$ |
|---|---|---|---|
| 0 | -1 | | |
| | | 2 | |
| 1 | 1 | | 1 |
| | | 3 | |
| 2 | 4 | | |

$h = x$

$f(x) = -1 + 2x + \frac{x(x-1)}{2}(1)$

$= \frac{1}{2}(x^2 + 3x - 2)$

$f(x) = 0 \Rightarrow x = \frac{-3 \pm \sqrt{17}}{2}$.

## Part - B

1. Using Lagrange's formula to calculate $f(3)$ from the following data

| $x$ | 0 | 1 | 2 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| $y$ | 1 | 14 | 15 | 5 | 6 | 19 |

2. Using Lagrange's formula, fit a polynomial to the data

| $x$ | 0 | 1 | 3 | 4 |
|---|---|---|---|---|
| $y$ | -12 | 0 | 6 | 12 |

3. Using Lagrange's interpolation find the polynomial through (0,0), (1,1) and (2,2)

4. Find the cubic spline approx. for the function $y = f(x)$ from the following data, given that $y_0'' = y_3'' = 0$.

| $x$ | -1 | 0 | 1 | 2 |
|---|---|---|---|---|
| $y$ | -1 | 1 | 3 | 35 |

5. Using Newton's backward difference formula to construct an interpolating polynomial of degree 3 for the data.

$f(-0.75) = -0.07181250$  $f(-0.5) = -0.024750$  $f(-0.25) = 0.33493750$

$f(0) = 1.10100$. Hence find $f(-1/3)$

6. Using Lagrange's interpolation formula find $f(x)$ from the following data

| $x$ | 1 | 2 | 3 | 4 | 7 |
|---|---|---|---|---|---|
| $f(x)$ | 2 | 4 | 8 | 16 | 128 |

7. From the following data, estimate the no. of persons earning weekly between 60 and 70 rupees

| wages | below 40 | 40-60 | 60-80 | 80-100 | 100-120 |
|---|---|---|---|---|---|
| No. of persons | 250 | 370 | 470 | 540 | 590 |

8. Find the cubic polynomial following table using Newton's divided difference formula and hence find f(4)

| | | | | |
|---|---|---|---|---|
| $x$ | 0 | 1 | 2 | 5 |
| $f(x)$ | 2 | 3 | 12 | 147 |

9. For the given values evaluate f(9) using Lagrange's formula

| | | | | | |
|---|---|---|---|---|---|
| $x$ | 5 | 7 | 11 | 13 | 17 |
| $y$ | 150 | 392 | 1452 | 2366 | 5202 |

10. Fit the cubic spline for the data

| | | | |
|---|---|---|---|
| $x$ | 1 | 2 | 3 |
| $y$ | -6 | -1 | 16 |

Hence evaluate y(1.5) given that $y_0'' = 0$, $y_2'' = 0$

11. obtain the root of f(x)=0 by Lagrange's inverse interpolation given that f(30) = -30, f(34) = -13, f(38) = 3, f(42) = 18.

12. The following values of x & y are given

| | | | | |
|---|---|---|---|---|
| $x$ | 1 | 2 | 3 | 4 |
| $y$ | 1 | 2 | 5 | 11 |

find the cubic spline and evaluate y(1.5)

13. From the following table of half-yearly premium for policies maturing at different ages, estimate the premium for policies maturing at age 46

| | | | | | |
|---|---|---|---|---|---|
| Age (x) | 45 | 50 | 55 | 60 | 65 |
| Premium (y) | 114.84 | 96.16 | 83.32 | 74.48 | 68.48 |

14. Using Newton's Divided difference formula find f(x) & f(6) from the following data.

| | | | | |
|---|---|---|---|---|
| $x$ | 1 | 2 | 7 | 8 |
| $y$ | 1 | 5 | 5 | 4 |

# UNIT III
# NUMERICAL DIFFERENTIATION AND INTEGRATION

***Differentiation using interpolation formulae***

***Numerical integration by trapezoidal rule***

***Simpson's 1/3 and 3/8 rules***

***Romberg's method***

***Two and Three point Gaussian quadrature formulas***

***Double integrals using trapezoidal and simpsons's rules.***

## Numerical Differentiation

- **Differentiation using Forward Interpolation formula(for equal interval)**
- **Differentiation using Backward Interpolation formula(for equal interval)**
- **Differentiation using Stirling's(Central Difference) Formula(for equal interval)**
- **Maximum and Minimum**
- **Differentiation using divided difference(for unequal interval)**

### Forward difference formula to compute the derivative

Newton's forward interpolation formula is

$$f(x_0 + rh) = y_0 + \Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \dots$$

(Here using of r for n is only for convenience)

Differentiating w.r.t. we get,

$$hf'(x_0 + rh) = \Delta y_0 + \frac{2r-1}{2}\Delta^2 y_0 + \frac{3r^2-6r+2}{6}\Delta^3 y_0 + \frac{2r^3-9r^2+11r-3}{12}\Delta^4 y_0 + \dots \quad \text{---} \quad (1)$$

$$h^2 f''(x_0 + rh) = \Delta^2 y_0 + (r-1)\Delta^3 y_0 + \frac{6r^2-18r+11}{12}\Delta^4 y_0 + \dots \quad \dots (2)$$

$$h^3 f'''(x_0 + rh) = \Delta 3 y_0 + \frac{2r-3}{2}\Delta^4 y_0 + \dots \quad \dots(3)$$

Similarly we can find the remaining derivatives.

If we want to find the derivatives at a point $x = x_0$, then $x_0 + rh = x_o$

i.e., r=0.

Hence on substituting this value of r=0 in the above formula (1), (2) and (3), we get

$$f'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 \dots\right]$$

$$f''(x_0) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \dots\right]$$

$$f'''(x_0) = \frac{1}{h^3}\left[\Delta 3 y_0 + \frac{3}{2}\Delta^4 y_0 + \dots\right] \text{ and so on.}$$

Note: If the x value is nearer to the starting of the given table we use Forward Interpolation formula

### Backward difference formula to compute the derivatives

Newton's backward difference formula is

$$f(x_n + rh) = y_n + \nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \frac{r(r+1)(r+2)(r+3)}{4!}\nabla^4 y_n + \dots$$

$$f'(x_n + rh) = \frac{1}{h}\left[\nabla y_n + \frac{(2r+1)}{2}\nabla^2 y_n + \frac{2r^2+6r+2}{6}\nabla^3 y_n + \frac{2r^3+6r^2+11r+3}{12}\nabla^4 y_n + \dots\right]$$

$$h^2 f''(x_n + rh) = \nabla^2 y_n + (r+1)\nabla^3 y_n + \frac{6r^2 + 18r + 11}{12}\nabla^4 y_n + \ldots$$

$$h^3 f'''(x_n + rh) = \nabla^3 y_n + \frac{2r+3}{2}\nabla^4 y_n + \ldots$$

Similarly we can find the remaining derivatives.

At the point $x = x_n$, i.e., $x_n + rh = x_n$, we have r=0.

$$f'(x_n) = \frac{1}{h}\left[\nabla y_n - \frac{1}{2}\nabla^2 y_n + \frac{1}{3}\nabla^3 y_n - \frac{1}{4}\Delta^4 y_n \ldots\right]$$

$$f''(x_n) = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n - \ldots\right]$$

$$f'''(x_n) = \frac{1}{h^3}\left[\nabla^3 y_n + \frac{3}{2}\nabla^4 y_n + \ldots\right] \text{ and so on.}$$

Note: If the x value is nearer to the end of the given table we use backward Interpolation formula

**Central difference formula for computing the derivatives**

We know that Stirling's central difference formula is

$$f(x_0 + rh) = y_0 + r\left(\frac{\Delta y_0 + \Delta y - 1}{2}\right) + \frac{r^2}{2!}\nabla^2 y - 1 + \frac{r(r^2-1)}{3!}\left(\frac{\Delta^3 y - 1 + \Delta^3 y - 2}{2}\right) +$$
$$\frac{r^2(r^2-1)}{4!}\nabla^4 y - 2 + \frac{r(r^2-1)(r^2-4)}{5!}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \ldots$$

(Here using of r for n is only for convenience)

$$f(x_0 + rh) = y_0 + r\left(\frac{\Delta y_0 + \Delta y - 1}{2}\right) + \frac{r^2}{2!}\nabla^2 y - 1$$

i.e., $$+\frac{(r^3 - r)}{3!}\left(\frac{\Delta^3 y - 1 + \Delta^3 y - 2}{2}\right) + \frac{(r^4 - r^2)}{4!}\nabla^4 y - 2$$

$$+\frac{(r^5 - 5r^3 + 4r)}{5!}(\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \ldots$$

Differentiating (1) w.r.t. 'r' we get,

$$hy'(x_0 + rh) = \left(\frac{\Delta y_0 + \Delta y - 1}{2}\right) + \Delta^2 y - 1 + \left(\frac{3r^2 - 1}{12}\right)(\Delta^3 y - 1 + \Delta^3 y - 2)$$
$$+\left(\frac{2r^3 - r}{12}\right)\Delta^4 y_{-2} + \left(\frac{5r^4 - 15r + 4}{5!}(\Delta^5{}_{y-2} + \Delta^5{}_{y-3})\right) + \ldots$$

$$\therefore y'(x_0 + rh) = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y - 1}{2}\right) + \Delta^2 y - 1 + \left(\frac{3r^2 - 1}{12}\right)(\Delta^3 y_{-1} + \Delta^3 y_{-2})\right.$$
$$\left. + \left(\frac{2r^3 - r}{12}\right)\Delta^4 y_{-2} + \left(\frac{(5r^4 - 15r^2 + 4)}{5!}(\Delta^5 y_{-2} + \Delta^5 y_{-3})\right) + \ldots\right]$$

...(2)

Differentiating (2) w.r.t. 'r', we get,

$$y''(x_0 + rh) = \frac{1}{h^2}\left[\Delta^2 y_{-1} + \frac{r}{2}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \left(\frac{6r^2 - 1}{12}\right)\Delta^4 y_{-2} + \left(\frac{20r^3 - 30r}{5!}\right)(\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \ldots\right]$$ **-----------(3)**

**Similarly we can find the remaining derivatives.**

**If we want to find the derivative at a point** $x = x_0$**, then** $x_0 + rh = x_0$

i.e., r=0

Substituting r=0 in (2) and (3) we get,

$$y'(x_0) = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y - 1}{2}\right) - \frac{1}{12}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \frac{1}{30}(\Delta^5 y_{-2} + \Delta^5 y_{-3} + \ldots)\right]$$

$$y''(x_0) = \frac{1}{h_2}\left[\Delta^2 y_{-1} - \frac{1}{12}\Delta^4 y_{-2} + \ldots\right]$$

**Similarly we can find the remaining derivatives.**

**Note:**

- If the x value is middle of the given table we use central Difference formula

**Maximum and Minimum:**

**Steps**

- Write the Newton's Forward difference formula y(x), $\frac{dy}{dx}$
- Write the forward difference table
- Find $\frac{dy}{dx}$
- Put $\frac{dy}{dx}$ =0 and find the value of x
- Find $\frac{d^2y}{dx^2}$
- For every value of x find $\frac{d^2y}{dx^2}$
- If $\frac{d^2y}{dx^2}$<0 , y is maximum at that x(maximum point)
- If $\frac{d^2y}{dx^2}$>0 , y is minimum at that x(minimum point)
- To find the maximum and minimum value substitute the maximum and minimum points in y(x) formula respectively.

## Problems based on Differentiation using Interpolation formula

1. Find the first, second and third derivatives of the function tabulated below at the point x=1.5

| x | 1.5 | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |
|---|---|---|---|---|---|---|
| f(x) | 3.375 | 7.0 | 13.625 | 24.0 | 38.875 | 59.0 |

## Solution

**The difference table is as follows:**

| **x** | **y=f(x)** | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|
| **1.5** $x_0$ | **3.375** $y_0$ | | | | |
| | | **3.625** $(\Delta y_0)$ | | | |
| **2.0** | **7.0** | | **3.0** $(\Delta^2 y_0)$ | | |
| | | **6.625** | | **0.75** $(\Delta^3 y_0)$ | |
| **2.5** | **13.625** | | **3.75** | | **0** |
| | | **10.375** | | **0.75** | |
| **3.0** | **24.0** | | **4.50** | | **0** |
| | | **14.875** | | **0.75** | |
| **3.5** | **38.875** | | **5.25** | | |
| | | **20.125** | | | |
| **4.0** | **59.0** | | | | |

**Here we have to find the derivative at the point x=1.5 which is the initial value of the table. Therefore by Newton's forward difference formula for derivatives at $x=x_0$, we have**

$$f'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \ldots\right]$$

**Here $x_0=1.5$, h=0.5**

$$\therefore \quad f'(1.5) = \frac{1}{0.5}\left[3.625 - \frac{1}{2}(3.0) + \frac{1}{3}(0.75) - \ldots\right]$$

$$f'(1.5) = 4.75$$

**At the point** $x = x_0$,

$$f''(x_0) = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - ...\right]$$

**Here $x_0$=1.5, h=0.5**

$$\therefore \quad f''(1.5) = \frac{1}{(0.5)^2}\left[3.0 - 0.75\right]$$

$$f''(1.5) = 9.0$$

**At the point** $x = x_0$,

$$f'''(x_0) = \frac{1}{h^3}\left[\Delta^3 y_0 + \frac{3}{2}\Delta^4 y_0\right]$$

$$f'''(1.5) = \frac{1}{(0.5)^3}(0.75) = 6.0$$

$$f'''(1.5) = 6.0$$

----------------------------------------------------------------------------------------------------------2.

Compute f'(0) and f''(4) from the data

| | | | | | |
|---|---|---|---|---|---|
| x | 0 | 1 | 2 | 3 | 4 |
| y | 1 | 2.718 | 7.381 | 20.086 | 54.598 |

**Solution:**

| x | y=f(x) | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|
| **0$x_0$** | **1$y_0$** | | | | |
| | | **1.718** $(\Delta y_0)$ | | | |
| **1** | 2.718 | | **2.945** $(\Delta^2 y_0)$ | | |
| | | 4.663 | | **5.097** $(\Delta^3 y_0)$ | |
| **2** | 7.381 | | 8.042 | | **8.668** $(\Delta^4 y_0)$ |
| | | 12.705 | | **13.765** $(\nabla^3 y_n)$ | $(\nabla^4 y_n)$ |
| **3** | 20.086 | | **21.807** $(\nabla^2 y_n)$ | | |
| | | **34.512** $(\nabla y_n)$ | | | |
| **4** | **54.598** | | | | |
| **$x_n$** | **$y_n$** | | | | |

Here we have to find f'(0) .ie.x=0 which is the starting of the given table. So we use the forward interpolation formula.

$$f'(x_0) = \frac{1}{h}\left[\Delta y_0 - \frac{1}{2}\Delta^2 y_0 + \frac{1}{3}\Delta^3 y_0 - \frac{1}{4}\Delta^4 y_0 \ldots\right]$$

$$f'(0) = \frac{1}{1}\left[1.718 - \frac{1}{2}2.945 + \frac{1}{3}5.097 - \frac{1}{4}8.668\right] = -0.2225$$

Here we have to find f''(4) .ie.x=4 which is the end of the given table. So we use the backward interpolation formula.

$$f''(x_n) = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n - \ldots\right]$$

$$f''(4) = \frac{1}{1^2}\left[21.807 + 13.765 + \frac{11}{12}8.668\right] = 43.5177$$

---

2. Find $\frac{dy}{dx}$ *and* $\frac{d^2y}{dx^2}$ at x = 51 from the following data.

| x | 50 | 60 | 70 | 80 | 90 |
|---|---|---|---|---|---|
| y | 19.96 | 36.65 | 58.81 | 77.21 | 94.61 |

Solution:

Here h=10. To find the derivatives of y at x=51 we use Forward difference formula taking the origin at $x_0$=50.

We have $r = \frac{x - x_0}{h} = \frac{51-50}{10} = 0.1$

$\therefore$ at x=51, r=0.1

$$\left(\frac{dy}{dx}\right)_{x=51} = \left(\frac{dy}{dx}\right)_{r=0.1} = \frac{1}{h}\left[\Delta y_0 + \frac{2r-1}{2}\Delta^2 y_0 + \frac{3r^2-6r+2}{6}\Delta^3 y_0 + \frac{2r^3-9r^2+11r-3}{12}\Delta^4 y_0 + \ldots\right]$$

The difference table is given by

| x | y=f(x) | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|
| **50** $x_0$ | **19.96** $y_0$ | | | | |
| | | **16.69** $(\Delta y_0)$ | | | |
| **60** | 36.65 | | **5.47** $(\Delta^2 y_0)$ | | |
| | | 22.16 | | **-9.23** $(\Delta^3 y_0)$ | |
| **70** | 58.81 | | -3.76 | | **11.09** $(\Delta^4 y_0)$ |
| | | 18.40 | | **2.76** | |
| **80** | 77.21 | | **-1.00** | | |
| | | **17.40** | | | |

| | | | | | |
|---|---|---|---|---|---|
| 90 | **94.61** | | | | |

$$\left(\frac{dy}{dx}\right)_{x=51} = \left(\frac{dy}{dx}\right)_{r=0.1} = \frac{1}{10}\left[16.69 + \frac{0.2-1}{2}(5.47) + \left[\frac{3(0.1)^2 - 6(0.1)+2}{6}\right](-9.23) + \left[\frac{2(0.1)^3 - 9(0.1)^2 + 11(0.1) - 3}{12}\right](11.99)\right]$$

$$= \frac{1}{10}\left[16.69 - 2.188 - 2.1998 - 1.9863\right] = 1.0316$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=51} = \left(\frac{d^2y}{dx^2}\right)_{r=0.1} = \frac{1}{h^2}\left[\Delta^2 y_0 + (r-1)\Delta^3 y_0 + \frac{6r^2 - 18r + 11}{12}\Delta^4 y_0 + \dots\right]$$

$$\left(\frac{d^2y}{dx^2}\right)_{x=51} = \left(\frac{d^2y}{dx^2}\right)_{r=0.1} = \frac{1}{100}\left[5.47 + (0.1-1)(-9.23) + \frac{6(0.1)^2 - 18(0.1) + 11}{12}(11.99)\right]$$

$$= \frac{1}{100}\left[5.47 + 8.307 + 9.2523\right] = 0.2303$$

---

3. Find the maximum and minimum value of y tabulated below.

| x | -2 | -1 | 0 | 1 | 2 | 3 | 4 |
|---|---|---|---|---|---|---|---|
| y | 2 | -0.25 | 0 | -0.25 | 2 | 15.75 | 56 |

**Solution:**

$$\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2r-1}{2}\Delta^2 y_0 + \frac{3r^2 - 6r + 2}{6}\Delta^3 y_0 + \frac{2r^3 - 9r^2 + 11r - 3}{12}\Delta^4 y_0 + \dots\right]$$

| x | y=f(x) | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ | $\Delta^5 y$ |
|---|---|---|---|---|---|---|
| -2 | 2 | | | | | |
| | | -2.25 | | | | |
| -1 | -0.25 | | 2.5 | | | |
| | | 0.25 | | -3 | | |
| 0 | 0 | | -0.5 | | 6 | |
| | | -0.25 | | 3 | | 0 |
| 1 | -0.25 | | 2.5 | | 6 | |
| | | 2.25 | | 9 | | 0 |
| 2 | 2 | | 11.5 | | 6 | |
| | | 13.75 | | 15 | | |
| 3 | 15.75 | | 26.5 | | | |
| | | 40.25 | | | | |
| 4 | 56 | | | | | |

Choosing $x_0=0$, $r = \frac{x-0}{1} = x$

$$\frac{dy}{dx} = \frac{1}{1}\left[-0.25 + \frac{2x-1}{2}(2.5) + \frac{3x^2 - 6x + 2}{6}(9) + \frac{2x^3 - 9x^2 + 11x - 3}{12}(6)\right]$$

$$= \frac{1}{1}\left[-0.25 + 2.5x - 1.25 + 4.5x^2 - 9x + 3 + x^3 - 4.5x^2 + 5.5x - 1.5\right]$$

$$\frac{dy}{dx} = x^3 - x$$

Now $\frac{dy}{dx} = 0 \Rightarrow x^3 - x = 0$

$\Rightarrow$ x = 0, x = 1, x = -1.

$$\frac{d^2y}{dx^2} = 3x^2 - 1$$

at x=0 $\frac{d^2y}{dx^2} = -ve$ at x=1 $\frac{d^2y}{dx^2} = +ve$ at x=-1 $\frac{d^2y}{dx^2} = +ve$

$\therefore$ y is maximum at x=0, minimum at x=1 and -1

$$\therefore y(x) = \left[ y_0 + x\Delta y_0 + \frac{x(x-1)}{2!}\Delta^2 y_0 + ... \right]$$

Maximum value =y(0) =0, Minimum value =y(1)=-0.25.

---

4. Consider the following table of data

| x | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 |
|---|---|---|---|---|---|
| f(x) | 0.9798652 | 0.9177710 | 0.8080348 | 0.6386093 | 0.3843735 |

Find f'(0.25),f'(0.6) and f'(0.95).

**Solution:**

Here h=0.2

- 0.25 is nearer to the starting of the given table. So we use Newton's forward interpolation formula to evaluate f'(0.25)
- 0.95 is nearer to the ending of the given table. So we use Newton's backward interpolation formula to evaluate f'(0.95)
- 0.6 is middle point of the given table. So we use Central Difference formula to evaluate f'(0.6)

The difference table

| x | y=f(x) | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|
| 0.2 $x_0$ | 0.9798652 $y_0$ | | | | |
| | | **-0.0620942** $(\Delta y_0)$ | | | |
| 0.4 | 0.9177710 | | **0.047642** $(\Delta^2 y_0)$ | | |
| | | -0.1097362 | | **-0.0120473** $(\Delta^3 y_0)$ | |
| 0.6 | 0.8080348 | | -0.0596893 | | $(\Delta^4 y_0)$ **-0.01310985** $(\nabla^4 y_n)$ |
| | | -0.1694255 | | - | |

| | | | | | |
|---|---|---|---|---|---|
| | | | | **0.02515715** $(\nabla^3 y_n)$ | |
| 0.8 | 0.6386093 | | **-0.08484645** $(\nabla^2 y_n)$ | | |
| | | **-0.25427195** $(\nabla y_n)$ | | | |
| 1.0 | 0.3843735 | | | | |
| $x_n$ | $\mathbf{y_n}$ | | | | |
| | | | | | |

**To find f'(0.25)**

**Newton's forward interpolation formula for derivative**

$$hf'(x_0+rh) = \Delta y_0 + \frac{2r-1}{2}\Delta^2 y_0 + \frac{3r^2-6r+2}{6}\Delta^3 y_0 + \frac{2r^3-9r^2+11r-3}{12}\Delta^4 y_0 + \ldots$$

$$f'(0.25) = \frac{1}{0.2}\left[\begin{array}{l} -0.0620942 + \dfrac{2(0.25)-1}{2}(-0.047642) \\ + \dfrac{3(0.25)^2-6(0.25)+2}{6}(-0.0120473) \\ + \dfrac{2(0.25)^3-9(0.25)^2+11(0.25)-3}{12}(-0.01310985) \end{array}\right]$$

=-0.2536(correct to four decimal places)

**To find f'(0.95)**

| x | y=f(x) | $\Delta y$ | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ |
|---|---|---|---|---|---|
| 0.2$x_0$ | 0.9798652($y_{-2}$) | | | | |
| | | **-0.0620942** $(\Delta y_{-2})$ | | | |
| 0.4 | 0.9177710 ($y_{-1}$) | | **0.047642** $(\Delta^2 y_{-2})$ | | |
| | | -0.1097362 $(\Delta y_{-1})$ | | **-0.0120473** $(\Delta^3 y_{-2})$ | |
| 0.6 | 0.8080348 ($y_0$) | | -0.0596893 $(\Delta^2 y_{-1})$ | | $(\Delta^4 y_{-2})$ **-0.01310985** |
| | | -0.1694255 $(\Delta y_0)$ | | **-0.02515715** $(\Delta^3 y_{-1})$ | |
| 0.8 | 0.6386093($y_1$) | | **-0.08484645** $(\Delta^2 y_0)$ | | |
| | | **-0.25427195** $(\Delta y_1)$ | | | |
| 1.0 | 0.3843735($y_2$) | | | | |

**Newton's backward interpolation formula for derivative**

$$f'(x_n + rh) = \frac{1}{h}\left[\nabla y_n + \frac{(2r+1)}{2}\nabla^2 y_n + \frac{2r^2+6r+2}{6}\nabla^3 y_n + \frac{2r^3+6r^2+11r+3}{12}\nabla^4 y_n + \dots\right]$$

$$r = \frac{x - x_n}{h} = \frac{0.95-1}{0.2} = -0.25$$

$$f'(0.95) = \frac{1}{0.2}\left[\begin{array}{l} -0.25427195 + \frac{2(-0.25)+1}{2}(-0.08484645) \\ + \frac{3(-0.25)^2+6(-0.25)+2}{6}(0.02515715) \\ + \frac{2(-0.25)^3+9(-0.25)^2+11(-0.25)+3}{12}(-0.01310985) \end{array}\right]$$

f'(0.95) =-1.71604

**To find f'(0.6)**

**Central Difference formula (Stirling's Formula)**

$$f'(x_0 + rh) = \frac{1}{h}\left[\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + r\Delta^2 y_{-1} + \left(\frac{3r^2-1}{12}\right)(\Delta^3 y_{-1} + \Delta^3 y_{-2}) + \left(\frac{2r^3-r}{12}\right)\Delta^4 y_{-2} + \left(\frac{5r^4-15r^2+4}{5!}\right)(\Delta^5 y_{-2} + \Delta^5 y_{-3}) + \dots\right]$$

$$f'(0.6) = \frac{1}{0.2}\left[\begin{array}{l} \left(\frac{-0.1694255-0.1097362}{2}\right) + (0.2)(-0.00596893) \\ + \left(\frac{3(0.2)^2-1}{12}\right)(-0.02515715-0.0120473) + \left(\frac{2(0.2)^3-0.2}{12}\right)(-0.01310985)\dots \end{array}\right]$$

f'(0.6)=-0.74295(correct to 5 decimal places)

---

5. Given the following data, find y'(6), y'(5) and the maximum value of y

| | | | | | | |
|---|---|---|---|---|---|---|
| x | 0 | 2 | 3 | 4 | 7 | 9 |
| y | 4 | 26 | 58 | 112 | 466 | 922 |

**Solution:**

Since the intervals are , we will use Newton's divided difference formula.

Divided Difference Table

| x | y=f(x) | $\Delta f(x)$ | $\Delta^2 f(x)$ | $\Delta^3 f(x)$ | $\Delta^4 f(x)$ |
|---|---|---|---|---|---|
| 0 | **4** | | | | |
| | | **11=f($x_0$,$x_1$)** | | | |
| 2 | **26** | | **7= f($x_0$,$x_1$,$x_2$)** | | |
| | | 32 | | **1= f($x_0$,$x_1$,$x_2$,$x_3$)** | |
| 3 | **58** | | 11 | | **0= f($x_0$,$x_1$,$x_2$,$x_3$, $x_4$)** |
| | | 54 | | **1** | |
| 4 | **112** | | **16** | | **0** |
| | | **118** | | **1** | |
| 7 | **466** | | **22** | | |
| | | **228** | | | |
| 9 | **922** | | | | |

By Newton's Divided Difference formula,

y =f(x)=f($x_0$)+(x-$x_0$)f($x_0$, $x_1$)+ (x-$x_0$) (x-$x_1$)f($x_0$, $x_1$,$x_2$)+……….

=4+(x-0)11+(x-0)(x-2)7+(x-0)(x-2)(x-3)1

=$x^3$+2$x^2$+3x+4

Therefore, f'(x) = $3x^2+4x+3$
f'(6)=135
f'(5)=98.

---

## Numerical Integration

- Single integral
  - → Trapezoidal
  - → Simpson's one-third rule
  - → Simpson's three-eighth rule
  - → Romberg method
  - → Two and Three point Gaussian Quadrature Formulas

- Double integral
  - → Trapezoidal rule
  - → Simpson's Rule

## Single Integral

### Trapezoidal Rule

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[(\text{sum of the first and last ordinates}) + 2(\text{sum of the remaining ordinates})\right]$$

**ie,** $$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right]$$

### Simpson's one third rule

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{3}\begin{bmatrix}(\text{sum of the first and last ordinates}) + 2(\text{sum of the remaining odd ordinates}) \\ +4(\text{sum of the remaining even ordinates})\end{bmatrix}$$

**ie,** $$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_3 + \ldots + y_{n-1}) + 4(y_2 + y_4 + \ldots + y_{n-2})\right]$$

### Simpson's three eight rule

$$\int_{x_0}^{x_n} f(x)dx = \frac{3h}{8}\begin{bmatrix}(\text{sum of the first and last ordinates}) \\ +3(\text{sum of the remaining ordinates which are not divisible by } 3) \\ +2(\text{sum of the remaining ordinates which are divisible by } 3)\end{bmatrix}$$

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \ldots + y_{n-1}) + 2(y_3 + y_6 + \ldots + y_{n-3})\right]$$

| Rule | Degree of y(x) | No.of intervals | Error | Order |
|---|---|---|---|---|
| **Trapezoidal Rule** | **One** | **Any** | $\lvert E\rvert < \frac{(b-a)h^2 M}{12}$ | $h^2$ |
| **Simpson's one third rule** | **Two** | **Even** | $\lvert E\rvert < \frac{(b-a)h^4 M}{180}$ | $h^4$ |
| **Simpson's three eight rule** | **three** | **Multiple of 3** | $\lvert E\rvert < \frac{3h^5}{8}$ | $h^5$ |

## Two and Three point Gaussian Quadrature Formulas

### Gaussian Two point formula

- If the Limit of the integral is -1 to 1 then we apply $\int_{-1}^{1} f(x)dx = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$. This formula is exact for polynomials upto degree 3.
- If $\int_a^b f(x)dx$ then x= $\frac{b-a}{2}t + \frac{b+a}{2}$ and dx = $\frac{b-a}{2}$ using these conditions convert $\int_a^b f(x)dx$ into $\int_{-1}^{1} f(t)dt$ and then we apply the formula $\int_{-1}^{1} f(t)dt = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

### Gaussian Three point formula

- If the Limit of the integral is -1 to 1 then we apply $\int_{-1}^{1} f(x)dx = \frac{5}{9}\left[ f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9} f(0)$. This formula is exact for polynomials upto degree 5.
- Otherwise $\int_a^b f(x)dx$ then x= $\frac{b-a}{2}t + \frac{b+a}{2}$ and dx = $\frac{b-a}{2}$ using these conditions convert $\int_a^b f(x)dx$ into $\int_{-1}^{1} f(t)dt$ and then we apply the formula $\int_{-1}^{1} f(t)dt = \frac{5}{9}\left[ f\left(-\sqrt{\frac{3}{5}}\right) + f\left(\sqrt{\frac{3}{5}}\right)\right] + \frac{8}{9} f(0)$

## Problems based on single integrals

**1. Using Trapezoidal rule evaluate** $\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx$

**Solution**

| x | 0 | 15 | 30 | 45 | 60 | 75 | 90 |
|---|---|---|---|---|---|---|---|
| y | 0 | 0.5087 | 0.7071 | 0.8408 | 0.9306 | 0.9828 | 1 |

**Trapezoidal rule**

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right]$$

$$\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx = \frac{\frac{\pi}{12}}{2}\left[(0+1) + 2(0.5087 + 0.7071 + 0.8408 + 0.9306 + 0.9828)\right]$$

**= 1.17024**

---

2. By dividing the range into ten equal parts, evaluate $\int_0^{\pi} \sin x\,dx$ by Trapezoidal rule and Simpson's Rule. Verify your answer with integration

Solution:

Range = π-0=π. Hence h= $\frac{\pi}{10}$

we tabulate below the values of y at different x's

| x | 0 | $\frac{\pi}{10}$ | $\frac{2\pi}{10}$ | $\frac{3\pi}{10}$ | $\frac{4\pi}{10}$ | $\frac{5\pi}{10}$ | $\frac{6\pi}{10}$ | $\frac{7\pi}{10}$ | $\frac{8\pi}{10}$ | $\frac{9\pi}{10}$ | $\pi$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| sinx | 0 | .309 | .5878 | .809 | .9511 | 1 | .9511 | .809 | .5878 | .309 | 0 |

**Trapezoidal rule**

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_2 + \ldots + y_{n-1})\right]$$

$$\int_0^{\frac{\pi}{2}} \sin x\,dx = \frac{\frac{\pi}{10}}{2}\left[(0+0) + 2(0.3090 + 0.5878 + 0.8090 + 0.9306 + 0.9511 + 1 + 0.9511 + 0.9306 + 0.8090 + 0.3090)\right]$$

= 1.9843 nearly

**Simpson's one third Rule:**

we use Simpson's one third rule only when the no.of intervals is even

here the no of intervals =10(even)

$$\int_{x_0}^{x_n} f(x)dx = \frac{h}{2}\left[(y_0 + y_n) + 2(y_1 + y_3 + \ldots + y_{n-1}) + 4(y_2 + y_4 + \ldots + y_{n-2})\right]$$

$$\int_0^{\frac{\pi}{2}} \sin x dx = \frac{\frac{\pi}{10}}{2}\left[(0+0)+2(0.9511+0.5878+0.9306+0.9511+0.9306)+4(0.3090+0.8090+1+0.8090+0.3090)\right]$$

**= 2.00091 nearly**

We use Simpson's three eight rule only when the no.of intervals divisible by 3
Here the no of intervals =10 which is not divisible by 3.
So we cannot use this method.

**By actual integration,**

$$\int_0^{\pi} \sin x dx = \left[-\cos x\right]_0^{\pi} = 2$$

Hence, Simpson's one third rule is more accurate than the Trapezoidal rule.

---

3. Evaluate (i) $\int_{-1}^{1}(3x^2+5x^4)dx$ and (ii) $\int_{0}^{1}(3x^2+5x^4)dx$ by Gaussian two and three point formulas

**Solution:**

**(i)** $\int_{-1}^{1}(3x^2+5x^4)dx$

**Gaussian two point formula**

Given interval is -1 and 1.
Hence we can apply

$$\int_{-1}^{1} f(x)dx = f\left(\frac{-1}{\sqrt{3}}\right)+f\left(\frac{1}{\sqrt{3}}\right)$$

$$\Rightarrow \int_{-1}^{1}(3x^2+5x^4)dx = 3\left(\frac{-1}{\sqrt{3}}\right)^2+5\left(\frac{-1}{\sqrt{3}}\right)^4+3\left(\frac{1}{\sqrt{3}}\right)^2+5\left(\frac{1}{\sqrt{3}}\right)^4 = 3.112$$

**Gaussian two point formula**

Given interval is -1 and 1.
Hence we can apply

$$\int_{-1}^{1} f(x)dx = \frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right)+f\left(\sqrt{\frac{3}{5}}\right)\right]+\frac{8}{9}f(0)$$

$$\int_{-1}^{1}(3x^2+5x^4)dx = \frac{5}{9}\left\{\left[3\left(-\sqrt{\frac{3}{5}}\right)^2+5\left(-\sqrt{\frac{3}{5}}\right)^4\right]+\left[3\left(\sqrt{\frac{3}{5}}\right)^2+5\left(\sqrt{\frac{3}{5}}\right)^4\right]\right\}+\frac{8}{9}(0)$$

$$= \frac{5}{9}\left\{\left[3\left(\frac{3}{5}\right)+5\left(\frac{9}{25}\right)\right]+\left[3\left(\frac{3}{5}\right)+5\left(\frac{9}{25}\right)\right]\right\}$$

$$\int_{-1}^{1}(3x^2+5x^4)dx = 4$$

(ii) $\int_0^1 (3x^2+5x^4)dx$

**Gaussian two point formula**

Here the interval is 0 to 1. So we use the formula

$$x=\frac{b-a}{2}t+\frac{b+a}{2} \text{ and } dx=\frac{b-a}{2}\,dt$$

i.e, $x=\frac{t+1}{2}$ and dx=$\frac{dt}{2}$

$$\Rightarrow \int_{-1}^{1}\left[3\left(\frac{t+1}{2}\right)^2+5\left(\frac{t+1}{2}\right)^4\right]dt=1.556$$

**Gaussian two point formula**

Here the interval is 0 to 1. So we use the formula

$$x=\frac{b-a}{2}t+\frac{b+a}{2} \text{ and } dx=\frac{b-a}{2}\,dt \quad \text{i.e, } x=\frac{t+1}{2} \text{ and } dx=\frac{dt}{2}$$

$$\Rightarrow \int_{-1}^{1}\left[3\left(\frac{t+1}{2}\right)^2+5\left(\frac{t+1}{2}\right)^4\right]\frac{dt}{2}=\frac{1}{2}\int_{-1}^{1}\left[3\left(\frac{t+1}{2}\right)^2+5\left(\frac{t+1}{2}\right)^4\right]dt$$

$$f(t)=\left[3\left(\frac{t+1}{2}\right)^2+5\left(\frac{t+1}{2}\right)^4\right]$$

$$\int_{-1}^{1} f(t)dt=\frac{5}{9}\left[f\left(-\sqrt{\frac{3}{5}}\right)+f\left(\sqrt{\frac{3}{5}}\right)\right]+\frac{8}{9}f(0)$$

$$f\left(-\sqrt{\frac{3}{5}}\right)=\left[3\left(\frac{-0.7745+1}{2}\right)^2+5\left(\frac{-0.7745+1}{2}\right)^4\right]=0.038138$$

$$f\left(\sqrt{\frac{3}{5}}\right)=\left[3\left(\frac{0.7745+1}{2}\right)^2+5\left(\frac{0.7745+1}{2}\right)^4\right]=4.28$$

$$f(0)=\left[3\left(\frac{0+1}{2}\right)^2+5\left(\frac{0+1}{2}\right)^4\right]=1.0625$$

$$\int_{-1}^{1}\left[3\left(\frac{t+1}{2}\right)^2+5\left(\frac{t+1}{2}\right)^4\right]dt=\frac{5}{9}(0.038138+4.28)+\frac{8}{9}(1.0625)$$

= 4(approximately)

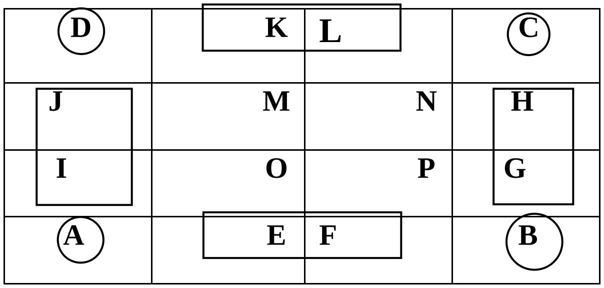
$$\frac{1}{2}\int_{-1}^{1}\left[3\left(\frac{t+1}{2}\right)^2+5\left(\frac{t+1}{2}\right)^4\right]dt=\frac{4}{2}=2$$

---

## Double Integral

### Trapezoidal Rule

Evaluate $\int_c^d \int_a^b f(x, y)\,dxdy$ where a, b, c, d are constants.




**I=** $\frac{hk}{4}$ **{[sum of values in]+2(sum of values in ▭ +4[sum of remaining values]}**

### Simpson's Rule

$$I = \frac{hk}{9}\left\{\begin{array}{l} (\text{sum of the values of f at four corners}) \\ +2(\text{sum of the values of f at the odd positions on the} \\ \text{boundary except the corners}) \\ +4(\text{sum of the values of f at the even positions on the boundary}) \\ +\{4(\text{sum of the values of f at the odd positions})+ \\ 8(\text{sum of the values of f at the even positions}) \\ \text{on the odd row f of the matrix except boundary rows}\}+ \\ \{8(\text{sum of the values of f at the odd positions})+ \\ 16(\text{sum of the values of f at the even positions}) \\ \text{on the even row f of the matrix}\} \end{array}\right\}$$

## Problems based on Double integrals

1. Evaluate $\int_1^{1.4} \int_2^{2.4} \frac{1}{xy}\,dxdy$ using Trapezoidal and Simpson's rule. Verify your result by actual integration.

**Solution:**

Divide the range of x and y into 4 equal parts

$$h = \frac{2.4-2}{4} = 0.1$$

$$k = \frac{1.4-1}{4} = 0.1$$

Get the values of f(x, y) = $\frac{1}{xy}$ at nodal points

| y\x | 2 | 2.1 | 2.2 | 2.3 | 2.4 |
|---|---|---|---|---|---|
| 1 | 0.5 | 0.4762 | 0.4545 | 0.4348 | 0.4167 |
| 1.1 | 0.4545 | 0.4329 | 0.4132 | 0.3953 | 0.3788 |
| 1.2 | 0.4167 | 0.3968 | 0.3788 | 0.3623 | 0.3472 |
| 1.3 | 0.3846 | 0.3663 | 0.3497 | 0.3344 | 0.3205 |
| 1.4 | 0.3571 | 0.3401 | 0.3247 | 0.3106 | 0.2976 |

(i) Trapezoidal Rule

$$I = \frac{(0.1)(0.1)}{4}\left\{ \begin{array}{l} (0.5 + 0.4167 + 0.3571 + 0.2976) + 2\left( \begin{array}{l} 0.4545 + 0.4167 + 0.3846 + 0.4762 + 0.4545 + \\ 0.4348 + 0.3788 + 0.3472 + 0.3205 + 0.3106 + \\ 0.3247 + 0.3401 \end{array} \right) \\ + 4(0.4329 + 0.4132 + 0.3953 + 0.3968 + 0.3788 + 0.3623 + 0.3663 + 0.3497 + 0.3344) \end{array} \right\}$$

**<u>I=0.0614</u>**

(ii) Simpson's Rule:

$$I = \frac{(0.1)(0.1)}{9}\left\{ \begin{array}{l} (0.5 + 0.4167 + 0.3571 + 0.2976) + 2(0.4167 + 0.4545 + 0.3472 + 0.3247) \\ + 4(0.3846 + 0.4545 + 0.4762 + 0.4348 + 0.3788 + 0.3205 + 0.3106 \\ + 0.3401 + 0.3788) + 8(0.3968 + 0.3623 + 0.3497 + 0.4132) + \\ 16(0.3663 + 0.3344 + 0.4329 + 0.3953) \end{array} \right\}$$

I= $\frac{0.01(55.2116)}{9}$

**<u>I=0.0613</u>**

---------------------------------------------------------------------------------------------------------

# Unit - 3

## Part - A

1. State Newton's forward different formula to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = x_0$

Ans:- $\left(\frac{dy}{dx}\right)_{x=x_0} = \frac{1}{h}\left[\Delta y_0 - \frac{\Delta^2 y_0}{2} + \frac{\Delta^3 y_0}{3} - \cdots\right]$ and

$$\left(\frac{d^2y}{dx^2}\right)_{x=x_0} = \frac{1}{h^2}\left[\Delta^2 y_0 - \Delta^3 y_0 + \frac{11}{12}\Delta^4 y_0 - \cdots\right]$$

---

2. Write the formula to compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = x_0 + ph$ for a given data $(x_i, y_i)$ $i = 0, 1, \ldots, n$.

Ans:- $\frac{dy}{dx} = \frac{1}{h}\left[\Delta y_0 + \frac{2p-1}{2}\Delta^2 y_0 + \frac{3p^2-6p+2}{6}\Delta^3 y_0 + \frac{2p^3-9p^2+11p-3}{12}\Delta^4 y_0 + \cdots\right]$

$$\frac{d^2y}{dx^2} = \frac{1}{h}\left[\Delta^2 y_0 + (p-1)\Delta^3 y_0 + \frac{6p^2-18p+11}{12}\Delta^4 y_0 + \cdots\right]$$

---

3. State Newton's backward interpolation formula to find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = x_n$.

Ans:- $\left.\frac{dy}{dx}\right|_{x=x_n} = \frac{1}{h}\left[\nabla y_n + \frac{\nabla^2 y_n}{2} + \frac{\nabla^3 y_n}{3} + \cdots\right]$

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_n} = \frac{1}{h^2}\left[\nabla^2 y_n + \nabla^3 y_n + \frac{11}{12}\nabla^4 y_n + \cdots\right].$$

---

4. Write the formula to compute $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ at $x = x_n + ph$ for the data $(x_i, y_i)$, $i = 0, 1, \ldots, n$.

Ans: $\left.\frac{dy}{dx}\right|_x = \frac{1}{h}\left[\nabla y_n + \frac{2p+1}{2}\nabla^2 y_n + \frac{3p^2+6p+2}{6}\nabla^3 y_n + \frac{2p^3+9p^2+11p+3}{12}\nabla^4 y_n + \cdots\right]$

$$\frac{d^2y}{dx^2} = \frac{1}{h^2}\left[\nabla^2 y_n + (p+1)\nabla^3 y_n + \frac{6p^2+18p+11}{12}\nabla^4 y_n + \cdots\right]$$

5. Find $\frac{dy}{dx}$ at $x=2$ from the following data.

| | | | |
|---|---|---|---|
| x | 2 | 3 | 4 |
| y | 26 | 58 | 112 |

<u>Ans</u>: $\Delta y_0 = 32$ $\Delta y_1 = 54$ $\Delta^2 y_0 = 22$

$$\frac{dy}{dx} = 32 - \frac{1}{2}(22) = 21.$$

6. Find $\frac{dy}{dx}$ at $x=6$ from following table

| | | | |
|---|---|---|---|
| x | 2 | 4 | 6 |
| y | 3 | 11 | 27 |

<u>Ans</u>: $\nabla y_n = 16$ $\nabla y_{n-1} = 8$ $\nabla^2 y_n = 16-8 = 8$.

$$\left.\frac{dy}{dx}\right|_{x=6} = \frac{1}{2}\left[16 + \frac{8}{2}\right] = 10.$$

7. A curve passing through the points (1,0), (2,1) and (4,5). Find the slope of the curve at $x=3$

<u>Ans</u>:

| x | y | $\Delta y$ | $\Delta^2 y$ |
|---|---|---|---|
| 1 | 0 | | |
| | | 1 | |
| 2 | 1 | | 1/3 |
| | | 2 | |
| 4 | 5 | | |

unequal intervals. So we use Newton's Divided difference formula

$$f(x) = f(x_0) + (x-x_0) f(x_0,x_1) + (x-x_0)(x-x_1) f(x_0,x_1,x_2)$$

$$= 0 + (x-1)(1) + (x-1)(x-2)\, 1/3 = x-1 + \frac{1}{3}(x^2 - 3x + 2)$$

$$f'(x) = 1 + \frac{2x}{3} - 1 = \frac{2x}{3}$$

Slope at $x=3$ is $\frac{2(3)}{3} = 2$

8. State the basic principle for deriving Simpson's 1/3 rule.

Ans: The curve passing through three consecutive points is replaced by a parabola.

9. State the order of error in Simpson's 1/3 rule.

Ans:- Error in Simpson's 1/3 rule is of order $h^4$.

10. Using Simpson's rule, find $\int_0^4 e^x\,dx$ given $e^0=1$, $e^1=2.72$, $e^2=7.39$, $e^3=20.09$ and $e^4=54.6$

Ans:- $\int_0^4 e^x\,dx = \frac{1}{3}\left[(1+54.6)+4(2.72+20.09)+2(7.39)\right]$

$= 53.873$

[ No. of intervals 4 = even. we use simpson's 1/3 rule]

11. A curve passes through (2,8), (3,27), (4,64) and (5,125). Find the area of the curve between x axis and the lines $x=2$ and $x=5$ by trapezoidal rule.

Ans:- $\int_2^5 y\,dx = \frac{1}{2}\left[(8+125)+2(27+64)\right] = 157.5$ Sq. units

12. Compute $\int_1^2 \frac{dx}{x}$ using simpson's rule with $h=0.25$

Ans:-

| x | 1 | 1.25 | 1.5 | 1.75 | 2 |
|---|---|---|---|---|---|
| y | 1 | 0.8 | 0.666 | 0.571 | 0.5 |

No. of intervals = 4 = even. we use simpson's 1/3 rule.

$\int_1^2 \frac{dx}{x} = \frac{0.25}{3}\left[(1+0.5)+4(0.8+0.571)+2(0.666)\right]$

$= 0.6931.$

13. Use Simpson's 3/8 rule with $h=0.5$ to evaluate $\int_0^1 \frac{dx}{1+x}$.

**Ans:-**

| $x$ | 0 | 0.5 | 1 |
|---|---|---|---|
| $y$ | 1 | $\frac{2}{3}$ | $\frac{1}{2}$ |

No. of intervals = 2 = not divisible by 3. So we cannot use simpson's 3/8 rule

---

14. Evaluate $\int_{-1}^{1} |x| dx$ with two subintervals by Simpson's 1/3 rule and by Trapezoidal rule.

**Ans:-** By Simpson's rule, $I = \frac{1}{3}[1+0+1] = \frac{2}{3}$.

By Trapezoidal rule $I = \frac{1}{2}[1+1] = 1$

---

15. If $I_1 = 0.775$, $I_2 = 0.7828$, find $I$ using Romberg's method.

**Ans:-**

By Romberg's method $I = I_2 + \left(\frac{I_2 - I_1}{3}\right) = 0.7802$.

---

16. Find $I = \int_{-1}^{1} \frac{dx}{1+x^2}$ by two point Gaussian formula.

**Ans:-**

$I = \frac{1}{1+1/3} + \frac{1}{1+1/3} = \frac{3}{2} = 1.5$ $\qquad I = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

---

17. Find $I = \int_0^1 \frac{dx}{1+x}$ by Gaussian two point formula

**Ans:-** $I = \int_{-1}^{1} \frac{dt}{t+3}$ using $x = \frac{1+t}{2}$ then $I = \frac{1}{3+\frac{1}{\sqrt{3}}} + \frac{1}{3-\frac{1}{\sqrt{3}}} = 0.6923$

---

18. Evaluate $\int_{-1}^{1} \cos x \, dx$ using two point Gaussian formula.

**Ans:-**

$\int_{-1}^{1} \cos x \, dx = 2\cos\left(\frac{1}{\sqrt{3}}\right) = 1.6758$.

# UNIT IV
# INITIAL VALUE PROBLEMS FOR ORDINARY DIFFERENTIAL EQUATIONS

Taylor series method

Euler methods

Runge-Kutta method for solving first and second order equations

Milne'spredictor and corrector method

Adam's predictor and corrector method.

## Introduction

An ordinary differential equation of order n is a relation of the form $F(x, y, y', y'', ..., y^{(n)}) = 0$ where $y = y(x)$ and $y^{(r)} = \frac{d^r y}{dx^r}$. The solution of this differential equation involves n constants and these constants are determined with the help of n conditions $y, y', y'', ..., y^{(n-1)}$ are prescribed at $x = x_0$, by

$y(x_0) = y_0, y'(x_0) = y_0', ... y^{(n-1)}(x_0) = y_0^{(n-1)}$

These conditions are called the *initial conditions* because they depend only on $x_0$.

*The differential equation together with the initial conditions is called an initial value problem.*

## Taylor's Series

## Point wise solution

If $y(x)$ is the solution of (1), then by Tailor series,

$$y(x) = y_0 + \frac{(x - x_0)}{1!} y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \frac{(x - x_0)^3}{3!} y_0''' + ...$$

Put $x_1 = x_0 + h$ where h is the step-size, we have

$$y(x_1) = y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + ... \quad (3)$$

once $y_1$ has been calculated form (1), $y_1', y_1'', y_1'''$ can be calculated from

$y' = f(x, y)$

Expanding $y(x)$ in a Taylor series about $x = x_1$, we get

$$y_2 = y_1 + \frac{h}{1!} y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + ...$$

Where $y_2 = y(x_2)$ and $x_2 = x_1 + h$

The Tailor algorithm is given as follows

$$y_{m+1} = y_m + \frac{h}{1!} y_m' + \frac{h^2}{2!} y_m'' + \frac{h^3}{3!} y_m''' + ...$$

Where $y_m^{(r)} = \frac{d^r y}{dx^r}$ at the point $(x_m, y_m)$ where $m = 0,1,2,\ldots$

## Problems based on Taylor's Series

Solve $y' = y^2 + x; y(0) = 1$ using Taylor series method and compute $y(0.1)$ and $y(0.2)$

**Solution**

Here $x_0 = 0, y_0 = 1$.

Given $y' = y^2 + x$ ; $y_0' = 1$

$y'' = 2yy' + 1$ ; $y_0'' = 3$

$y''' = 2yy'' + 2y'^2$ ; $y_0''' = 8$

$y^{iv} = 6y'' + 2yy'''$ ; $y_0^{iv} = 34$ etc.,

To find

Here $h = 0.1$

By Taylor algorithm

$$y_1 = y_0 + \frac{h}{1!} y_0^{'} + \frac{h^2}{2!} y_0^{''} + \frac{h^3}{3!} y_0^{'''} + ...$$

$$\therefore y_1 = 1 + (0.1)(1) + \frac{0.01}{2}(3) + \frac{0.001}{6}(8) + \frac{0.001}{24}(34)$$

$$y(0.1) = 1.1164$$

To find $y(0.2)$

$x_2 = x_1 + h$ where $x_2 = 0.2$

$$y_2 = y_1 + \frac{h}{1!} y_1^{'} + \frac{h^2}{2!} y_1^{''} + \frac{h^3}{3!} y_1^{'''} + ...$$

$$y_1^{'} = 0.1 + (1.1164)^2 = 1.3463$$

$$y_1^{''} = 1 + 2(1.1164)(1.3463) = 4.006$$

$$y_1^{'''} = 2(1.1164)(4.006) + 2(1.3463)^2$$

$$= 12.5696$$

$$y(0.2) = 1.1164 + (0.1)(1.3463) + \frac{0.01}{2}(4.006) + \frac{0.0001}{6}(12.5696)$$

$$\Rightarrow y(0.2) = 1.2732$$

---

2. Evaluate $y(0.1)$ and $y(0.2)$, correct to four decimal places by Taylor series method, if $y(x)$ satisfies $y' = xy + 1, y(0) = 1$

**Solution**

Here $x_0 = 0, y_0 = 1$

| | | |
|---|---|---|
| $y' = xy + 1$ | ; | $y_0^{'} = 1$ |
| $y'' = xy' + y$ | ; | $y_0^{''} = 1$ |
| $y''' = xy'' + 2y^{'}$ | ; | $y_0^{''} = 2$ |
| $y^{iv} = xy''' + 3y''$ | ; | $y_0^{iv} = 3$ |

To find $y(0.1)$

By Taylor algorithm

$$y_1 = y_0 + \frac{h}{1!} y_0^{'} + \frac{h^2}{2!} y_0^{''} + \frac{h^3}{3!} y_0^{'''} + ...$$

Let $x_1 = x_0 + h$ where $h = 0.1$

$\therefore x_1 = 0.1$

$$\therefore y_1 = 1 + (0.1) + \frac{0.01}{2}(1) + \frac{0.001}{3}(2) + \frac{0.0001}{24}(3)$$

$$= 1.1057$$

To find $y(0.2)$

Let $y_2 = y(x_2)$ where $x_2 = x_1 + h$

$\therefore x_2 = 0.2$

By Taylor algorithm,

$$y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \dots$$

$y_1' = 1 + (0.1)(1.1057) = 1.11057$

$y_1'' = (0.1)(1.11057) + 1.1057 = 1.216757$

$y_1''' = (0.1)(1.216757) + 2(1.11057) = 2.3428157$

$$\therefore y_2 = 1.1057 + (0.1)(1.11057) + \frac{0.01}{2}(1.216757) + \frac{0.0001}{6}(2.342815)$$

$\Rightarrow y_2 = 1.2178$

Hence $y(0.1) = 1.1057$ and $y(0.2) = 1.2178$ .

---

3.Solve by Taylor series method, $y' = xy + y^2, y(0) = 1$ at $x = 0.1$ and $0.2$, , correct to four decimal places.

**Solution**

Given $x_0 = 0, y_0 = 1$

$y' = xy + y^2$ ; $y_0' = 1$

$y'' = y + xy' + 2yy'$ ; $y_0'' = 3$

$y''' = 2y' + xy'' + 2(y')^2 + 2yy''$ ; $y_0''' = 10$

$y^{iv} = 3y'' + xy''' + 6y'y'' + 2yy'''$ ; $y_0^{iv} = 47$

To find $y_1 = y(0.1)$

Let $x_1 = x_0 + h$. Here $h = 0.1$

$x_1 = x_0 + h \Rightarrow x_1 = 0.1$

By Taylor algorithm

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \dots$$

$$\therefore y_1 = 1 + (0.1)(1) + \frac{0.01}{2}(3) + \frac{0.001}{6}(10) + \frac{0.0001}{24}(47)$$

$= 1.1 + 0.015 + 0.0017 + 0.0002$

$\therefore y(0.1) = 1.1169$

To find $y(0.2)$

$y_2 = y(x_2)$ where $x_2 = x_1 + h$

Here $x_1 = 0.1, h = 0.1, x_2 = 0.02$

By Taylor algorithm,

$$y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \dots$$

$x_1 = 0.1, y_1 = 1.1169$

$y_1' = x_1 y_1 + y_1^2 \Rightarrow y_1' = 1.35925$

$y_1'' = y_1 + x_1 y_1' + 2y_1 y_1' \Rightarrow y_1' = 1.35925$

$= 1.1169 + (0.1)(1.35925) + 2(1.1169)(1.35925)$
$= 4.2891$
$y_1''' = 2y_1' + x_1 y_1'' + 2(y_1')^2 + 2y_1 y_1''$
$= 2.7185 + 0.42891 + 3.6951 + 6.5810$
$= 16.4235$
$y_1^{iv} = 3y_1'' + x_1 y_1''' + 6y_1 y_1'' + 2y_1 y_1'''$
$= 212.8673 + 1.64235 + 34.9797 + 36.6868$
$= 86.17615$

$\therefore y_2 = 1.1169 + (0.1)(1.35925) + \frac{0.01}{2}(4.2891) + \frac{0.001}{6}(16.4235) + \frac{0.0001}{24}(86.17615)$
$= 1.1169 + 0.135925 + 0.02144 + 0.00274 + 0.00036$
$\therefore y(0.2) = 1.2774$
Hence $y(0.1) = 1.1169$ and $y(0.2) = 1.2774$.

---

## Taylor series method for simultaneous first order differential equation

We can solve the equations of the form $\frac{dy}{dx} = f(x, y, z); \frac{dy}{dx} = g(x, y, z)$ with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$.

The values of y and z at $x_1 = x_0 + h$ are given by Taylor algorithm,

$$y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots$$

$$z_1 = z_0 + \frac{h}{1!} z_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0''' + \ldots$$

The derivatives on R.H.S of the above expressions are found at $x = x_0$ using the given equations.

Similarly $y_2$ and $z_2$ corresponding to $x_2 = x_1 + h$ are calculated by Taylor series method.

## Problems

1.Evaluate $x(0.1)$ and $y(0.1)$ given $\frac{dx}{dt} = 1 + ty; \frac{dy}{dt} = -tx$ given $x = 0, y = 1$ at $t = 0$ by Taylor series method.

## Solution

Given $t_0 = 0, x_0 = 0, y_0 = 1$

| | |
|---|---|
| $x' = 1 + ty$ | $y' = -tx$ |
| $x'' = y + ty'$ | $y'' = -[x + tx']$ |
| $x''' = 2y' + ty''$ | $y''' = -[2x' + tx'']$ |
| $x^{iv} = 3y'' + ty'''$ | $y^{iv} = -[3x'' + tx''']$ |

Then

| | |
|---|---|
| $x_0' = 1$ | $y_0' = 0$ |

$x_0^{''}=1$ $\quad$ $y_0^{''}=0$

$x_0^{'''}=0$ $\quad$ $y_0^{'''}=-2$

$x_0^{iv}=0$ $\quad$ $y_0^{iv}=-3$

By Taylor algorithm, we have

$$y_1=y_0+\frac{h}{1!}y_0^{'}+\frac{h^2}{2!}y_0^{''}+\frac{h^3}{3!}y_0^{'''}+\ldots$$

$$\therefore y(0.1)=1+(0.1)(0)+\frac{0.01}{2}(0)+\frac{0.001}{6}(-2)+\frac{0.0001}{24}(-3)$$

$$\therefore y(0.1)=0.9997$$

$$x(0.1)=x_1=x_0+\frac{h}{1!}x_0^{'}+\frac{h^2}{2!}x_0^{''}+\frac{h^3}{3!}x_0^{'''}+\ldots$$

$$=0+(0.1)(1)+\frac{0.01}{2}(1)+\ldots..$$

$$=0.105$$

---

2. Find $x(0.2)$ and $y(0.2)$ using Taylor series method given that $\frac{dx}{dt}=xy+2t;\frac{dy}{dt}=2ty+x$, with initial conditions $x=1,y=1$ at $t=0$.

**Solution**

Given $t_0=0,x_0=1,y_0=-1$

Taking $h=0.2,t_1=t_0+h\Rightarrow t_1=0.2$

$x'=xy+2t$ $\quad$ $y'=2ty+x$

$x''=xy'+x'y+2$ $\quad$ $y''=2ty'+2y+x'$

$x'''=xy''+2x'y+x''y$ $\quad$ $y'''=4y'+2ty''+x''$

$x_0^{'}=-1$ $\quad$ $y_0^{'}=1$

$x_0^{''}=4$ $\quad$ $y_0^{''}=-3$

$x_0^{'''}=-9$ $\quad$ $y_0^{'}=8$

By Taylor algorithm, we have

$$x_1=x_0+\frac{h}{1!}x_0^{'}+\frac{h^2}{2!}x_0^{''}+\frac{h^3}{3!}x_0^{'''}+\ldots$$

$$y_1=y_0+\frac{h}{1!}y_0^{'}+\frac{h^2}{2!}y_0^{''}+\frac{h^3}{3!}y_0^{'''}+\ldots$$

$$\therefore x(0.2)=1+(0.2)(-1)+2(0.2)^2-\frac{3}{2}(0.2)^3+\ldots$$

$$\therefore x(0.2)=0.796$$

$$y_1=y(0.2)=-1+0.2-(1.5)(0.04)+\frac{4}{3}(0.008)$$

$$=-0.8493$$

Hence $x(0.2)=0.796$ and $y(0.2)=-0.8493$

---

## Taylor series method for second order differential equations

Consider the differential equation $\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$ with the initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_0'$, where $y_0, y_0'$ are known values.

This equation can be reduced into a set of simultaneous equations, by putting $y' = p$

$\therefore$ we have $y' = p, y(x_0) = y_0$ (1)

And $p' = f(x, y, p), p(x_0) = p_0 = y_0'$ (2)

Successively differentiating $y''$, the expression for $y''', y^{iv}$ etc., are known.

By Taylor series method, we find

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots$$

$$= y_0 + hp_0 + \frac{h^2}{2!}p_0'' + \frac{h^3}{3!}p_0''' + \ldots$$

Also by Taylor series method, we have

$$p_1 = y_1' = p_0 + \frac{h}{1!}p_0' + \frac{h^2}{2!}p_0'' + \frac{h^3}{3!}p_0''' + \ldots$$

Then the values $y_1'', y_1''', y_1^{iv}$ are found from $y_1'', y_1'''$ etc

$$y_2 = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \ldots$$

Thus we calculate $y_1, y_2, \ldots$

**Problems**

1.Find $y(0.2)$ and $(0.4)$ given $y'' = xy$ if $y(0) = 1, y'(0) = 1$ by Taylor series method.

**Solution**

Given $y'' = xy, x_0 = 0, y_0 = 1$ and $y'(0) = 1$

Then we have

$y''' = xy' + y$

$y^{iv} = xy'' + 2y'$

$y^{v} = 3y'' + xy'''$ etc

$\therefore y_0'' = 0, y_0''' = 1, y_0^{(iv)} = 2, y_0^{(v)} = 0$

By Taylor Algorithm,

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots$$

Taking, $h = 0.2, x_1 = x_0 + h \Rightarrow x_1 = 0.2$

$$y_1 = y(x_1) = 1 + (0.2)(1) + \frac{0.04}{2}(0) + \frac{0.008}{6}(1) + \frac{0.0016}{24}(2)$$

$\therefore y(0.2) = 1.2014$

To find $y_1'$

Set $p = y'$. Then $p' = xy$

$\therefore p'' = xy' + y;\ p''' = xy'' + 2y';\ p^{iv} = 3y'' + xy'''$

$\therefore p_0 = 1,\ p_0' = 0,\ p_0'' = 1,\ p_0''' = 2,\ p_0^{iv} = 0$

By Taylor Algorithm,

$$p_1 = p_0 + \frac{h}{1!}p_0' + \frac{h^2}{2!}p_0'' + \frac{h^3}{3!}p_0''' + \dots$$

$$\therefore p_1 = 1 + (0.2)(0) + \frac{0.04}{2}(1) + \frac{0.08}{6}(2)$$

$\Rightarrow y_1' = 1 + 0.02 + 0.0027$

$= 1.0227$

Let $x_2 = x_1 + h$. Since $h = 0.2, x_1 = 0.2 \Rightarrow x_2 = 0.4$

By Taylor series method,

$$\therefore y(x_2) = y(0.4) = y_1 + \frac{h}{1!}y_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \dots$$

$y'' = xy \Rightarrow y_1'' = x_1 y_1 = (0.2)(1.2014) = 0.24028$

$y''' = xy' + y \Rightarrow y_1''' = (0.2)(1.0227) + 1.2014 = 1.40594$

$y^{iv} = 2y' + xy'' \Rightarrow y_1^{iv} = (0.2)(0.024028) + 2(1.0227) = 0.00048 + 2.0454 = 2.0459$

$$\therefore y_2 = 1.2014 + (0.2)(1.0227) + \frac{0.04}{2}(0.024028) + \frac{0.008}{6}(1.40594) + \frac{0.0016}{24}(2.0459)$$

$\Rightarrow y(0.4) = 1.4084$

Hence $y(0.2) = 1.2014$ and $y(0.4) = 1.4084$

---

2.Find y at $x = 1.1, 1.2$ given $y'' = x^3 - y^2 y', y(1)$ and $y'(1) = 1$, correct to four decimal places using Taylor series method.

**Solution**

Given $x_0 = 1, y_0 = 1; y_0' = 1$

$y'' = x^3 - y^2 y' \Rightarrow y_0'' = 0$

$y''' = 3x^2 - 2y(y')^2 - y^2 y'' \Rightarrow y_0'' = 1$

$y^{iv} = 6x - 2y^3 - 4yy'y'' - 2yy'y'' - y^2 y'''$

$\Rightarrow y_0^{iv} = 6 - 2 - 0 - 0 - 1 = 3$

$\therefore y_0' = 0, y_0'' = 0, y_0''' = 1, y_0^{(iv)} = 3$

By Taylor series method,

$$y_1 = y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \dots$$

Taking, $h = 0.1, x_0 = 1, x_1 = x_0 + h = 0.1$

We have

$$\therefore y(1.1) = 1 + (0.1)(1) + \frac{0.01}{2}(0) + \frac{0.001}{6}(1) + \frac{0.0001}{24}(3)$$

$\therefore y(1.1) = 1.1002$

To find $y_1^{'}$

Set $p = y'$. Then $p' = x^3 - y^2 y'$, $p_0 = y_0^{'} = 1$

$p_0^{1} = 1 - 1 = 0,\ p_0^{''} = 1,\ p_0^{'''} = 3$ etc.,

$$\therefore p_1 = p_0 + \frac{h}{1!} p_0^{'} + \frac{h^2}{2!} p_0^{''} + \frac{h^3}{3!} p_0^{'''} + \ldots$$

$$\Rightarrow p_1 = 1 + (0.1)(0) + \frac{0.01}{2}(1) + \frac{0.01}{6}(2)$$

$\therefore p_1 = 1.0053 \Rightarrow y_1^{'} = 1.0053$

$y'' = x^3 - y^2 y' \Rightarrow y_1^{''} = (1.1)^3 - (1.1002)^2 (1.0053)$

$= 0.11415$

$y''' = 3xy^2 - 2y(y')^2 - y^2 y'' \Rightarrow y_1^{'''} = 3(1.1)^3 - 2(1.1002)(1.0053)^2 - (1.1002)^2 (0.11415)$

$= 1.268$

By Taylor Algorithm,

$$y_2 = y_1 + \frac{h}{1!} y_1^{'} + \frac{h^2}{2!} y_1^{''} + \frac{h^3}{3!} y_1^{'''} + \ldots$$

Where $y_2 = y(x_2), x_2 = 1.2$

$$= 1.1002 + (0.1)(1.0053) + \frac{0.01}{2}(0.11415) + \frac{0.01}{6}(1.268)$$

$\Rightarrow y(1.2) = 1.2015$

Hence $y(1.1) = 1.1002$ and $y(1.2) = 1.2015$

---

## Euler method

Let $y_1 = y(x_i)$, where $x_1 = x_0 + h$

Then $y_1 = y(x_0 + h)$. Then by Taylor's series,

$$y_1 = y(x_0) + \frac{h}{1!} y'(x_0) + \frac{h^2}{2!} y''(x_0) + \ldots \qquad (1)$$

Neglecting the terms with $h^2$ and higher powers of h, we get from (1),

$$y_1 = y_0 + hf(x_0, y_0) \qquad (2)$$

Expression (2) gives an approximate value of y at $x_1 = x_0 + h$.

Similarly, we get $y_2 = y_1 + hf(x_1, y_1)$ for $x_2 = x_1 + h$.

$\therefore$ for any m, $y_{m+1} = y_m + hf(x_m, y_m), m - 0,1,2,\ldots$ (3)

In Euler's method, we use (3) to compute successively $y_1, y_2, \ldots$ etc., with an Error $= 0(h^2)$

## Modified Euler method

The algorithm presented already in Modified Euler method in unit IV is sometimes referred as Improved Euler Method.

Therefore a different algorithm for Modified Euler method to solve

$\frac{dy}{dx} = f(x, y), y(x_0) = y_0$ is explained with illustrations.

**Explanation: Modified Euler Method**

$$y(x+h) = y(x) + h\left[ f\left( x + \frac{h}{2}, y + \frac{h}{2} f(x,y) \right) \right]$$

(or) $y_{n+1} = y_n + h\left[ f\left( x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x,y) \right) \right]$

Where $y_{n+1} = y(x_n + h)$ and h is the step – size

**Problems based on Euler's Method**

1,Using Euler's method, compute y in the range $0 \le x \le 0.5$, if y satisfies $\frac{dy}{dx} = 3x + y^2, y(0) = 1$.

**Solution**

Here $f(x,y) = 3x + y^2, x_0 = 0, y_0 = 1$

By Euler's, method

$y_{n+1} = y_n + hf(x_n, y_n), n = 0,1,2,...$

Choosing $h = 0.1$, we compute the values of y using (1)

$y_{n+1} = y_n + hf(x_n, y_n), n = 0,1,2,...$

$y(0.1) = y_1 = y_0 + hf(x_0, y_0) = 1 + (0.1)[3(0) + (1)^2]$

$= 1.1$

$y(0.2) = y_2 = y_1 + hf(x_1, y_1) = 1.1 + (0.1)[0.3 + (1.1)^2]$

$= 1.251$

$y(0.3) = y_3 = y_2 + hf(x_2, y_2) = 1.251 + (0.1)[0.6 + (1.251)^2]$

$= 1.4675$

$y(0.4) = y_4 = y_3 + hf(x_3, y_3) = 1.4675 + (0.1)[0.9 + (1.4675)^2]$

$= 1.7728$

$y(0.5) = y_5 = y_4 + hf(x_4, y_4) = 1.7728 + (0.1)[1.2 + (1.7728)^2]$

$= 2.2071$

---

2.Given $\frac{dy}{dx} = \frac{y-x}{y+x}$, with $y = 1$ for $x = 0$. Find y approximately for $x = 0.1$ by Euler's method in five steps.

**Solution**

Given $y_0 = 1, x_0 = 0$, choosing $h = 0.002$,

$x_i = x_0 + ih, i = 1,2,3,4,5$

To find $y_1, y_2, y_3, y_4$ and $y_5$ where $y_i = y(x_i)$

Here $f(x,y) = \frac{y-x}{y+x}$

Using $y_{n+1} = y_n + hf(x_n, y_n), n = 0,1,2,...$

We get

$$y_1 = y_0 + hf(x_0, y_0) = 1 + (0.02)\left[\frac{1-0}{1+0}\right]$$

$$= 1.02$$

$$y_2 = y_1 + hf(x_1, y_1) = 1.02 + (0.02)\left[\frac{1.02 - 0.02}{1.02 + 0.02}\right]$$

$$= 1.0392$$

$$y_3 = y_2 + hf(x_2, y_2) = 1.0392 + (0.02)\left[\frac{1.0392 - 0.04}{1.0392 + 0.04}\right]$$

$$= 1.0577$$

$$y_4 = y_3 + hf(x_3, y_3) = 1.0577 + (0.02)\left[\frac{1.0577 - 0.06}{1.0577 + 0.06}\right]$$

$$= 1.0756$$

$$y_5 = y_4 + hf(x_4, y_4) = 1.0756 + (0.02)\left[\frac{1.0756 - 0.08}{1.0756 + 0.08}\right]$$

$$= 1.0928$$

Hence $y = 1.0928$ and $x = 0.1$

---

3.Compute y at $x = 0.25$ by Modified Euler method given $y' = 2xy, y(0) = 1$

**Solution**

Here $f(x, y) = 2xy, x_0 = 0, y_0 = 1$

Choose $h = 0.25, x_1 = x_0 + h = 0.25$

By Modified Euler method

$$y_1 = y_0 + h\left[f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right)\right]$$

$$\therefore f(x_0, y_0) = 2(0)(1) = 0$$

$$f\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2} f(x_0, y_0)\right) = f(0.125, 1) = 0.25$$

$$\therefore y_1 = 1 + (0.25)(0.25) = 1.0625$$

Hence $\therefore y(0.25) = 1.0625$ .

---

4.Using Modified Euler method, find $y(0.1)$ and $y(0.2)$ given $\frac{dy}{dx} = x^2 + y^2, y(0) = 1$.

**Solution**

Here $x_0 = 0, y_0 = 1, h = 0.1$

$$f(x, y) = x^2 + y^2, y_0 + \frac{h}{2} f(x_0, y_0) = 1 + \frac{0.1}{2}(0 + 1) = 1.05$$

$$y_1 = y_0 + hf\left(x_0 + \frac{h}{2}, y_0 + \frac{h}{2}(x_0, y_0)\right) = 1 + 0.1 f(0.05, 1.05)$$

$= 1 + 0.1\left[0.05^2 + (1.05)^2\right] = 1.1105$

$y(0.1) = 1.1105$

$f(x_1, y_1) = f(0.1, 1.1105) = (0.1)^2 + (1.1105)^2 = 1.24321$

$y_1 + \frac{h}{2} f(x_1, y_1) = 1.1105 + (0.05)(1.24321) = 1.17266$

$\therefore y_2 = y_1 + hf\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} f(x_1, y_1)\right)$

$= 1.1105 + (0.1) f(0.15, 1.17266)$

$= 1.1105 + (0.1)\left[(0.15)^2 + (1.17266)^2\right]$

$\therefore y(0.2) = y_2 = 1.2503$

$y(0.2) = 1.2503$

---

**Fourth-order Range-Kutta method**

This method is commonly used for solving the initial value problem $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$

**Working Rule**

The value of $y_1 = y(x_1)$ where $x_1 = x_0 + h$ where h is the step-size is obtained as follows. We calculate successively.

$k_1 = hf(x_0, y_0)$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$k_4 = hf(x_0 + h, y_0 + k_3)$

Finally compute the increment

$\Delta y = \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

The approximate value of $y_1$ is given by

$y_1 = y_0 + \Delta y \Rightarrow y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

Error in R-K fourth order method $= 0(h^5)$

In general, the algorithm can be written as

$y_{m+1} = y_m + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$ where

$k_1 = hf(x_m, y_m)$

$k_2 = hf\left(x_m + \frac{h}{2}, y_m + \frac{k_1}{2}\right)$

$$k_3 = hf\left(x_m + \frac{h}{2}, y_m + \frac{k_2}{2}\right)$$

$$k_4 = hf(x_m + h, y_m + k_3) \qquad \text{where } m = 0,1,2,\ldots$$

**Runge-Kutta method for second order differential equations**

Consider the second order differential equation $\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$ with initial conditions $y(x_0) = y_0$ and $y'(x_0) = y_0'$. This can be reduced to a system of simultaneous linear first order equations, by putting $z = \frac{dy}{dx}$. Then we have,

$$\frac{dy}{dx} = z \text{ with } y(x_0) = y_0$$

$$\frac{dz}{dx} = f(x, y, z), \text{ with } z(x_0) = y'(x_0) = y_0'$$

i.e., $\frac{dy}{dx} = g(x, y, z)$, where $g(x, y, z) = z$

and $\frac{dz}{dx} = f(x, y, z)$ with initial conditions $y(x_0) = y_0$ and $z(x_0) = z_0$ where $z_0 = y_0'$

Now, starting from $(x_0, y_0, z_0)$, the increments $\Delta y$ and $\Delta z$ in y and z are given by (h-step size)

$$k_1 = hg(x_0, y_0, z_0) \qquad l_1 = hf(x_0, y_0, z_0)$$

$$k_2 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) \qquad l_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right)$$

$$k_3 = hg\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) \qquad l_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right)$$

$$k_4 = hg(x_0 + h, y_0 + k_3, z_0 + l_3) \qquad l_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3)$$

$$\Delta y = \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4] \qquad \Delta z = \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$$

Then for $x_1 = x_0 + h$, the values of y and z are $y_1 = y_0 + \Delta y$ and $z_1 = z_0 + \Delta z$ respectively. By repeating the above algorithm the value of y at $x_2 = x_1 + h$ can be found.

**Problems based on RK Method**

**1.**The value of y at $x = 0.2$ if y satisfies $\frac{dy}{dx} = x^2 y + x, y(0) = 1$ taking $h = 0.1$ using Runge-Kutta method of fourth order.

**Solution**

Here $f(x, y) = x^2 y + x, x_0 = 0, y_0 = 1.$

Let $x_1 = x_0 + h$, choosing $h = 0.1, x_1 = 0.1.$

Then by R-K fourth order method,

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$k_1 = hf(x_0, y_0) = (0.1)[0 + 0] = 0$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = hf(0.05, 1) = 0.00525$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = hf(0.05, 1.0026) = 0.00525$

$k_4 = hf(x_0 + h, y_0 + k_3) = hf(0.1, 1.00525) = 0.0110050$

$y_1 = 1 + \frac{1}{6}[0 + 2(0.00525 + 0.00525) + 0.011005] = 1.0053$

$y(0.1) = 1.0053$

To find $y_2 = y(x_2)$ where $x_2 = x_1 + h$. Then $x_2 = 0.2$

$y_2 = y_1 + \frac{1}{6}[k_1 + 2(k_2 + k_3) + k_4]$

$k_1 = hf(x_1, y_1) = (0.1)[0.1 + (0.01)(1.0053)] = 0.0110$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1)[0.15 + (0.15)^2(1.0108)] = 0.01727$

$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = (0.1)[0.15 + (0.15)^2(1.013935)] = 0.01728$

$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1)[0.2 + (0.2)^2(1.02258)] = 0.02409$

$\therefore y_2 = 1.0053 + \frac{1}{6}[0.0110 + 2(0.01727 + 0.01728) + 0.02409] = 1.0227$

$\therefore y(0.2) = 1.0227$

Hence $y(0.2) = 1.0227$ .

---

**2.** Apply Runge-Kutta method to find an approximate value of y for x = 0.2 in steps of 0.1 if $\frac{dy}{dx} = x + y^2, y(0) = 1$, correct to four decimal places.

**Solution**

Here $f(x, y) = x + y^2, x_0 = 0, y_0 = 1$

We choose $h = 0.1$

$y_1 = y_o + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

$k_1 = hf(x_0, y_0) = (0.1)[0 + 1^2] = 0.1$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)[0.05 + (1.05)^2] = 0.1152$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)[0.05 + (1.0576)^2] = 0.1168$

$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)[0.1 + (1.1168)^2] = 0.1347$

$\therefore y_1 = 1 + \frac{1}{6}\left[0.1 + 2(0.1152 + 0.1168) + 0.1347\right]$

$\therefore y_1 = 1.1165$

Hence $y(0.1) = 1.1165$.

---

3.Use Runge-Kutta method to find y when x=1.2 in steps of 0.1, given that $\frac{dy}{dx} = x^2 + y^2$ and $y(1) = 1.5$.

**Solution**

Given $f(x, y) = x^2 + y^2, x_0 = 1, y_0 = 1.5$

Let $x_1 = x_0 + h$, we choose $h = 0.1$

$y_1 = y_o + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

$k_1 = hf(x_0, y_0) = (0.1)\left[1 + (1.5)^2\right] = 0.325$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.1)\left[(1.05)^2 + (1.6625)^2\right] = 0.3866$

$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.1)\left[(1.05)^2 + 2.8673\right] = 0.397$

$k_4 = hf(x_0 + h, y_0 + k_3) = (0.1)\left[(1.1)^2 + (1.897)^2\right] = 0.4809$

$\therefore y_1 = 1.5 + \frac{1}{6}\left[0.325 + 2(0.3866 + 0.397) + 0.4809\right] = 1.8955$

$\Rightarrow y_1 = 1.8955$

To compute $y(1.2)$:

$y_2 = y(x_2)$ where $x_2 = x_1 + h = 1.2$, since $h = 0.1$

$y_2 = y_1 + \frac{1}{6}\left[k_1 + 2(k_2 + k_3) + k_4\right]$

$k_1 = hf(x_1, y_1) = (0.1)\left[(1.1)^2 + (1.8955)^2\right] = 0.4803$

$k_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1}{2}\right) = (0.1)\left[(1.15)^2 + (2.1356)^2\right] = 0.5883$

$k_3 = hf\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2}{2}\right) = (0.1)\left[(1.3225) + (2.1897)^2\right] = 0.6117$

$k_4 = hf(x_1 + h, y_1 + k_3) = (0.1)\left[1.44 + 6.286\right] = 0.7726$

$\therefore y_2 = 1.8955 + \frac{1}{6}\left[0.4803 + 2(0.5883) + 2(0.6117) + 0.7726\right] = 2.5043$

Hence $y(1.2) = 2.5043$.

---

4.Solve the equation $\frac{dy}{dx} = xz + 1, \frac{dz}{dx} = -xy$ for x = 0.3 and 0.6. Given that $y = 0, z = 1$ when $x = 0$

**Solution**

Here $f_1(x,y,z) = 1 + xz, x_0 = 0, y_0 = 0, z_0 = 0$ and $h = 0.3$

To find $y(0.3)$ and $z(0.3)$

$k_1 = hf(x_0, y_0) = (0.3)(0+1) = 0.3$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.3)[0.15 + 1] = 0.3450$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.3)[0.15 \times 0.9966 + 1] = 0.3448$$

$k_4 = hf(x_0 + h, y_0 + k_3) = (0.3)[0.3 \times 0.99224 + 1] = 0.3893$

$l_1 = hf(x_0, y_0, z_0) = (0.3)(0) = 0$

$$l_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = (0.3)[-0.15 \times 0.15] = -0.00675$$

$$l_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = (0.3)[-0.15 \times 0.1725] = -0.00776$$

$l_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) = (0.3)[-0.3 \times 0.3448] = -0.031036$

$$y(0.3) = y_1 = y_0 + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$$

$$= 0 + \frac{1}{6}[0.3 + 2(0.3450 + 0.3448) + 0.3893]$$

$\therefore y(0.3) = 0.3448$

$$z(0.3) = z_1 = z_0 + \frac{1}{6}[l_1 + 2l_2 + 2l_3 + l_4]$$

$$\Rightarrow z(0.3) = 1 - \frac{1}{6}[0 + 2(0.00675 + 0.00776) + 0.031036]$$

$\therefore z(0.3) = 0.9899$

To find y at x = 0.6, the starting values are $x_1 = 0.3, y_1 = 0.3448, z_1 = 0.9899$ and $h = 0.3$

$k_1 = hf(x_0, y_0) = (0.3)[1 + (0.3)(0.9899)] = 0.3891$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.3)[1 + (0.45)(0.9744)] = 0.4315$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.3)[1 + (0.45)(0.9535)] = 0.4287$$

$k_4 = hf(x_0 + h, y_0 + k_3) = (0.3)[1 + (0.6)(0.9142)] = 0.4645$

$l_1 = hf(x_0, y_0, z_0) = (0.3)[-(0.3)(0.3448)] = -0.0310$

$$l_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = (0.3)[-(0.45)(0.53935)] = -0.0728$$

$$l_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = (0.3)[-(0.45)(0.56055)] = -0.0757$$

$l_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) = (0.3)[-(0.6)(0.7735)] = -0.1392$

$$y_2 = y_1 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$= 0.3448 + \frac{1}{6}\left[0.3891 + 2(0.4315 + 0.4287) + 0.4645\right]$$

$$y(0.6) = 0.7738$$

$$z_2 = z_1 + \frac{1}{6}\left[l_1 + 2l_2 + 2l_3 + l_4\right]$$

$$= 0.9899 - \frac{1}{6}\left[0.0310 + 2(0.0728 + 0.0757) + 0.1392\right]$$

$$\therefore z(0.3) = 0.9210$$

---

Using R-K method of fourth order solve $y'' = x(y')^2 - y^2$ for x=0.2, given that y=1 and $y' = 0$ when x = 0.

**Solution**

Let $y' = z$ then $y'' = z'$

Hence the given equation reduces to the form,

$$\frac{dy}{dx} = z \text{ and } \frac{dz}{dx} = xz^2 - y^2$$

Given $x_0 = 0, y_0 = 1, z_0 = 0$ and $h = 0.2$

Take $f_1(x, y, z) = z,\ f_2(x, y, z) = xz^2 - y^2$

$$k_1 = hf(x_0, y_0) = (0.2)(0) = 0$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right) = (0.2)(-0.1) = -0.02$$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right) = (0.2)\left[-0.999\right] = -0.01998$$

$$k_4 = hf(x_0 + h, y_0 + k_3) = (0.3)\left[-0.1958\right] = -0.03916$$

$$l_1 = hf(x_0, y_0, z_0) = (0.2)\left[0 - 1^2\right] = -0.2$$

$$l_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, z_0 + \frac{l_1}{2}\right) = (0.2)\left[(0.1)(-0.1)^2 - (0.99)^2\right] = -0.1998$$

$$l_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, z_0 + \frac{l_2}{2}\right) = (0.2)\left[(0.1)(-0.0999)^2 - (0.99)^2\right] = -0.1958$$

$$l_4 = hf(x_0 + h, y_0 + k_3, z_0 + l_3) = (0.2)\left[(0.2)(-0.1958)^2 - (0.98)^2\right] = -0.1905$$

$$y_1 = y_0 + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$$

$$= 1 - \frac{1}{6}\left[0.2 + 2(0.02 + 0.01998) + 0.03916\right]$$

$$\therefore y(0.2) = 0.9801$$

Also $z(0.2) = 0 - \frac{1}{6}\left[0.2 + 2(0.1998 + 0.1958) + 0.1905\right] = -0.1969$

---

## Multi-Step Methods (Predictor-Corrector Methods)

## Introduction

Predictor-corrector methods are methods which require function values at $x_n, x_{n-1}, x_{n-2}, x_{n-3}$ for the computation of the function value at $x_{n+1}$. A predictor is used to find the value of y at $x_{n+1}$ and then the corrector formula is used to improve the value of $y_{n+1}$.

The following two methods are discussed in this chapter

(1) Milne's Method (2) Adam's Method

## Milne's Predictor-corrector method

Consider the initial value problem $\frac{dy}{dx} = f(x, y), y(x_0) = y_0$. Assume that $y_0 = y(x_0), y_1 = y(x_1), y_2 = y(x_2)$ and $y_3 = y(x_3)$ where $x_{i+1} = x_i + h, i = 0,1,2,3$ are known, these are the starting values.

***Milne's predictor formula***

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right] \text{ and}$$

***Milne's corrector formula***

$$y_{4,c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right] \text{ where } y_4' = f(x_4, y_{4,p})$$

---

## Problems based on Predictor-Corrector method

**1.** Using Milne's method, compute y(0.8) given that $\frac{dy}{dx} = 1 + y^2, y(0) = 1,\quad y(0.2) = 0.2027$, $y(0.4) = 0.4228$ and $y(0.6) = 0.6841$

## Solution

we have the following table of values

| x | y | $y' = 1 + y^2$ |
|---|---|---|
| 0 | 0 | 1.0 |
| 0.2 | 0.2027 | 1.0411 |
| 0.4 | 0.4228 | 1.1787 |
| 0.6 | 0.6481 | 1.4681 |

$\therefore x_0 = 0, x_1 = 0.2, x_2 = 0.4, x_3 = 0.6$

$y_0 = 0, y_1 = 0.2027, y_2 = 0.4228, y_3 = 0.6841$

$y_0' = 1, y_1' = 1.0411, y_2' = 1.1787, y_3' = 1.4681$

*To find* y (0.8):

$x_4 = 0.8$. Here h = 0.2

By Milne's predictor formula,

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$= 0 + \frac{0.8}{3}\left[2(1.0411) - 1.1787 + 2(1.4681)\right]$$

$\therefore y_{4,p} = 1.0239$

$y_4' = f(x_4, y_4) = 1 + (1.0239)^2$

$= 2.0480$

By Milne's corrector formula,

$$y_{4,c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$= 0.4228 + \frac{0.2}{3}\left[1.1787 + 4(1.4681) + 2.0480\right]$$

$$\therefore y(0.8) = 1.0294$$

---

2.Given $y' = x^2 - y, y(0) = 1, y(0.1) = 0.9052, y(0.2) = 0.8213$, find y(0.3) by Taylor series method. Also fine y(0.4) by Milne's method

<u>**Solution**</u>

Given $x_0 = 0$, $y_0 = 1$

$x_1 = 0.1$, $y_1 = 0.9052$

$x_2 = 0.2$, $y_2 = 0.8213$

$x_3 = 0.3$, $y_3 = y(x_3)$

By Taylor algorithm

$$y_3 = y_2 + hy_2' + \frac{h^2}{2!}y_2'' + \ldots$$

$y' = x^2 - y \Rightarrow y'' = 2x - y'$

$y''' = 2 - y'', y^{iv} = y'''$ etc

$\therefore y_2' = (0.2)^2 - 0.8213 = -0.7813$

$y_2'' = 2(0.2) + 0.7813 = 1.1813$

$y_2''' = 2 - 1.1813 = 0.8187$

$y_2^{iv} = -8187$

$$\therefore y_3 = 0.8213 - (0.1)(0.7813) + \frac{0.01}{2}(1.1813) + \frac{0.001}{6}(0.8187) - \frac{0.0001}{24}(0.8187)$$

$\therefore y(0.3) = 0.7492$

For $x_3 = 0.3 y_3 = 0.7492$ and $y_3' = 0.09 - 0.7492 = -0.6592$

Also $y_0' = -1, y_1' = (0.01) - 0.905 = -0.8952$ and $y_2' = -0.7813$

By Milne's method

$$y_{4,p} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$y_{4,p} = 1 - \frac{0.4}{3}\left[2(0.8952) - 0.7813 + 2(0.6952)\right]$$

$y_{4,p} = 0.6897$

$y_4' = (0.16) - (0.6897) = -0.5297$

By Correctors formula,

$$y_{4,c} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$y_{4,c} = 0.8213 - \frac{0.1}{3}\left[0.7813 + 4(0.6592) + 0.5297\right]$$

$$y_{4,c} = 0.6897 \Rightarrow y(0.4) = 0.6897$$

---

## Adam-Bash Forth Predictor-Corrector Method

Using Newton's backward difference interpolation formula, we derive a set of predictor and corrector formulae. This method also requires past four values to estimate the fifth value.

Adam's predictor formula

$$y_{n+1,p} = y_n + \frac{h}{24}\left[55y_n' - 59y_{n-1}' + 37y_{n-2}' - 9y_{n-3}'\right]$$

Adam's corrector formula

$$y_{n+1,c} = y_n + \frac{h}{24}\left[9y_{n+1}' + 19y_n' - 5y_{n-1}' + y_{n-2}'\right]$$

The errors in these formulae are

$\frac{251}{720}h^5 f^{iv}(\xi)$ and $-\frac{19}{720}h^5 f^{iv}(\xi)$ respectively.

In particular,

$$y_{4,p} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

And

$$y_{4,c} = y_3 + \frac{h}{24}\left[9y_4' + 19y_3' - 5y_2' + y_1'\right]$$

---

Given $y' = 1 + y^2$, $y(0) = 0$, $y(0.2) = 0.2027$, $y(0.4) = 0.4228$, $y(0.6) = 0.6841$, estimate $y(0.8)$ using Adam's method.

**Solution**

Form the given data

| | | |
|---|---|---|
| $x_0 = 0$, | $y_0 = 0$, | $y_0' = 1$ |
| $x_1 = 0.2$ | $y_1 = 0.2027$ | $y_1' = 1.0411$ |
| $x_2 = 0.4$ | $y_2 = 0.4228$ | $y_2' = 1.1786$ |
| $x_3 = 0.6$ | $y_3 = 0.6841$ | $y_3' = 1.4680$ |

To fine $y_4$ for $x_4 = 0.8$. Here h =0.2

$$y_{4,p} = y_3 + \frac{h}{24}\left[55y_3' - 59y_2' + 37y_1' - 9y_0'\right]$$

$$y_{4,p} = 0.6841 + \frac{0.2}{24}\left[55(1.4680) - 59(1.1786) + 37(1.0411) - 9\right]$$

$$y_{4,p} = 1.0235$$

$$y_{4,c} = y_3 + \frac{h}{24}\left[9y_4^{'} + 19y_3^{'} - 5y_2^{'} + y_1^{'}\right]$$

$$y_4^{'} = 1 + (1.0235)^2 = 2.0475$$

$$\therefore y_{4,c} = 0.6481 + \frac{0.2}{24}\left[9(2.0475) + 19(1.4680) - 5(1.1786) + 1.0411\right]$$

$$y_{4,c} = 1.0297$$

$$\therefore y(0.8) = 1.0297$$

---

## Unit - IV

### Part - A

1. Write the fourth ~~down~~ order Taylor algorithm.

Ans: Let $y' = f(x, y)$ with $y(x_0) = y_0$. Then the Taylor algorithm is given by

$$y(x_1) = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{(iv)} + \dots \quad \text{Where } x_1 = x_0 + h$$

and $y_0^{(r)} = \frac{d^r y}{dx^r}$ at $(x_0, y_0)$

---

2. What are the merits and demerits of Taylor series method of solution?

Ans: It is powerful single step method. It is the best method if the expression for higher order derivatives are simpler

The major demerit of this method is evaluation of higher order derivatives becomes tedious for complicated algebraic expressions.

---

3. Given $y' = x + y$, $y(0) = 1$. find $y(0) = 1$ by Taylor Series method.

Ans: $y' = x + y \quad y'' = 1 + y' \quad y''' = 0 + y''$

$x_0 = 0 \quad y_0 = 0$ then $y(0.1) = y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \dots$

$$y(0.1) = 1 + 0.1 + \frac{0.01}{2}(2) + \frac{0.001}{6}(2) = 1.1103.$$

---

4. Find $y(0.1)$ by Euler's method, given that $\frac{dy}{dx} = 1 - y \quad y(0) = 0$

Ans: $y_1 = y_0 + h f(x_0, y_0) = 0 + (0.1)(1.0) = 0.1 \Rightarrow y(0.1) = 0.1$

5. State algorithm for modified Euler's method, to solve $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$.

Ans:- $y_{n+1}^{(1)} = y_n + h f(x_n, y_n)$

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_{n+1}^{(1)})\right]$$

Where $n = 0, 1, 2, \ldots$ and $x_{n+1} = x_n + h$.

6. What are the distinguished properties of Rungekutta method?

Ans:- These methods do not require the higher order derivatives and requires only the function values at different points.

To evaluate $y_{n+1}$, we need only $y_n$ but not previous of y's.

The solution by these methods agree with Taylor Series solution upto the terms of $h^r$ where r is the order of R-k method.

7. Which is better Taylor Series method or R-k method? Why?

Ans: R-k method is better. since higher order derivatives of y are not required. Taylor series method involves use of higher order derivatives which may be difficult in case of Complicated algebraic functions.

8. State the order of error in R-k method of fourth order

Ans:- Error $O(h^5)$, where $h$ is the interval of differencing.

9. Write the predictor - Error and Corrector - Error in Milne's Method.

Ans :-

$$\text{Predictor - Error} = \frac{14}{45} h^5 f^{(iv)}(\xi)$$

$$\text{Corrector - Error} = -\frac{h^5}{90} y^{(iv)}(\xi)$$

---

10. Distinguish single - step and Multi - step method.

Ans :-

Single step methods :- To find $y_{n+1}$ the information at $y_n$ is enough.

Multi - step methods :- To find $y_{n+1}$, the past four values $y_{n-3}, y_{n-2}, y_{n-1}, y_n$ are needed.

---

## Part-B

1. Solve $\frac{dy}{dx} = y - x^2$, $y(0)=1$, find (i) $y(0.1)$ and $y(0.2)$ by R-K method (ii) $y(0.3)$ by Euler's method (iii) $y(0.4)$ by Milne's predictor corrector method.

2. Solve $y'' - 0.1(1-y^2)y' + y = 0$ subject to $y(0)=0$, $y'(0)=1$ using 4th order R-K method (i) Find $y(0.2)$ and $y'(0.2)$ use step size $x=0.2$.

3. Use R-K method to obtain an approx. solution to the differential equation $\frac{dy}{dx} = y - x + 5$ at the points $x = 2.1$, $2.2$, $2.3$ with initial condition.

4. Tabulate $y(0.1)$, $y(0.2)$ and $y(0.3)$ using Taylor series method given that $y' = y^2 + x$ and $y(0) = 1$.

5. Solve the initial value problem $\frac{dy}{dx} = x - y^2$, $y(0)=1$ to find $y(0.4)$ by Adam's method. $h = 0.1$

6. Using Adam's method find $y(0.4)$ given that $y' = xy/2$, $y(0)=1$, $y(0.1) = 1.01$, $y(0.2) = 1.022$, $y(0.3) = 1.023$.

7. Using Taylor's Series method solve $\frac{dy}{dx} = xy + y^2$, $y(0)=1$ ~~and~~ at $x = 0.1, 0.2$ and $0.3$ Continue the solution at $x = 0.4$ by Milne's predictor-corrector formula.

8. Consider the initial value problem $\frac{dy}{dx} = y - x^2 + 1$, $y(0) = 0.5$ find (i) $f(0.2)$ using the modified Euler method (ii) 4th order R-K method find $y(0.4)$ and $y(0.6)$ (iii) find $y(0.8)$ using Adam's predictor Corrector formula. ~~find ye~~