UNIT - I (1)

1. Define subspaces.

Let V be a vector spaces over a field F. A non-empty subset W of V is called subspace of V if W itself is a vector space over F under the operations of V.

2. Prove that the intersection of two sub-spaces of a vector space is a subspace.

Soln: Let A and B be two subspace of a vector space V over a field F.

We claim that $A \cap B$ is a subspace of V.

Clearly, $0 \in A \cap B$ and hence $A \cap B$ is not empty. Now, let $u, v \in A \cap B$ and $\alpha, \beta \in F$

Then, $u, v \in A$ and $u, v \in B$

$\therefore \alpha u + \beta v \in A$ and $\alpha u + \beta v \in B$ [$\because$ A&B are subspaces]

$\therefore \alpha u + \beta v \in A \cap B$

Hence $A \cap B$ is a subspace of V.

3. Let V be a vector space over a field F. Let A and B be subspaces of V then

$\frac{A+B}{A} \cong \frac{B}{A \cap B}$.

Soln: W.K.T. $A+B$ is a subspace of V containing A.

Hence $\frac{A+B}{A}$ is also a vector space over F.

An element of $\frac{A+B}{A}$ is of the form $A+(a+b)$ where $a \in A$ and $b \in B$. But $A+a=A$

Hence an element of $\frac{A+B}{A}$ is of the form $A+b$.

Now, consider $f: B \to \frac{A+B}{A}$ defined by $f(b) = A+b$.

Clearly $f$ is onto.

Also $f(b_1+b_2) = A+(b_1+b_2)$

$= (A+b_1)+(A+b_2)$

$= f(b_1)+f(b_2)$

and $f(\alpha b_1) = A+\alpha b_1$

$= \alpha(A+b_1)$

$= \alpha f(b_1)$

Hence $f$ is a linear transformation

Let $K$ be the kernel of $f$.

Then, $K = \{b \mid b \in B, A+b = A\}$

Now, $A+b = A$ iff $b \in A$ ∴ Hence $K = A \cap B$

$$\therefore \frac{B}{A \cap B} \cong \frac{A+B}{A}$$

4. Define Homomorphism.

Let $V$ and $W$ be vector space over a field $F$. A mapping $T: V \to W$ is called homomorphism if a) $T(u+v) = T(u)+T(v)$ and

b) $T(\alpha u) = \alpha T(u)$ where $\alpha \in F$ and $u, v \in V$.

5. Let V be a vector space over F. A non-empty subset W of V is a subspace of V iff W is closed with respected to vector addition and scalar multiplication in V.

Proof: Let W is a subspace of V

Then W itself is a vector space and hence W is closed with respect to vector addition and scalar multiplication.

Conversely let W be a non-empty subset of V such that $u, v \in W \Rightarrow u+v \in W$ and

$u \in W$ and $\alpha \in F \Rightarrow \alpha u \in W$.

We prove that W is a subspace of V

Since W is non-empty, there exists an element $u \in W$.

$$0u = 0 \in W$$

Also $v \in W \Rightarrow (-1)v = -v \in W$

Thus W contain 0 and the additive inverse of each of its element.

Hence W is an additive subgroup of V

Also $u \in W$ and $\alpha \in F \Rightarrow \alpha u \in W$.

Since the elements of W are the elements of V the other axioms of a vector space are true in W.

∴ Hence W is a subspace of V.

UNIT - II

1. Any subset of linearly independent set is linearly independent.

Soln Let V be a vector space over a field F.

Let $S = \{v_1, v_2, \ldots v_n\}$ be a linearly independent set.

Let S' be a subset of S without loss of generality we take $S' = \{v_1, v_2, \ldots v_k\}$ where $k \leq n$.

Suppose S' is a linearly dependent set. Then there exists $\alpha_1, \alpha_2, \ldots \alpha_k$ in F not all zero. such that $\alpha_1 v_1 + \alpha_2 v_2 + \ldots \alpha_k v_k = 0$

Hence $\alpha_1 v_1 + \alpha_2 v_2 + \ldots + \alpha_k v_k + 0v_{k+1} + \ldots + 0v_n = 0$ is a non-trivial linear combination giving the zero vector.

Here S is a linearly dependent set which is a contradiction.

Hence S' is a linearly independent.

2. Let $T: V \to W$ be a linear transformation. Then $\dim V = \text{rank } T + \text{nullity } T$.

Proof: W.K.T $V/\ker T = T(V)$

$$\dim V - \dim(\ker T) = \dim(T(V))$$

$$\dim V - \text{nullity } T = \text{rank } T$$

$$\dim V = \text{nullity } T + \text{rank } T.$$

3. Define dimension.

Let V be a finite dimensional vector space over a field F. The number of element in any basis of V is called the dimension of V and is denoted by dim V.

4. Any set containing a linearly dependent is also linearly dependent.

Proof: Let V be a vector space. Let S be a linearly dependent set. Let $S' \supseteq S$.

If $S'$ is linearly independent S is also linearly independent.

By the theorem. Any subset of a linearly independent set is linearly independent which is a contradiction.

Hence $S'$ is a linearly dependent.

5. Define rank and nullity.

Let $T: V \to W$ be a linear transformation. Then the dimension of $T(V)$ is called the rank of T. The dimension of ker T is called the nullity of T.

UNIT - III

1. $V_n(C)$ is a complex inner product space with inner product defined by.

$\langle x, y\rangle = x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n$ where

$x = (x_1, x_2, \ldots x_n)$ and $y = (y_1, y_2, \ldots y_n)$

soln: Let $x, y, z \in V_n(C)$ and $\alpha \in C$

i) $\langle x+y, z\rangle = (x_1+y_1)\bar{z}_1 + (x_2+y_2)\bar{z}_2 + \cdots + (x_n+y_n)\bar{z}_n$

$= (x_1\bar{z}_1 + x_2\bar{z}_2 + \cdots + x_n\bar{z}_n) + (y_1\bar{z}_1 + y_2\bar{z}_2 + \cdots + y_n\bar{z}_n)$

$= \langle x, z\rangle + \langle y, z\rangle$

ii) $\langle \alpha x, y\rangle = \alpha x_1\bar{y}_1 + \alpha x_2\bar{y}_2 + \cdots + \alpha x_n\bar{y}_n$

$= \alpha(x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n)$

$= \alpha\langle x, y\rangle$

iii) $\overline{\langle y, x\rangle} = \overline{y_1\bar{x}_1 + y_2\bar{x}_2 + \cdots + y_n\bar{x}_n}$

$= \bar{y}_1 x_1 + \bar{y}_2 x_2 + \cdots + \bar{y}_n x_n$

$= x_1\bar{y}_1 + x_2\bar{y}_2 + \cdots + x_n\bar{y}_n$

$= \langle x, y\rangle$

iv) $\langle x, x\rangle = x_1\bar{x}_1 + \cdots + x_n\bar{x}_n$

$= |x_1|^2 + |x_2|^2 + \cdots + |x_n|^2 \geq 0$

and $\langle x, x\rangle = 0$ iff $x = 0$

2. Define orthogonality.

Let $V$ be an inner product space and let $x, y \in V$. $x$ is said to be orthogonal to $y$ if $\langle x, y\rangle = 0$

3. Schwartz's equality.

The inequality is trivially true when $x=0$ or $y=0$. Hence let $x\neq 0$ and $y\neq 0$

Consider $z = y - \frac{\langle y,x\rangle}{\|x\|^2}x$

Then, $0 \le \langle z,z\rangle$

$$= \langle y - \frac{\langle y,x\rangle}{\|x\|^2}x,\ y - \frac{\langle y,x\rangle}{\|x\|^2}x\rangle$$

$$= \langle y,y\rangle - \frac{\overline{\langle y,x\rangle}}{\|x\|^2}\langle y,x\rangle - \frac{\langle y,x\rangle}{\|x\|^2}\langle x,y\rangle$$

$$+ \frac{\langle y,x\rangle\overline{\langle y,x\rangle}}{\|x\|^2\ \|x\|^2}\langle x,x\rangle$$

$$= \|y\|^2 - \frac{\overline{\langle y,x\rangle}\langle y,x\rangle}{\|x\|^2} - \frac{\langle y,x\rangle\langle x,y\rangle}{\|x\|^2} + \frac{\langle y,x\rangle\overline{\langle y,x\rangle}}{\|x\|^2}$$

$$= \|y\|^2 - \frac{\overline{\langle x,y\rangle}\langle x,y\rangle}{\|x\|^2}$$

$\therefore 0 \le \|x\|^2\|y\|^2 - |\langle x,y\rangle|^2$

$\therefore |\langle x,y\rangle| \le \|x\|\|y\|$ is a schwartz's inequality.

4. Let V be a finite dimensional inner product space. Let W be a subspace of V. Then $(W^\perp)^\perp = W$.

Sol: Let $w \in W$. Then for any $u \in W^\perp$ $\langle w,u\rangle = 0$

Hence $w \in (W^\perp)^\perp$. Thus $W \subseteq (W^\perp)^\perp \rightarrow (1)$

Now by thrm $V = W \oplus W^\perp$

Also $V = W^\perp \oplus (W^\perp)^\perp$.

Hence $\dim W = \dim (W^\perp)^\perp \rightarrow (2)$

From (1) & (2) we get $W = (W^\perp)^\perp$.

(8)

5. If $S$ is any subset of $V$ then $S^{\perp}$ is a subspace of $V$.

Soln: Clearly $0 \in S^{\perp}$ and hence $S^{\perp} \neq \phi$

Now, let $x, y \in S^{\perp}$ and $\alpha, \beta \in F$

Then $\langle x, u \rangle = \langle y, u \rangle = 0 \ \forall u \in S$

$\therefore \langle \alpha x + \beta y, u \rangle = \alpha \langle x, u \rangle + \beta \langle y, u \rangle = 0 \ \forall u \in S$

$\therefore \alpha x + \beta y \in S^{\perp}$. Hence $S^{\perp}$ is a subspace of $V$.

UNIT - IV

1. Let $A$ be an $m \times n$ matrix, $B$ an $n \times p$ matrix and $C$ an $p \times q$ matrix. Then $A(BC) = (AB)C$.

Soln: Let $A = (a_{ij})$, $B = (b_{ij})$ and $C = (c_{ij})$ let us find the $rs^{th}$ entry in $A(BC)$.

The $r^{th}$ row in $A$ is $a_{r1}, a_{r2}, \ldots a_{rn}$. The $s^{th}$ column in $BC$ consists of the elements $\sum b_{1j}c_{js}$ ...... $\sum b_{nj}c_{js}$. Hence, the $rs^{th}$ entry in $A(BC)$ is $a_{r1}\sum b_{1j}c_{js} + \ldots + a_{rn}\sum b_{nj}c_{js}$

$$= \sum_{i=1}^{n} a_{ri} \sum_{j=1}^{p} b_{ij}c_{js} = \sum_{i=1}^{n}\sum_{j=1}^{p} a_{ri}b_{ij}c_{js}$$

Let us now find the $rs^{th}$ entry in $(AB)C$

The $r^{th}$ row in $AB$ is $\sum a_{ri}b_{i1}, \sum a_{ri}b_{i2}, \ldots \sum a_{ri}b_{ip}$

The $s^{th}$ column in $C$ is $c_{1s}, c_{2s}, \ldots c_{ps}$

Hence the $rs^{th}$ entry in $(AB)C$ is

$(\sum a_{ri}b_{i1})c_{1s} + (\sum a_{ri}b_{i2})c_{2s} + \ldots + (\sum a_{ri}b_{ip})c_{ps} =$

$$\sum_{i=1}^{n}\sum_{j=1}^{p} a_{ri}b_{ij}c_{js}$$

Thus $A(BC) = (AB)C$.

2. Let A and B be two $m \times n$ matrices. Then (9)

i) $(A^T)^T = A$ ii) $(A+B)^T = A^T + B^T$

i) The $(i,j)^{th}$ entry of $(A^T)^T$ = $(j,i)^{th}$ entry of $A^T$

= $(i,j)^{th}$ entry of A

$\therefore (A^T)^T = A$

ii) The $(i,j)^{th}$ entry of $(A+B)^T$

= $(j,i)^{th}$ entry of $A+B$

= $(j,i)^{th}$ entry of A + $(j,i)^{th}$ entry of B

= $(i,j)^{th}$ entry of $A^T$ + $(i,j)^{th}$ entry of $B^T$

= $(i,j)^{th}$ entry of $(A^T + B^T)$

$\therefore (A+B)^T = A^T + B^T$.

3. A square matrix A is symmetric iff $A = A^T$.

Proof: Let A be a symmetric matrix.

Then the $(i,j)^{th}$ entry of A = $(j,i)^{th}$ entry of A

= $(i,j)^{th}$ entry of $A^T$

Hence $A = A^T$

Conversely let $A = A^T$

Then $(i,j)^{th}$ entry of A = $(i,j)^{th}$ entry of $A^T$

= $(j,i)^{th}$ entry of A

Hence A is symmetric

4. Let A be any square matrix of order n. Then $(\text{adj } A)A = A(\text{adj } A) = |A| I$. where I is the identity matrix of order n.

Soln: The $(i,j)^{th}$ element of $[A(\text{adj } A)]$

$$= \sum_{k=1}^{n} a_{ik} A_{jk}$$

$$= \begin{cases} 0 & \text{if } i \neq j \\ |A| & \text{if } i = j \end{cases}$$

$$A(\text{adj } A) = \begin{vmatrix} |A| & 0 & \cdots & 0 \\ 0 & |A| & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & |A| \end{vmatrix} = |A| I$$

Similarly $= (\text{adj } A)A = |A| I$

Hence $(\text{adj } A)A = A(\text{adj } A) = |A| I$.

5. Define Elementary matrix.

A matrix obtained from the identity matrix by applying a single elementary row or column operation is called an elementary matrix.

UNIT - V

1. Define characteristics matrix.

Let A be any square matrix of order n and let I be the identity matrix of order n. Then the matrix polynomial given by $A - xI$ is called the characteristics matrix of A.

2. Show that matrix $A = \begin{bmatrix} 2 & -3 & 1 \\ 3 & 1 & 3 \\ -5 & 2 & -4 \end{bmatrix}$ satisfies the eqn

$A(A-I)(A+2I) = 0$.

Soln:

The characteristic polynomial of A is

$$|A-\lambda I| = \begin{vmatrix} 2-\lambda & -3 & 1 \\ 3 & 1-\lambda & 3 \\ -5 & 2 & -4-\lambda \end{vmatrix}$$

$$= -\lambda^3 - \lambda^2 + 2\lambda$$

By Cayley-Hamilton theorem, $-A^3 - A^2 + 2A = 0$

(i.e) $A^3 + A^2 - 2A = 0$ Hence $A(A^2 + A - 2I) = 0$

$\therefore A(A+2I)(A-I) = 0$

8. Verify Cayley Hamilton's theorem for the matrix $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

Soln:-

The characteristic eqn of A is $|A - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$\therefore (1-\lambda)(3-\lambda) - 8 = 0$

$$\lambda^2 - 4\lambda - 5 = 0$$

We have $A^2 - 4A - 5I = 0$

Now, $A^2 = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$

$$= \begin{pmatrix} 9 & 8 \\ 16 & 17 \end{pmatrix}$$

$4A = \begin{pmatrix} 4 & 8 \\ 16 & 12 \end{pmatrix}$ $\quad 5I = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}$

$A^2 - 4A - 5I = \begin{pmatrix} 9 & 8 \\ 16 & 17 \end{pmatrix} - \begin{pmatrix} 4 & 8 \\ 16 & 12 \end{pmatrix} - \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix} = 0$

Hence Cayley Hamilton's theorem is verified.

4. The eigen value of A and its transpose $A^T$ are the same.

Proof: It is enough if we prove A and $A^T$ have the same characteristics polynomial.

Since for any square M, $|M| = |M^T|$ we have,

$$|A - \lambda I| = |(A - \lambda I)^T| = |A^T - (\lambda I)^T| = |A^T - \lambda I|$$

Hence the result.

5. Find the sum of the square of eigen value of $A = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$

Soln:

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigen value of A.

$$A^2 = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 9 & 5 & 38 \\ 0 & 4 & 42 \\ 0 & 0 & 25 \end{pmatrix}$$

$$\left.\begin{matrix}\text{Sum of the eigen value} \\ \text{of } A^2\end{matrix}\right\} = \text{Trace of } A^2$$

$$= 9 + 4 + 25$$

$$\text{(i.e) } \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 38.$$

Sum of the square of the eigen values of $A = 38$.