Complex-Assignment - I
Analysis

Two Marks :

1) Is every ball is an open set :

Taking an open ball $B(x_0, r)$ in the metric space with centre x_0 & radius R.

To (p.T) $B(x_0, r)$ is a neighbourhood of each of its Points.

$\delta > 0$ Such that $B(x, \delta) \subseteq B(x_0, r)$

Let $x \in B(x_0, r)$

$d(x, x_0) < r$ — ①

$0 < r - d(x, x_0)$

$r - d(x, x_0) > 0$

$\delta > 0 \Rightarrow \delta = r - d(x, x_0)$

Let $y \in B(x, \delta)$

$\Rightarrow d(x, y) < \delta$ — ②

from ① & ②

$d(x, x_0) < r - d(x, y) < \delta$

$d(x_0, y) \leq d(x_0, x) + d(x, y)$

$< d(x_0, x) + \delta$

$\leq d(x_0, x) + r - d(x, x_0)$

$= r$

$d(x_0, y) < r$

$y \in B(x_0, r) \Rightarrow B(x, \delta) \subseteq B(x_0, r)$

$\therefore B(x, \delta)$ is a neighbourhood of each of its Points of $B(x_0, r)$

$\therefore B(x_0, r)$ is open Set

Hence Proved.

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2) A Set is totally bounded:

A set x is totally bounded if for all $\epsilon > 0$ x can be covered by a finite no. of balls of radius ϵ.

There exist balls $B(x_0, \epsilon), B(x_1, \epsilon) \ldots B(x_n, \epsilon)$ such that

$x \leq B(x_0, \epsilon) \cup B(x_1, \epsilon) \cup \ldots \cup B(x_n, \epsilon)$

3) Define a Homothetic transformation:

The transformation $w = bz$ where b is a non-zero complex number represents a rotation through an angle $\arg b$ followed by a magnification or a contraction by the factor $|b|$.

4) Distingish b/w translation & rotation:

$w = z + b$

Consider the transformation $w = z + b$, if $z = x + iy$, $w = u + iv$, $b = b_1 + ib_2$ then the image of the points (x, y) in the z-plane is the point $(x + b_1, y + b_2)$ in w-plane under this transformation the image of any region is simply a translation of the region.

Rotation:

$w = az$, where $|a| = 1$. Consider the translation $w = az$, where $|a| = 1$

Let $z = re^{i\theta}$ & $a = e^{i\alpha}$, so that $|a| = 1$,

$$
\begin{aligned}
w &= az \\
&= e^{i\alpha}(re^{i\theta}) \\
&= re^{i(\theta+\alpha)}
\end{aligned}
$$

A Point with Polar Co-ordinates (r_1, θ) in the z-plane is mapped to the Point $(r, \theta+\alpha)$ in the w-Plane.

5) Define : inversion :

$w = 1/z$. Consider the translation $w = \frac{1}{z}$

Put $z = re^{i\theta}$. $w = \frac{1}{re^{i\theta}}$. This translation can be expressed as a Product of two translation.

$$T_1(z) = \frac{1}{re^{i\theta}} ; T_2(z) = re^{i\theta} = \bar{z} \text{ for}$$

$$(T_1 \circ T_2) = T_1(T_2(z))$$
$$= T_1(re^{-i\theta})$$
$$= (1/r)e^{-i\theta})$$
$$= \frac{1}{z}$$

The translation $T_1(z) = \frac{1}{r}e^{i\theta}$ represent the with respect to the circle $|z| = 1$ & $T_2(z) = \bar{z}$ represent reflection about the real axis.

6) Recifiable :

The length of an arc can also be defined as the least upper bound of all sums.

$$|z(t_1) - z(t_0)| + |z(t_2) - z(t_1)| + \dots + |z(t_n) - z(t_{n-1})| \quad \text{—} \; ①$$

where $a = t_0 < t_1 < t_2 \dots < t_n = b$.

If this least bound is finite.

7) Cauchy's thrm in a disk:

If $f(z)$ is analytic in an open disk Δ. $\int_{\gamma} f(x) dz = 0$ for every closed curve γ in Δ.

8) Compute $\int_{|z|=1} e^{z} z^{-1} dz$

Solution: $\int_{|z|=1} \frac{e^{z}}{z} dz = 2\pi ; f(0)$

where $f(z) e^{z}$

$\therefore f(0) = e^{(0)} = 1$

$\Rightarrow e^{z} z^{-1} = \frac{e^{z}}{z}$

$= 2\pi i f(0)$

$= 1$

9) Liouville's theorem:

A function which is analytic and bounded in the whole plane must reduse to a constant ie bounded entire function are constant.

10) Index of a Point:

The index of a Point with ~~represent~~ respect to the curve γ by the equation

$$n(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a}$$

Index of a Point is also called the winding no. of. γ with respect to a.

11) Define zero & Pole:

Zero :

A zero as a function. z is called a zero of $f(z)$ if $f(a)=0$. further if $f(a)=0$ and $f^n(a) \neq 0$. Then $z=a$ is said to be a zero of order n.

Pole :

A Singularity $z=a$ is said to be a Pole of $f(z)$. If $\lim_{z \to a} f(z) = \infty$, a Pole $z=a$ is said to be of order m.

If $\lim_{z \to a} (z-a)^m f(x) \neq 0$ when $m=1$ then the Pole is called Simple Pole.

12). Maximum modulus Principle :

If $f(z)$ is analytic and non-constant in a region Ω then its absolute value $|f(z)|$ has no maximum in Ω.

13) Memorphic function :

A memorphic function is a single valued function that is analytic in all but possibly a discrete subset of its domain & at those singularities it must go to infinity like a Polynomial. ie These exceptional Points must be Poles and not essential singularities.

14) State Taylors theorem:

If $f(z)$ is analytic in a region Ω containing a its possible to write

$$f(z) = f(a) + \frac{f'(a)}{\lfloor 1}(z-a) + \frac{f''(a)}{\lfloor 2}(z-a)^2 + \cdots$$

$$+ \frac{f^{(n-1)}(a)}{\lfloor n-1}(z-a)^{n-1} + f_n(z)(z-a)^n$$ where

$f_n(z)$ is analytic in Ω & $f_n(z) = \frac{1}{2\pi i}\int_C \frac{f(\xi)\,d\xi}{(\xi-a)^n(\xi-z)}$ where C is any circle with centre at a.

15) Removable Singularities:

A Point $z=a$ lying inside a region Ω is said to be a removable singular point of a function $f(z)$ which is analytic except at $z=a$ if $\lim\limits_{z \to a} (z-a) f(z) = 0$.

16) Find the residue of $\cot z$ at its Pole:

$z=0$ is a simple Pole for $\cot z$

Let $f(z) = \frac{\cos z}{\sin z}$

$= \frac{h(z)}{k(z)}$

$$\therefore \operatorname{Res}\{f(z); 0\} = \frac{h(0)}{k'(0)} = \frac{\cos(0)}{\cos(0)}$$

$$= 1.$$

17) Residue for $\frac{e^z}{(z-a)(z-b)}$

Solution :

$$f(z) = \frac{e^z}{(z-a)(z-b)}$$

$f(z)$ has simple Poles at a, b.

The Pole is a, b

$$Res\{f(z); a\} = \lim_{z \to a} (z-a)\left(\frac{e^z}{(z-a)(z-b)}\right)$$

$$= \frac{e^a}{a-b}$$

by residue theorem

$$\int_c f(z)dz = \frac{2\pi i e^a}{a-b}$$

$$\int_c \frac{e^z}{(z-a)(z-b)} dz = 2\pi i \frac{e^a}{a-b}$$

18) The Rouche's Theorem :

Let γ be homologous to zero in Ω Such that $n(\gamma, z)$ is either zero (or) one for Points z not in γ suppose that $f(z)$ & $g(z)$ are analytic in Ω and satisfy the inequalities $|f(z) - g(z)| < |f(z)|$ on γ.

Then $f(z)$ & $g(z)$ have the same no. of zero's enclosed by γ.

19) Argument Principle:

If $f(z)$ is meromorphic function in Ω with zero's a_j & the Poles b_k

then $\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}dz = \sum_j n(\gamma, a_j) - \sum_k n(\gamma, b_k)$

for every cycle which is homologous to zero & does not Pass through any of the zero's & the Poles.

20) Multiply Connected Region:

A region Ω is said to be multiply connected if it is not simply connected.

21) P.T the arithmetic mean of a harmonic function over concentric circles $|z| = r$ is a linear function of $\log r$.

The arithmetic of a harmonic fun over concentric circles $|z| = r$ is a linear function of $\log r$.

$$\frac{1}{2\pi}\int_{|z|=r} u\,d\theta = \alpha \log r + \beta$$

where α & β constant.

If u is harmonic in a disk then the arithmetic mean is constant.

22) Lawrent Series:

If $f(z)$ is analytic is the annulus $R_1 < |z-a| < R_2$ then, $f(z)$ can be developed in to a series of the form $f(z) = \sum_{-\alpha}^{\alpha} A_n (z-a)^n$.

23) Schwartz's theorem:

The function $P_U(z)$ is harmonic for $|z|<1$ & $\lim_{z \to e^{i0}} P_U(z) = U(\theta_0)$ Provided that U is continuous at θ_0.

24) Series expansion for arc sinz

$$\text{arc sin} z = z + \frac{1}{2}\frac{z^3}{3} + \frac{1.3}{2.4}\frac{z^5}{5} + \cdots$$

25) Harmonic function:

A real valued function $u(z)$ or $u(x,y)$ defined,

i) $u, \frac{\partial^2 v}{\partial x^2}$ & $\frac{\partial^2 v}{\partial y^2}$ are Constant

ii) U satisfied the Laplace equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$