Partial Differential Equation

Assignment - I

1) Eliminate the constants a & b from the equation
$z = (x+a)(y+b)$

Given that,

$$z = (x+a)(y+b) \quad \text{--- (1)}$$

Differentiate equ (1) Partially w.r.to x

$$\frac{\partial z}{\partial x} = y+b$$

Differentiate (1) Partially w.r.to y

$$\frac{\partial z}{\partial y} = x+a \quad \text{--- (3)}$$

Sub (2) & (3) in (1)

$$z = pq$$

This is the required PDE.

2) Eliminate the arbitrary constant a & b from
$2z = (ax+y)^2+b$

Given that,

$$2z = (ax+y)^2+b \quad \text{--- (1)}$$

Differentiate (1) Partially w.r.to x

$$2\frac{\partial z}{\partial x} = a \cdot 2(ax+y)$$

$$\cancel{2}p = \cancel{2}(ax+y) \cdot a$$

$$p = (ax+y) \cdot a \quad \text{--- (2)}$$

Differentiate equ (1) w.r.to y

$$\cancel{2}\frac{\partial z}{\partial y} = \cancel{2}(ax+y)$$

$$q = ax+y \quad \text{--- (3)}$$

Multiply equ (2) by x

$$px = ax(ax+y)$$

Multiply ③ by y

$$qy = y(ax+y)$$

$$px + qy = ax(ax+y) + y(ax+y)$$

$$= (ax+y)(ax+y)$$

$$px + qy = q^2.$$

3) Singular Integral:

Two Parameter System

$f(x, y, z, a, b) = 0$ its is also a solution of equation,

$f(x, y, z, p, q) = 0$ is called the singular integral equation.

4) Along every Characteristic strip of the equation $F(x, y, z, p, q) = 0$, the function $f(x, y, z, p, q)$ is constant.

Proof:

Along characteristic strip, we have

$$\frac{\partial}{\partial t} F\{x(t), y(t), z(t), p(t), q(t)\}$$

$$= F_x x' + F_y y' + F_z z' + F_p p' + F_q q'$$

$$= F_x F_p + F_y F_q + F_z(pF_p + qF_q) - F_p(F_x + pF_z) - F_q(F_y + qF_z)$$

$$= 0$$

$\therefore F(x, y, z) = k$, a constant along the strip.

5) Find the complete integral of $p^2x+q^2y=z$

Solution :

Given equ is

$$f(x,y,z,p,q) = p^2x+q^2y-z \quad \text{---} (1)$$

$$\frac{dp}{f_x+pf_z} = \frac{dq}{f_y+qf_z} = \frac{dz}{-pf_p-qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_p}$$

$f_x = p^2$; $f_y = q^2$; $f_z = -1$; $f_p = 2px$; $f_q = 2qy$

$$= p^2\frac{dx}{-2px}$$

$$\frac{2px\,dp+p^2dx}{2px(-p-p^2)+p^2(-2px)} = \frac{dz}{-p(2px)-q(2qy)} = \frac{dy}{-2qy}$$

$$\Rightarrow \frac{d(p^2x)}{-2p^2x} = \frac{d(q^2y)}{-2q^2y}$$

$$\log(p^2x) = \log(q^2y)+\log a$$

$$\log(p^2x)-\log(q^2y) = \log a$$

$$\log(p^2x) = \log a q^2y$$

$$p^2x = aq^2y \quad \text{---} (3)$$

Using (3) in (1)

$$aq^2y+q^2y-z=0$$

$$aq^2y+q^2y=z$$

$$q^2y(a+1)=z$$

$$q^2y = \frac{z}{1+a}$$

$$q^2 = \frac{z}{(1+a)y}$$

$$q = \left(\frac{z}{(1+a)y}\right)^{1/2} \quad \text{---} (4)$$

Sub (4) in (3)

$$p^2x = aq^2y$$

$$p = \left[\frac{az}{(1+a)x}\right]^{1/2}$$

$$dz = p\,dx + q\,dy$$

$$dz = \left\{\frac{az}{(1+a)dx}\right\}^{1/2} dx + \left\{\frac{z}{(1+a)y}\right\}^{1/2} dy$$

$$\frac{dz}{z^{1/2}} = \sqrt{\frac{1}{1+a}}\left\{\frac{\sqrt{a}\,dx}{x^{1/2}} + \frac{dy}{y^{1/2}}\right\}$$

$$(1+a)^{1/2}\sqrt{z} = \sqrt{a}\cdot\sqrt{x}$$

$$= \sqrt{y} + b.$$

6) Find the complete Integral of the equation $(p+q)(z - xp - yq) = 1$

Solution :

Given equation

$$(p+q)(z - xp - yq) = 1$$

$$z - xp - yq = \frac{1}{p+q}$$

$$z = \frac{1}{p+q} + px + qy$$

This is an complete integral is

$$z = ax + by + \frac{1}{a+b}$$

7) Find the complete integral of $q = px + p^2$

Solution :

Let $f(x, y, z, p, q) = q - px - p^2$ —— ①

The auxilliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \quad \text{—— ②}$$

$f_x = -p$; $f_y = 0$; $f_z = 0$

$f_p = -x - 2p$; $f_q = 1$.

Consider 2nd function, we get

$$dq = 0$$

$$q = a \quad \text{--- (3)}$$

Put $q = a$ in (1)

$$p^2 + px - a = 0$$

$$p = \frac{1}{2}\left[-x \pm \sqrt{x^2+4a}\right] \quad \text{--- (4)}$$

Using (3) & (4) in $dz = pdx + qdy$

$$dz = \frac{1}{2}\left[-x \pm \sqrt{x^2-4a}\right] + ady$$

Sing $\int dz = \int -\frac{1}{2}x \pm \int \frac{1}{2}\sqrt{x^2+4a}\, dx + \int ady$

$$\int dz = -\frac{1}{2}\int x dx \pm \frac{1}{2}\int\sqrt{x^2+4a}\, dx + a\int dy$$

$$z = -\frac{1}{2}\frac{x^2}{2} \pm \frac{1}{2}\left[\frac{x}{2}\sqrt{x^2+4a} + 2a\log\left\{x + \sqrt{4+4a}\right\}\right] + ay + b$$

8) Define: Two-dimensional biharmonic Equation:

Fourth order linear Partial differential equation $\frac{\partial^4 \phi}{\partial x^4} + 2\frac{\partial^4 \phi}{\partial x^2 \partial y^2} + \frac{\partial^4 \phi}{\partial y^4} = 0$ --- (5)

which may be written as $\nabla^4 \phi = 0$.

9) Find the P.I of the equation $(D^2 - D')z = 2y - x^2$

Solution:

$$P.I = \frac{1}{D^2 - D'}(2y - x^2)$$

$$= \frac{1}{-D'\left(\frac{-D^2}{D'} + 1\right)}$$

$$= \frac{1}{-D'}\left(1 - \frac{D^2}{D'}\right)^{-1}(2y - x^2)$$

$(1-x)^{-1} = 1+x+x^2+\cdots$

$$\left(1-\frac{D^2}{D'}\right)^{-1} = 1+\frac{D^2}{D'}+\frac{D^4}{D'^2}+\cdots$$

$$P.I = -\frac{1}{D'}\left[1+\frac{D^2}{D'}+\frac{D^4}{D'^2}+\cdots\right](2y-x^2)$$

$$= \left[-\frac{1}{D'}-\frac{D^2}{D'^2}-\frac{D^4}{D'^3}-\cdots\right](2y-x^2)$$

$$= -\frac{1}{D'}(2y-x^2)-\frac{D^2}{D'^2}(2y-x^2)-\cdots$$

$$= -\left(\frac{2y^2}{2}-x^2y\right)-\frac{1}{D'^2}(-2)$$

$$= -(y^2-x^2y)+2\left(\frac{1}{D'^2}\right)(1)$$

$$= -y^2+x^2y+2\left(\frac{y^2}{2}\right)$$

$$= -y^2+x^2y+y^2$$

$$P.I = x^2y.$$

10) Find the P.I of the equ $(D^2-D')z = e^{2x+y}$

Solution:

$$P.I = \frac{1}{D^2-D'}e^{2x+y}$$

$$F(a,b) = \frac{1}{D^2-D'} = \frac{1}{4-1} = \frac{1}{3}$$

$D=a=2$

$D'=0$; $\quad P.I = \frac{1}{3}e^{2x+y}$.

11) Uniform non-linear Equation:

Assume that a Ist integral of the equ $Rr+Ss+Tt+U(rt-s^2)V$ — ⑧ is of formal. For any function z of x & y

We've relations.

$dp = r\,dx + s\,dy$, $dq = s\,dx + t\,dy$.

12) Exterior Neumann Problem:

If f is a continuous function prescribed at each point of the boundary S of a bounded simply connected region V find a function $\psi(x, y, z)$ satisfying $\nabla^2 \psi = 0$ outside V & $\frac{\partial \phi}{\partial n} = f$ on S.

13) Interior Dirichlet Problem:

If f is a continuous function prescribed on the boundary S of some finite region on the boundary S of some finite V, determine a function $\psi(x, y, z)$ such that $\nabla^2 \phi = 0$ in V it's normal derivative $\frac{\partial \psi}{\partial n}$ coincides with f at every point of S.

14) Exterior Dirichlet Problem:

If f is a continuous function prescribed on the boundary S of a finite simply connected region V, determine a function $\psi(x, y, z)$ which satisfy $\nabla^2 \psi = 0$ outside V and is such that $\psi = f$ on S.

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15) Separation of Variables:

Laplace equation takes the form

$$\frac{\partial^2\psi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta} +$$

$$\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\theta^2} = 0 \quad (1)$$

this equation is separable with solution of the form

$$\left\{A_n r^n + \frac{B_n}{r^{n+1}}\right\}\theta(\cos\theta)e^{\pm im\phi}.$$

16) Boundary Value Problem:

The function $\psi = \psi_1 - \psi_2$ such that

$\nabla^2\phi = 0$ within V and $\psi = 0$

$$\therefore |\psi(x, y, z)| < \frac{c}{r}$$

$\frac{\partial\psi}{\partial x} + (k+1)\phi = f$ at every Point of S.

17) Lagrange's equation:

A quasi Partial differential equation of Order one is of the form $Pp + Qq = R$, where P, Q & R are function of x, y, z such that a Partial differential equation

18) Poisson's equation in cartiesion form:

$\nabla^2 \phi + 4 \pi \rho = 0$, ρ is 0 An reduces to simple form,

$$\nabla^2 \phi = 0.$$