Function Analysis - P16MA41

unit - I

1. write the basis for the linear space P of all polynomial with real co-eff on [0,1]

$D(P) = dP/dx$ is a linear transformation of P into itself for $P_1 P_2 \in P$

$$D(P_1 + P_2) = d/dx\,(P_1 + P_2)$$
$$= dP_1/dx + dP_2/dx$$
$$= D(P_1) + D(P_2)$$
$$P(\alpha P_1) = dP(dx(\alpha P_1) = \frac{\alpha dP_1}{dx} = \alpha D(P_1)$$

Since differentiation is with respected to x and α being in f as constant

2. Congurent modulo:

Let I be a ideal in a ring R. we use I do define an equivalence relation in R as follows. two elements x and y in R said to be congurent modulo I. written $x \equiv y \pmod I$ if $x-y$ is in I

3. Dimension of a linear space:

Let L be a linear space and let $S = \{x, x_2 \ldots x_n\}$ be a finite non-emply set of vector in linear space to be linear dependent.

If $\exists$ a scalar $\alpha_1, \alpha_2 \dots \alpha_n$ not all of which are zero such that $\alpha_1 x_1 + \alpha_2 x_2 + \dots + \alpha_n x_n = 0$

If S is not linearly dependent then. S is called linear dependent

4. State open mapping theorem:

If B and B' are banach spaces and if T is continuous linear transformation of B is on to B' then T is an open mapping

5. Give any three subspaces

Subspaces: A non-empty subset M of L is called a subspace (or) a linear subspaces of L if M is a linear space own right with respect to the linear operations defi. in L

Proper subspace: If the subspace M is a proper subset of L Then it is called a proper subspace of L. The $\{0\}$ space s the full space itself are always subspace of L

Unit - II

6 Hillbert space:

An linear product space which complete in the non - induded by the linear product is called a Hillbert space

It is denoted by "H"

A Hillber space is a complete be each Space . whose non - arrivers from an inner Product

ie] which there is define a complex function (x, y) of vectors (x, y) with the following properties

i] $(\alpha x + \beta y, z) = \alpha(x, z) + \beta(y, z)$

ii] $\overline{(x, y)} = (y, x)$

iii] $(x, x) = \|x\|^2$

iv] $(x, \alpha y + \beta z) = \bar{\alpha}(x, y) + \bar{\beta}(x, z)$

7 Orthogonal complement:

A vector x is said to be orthogonal to a non-empty set s

If $x \perp y$ for every y in 's' and the orthogonal complement of 's' denoted by '$s^{\perp}$' is the set of all vectors orthogonal to 's'

ie] $s^{\perp} = \{ y \in s \,/\, x \perp y$ for every $x \in s \}$

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8. Prove that $(T_1 + T)^* = T_1^* + T_2^*$

Let $x, y \in H$

$(x, (T_1 + T_2)^* y) = (T_1 + T_2) x, y$

$= (T_1 x + T_2 x, y)$

$= (T_1 x, y) + (T_2 x, y)$

$= (x, T_1^* y) + (x, T_2^* y)$

$(T_1 + T_2)^* y = T_1^* y + T_2^* y$

9 When $x \to y$ find the value of $\|x+y\|^2$

$\|x+y\|^2$

$\Rightarrow \langle x+y, x+y \rangle + \langle x-y, x-y \rangle$

$\Rightarrow \langle x, x \rangle + \langle x, y \rangle + \langle y, x \rangle + \langle y, y \rangle +$
$\langle x, x \rangle - \langle x, y \rangle - \langle y, x \rangle + \langle y, y \rangle$

$\Rightarrow 2\langle x, x \rangle + 2\langle x, y \rangle - 2\langle y, x \rangle +$
$2\langle y, y \rangle$

$\Rightarrow 2\langle x, x \rangle + 2\langle y, y \rangle$

$\Rightarrow 2[\langle x, x \rangle + \langle y, y \rangle]$

$\Rightarrow 2[\|x+y\|^2]$

10 Normed linear spaces:

A normed linear space is a linear space N in which to each vector x this correspond a real number depoted by $\|x\|$ and norm of x in $\ni$ manner that

i] $\|x\| \geq 0$ & $\|x\| = 0 \Leftrightarrow x = 0$

ii] $\|x+y\| \leq \|x\| + \|y\|$ iii] $\|\alpha x\| = |\alpha| \|x\|$

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A banach space is a complete normed linear space

Unit - III

11 State spectrum theorem:

Let T be a arbitrary linear operator on a H and $\lambda_1, \lambda_2 \ldots \lambda_m$ be the set of complex eigen values of T with eigen space $M_1, M_2 \ldots M_m$ then following statement are equivalent

i] The subspace M_i's are pairwise orthogonal and span H

ii] The projection P_i's are pairwise orthogonal $I = \sum_{j=1}^{n} P_i$, $T = \sum_{i=1}^{M} \lambda_i P_i$

iii] T is normal

12. Define spectrum:

M is the null space of the operator $(T - \lambda I)$ The set of all eigen values of T is called spectrum

15. Prove that G is an open set

Proof:

Let x_0 be an element in G and Let x be an element in A

Then $\| x - x_0 \| < \frac{1}{\| x_0^{-1} \|}$

It is clear that

$$\| x_0^{-1} x^{-1} \| = \| x_0^{-1} (x - x_0) \|$$
$$\leq \| x_0^{-1} \| \, \| x - x_0 \|$$
$$\leq \| x_0^{-1} \| \cdot \frac{1}{\| x_0^{-1} \|}$$
$$\| x_0^{-1} x^{-1} \| \leq 1$$

$x = x_0 x_0^{-1} x$ is in G

It follows that x is also in G

$\therefore$ G is open set

Hence the proved.

14. Complete orthonormal set of vectors in a Hilbert space H

In a Hilbert space H is a non-empty subset of H which consists of mutually orthogonal unit vectors. that is it a non-empty subset $\{e_i\}$ of H

i) $i \neq j \Rightarrow e_i \perp e_j$
ii) $\| e_i \| = 1$ for every i

15. Define a projection on Hilbert space H

A projection on H can be defined an operator P which satisfies the condition $P^2 = P$

$P^* = P$ The operator 0 & I are projection's and they are distinct $\Leftrightarrow$ $H \neq \{0\}$

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Unit - IV

16 $\sigma(x^n) = \sigma(x)^n$

Proof:

Let λ be a non-zero complex number and $\lambda_1, \lambda_2 \ldots \lambda_n$ its distinct n^{th} roots so that $x^n - \lambda_1 = [x - \lambda_1][x - \lambda_2'] \ldots [x - \lambda_n']$

The statement of the lemma follows easily from the fact that $x^n - \lambda'$ is singular $\Leftrightarrow$ $x - \lambda_i I$ is singular for atleast one i

17 Banach .algebra:

A banach algebra is a complex banach space which is also algebra with identity 1 and in which the multiplicative structure is related to the norm by the follow recuirement

[i] $\|xy\| \leq \|x\| \|y\|$

[ii] $\|1\| = 1$

This are called the banach algebra.

18. State and prove the resolvent equation:

Let T be a opeator an a non-trivial hilbert space

The resolvant of x is the function with values of A is defined on $P(x)$ by $x(\lambda) = (x - \lambda_1)^{-1}$

$X(\lambda)$ is continuous function of λ and the fact that $x(\lambda) = \lambda^{-1}(x/\lambda - 1)^{-1}$ for $\lambda \neq 0$

$$\Rightarrow x(\lambda) \to 0 \text{ as } \lambda \to 0$$

If λ and μ are both in $e(x)$

$$x(\lambda) = x(\lambda)(x - \mu 1) x(\mu)$$

$$= x(\lambda)[x - \lambda_1 + (\lambda - \mu_1)x(\mu)]$$

$$= 1 + (\lambda - \mu 1)x(\lambda)] x/\mu$$

$$= x(\mu) + (\lambda - \mu) x(\lambda) x(\mu)$$

$$x(\lambda) = x(\mu) + (\lambda - \mu) x(\lambda) x(\mu)$$

So this relation is called the resovant equation.

39 When do you say an operator U on H is unitary

An operator U on H which satisfies the equation $UU^* = U^*U = I$ is said to be unitary operator

Unit - V

20 If x is a normal element in a B^*- algebra then $\|x^2\| = \|x\|^2$

Proof:

It is obvious that $\|x^2\| \leq \|x\|^2 \rightarrow (1)$
The inequality in one other direction is a consequence

$$\|x^*\|^2 \|x\|^2 = (\|x^*\| \, \|x\|)^2$$
$$= \|x^* x\|^2 = \|(x^* x)^* x^* x\|$$
$$= \|x^* x x^* x\|$$
$$= \|x^*(x x^*) x\|$$
$$= \|x^* x^* x x\|$$
$$= \|(x^*)^2 x^2\|$$
$$\|x^*\|^2 \|x\|^2 = \|(x^*)\|^2 \|x^2\|$$
$$\leq \|x^*\|^2 \|x^2\|$$
$$\|x\|^2 \leq \|x^2\| \rightarrow (2)$$

from (1) & (2) we get

$$\|x^2\| = \|x\|^2$$

Hence the proved.

Q1 State Gelfand Neumark theorem

If A is a commutative B^*-algebra then the Gelfand mapping $x \to \hat{x}$ is an isometric $*$ isomorphism of A onto the commutative B^*-algebra $\mathcal{C}(M)$

Q2 Gelfand mapping:

If M is a maximal ideal in A then the Banach algebra A/M is a division algebra and therefore equals the Banach algebra $\mathbb{C}$ of complex numbers.

The natural homomorphism $x \to x+M$ of A onto $A/M = \mathbb{C}$ assigns to each element x in A a complex number $x(M)$ defined by $x(M) = x+M$ and the mapping $x \to x(M)$ has the following properties.

i] $(x+y)(M) = x(M) + y(M)$

ii] $(\alpha x)(M) = \alpha x(M)$

iii] $(xy)(M) = x(M)\, y(M)$

iv] $x(M) = 0 \Leftrightarrow x \in M$

v] $1(M) = 1$

vi] $|x(M)| \leq \|x\|$

23. Multiplicative function:

M is maximal for proper ideal I then $f(I)$ could be a not trivial ideal in C

Since f and f_M are multiplicative function with the same null space

$\therefore f = f_M$

24 Define the resolvent of an element $x \in A$

Let f be a fundamental on H an element of the conjugate space A^* and define $F(\lambda)$ by

$$f(\lambda) = f[x(\lambda)]$$

now $f(\lambda)$ is a complex function which is define and continuous on the resolvant element of $e(x)$

25. Define $\sigma(x)$ for $x \in A$ an algebra

If X has a only one point Then $\sigma(x)$ can be identity with the simplest of all banach algebra the algebra of complex number.