Advanced Probability Theory.

2-Mark.

1. Define Borel field and ex.

The Borel field is the minimal field containing any one of the following class

$\mathcal{C}_2 = \{(-\infty, x], x \in R\}$

$\mathcal{C}_3 = \{(a,b], a<b, a,b \in R\}$

$\mathcal{C}_4 = \{(a,b], a<b, a,b \in R\}$

$\mathcal{C}_5 = \{[a,b), a<b, a,b \in R\}$

$\mathcal{C}_6 = \{[x,\infty), x \in R\}$ etc

Thus $\mathcal{B}$ contains all subsets of R of the above form & their complements countable union & intersection.

However $\mathcal{B}$ is not the class of all subset of R.

The value $X(\omega)$ may be observal while ω is Ω.

2. X is a random variable iff $X^{-1}(\mathcal{C}) \subset \mathcal{A}$ where $\mathcal{C}$ is any class of subsets of R which generator B.

we've to P.T

$X^{-1}(\mathcal{C}) \subset \mathcal{A} \Rightarrow X^{-1}(\mathcal{B}) \subset \mathcal{A} \longrightarrow (1)$

$\therefore \mathcal{C} \subset \mathcal{B}$ & $X^{-1}(\mathcal{B}) \subset \mathcal{A}$

$X^{-1}(\mathcal{C}) \subset \mathcal{A}$

The prove the converse since A is a σ- field. &

$X^{-1}(\mathcal{E}) \subset A \iff \sigma(X^{-1}(\mathcal{E})) \subset A$

$\iff X^{-1}(\sigma(\mathcal{E})) \subset A$

$\iff X^{-1}(B) \subset A$

H.T.P.

③. Elementary function.

A contable linear combination of in fun/- is called an elementary function.

④. minimum field.

consider an arbitrary class $\mathcal{E}$ of sets the smallest field containing $\mathcal{E}$ is called the minimal field containing $\mathcal{E}$ (or) the field generated by $\mathcal{E}$.

⑤ with the usual notations prove that $P(A^C) = 1 - P(A)$

since $A + A^C = \Omega$

$P(\Omega) = P(A) + P(A^C)$

$P(\Omega) = 1$

Hence $P(A^C) = 1 - P(A) \longrightarrow ①$

If $A \subset B$, $B = A + BA^C$ and hence from

$P(B) = P(A) + P(BA^C)$

⑥ Define the hypergeometric distribution:.

From a lot containing n items suppose r items are chosen. This may be done in ways.

The no. of defectives among the r items chosen may the k $(=0,1,2\ldots n)$ Assuming that the $\binom{n}{r}$ outcomes.

The sample is $\binom{n_1}{k}\binom{n_2}{r-k}$ where $n_2 = n-n_1$ is the no. of non-defective items.

$\binom{n_1}{k}$ ways and the remaining $(r-k)$ non-defectives can be chosen from the n_2.

7. If $A \subset B$ then prove that $P(B) \geq P(A)$.

$P(BA^c) \geq 0$

$\therefore A \subset B \Rightarrow P(B) \geq P(A)$ —— (2)

Establishing monotonicity of P.

$P(BA^c) = P(B-A)$

$= P(B) - P(A)$.

H.T.P.

8. Empirical distribution function.

We have stated that if k values of x are less than to x $(X \leq x)$ out of n observed values 'n' then k/n is estimated of the D.F of X.

9. Distribution function:

Let X be real random variable Probabilitic space.

$(\Omega, -, P)$ for $x \in R$.

i) $P[X \leq x] = F_X(x)$

ii) $P_X(a,b) = P(a < b \leq b)$

$F_X(b) - F_X(a)$ $(b > a)$

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Then $F_X(x)$ called the distribution function.

⑩. Separating dense subset dense.

A set D is said to be dense in R if any point of R is either in D or is a limit point of D.

⑪. k variable (or) multi variable distribution function.

Suppose $X_1, X_2 \dots X_k$ is a k-dimensional vector random variable on Ω to R^k then sets of form.

$$[-\infty < x_1 \le x_1, -\infty, x_2 \le x_2 \dots -\infty < x_k \le X_k]$$

$$= \bigcap_{i=1}^{k} [-\infty < X_i \le x_i] \longrightarrow ①$$

'pr' of ① denoted by $F(x_1 \dots x_k)$ and is called the distribution function $(x_1, x_2 \dots X_k)$

$$0 = [F(-\infty, x_2 \dots x_k)]$$

$$= F(-\infty, x_3 \dots x_k)$$

$$F(-\infty, X_k \dots X_{k-1}, \infty)] \longrightarrow ②$$

$$1 = F(\infty \dots \infty)$$

$F(x_1 \dots X_k)$ to be a d.f of the k-dimensional vector r.v.

⑫. C_r - inequality.

$$E|X+Y|^r \le C_r E|X|^r + C_r E|Y|^r$$

$$C_r = 1 \text{ if } r \le 1$$

$$= 2^{r-1} \text{ if } r \ge 1$$

(13) The Lebesgue - Stieltjes integral of X on R.

If Fx possesses a derivative f(x) then

$$Ex = \int_0^\infty x f(x) dx - \int_{-\infty}^0 (-x) f(x) dx$$

where the integral is the usual Rieman integral.

(14) If z is a complete r.v $|Ez| \le E|z|$

let $z = \rho e^{i\alpha}$ $\rho\ \alpha$ being random

let $Ez = re^{it}$ then

$$|Ez| = r = e^{-it} E(\rho e^{i\alpha})$$

$$= E\rho e^{i(\alpha - t)}$$

$$= E\rho \cos(\alpha - t) + iE\rho \sin(\alpha - t)$$

$$= E(\rho \cos(\alpha - t)) \quad \because \text{LHS is real}$$

$$\le E\rho = E|z|$$

(15) If z is complex z is integrable iff |z| is integrable.

From Lemma

$$|Ez| \le E|z|$$

This implies that |z| is integrable => z is integrable

conversely let z be integrable.

$$|z| = \sqrt{x^2 + y^2} \le |x| + |y|$$

$$E|z| \le E(|x| \times |y|)$$

Since this implies that |x| and |y| are integral

$E|z| < \infty$ If z is integrable.

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(16). Holder's Inequality.

$E|XY| \le E^{1/r}|X|^r \, E^{1/s}|Y|^s$ where $r>1$ and $r^{-1}+s^{-1}=1$

~~were~~ and also where $E^{1/p}|X|^r$ denotes $\sqrt[p]{E|X|^r}$.

(17). If $X_n \xrightarrow{P} X$ then (P.T) $aX_n \xrightarrow{P} aX$

IF $a=0$ the result is trivially true

suppose $a \ne 0$ For every $\varepsilon > 0$

$P[|aX_n - aX| \ge \varepsilon] = P[|a||X_n - X| \ge \varepsilon]$.

$= P\left[|X_n - X| \ge \frac{\varepsilon}{|a|}\right] \to 0$

as $n \to \infty$

Hence (i) satisfied.

(18) convergence in mean (or) mean-square.

A sequence of r.v's $\{X_n\}$ is said to converg to X in the r^{th} mean denoted as $X_n \xrightarrow{r} X$ if $E|X_n - X|^r \to 0$ as $n \to \infty$

for $r=2$ it is called convergence in quadratic mean or mean square and for $r=1$ convergence in the first mean.

(19). L-space.

consider the space of all r.v's such that $E|X|^r < \infty$.

This is called the L-space.

(20). Define a random variable with an example.

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Suppose Ω is the sample space with sample points ω, we are interested in a value $X(\omega)$ associated with ω, & not in ω itself.

The value $X(\omega)$ may be observal while ω is Ω.